REMARKS ON RANK-ONE CONVEXITY AND QUASICONVEXITY

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Remarks on rank-one convexity and quasiconvexity

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1. Introduction

The purpose of this paper is to make some remarks connected with the open question of finding verifiable hypotheses that are necessary and sufficient for the weak lower semicontinuity of multiple integrals in the calculus of variations.

We consider integrals of the form

$$I(u) = \int_\Omega f(Du(x))\,dx\ , \tag{1.1}$$

where $\Omega \subset \mathbb{R}^m$ is bounded and open, $u:\Omega \longrightarrow \mathbb{R}^n$, and $f:M^{n\times m} \longrightarrow \mathbb{R}$ is continuous. Here $M^{n\times m} \cong \mathbb{R}^{nm}$ denotes the space of real $n\times m$ matrices. It is well known (Morrey [7]) that I is sequentially weak* lower semicontinuous in the Sobolev space $W^{1,\infty}(\Omega;\mathbb{R}^m)$ if and only if f is *quasiconvex*, that is

$$\int_E f(Du(x))\,dx \geq \int_E f(A)\,dx = \text{meas}(E)\cdot f(A) \tag{1.2}$$

for every (or any one fixed) bounded open subset $E \subset \mathbb{R}^m$, every $A \in M^{n\times m}$, and every $u \in Ax + W_0^{1,\infty}(E;\mathbb{R}^n)$. Furthermore, quasiconvexity of f is sufficient for I to be sequentially weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^n)$, $p \geq 1$, if f satisfies the growth condition

$$0 \leq f(A) \leq C(1 + |A|^p)\ , \text{ for all } A \in M^{n\times m} \tag{1.3}$$

for some constant C (Acerbi & Fusco [1]; see also Ball & Zhang [5]). Quasiconvexity of f is known to imply that f is *rank-one convex*, i.e.

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B) \tag{1.4}$$

whenever $\lambda \in [0,1]$ and $A,B \in M^{n\times m}$ with $\text{rank}(A - B) = 1$. For $f \in C^2$ rank-one convexity is equivalent to the *Legendre-Hadamard* (or *ellipticity*) condition

$$\frac{\partial^2 f(A)}{\partial A_{i\alpha}\partial A_{j\beta}} a_i n_\alpha a_j n_\beta \geq 0\ , \text{ for all } A \in M^{n\times m},\ a \in \mathbb{R}^n,\ n \in \mathbb{R}^m. \tag{1.5}$$

The pointwise form of rank-one convexity contrasts with that of the quasiconvexity condition (1.2), which appears to be far less tractable. It was suggested by Morrey [8 p 122] that rank-one convexity might in fact be equivalent to quasiconvexity, but despite strenuous efforts no-one has succeeded in proving or disproving this.

Attempts to attack the problem of whether rank-one convexity implies quasiconvexity generally fall into two categories. Either the problem is studied for a special class of integrands f, or the class of admissible mappings u in (1.2) is restricted. Here we make remarks of both types. In Section 2 we discuss for $m=n=2$ the case of separately convex integrands $f = f(A_{11}, A_{22})$ and certain modifications of these, while in Section 3 we consider (following Sivaloganathan [9]) admissible mappings u of the form $u(x) = U(\vartheta(x), x)$, where $U(t,\cdot)$ is a one-parameter family of solutions to the the Euler-Lagrange equations for I.

For a review of the 'does rank-one convexity imply quasiconvexity' problem the reader is referred to Ball [2]. A more recent paper (Ball [4]) contains further references, together with a counterexample to Morrey's conjecture involving an integrand that takes infinite values except on a small set of matrices.

2. Separately convex integrands

Let $H(w,z)$ be a continuous, real-valued, separately convex function. Let $w^{(j)} \overset{*}{\longrightarrow} w$, $z^{(j)} \overset{*}{\longrightarrow} z$ in $L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is bounded, open and strongly Lipschitz, and suppose further that

$$\partial w^{(j)}/\partial x_2 \text{ and } \partial z^{(j)}/\partial x_1 \text{ are relatively compact in } H^{-1}(\Omega). \tag{2.1}$$

Assume also that an appropriate subsequence of $(w^{(j)}, z^{(j)})$, again denoted by j, has been extracted so that the corresponding Young measure $(\nu_x)_{x\in\Omega}$ exists, i.e. so that

$$g(w^{(j)}, z^{(j)}) \overset{*}{\longrightarrow} \langle \nu_x, g\rangle = \int_{\mathbb{R}^2} g(\lambda_1, \lambda_2)\, d\nu_x(\lambda_1, \lambda_2) \quad \text{in } L^\infty(\Omega) \tag{2.2}$$

for every continuous $g:\mathbb{R}^2 \longrightarrow \mathbb{R}$. For the definition and properties of the Young measure see, for example, Tartar [12], Ball [3]. In the above situation, ν_x is a probability measure on $\mathbb{R}^2$, for almost every $x \in \Omega$, whose support is contained in a compact set of $\mathbb{R}^2$ which is independent of x.

We investigate the following questions:

(*a*) Is the Young measure $(\nu_x)_{x\in\Omega}$ associated with $(w^{(j)}, z^{(j)})$ a tensor product?

(*b*) Is it true that

$$\int_\Omega H(w,z)\, dx \le \liminf_{j\to\infty} \int_\Omega H(w^{(j)}, z^{(j)})\, dx \; ? \tag{2.3}$$

The connection of these questions with the 'does rank-one convexity imply quasiconvexity' problem is the following: consider a sequence $u^{(j)} = (u_1^{(j)}, u_2^{(j)})$ such that

$$u^{(j)} \overset{*}{\longrightarrow} u \quad \text{in } W^{1,\infty}(\Omega;\mathbb{R}^2), \tag{2.4}$$

with

$$\partial u_1^{(j)}/\partial x_2 \longrightarrow 0, \quad \partial u_2^{(j)}/\partial x_1 \longrightarrow 0 \quad \text{a.e. in } \Omega, \tag{2.5}$$

or more generally with

$$\partial u_1^{(j)}/\partial x_2 \text{ and } \partial u_2^{(j)}/\partial x_1 \text{ relatively compact in } L^2(\Omega), \tag{2.6}$$

and define

$$w^{(j)} = \partial u_1^{(j)}/\partial x_1, \quad z^{(j)} = \partial u_2^{(j)}/\partial x_2. \tag{2.7}$$

In this case (2.1) holds. We thus investigate here the problem of the sequential weak* lower semicontinuity of I defined by (1.1) for the special class of integrands $f(Du) = H(\partial u_1/\partial x_1, \partial u_2/\partial x_2)$ with H separately convex, restricting further our attention to sequences satisfying (2.5) or (2.6).

To say that ν_x is a tensor product means that $\nu_x = \mu_x^1 \otimes \mu_x^2$ for a.e. $x \in \Omega$, where μ_x^1, μ_x^2 are probability measures on $\mathbb{R}$; equivalently,

$$\phi(w^{(j)})\psi(z^{(j)}) \overset{*}{\longrightarrow} \langle \mu_x^1, \phi\rangle\langle \mu_x^2, \psi\rangle \quad \text{in } L^\infty(\Omega) \tag{2.8}$$

for each pair $\phi, \psi \in C(\mathbb{R})$.

If condition (2.1) is replaced by the stronger condition that

$$\partial w^{(j)}/\partial x_2 \text{ and } \partial z^{(j)}/\partial x_1 \text{ are bounded in } L^2(\Omega), \tag{2.9}$$

(this condition takes the form that $\partial^2 u_1^{(j)}/\partial x_1 \partial x_2$ and $\partial^2 u_2^{(j)}/\partial x_1 \partial x_2$ are bounded in $L^2(\Omega)$ in the setting (2.4),(2.7) described above) then the answers to both (a), (b) are positive (Tartar [11]). In this case (b) follows directly from (a) and from

$$w(x) = \int_{\mathbb{R}^2} \lambda_1\, d\nu_x(\lambda_1, \lambda_2), \quad z(x) = \int_{\mathbb{R}^2} \lambda_2\, d\nu_x(\lambda_1, \lambda_2)$$

using Fubini's theorem and Jensen's inequality twice (first in λ_2 for λ_1 fixed and then in λ_1). However, Example 2.1 below shows that in general (a) is false.

Example 2.1

Let $m = n = 2$ and

$$u_1^{(j)} = j^{-2} f(j^2 x_1, j x_2)\ , \quad u_2^{(j)} = j^{-3} g(j^2 x_1, j^3 x_2)\ , \tag{2.10}$$

where

$$f(r,s) = \sin r \cos s, \quad g(r,s) = \cos r \sin s\ . \tag{2.11}$$

Then $u^{(j)} \overset{*}{\longrightarrow} 0$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$ and (2.5) holds since $|\partial u_1^{(j)}/\partial x_2| \le j^{-1}$ and $|\partial u_2^{(j)}/\partial x_1| \le j^{-1}$. It follows that (2.1) holds for $w^{(j)}$ and $z^{(j)}$ defined by (2.7). We choose $\phi(\lambda_1) = (\lambda_1)^2$, $\psi(\lambda_2) = (\lambda_2)^2$. Simple calculations show that

$$\phi(\partial u_1^{(j)}/\partial x_1) = \cos^2(j^2x_1)\,\cos^2(jx_2) \xrightarrow{\;*\;} \tfrac{1}{4}\,, \tag{2.12}$$

$$\psi(\partial u_2^{(j)}/\partial x_2) = \cos^2(j^2x_1)\,\cos^2(j^3x_2) \xrightarrow{\;*\;} \tfrac{1}{4}\,, \tag{2.13}$$

$$\phi(\partial u_1^{(j)}/\partial x_1)\psi(\partial u_2^{(j)}/\partial x_2) =$$

$$= \cos^4(j^2x_1)\,\cos^2(jx_2)\,\cos^2(j^3x_2) \xrightarrow{\;*\;} \tfrac{3}{32} \neq \tfrac{1}{4}\cdot\tfrac{1}{4} \tag{2.14}$$

in $L^\infty(\Omega)$, so that by (2.8) $(\nu_x)_{x\in\Omega}$ is not a tensor product.

More generally, we can take $f = f(r,t)$, $g = g(r,s)$ in Example 2.1 to be any smooth functions with Df, Dg periodic with respect to $(0,R)\times(0,T)$ and $(0,R)\times(0,S)$ respectively. Then (2.5) is satisfied and

$$\partial u_1^{(j)}/\partial x_1 = f_r(j^2x_1, jx_2)\,, \qquad \partial u_2^{(j)}/\partial x_2 = g_s(j^2x_1, j^3x_2)\,,$$

so that (see Lemma A.2 of the appendix)

$$\partial u_1^{(j)}/\partial x_1 \xrightarrow{\;*\;} \frac{1}{RT}\int_0^R\!\!\int_0^T f_r(r,t)\,dr\,dt \quad \text{in } L^\infty(\Omega), \tag{2.15}$$

$$\partial u_2^{(j)}/\partial x_2 \xrightarrow{\;*\;} \frac{1}{RS}\int_0^R\!\!\int_0^S g_s(r,s)\,dr\,ds \quad \text{in } L^\infty(\Omega), \tag{2.16}$$

and

$$H(\partial u_1^{(j)}/\partial x_1, \partial u_2^{(j)}/\partial x_2) = H(f_r(j^2x_1, jx_2), g_s(j^2x_1, j^3x_2))$$

$$\xrightarrow{\;*\;} \frac{1}{RST}\int_0^R\!\!\int_0^S\!\!\int_0^T H(f_r(r,t), g_s(r,s))\,dr\,ds\,dt \quad \text{in } L^\infty(\Omega) \tag{2.17}$$

for any smooth function $H = H(w,z)$. In particular

$$\phi(\partial u_1^{(j)}/\partial x_1)\psi(\partial u_2^{(j)}/\partial x_2) \xrightarrow{\;*\;} \frac{1}{RST}\int_0^R\!\!\int_0^S\!\!\int_0^T \phi(f_r(r,t))\psi(g_s(r,s))\,dr\,ds\,dt \quad \text{in } L^\infty(\Omega). \tag{2.18}$$

For $(\nu_x)_{x\in\Omega}$ to be a tensor product the right-hand side of (2.18) would have to equal

$$\frac{1}{RT}\int_0^R\!\!\int_0^T \phi(f_r(r,t))\,dr\,dt\cdot\frac{1}{RS}\int_0^R\!\!\int_0^S \psi(g_s(r,s))\,dr\,ds\,. \tag{2.19}$$

Choosing as before $\phi(\lambda_1) = (\lambda_1)^2$, $\psi(\lambda_2) = (\lambda_2)^2$ and f,g with supports having disjoint projections on $(0,R)$ we see that both integrals in (2.19) are in general nonzero while the integral in (2.18) is zero. This again proves that $(\nu_x)_{x\in\Omega}$ is not a tensor product.

We now turn to question (b). The special case corresponding to (2.4),(2.6),(2.7) amounts to asking:

$(b)'$ If $u^{(j)} \xrightarrow{\;*\;} u$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$, with $\partial u_1^{(j)}/\partial x_2$ and $\partial u_2^{(j)}/\partial x_1$ relatively compact in $L^2(\Omega)$, does the lower semicontinuity property

$$\int_\Omega H(\partial u_1/\partial x_1, \partial u_2/\partial x_2)\,dx \le \liminf_{j\to\infty} \int_\Omega H(\partial u_1^{(j)}/\partial x_1, \partial u_2^{(j)}/\partial x_2)\,dx \tag{2.20}$$

hold for any continuous separately convex function H?

It is in some sense natural to search for a counterexample to $(b)'$ of the type (2.10) with Df, Dg periodic with respect to $(0,R)\times(0,T)$ and $(0,R)\times(0,S)$ respectively. In this case, we see from (2.15)-(2.17) that (2.20) becomes

$$H\left(\frac{1}{RT}\int_0^R\!\!\int_0^T f_r(r,t)\,dr\,dt, \frac{1}{RS}\int_0^R\!\!\int_0^S g_s(r,s)\,dr\,ds\right) \le$$

$$\frac{1}{RTS}\int_0^R\!\!\int_0^T\!\!\int_0^S H(f_r(r,t), g_s(r,s))\,dr\,ds\,dt\,. \tag{2.21}$$

But we will prove that (2.21) holds true, so that this idea does not furnish a counterexample. Indeed, for any fixed r and t Jensen's inequality yields

$$H(f_r(r,t), \frac{1}{S}\int_0^S g_s(r,s)\,ds) \le \frac{1}{S}\int_0^S H(f_r(r,t), g_s(r,s))\,ds. \tag{2.22}$$

Since Dg is a periodic function, the function G defined by $G(r,s) = g(r,s) - ar - bs$ with

$$a = \frac{1}{RS}\int_0^R\!\!\int_0^S g_r(r,s)\,dr\,ds, \quad b = \frac{1}{RS}\int_0^R\!\!\int_0^S g_s(r,s)\,dr\,ds$$

is a periodic function. Thus

$$\frac{1}{S}\int_0^S g_s(r,s)\,ds = \frac{1}{S}\,[g(r,S) - g(r,0)]$$

$$= \frac{1}{S}\,[G(r,S) + ar + bS - G(r,0) - ar] = b, \tag{2.23}$$

and (2.22), (2.23) yield

$$\frac{1}{RT}\int_0^R\!\!\int_0^T H(f_r(r,t), b)\,dr\,dt \le \frac{1}{RTS}\int_0^R\!\!\int_0^T\!\!\int_0^S H(f_r(r,t), g_s(r,s))\,dr\,ds\,dt\,.$$

Applying Jensen's inequality to the left-hand side and using (2.23) proves (2.21).

In fact, for every 'natural' modification of the sequence (2.10) that we have considered, involving oscillations with a finite (or in some cases infinite) number of scales, it always appears possible to apply Jensen's inequality to establish the analogue of (2.21). As far as we are aware both (b) and $(b)'$ are still open questions. In particular, there are many simple integrands H that can be written down for which the answers to (b) and (b)' are not known. Examples include

$$H(w,z) = w^+z^+, \tag{2.24}$$

where $w^+ = \max(w,0)$ (cf. Tartar [13]), and

$$H(w,z) = |w|^\alpha |z|^\beta, \quad \alpha\neq\beta, \ \alpha,\beta\geq 1. \tag{2.25}$$

These integrands are separately convex but are not *polyconvex*, in the sense that they cannot be written in the form

$$H(w,z) = h(w,z,wz), \tag{2.26}$$

where h is convex. For both (2.24) and (2.25) the latter statement is not hard to prove by applying the assumed convexity of h to the convex combination

$$\left(\tfrac{1}{3}(1+\tau), \tau/(2\tau-1), \tfrac{1}{3}(1+\tau)\tau/(2\tau-1)\right) =$$
$$= \tfrac{1}{3}\Big((1,0,0) + (0,1,0) + (\tau,(1+\tau)/(2\tau-1),(1+\tau)\tau/(2\tau-1)\Big) \tag{2.27}$$

with $-1 < \tau < 0$ and τ tending to 0 or -1. If H *is* polyconvex, then the lower semicontinuity property (b) holds by virtue of the convexity of h and the sequential weak continuity of wz under the differential constraints (2.1) (cf. Tartar [11]).

Motivated by the work described here, Šverák [10] has remarked that if H is C^2 and separately convex and if $M_0 = \frac{1}{2}\sup |H_{wz}| < \infty$, then

$$f(Du) = H(\partial u_1/\partial x_1, \partial u_2/\partial x_2) + M\,[(\partial u_1/\partial x_2)^2 + (\partial u_2/\partial x_1)^2] \tag{2.28}$$

is rank-one convex when $M \geq M_0$. Approximating t^+ by a smooth convex function T_ε satisfying

$$T_\varepsilon(t) = 0 \text{ if } t \leq 0, \quad T_\varepsilon(t) = t - \varepsilon/2 \text{ if } t \geq \varepsilon,$$

we deduce in particular that (cf. (2.24))

$$f(Du) = (\partial u_1/\partial x_1)^+(\partial u_2/\partial x_2)^+ + M\,[(\partial u_1/\partial x_2)^2 + (\partial u_2/\partial x_1)^2] \tag{2.29}$$

is rank-one convex for $M \geq \frac{1}{2}$, furnishing a particularly simple example of a rank-one convex function that is not known to be quasiconvex.

Finally we remark that Ball & Zhang [5] have used sequences similar to (2.10) to construct an example of a bounded sequence $u^{(j)}$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, such that the Young measure $(\nu_x)_{x\in\Omega}$ corresponding to $Du^{(j)}$ is independent of x but is not realisable as the Young measure of any sequence $D\phi(jx)$ corresponding to a mapping $\phi \in W^{1,1}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ whose gradient is periodic with respect to $(0,R)\times(0,S)$.

3. On a result of Sivaloganathan

In this section we simplify the proof of an interesting result of Sivaloganathan [9] which shows in particular that rank-one convexity of f implies the quasiconvexity condition (1.2) for a suitably restricted class of mappings u.

We consider for a moment integrals of the form

$$J(u) = \int_\Omega F(x,u,Du)\,dx, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^m$ is bounded, open and strongly Lipschitz, $u:\Omega \longrightarrow \mathbb{R}^n$, and F: $\bar{\Omega}\times\mathbb{R}^n\times M^{n\times m} \longrightarrow \mathbb{R}$ is C^2. We suppose we are given a one-parameter family

$$u_t(x) = U(t,x), \quad t \in E, \ E \text{ an interval of } \mathbb{R} \tag{3.2}$$

of solutions $u = u_t$ to the Euler-Lagrange equations

$$\operatorname{div} D_A F(x,u,Du) = D_u F(x,u,Du) \tag{3.3}$$

corresponding to (3.1). Here $D_A F, D_u F$ denote the derivatives of $F(x,u,A)$ with respect to A,u respectively. We suppose that

$$U,\ D_t U,\ D_x U,\ D^2_{xx} U,\ D^2_{xt} U \text{ are continuous and bounded on } E \times \bar{\Omega}.$$

Given a function $\vartheta:\bar{\Omega}\longrightarrow E$ which is continuously differentiable on $\bar{\Omega}$ with $\vartheta|_{\partial\Omega} = \lambda$, λ constant, we consider the mapping $u_\vartheta(x) = U(\vartheta(x),x)$ obtained by replacing the parameter t by the function $\vartheta(\cdot)$. Similarly, we have $u_\lambda(x) = U(\lambda,x)$.

Theorem 3.1 (Sivaloganathan [9])
Let $F(x,u,\cdot)$ be rank-one convex. Then $J(u_\vartheta) \geq J(u_\lambda)$.

Proof.

First note that

$$Du_\vartheta(x) = D_x U(\vartheta,x) + D_t U(\vartheta,x) \otimes D\vartheta(x), \tag{3.4}$$

so that by the rank-one convexity of $F(x,u,\cdot)$

$$F(x,u_\vartheta,Du_\vartheta) - F(x,u_\lambda,Du_\lambda) \geq h(x), \tag{3.5}$$

where

$$h(x) = F(x,u_\vartheta,D_x U(\vartheta,x)) + D_A F(x,u_\vartheta,D_x U(\vartheta,x)) : D_t U(\vartheta,x) \otimes D\vartheta - F(x,u_\lambda,Du_\lambda). \tag{3.6}$$

We claim that

$$h(x) = \operatorname{div}\left(\int_\lambda^{\vartheta(x)} [D_A F(x,U(t,x),D_x U(t,x))]^T D_t U(t,x)\,dt\right). \tag{3.7}$$

Indeed since F, U and ϑ are assumed to be sufficiently smooth, the right-hand side of (3.7) equals

$$\int_{\lambda}^{\vartheta(x)} \Big[D_t U(t,x)\cdot \mathrm{div}\, [D_A F(x,U(t,x),D_x U(t,x))] +$$

$$+ D_A F(x,U(t,x),D_x U(t,x)) : D^2_{tx} U(t,x) \Big] dt +$$

$$+ D_A F(x,U(\vartheta,x),D_x U(\vartheta,x)) : D_t U(\vartheta,x) \otimes D\vartheta,$$

and on using the Euler-Lagrange equation for $U(t,x)$ this equals

$$\int_{\lambda}^{\vartheta(x)} D_t \Big[F(x,U(t,x),D_x U(t,x)) \Big] \, dt + D_A F(x,U(\vartheta,x),D_x U(\vartheta,x)) : D_t U(\vartheta,x) \otimes D\vartheta$$

which is nothing but $h(x)$ defined by (3.6).

The theorem then follows from (3.5) since $\int_\Omega h(x)\, dx = 0$ in view of (3.7) and the boundary condition $\vartheta|_{\partial\Omega} = \lambda$.

□

The above proof follows the same pattern as that of Sivaloganathan, except that he proves that $\int_\Omega h(x)\, dx = 0$ by showing that h, considered as a function of $(x,\vartheta,D\vartheta)$, is a null Lagrangian, whereas we explicitly exhibit the corresponding divergence.

As remarked by Sivaloganathan, a special case of the above result is that rank-one convexity of $f = f(Du)$ implies that the quasiconvexity condition (1.2) holds for radial mappings $u: B(0,1) \longrightarrow \mathbb{R}^n$, where $B(0,1) = \{x \in \mathbb{R}^n : |x|<1\}$. (See also Ball [2].) A radial mapping u is a mapping of the form

$$u(x) = \frac{r(R)}{R} x, \tag{3.8}$$

for some function $r: [0,1] \longrightarrow [0,\infty)$, where $R = |x|$. Indeed, setting $F(x,u,A) = f(A)$ and $U(t,x) = tx$ in Theorem 3.1 we obtain

$$\int_{B(0,1)} f(D(\vartheta(x)x))\, dx \geq \int_{B(0,1)} f(\lambda 1)\, dx . \tag{3.9}$$

Choosing $\vartheta(x) = \frac{r(R)}{R}$ to be smooth in x, with r satisfying $r(0) = 0$, $r(1) = \lambda$, it follows from (3.9) that the quasiconvexity condition (1.2) holds for smooth radial mappings u. An easy approximation argument then shows that (1.2) holds for general Lipschitz radial mappings u with $u_{|\partial\Omega} = \lambda x$ and continuous rank-one integrands f.

Theorem 3.1 contains a classical result of the field theory of the one-dimensional calculus of variations (see, for example, Bolza [6]) to the effect that a smooth solution u of the Euler-Lagrange equation furnishes a strong relative minimizer of I provided it can be embedded in a field of extremals. The parameter ϑ then corresponds to the unique element of the field passing through the point $(x,v(x))$, where v is an arbitrary function satisfying the same boundary conditions as u and whose graph lies in the region covered by the field.

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Appendix. On periodic functions whose arguments oscillate with different speeds.

Lemma A.1

Let $f(r,s)$ be a bounded real-valued function on $\mathbb{R}^2$, which is periodic with respect to $(0,R)\times(0,S)$, is measurable in s for each r, and satisfies

$$|f(r,s) - f(r',s)| \le \omega(|r - r'|), \tag{A.1}$$

for all r, r' and a.e. s, where ω is a continuous non-decreasing function with $\omega(0) = 0$. Consider for $j = 1,2,\dots$ and $\vartheta > 0$ the function f_j defined on $\mathbb{R}$ by

$$f_j(x) = f(jx, j^{1+\vartheta}x) . \tag{A.2}$$

Then f_j belongs to $L^\infty(\mathbb{R})$ and satisfies

$$f_j \xrightarrow{\ *\ } \frac{1}{RS}\int_0^R\int_0^S f(r,s)\ dr\ ds \quad \text{in } L^\infty(\mathbb{R}) . \tag{A.3}$$

Proof.

Since f is of Carathéodory type, the function $y \longmapsto f(j^{-\vartheta}y, y)$ is measurable on $\mathbb{R}$. This proves that f_j belongs to $L^\infty(\mathbb{R})$.

In order to prove (A.3) it is sufficient to show that

$$X_j \overset{\text{def}}{=} \int_a^b f_j(x)\ dx \longrightarrow \frac{b-a}{RS}\int_0^R\int_0^S f(r,s)\ dr\ ds \tag{A.4}$$

for any $a < b$, since the step functions are dense in $L^1(\mathbb{R})$.

Splitting (a,b) into intervals $I_{n,j}$ of length $S/j^{1+\vartheta}$, defined for $n = 1,2,\dots$ by

$$I_{n,j} = \{x \in \mathbb{R}:\ nS/j^{1+\vartheta} \le x < (n+1)S/j^{1+\vartheta}\},$$

we obtain

$$X_j = \sum_n \int_{I_{n,j}} f(jx, j^{1+\vartheta}x)\ dx + 0_1(1/j^{1+\vartheta}), \tag{A.5}$$

where denoting by $E(t)$ the integer part of t, i.e. $E(t) = \max\,\{n \in \mathbb{Z} : n \le t\}$, the summation in n is understood as ranging in the interval

$$E(aj^{1+\vartheta}/S) \le n \le E(bj^{1+\vartheta}/S), \tag{A.6}$$

and where the remainder $0_1(1/j^{1+\vartheta})$ satisfies

$$|0_1(1/j^{1+\vartheta})| \le 2S/j^{1+\vartheta}\|f\|_{L^\infty}. \tag{A.7}$$

Using the continuity property (A.1) of f we have

$$\int_{I_{n,j}} f(jx, j^{1+\vartheta}x)\ dx = \int_{I_{n,j}} f(jnS/j^{1+\vartheta}, j^{1+\vartheta}x)\ dx + 0_2^{n,j} \tag{A.8}$$

with

$$|0_2^{n,j}| \le (S/j^{1+\vartheta})\cdot\omega(S/j^{1+\vartheta}) . \tag{A.9}$$

Let

$$0_2(\omega(S/j^{1+\vartheta})) = \sum_n 0_2^{n,j} , \tag{A.10}$$

$$\hat{f}(r) = \frac{1}{S}\int_0^S f(r,s)\ ds . \tag{A.11}$$

Performing in the right-hand side of (A.8) the change of variable $y = j^{1+\vartheta}x - nS$, we obtain from (A.5),(A.8),(A.9), since f is periodic in s, that

$$X_j = \sum_n S/j^{1+\vartheta}\hat{f}(jnS/j^{1+\vartheta}) + 0_2(\omega(S/j^{1+\vartheta})) + 0_1(1/j^{1+\vartheta}) , \tag{A.12}$$

where

$$\begin{aligned} |0_2(\omega(S/j^{1+\vartheta}))| &\le \sum_n |0_2^{n,j}| \\ &\le [E(bj^{1+\vartheta}/S) - E(aj^{1+\vartheta}/S)]\cdot(S/j^{1+\vartheta})\cdot\omega(S/j^{1+\vartheta}) \\ &\le (b-a)\ \omega(S/j^{1+\vartheta}). \end{aligned} \tag{A.13}$$

Define now

$$\hat{X}_j = \int_a^b \hat{f}(jx)dx . \tag{A.14}$$

Since $\hat{f}$ satisfies the hypotheses satisfied by f, and in particular (A.1), and since $(\hat{f})^{\hat{}} = \hat{f}$ because $\hat{f}$ is independent of s, the above computations yield

$$\hat{X}_j = \sum_n S/j^{1+\vartheta}\hat{f}(jnS/j^{1+\vartheta}) + \hat{0}_2(\omega(S/j^{1+\vartheta})) + \hat{0}_1(1/j^{1+\vartheta}) \tag{A.15}$$

with $\hat{0}_1$ and $\hat{0}_2$ satisfying (A.7) and (A.13) respectively. This implies that

$$X_j - \hat{X}_j \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty. \tag{A.16}$$

Since $\hat{f}$ is an $L^\infty(\mathbb{R})$ function which is periodic with respect to

$(0,T)$, by a well known result we have that

$$\hat{X}_j \longrightarrow \frac{b-a}{R}\int_0^R \hat{f}(r)\,dr \,. \tag{A.17}$$

(This last result moreover follows from the above computation by splitting the interval (a,b) into intervals $(nT/j,\ (n+1)T/j)$: a computation similar to (A.5), (A.7) and the change of variable $y = jx - nT$ in $\int_{nT/j}^{(n+1)T/j} f(jx)\,dx$ gives the result.) The definition (A.11) of $\hat{f}$, (A.16) and (A.17) together prove (A.4), and the lemma follows.

□

In the case when $\vartheta = 0$ and $S = R$ the proof used to establish (A.17) implies that

$$f_j \overset{\bullet}{\longrightarrow} \frac{1}{R}\int_0^R f(r,r)\,dr \quad \text{in } L^\infty(\Omega). \tag{A.18}$$

This formula is quite different from (A.3).

A similar proof to that of Lemma A.1, using the splitting of the rectangle $(a_1,b_1)\times(a_2,b_2)$ into rectangles of size $(S_1/j^{1+\vartheta_1}) \times (S_2/j^{1+\vartheta+\vartheta_2})$, allows one to prove the following lemma, which can also be extended to any finite number of independent variables and speeds.

Lemma A.2

Let $f(r_1,s_1,r_2,s_2)$ be a bounded real-valued function on $\mathbb{R}^4$ which is periodic with respect to $(0,R_1)\times(0,S_1)\times(0,R_2)\times(0,S_2)$, is measurable in (s_1,s_2) for each (r_1,r_2), and satisfies

$$|f(r_1,s_1,r_2,s_2) - f(r_1',s_1,r_2',s_2)| \le \omega(|r_1 - r_1'|) + \omega(|r_2 - r_2'|) \tag{A.19}$$

for all (r_1,r_2) and a.e. (s_1,s_2), where ω is a continuous non-decreasing function with $\omega(0) = 0$. Consider for $j = 1,2,\ldots$ and $\vartheta \ge 0$, $\vartheta_1 > 0$, $\vartheta_2 > 0$ the function f_j defined on $\mathbb{R}^2$ by

$$f_j(x_1,x_2) = f(jx_1, j^{1+\vartheta_1}x_1, j^{1+\vartheta}x_2, j^{1+\vartheta+\vartheta_2}x_2) \,. \tag{A.20}$$

Then f_j belongs to $L^\infty(\mathbb{R}^2)$ and satisfies

$$f_j \overset{\bullet}{\longrightarrow} \frac{1}{R_1S_1R_2S_2}\int_0^{R_1}\!\!\int_0^{S_1}\!\!\int_0^{R_2}\!\!\int_0^{S_2} f(r_1,s_1,r_2,s_2)\,dr_1ds_1dr_2ds_2 \quad \text{in } L^\infty(\mathbb{R}^2). \tag{A.21}$$

Note that ϑ is assumed to be non-negative (in contrast with ϑ_1 and ϑ_2 which are assumed to be positive).