ON THE CALCULUS OF VARIATIONS AND SEQUENTIALLY WEAKLY CONTINUOUS MAPS

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1. Introduction

Consider the problem of finding a function $u: \Omega \to \mathbb{R}^n$ minimizing

$$I(u,\Omega) = \int_\Omega f(x,u(x),\nabla u(x))\,dx \qquad (1)$$

subject to certain constraints, such as boundary conditions. In (1) Ω is a bounded open subset of $\mathbb{R}^m$, $f: \Omega \times \mathbb{R}^n \times M^{n\times m} \to \mathbb{R}$ (where $M^{n\times m}$ denotes the linear space of real $n\times m$ matrices), $x = (x_1,\ldots,x_m)$ and $dx = dx_1\ldots dx_m$.

In the direct method of the calculus of variations it is customary to seek conditions on f such that $I(u,\Omega)$ is sequentially weakly lower semicontinuous on a subset K of a suitable Banach space X (i.e. $u_r \rightharpoonup u$ in K implies $I(u,\Omega) \leqslant \liminf_{r\to\infty} I(u_r,\Omega)$). X is usually a space of Sobolev type. If I is bounded below on K and certain growth conditions are satisfied then the existence of a minimizer is assured.

The purpose of this paper is to show that the study of sequentially weakly <u>continuous</u> maps leads quickly to conditions on f guaranteeing lower <u>semi</u>continuity of $I(u,\Omega)$, and thus to new existence theorems for nonlinear elliptic systems such as those arising in nonlinear elasticity.

Notation: The spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$ are defined in the usual way (cf Adams [1]). We deal throughout with vector and matrix functions $w = (w_i)_{1\leqslant i \leqslant r}$. If Y is a Banach space and r a positive integer we define Y_r to be the Cartesian product $\prod_{i=1}^{r} Y$ equipped with the norm $\|w\|_{Y_r} = \sum_{i=1}^{r} \|w_i\|_Y$. $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{+\infty\}$. We employ the summation convention throughout.

2. The L^∞ case

To gain intuition we first consider maps between L^p spaces which arise from pointwise evaluation by a function. Corollary 1.1 characterizes maps of this type which are sequentially weakly continuous.

Theorem 1

Let $\phi: \mathbb{R}^n \to \mathbb{R}$ satisfy $\phi(u(\cdot)) \in L^1(\Omega)$ whenever $u \in L_n^\infty(\Omega)$. Then

$$J(u) \overset{\text{def}}{=} \int_\Omega \phi(u(x))\,dx$$

is sequentially weak * lower semicontinuous on $L_n^\infty(\Omega)$ if and only if ϕ is convex.

Proof

Suppose J is sequentially weak * lower semicontinuous. Let $a,b \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Let Q be the unit cube $\{x\in\mathbb{R}^m: 0 \leqslant |x_i| < \frac{1}{2}\}$ and define $v \in L_n^\infty(Q)$ by $v(x) = a$ if $x \in A_1$, $v(x) = b$ if $x \in A_2$, where $Q = A_1 \cup A_2$, $\mu(A_1) = \lambda$, $\mu(A_2) = 1-\lambda$, and μ denotes m-dimensional Lebesgue measure. Tesselate $\mathbb{R}^m$ by disjoint congruent open cubes Q_j with centre x_j and side $1/k$. For $i = 1,2$ let $E_{k,i} = \bigcup_j (x_j + \frac{1}{k} A_i)$. Define a sequence $u_k \in L_n^\infty(\Omega)$ $(k = 1,2\ldots)$ by $u_k(x) = v(k(x-x_j))$ if $x \in Q_j \cap \Omega$. If $E \subseteq \Omega$ is measurable and $c \in \mathbb{R}^n$ then

$$\int_\Omega u_k\cdot c\,\chi_E(x)\,dx = \int_E u_k\cdot c\,dx = \mu(E\cap E_{k,1})\,a\cdot c + \mu(E\cap E_{k,2})\,b\cdot c,$$

which as $k \to \infty$ tends to

$$\mu(E)[\lambda a+(1-\lambda)b]\cdot c = \int_\Omega [\lambda a+(1-\lambda)b]\cdot c\,\chi_E(x)\,dx.$$

Since finite linear combinations of functions of the form $\underset{\sim}{c}\,\chi_E$ are dense in $L^1_n(\Omega)$, and since $\|\underset{\sim}{u}\|_{L^\infty_n(\Omega)}$ is bounded, it follows that $\underset{\sim}{u}_k \xrightarrow{*} \lambda \underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$ in $L^\infty_n(\Omega)$. Hence

$$\phi(\lambda \underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}) \leqslant \liminf_{k\to\infty} \frac{1}{\mu(\Omega)} \int_\Omega \phi(\underset{\sim}{u}_k(\underset{\sim}{x}))\,d\underset{\sim}{x}$$

$$= \lim_{k\to\infty} \left[\frac{\mu(\Omega\cap E_{k,1})}{\mu(\Omega)}\,\phi(\underset{\sim}{a}) + \frac{\mu(\Omega\cap E_{k,2})}{\mu(\Omega)}\,\phi(\underset{\sim}{b})\right]$$

$$= \lambda\phi(\underset{\sim}{a}) + (1-\lambda)\,\phi(\underset{\sim}{b}),$$

so that ϕ is convex.

Conversely, let ϕ be convex, so that in particular ϕ is continuous. For $c,d\in\mathbb{R}$ the set $K(c,d) = \{\underset{\sim}{u}\in L^1_n(\Omega): \|\underset{\sim}{u}\|_{L^\infty_n(\Omega)} \leqslant c,\ J(\underset{\sim}{u}) \leqslant d\}$ is closed in $L^1_n(\Omega)$ (by the bounded convergence theorem) and convex, hence weakly closed. Thus J is sequentially weak * lower semicontinuous. □

Corollary 1.1

Let ϕ be as above. Then ϕ: $(L^\infty_n(\Omega)$, weak *$) \longrightarrow (L^1(\Omega)$, weak$)$ is sequentially continuous if and only if ϕ is affine i.e. $\phi(\underset{\sim}{u}) = \alpha + \underset{\sim}{k}.\underset{\sim}{u}$ for constant $\alpha,\underset{\sim}{k}$.

Proof

If ϕ is affine the stated continuity property holds trivially. The converse follows by applying Theorem 1 to ϕ and $-\phi$. □

Remark: Theorem 1 is closely related to many known lower semicontinuity results. Note, however, that no assumption is made about continuity of ϕ.

3. The $W^{1,\infty}$ case

Consider now a function ϕ: $M^{n\times m} \to \mathbb{R}$ satisfying $\phi(F(\cdot)) \in L^1(\Omega)$ whenever $F\in L^\infty_{mn}(\Omega)$. For $\underset{\sim}{u}$: $\mathbb{R}^m \to \mathbb{R}^n$ we pose the question: For which ϕ is the map $\underset{\sim}{u} \mapsto \phi(\nabla\underset{\sim}{u}(\cdot))$ sequentially continuous from

$(W^{1,\infty}_n(\Omega)$, weak *$) \to (L^1(\Omega)$, weak$)$? (By the weak * topology on $W^{1,\infty}_n(\Omega)$ we mean the topology induced by the canonical embedding of $W^{1,\infty}_n(\Omega)$ into a finite product of $L^\infty(\Omega)$ spaces, each being endowed with the weak * topology). Bearing Corollary 1.1 in mind one might think that only affine ϕ are possible. However this is not the case unless m = 1 or n = 1. The actual situation is characterized by the following result of Morrey [6].

Theorem 2

Let ψ: $\Omega \times \mathbb{R}^n \times M^{n\times m} \to \mathbb{R}$ be continuous. Define

$$J(\underset{\sim}{u}) = \int_\Omega \psi(\underset{\sim}{x},\underset{\sim}{u}(\underset{\sim}{x}),\ \nabla\underset{\sim}{u}(\underset{\sim}{x}))\,d\underset{\sim}{x}.$$

Then J is sequentially weak * lower semicontinuous on $W^{1,\infty}_n(\Omega)$ if and only if ψ is quasiconvex i.e. for each fixed $\underset{\sim}{x}_0\in\Omega, \underset{\sim}{u}_0\in\mathbb{R}^n, F_0\in M^{n\times m}$, and for every bounded open subset D of $\mathbb{R}^m$ the inequality

$$\int_D \psi(\underset{\sim}{x}_0,\underset{\sim}{u}_0,F_0+\nabla\underset{\sim}{\zeta}(\underset{\sim}{x}))\,d\underset{\sim}{x} \geqslant \int_D \psi(\underset{\sim}{x}_0,\underset{\sim}{u}_0,F_0)\,d\underset{\sim}{x} = \mu(D)\,\psi(\underset{\sim}{x}_0,\underset{\sim}{u}_0,F_0) \quad (2)$$

holds for all $\underset{\sim}{\zeta}\in C^\infty_0(D)$.

Corollary 2.1

Let ψ: $\Omega \times \mathbb{R}^n \times M^{n\times m} \to \mathbb{R}$ be continuous. The map $\underset{\sim}{u} \mapsto \psi(\cdot,\underset{\sim}{u}(\cdot),\nabla\underset{\sim}{u}(\cdot))$ is sequentially continuous from $(W^{1,\infty}_n(\Omega)$, weak *$) \to (L^1(\Omega)$, weak$)$ if and only if for each fixed $\underset{\sim}{x}_0\in\Omega, \underset{\sim}{u}_0\in\mathbb{R}^n, F_0\in M^{n\times m}$, and for every bounded open subset D of $\mathbb{R}^m$,

$$\int_D \psi(\underset{\sim}{x}_0,\underset{\sim}{u}_0,F_0+\nabla\underset{\sim}{\zeta}(\underset{\sim}{x}))\,d\underset{\sim}{x} = \mu(D)\,\psi(\underset{\sim}{x}_0,\underset{\sim}{u}_0,F_0) \quad (3)$$

for all $\underset{\sim}{\zeta}\in C^\infty_0(D)$.

Proof of Corollary

Suppose $\underset{\sim}{u} \mapsto \psi(\cdot,\underset{\sim}{u}(\cdot),\nabla\underset{\sim}{u}(\cdot))$ has the stated continuity property. Applying Theorem 2 to $\pm\psi$ we obtain (3). Conversely let ψ satisfy (3) and let $\underset{\sim}{u}_r \xrightarrow{*} \underset{\sim}{u}$ in $W^{1,\infty}_n(\Omega)$. Then the sequence $\psi(\cdot,\underset{\sim}{u}_r(\cdot),\nabla\underset{\sim}{u}_r(\cdot))$ is bounded in $L^\infty(\Omega)$, so that in particular there exists a subsequence u_μ of u_r such that $\psi(\cdot,u_\mu(\cdot), \nabla u_\mu(\cdot)) \overset{*}{\longrightarrow} \Theta$ in $L^\infty(\Omega)$. Let $\alpha: \mathbb{R}^m \to \mathbb{R}$ be continuous, and define $\psi_1(x,a,F) = \pm\psi(x,a,F)\alpha(x)$. Then ψ_1 is quasiconvex, so that by Theorem 2

$$\int_\Omega \psi(x,u_\mu(x),\nabla u_\mu(x))\alpha(x)dx \to \int_\Omega \psi(x,u(x),\nabla u(x))\alpha(x)dx.$$

The arbitrariness of α implies that $\Theta = \psi(\cdot,u(\cdot),\nabla u(\cdot))$, and hence

$$\psi(\cdot,u_r(\cdot),\ \nabla u_r(\cdot)) \overset{*}{\longrightarrow} \psi(\cdot,u(\cdot),\nabla u(\cdot)) \text{ in } L^\infty(\Omega)$$

which is stronger than the required conclusion. □

For the relationship of quasiconvexity to ellipticity see [2,6].

4. The null-space of the Euler-Lagrange operator

Let $\psi: \mathbb{R}^m \times \mathbb{R}^n \times M^{n\times m}$ be C^1. We say that ψ belongs to the null-space N of the Euler-Lagrange operator if and only if

$$\int_D \left(\frac{\partial\psi}{\partial u^i}\zeta^i + \frac{\partial\psi}{\partial u^i_{,\alpha}}\zeta^i_{,\alpha}\right) dx = 0 \qquad (4)$$

for every bounded open set $D \subseteq \mathbb{R}^m$ and for all $u \in C^1(\bar{D}), \zeta \in C_0^\infty(D)$.

Theorem 3

Let $\psi: \Omega \times \mathbb{R}^n \times M^{n\times m} \to \mathbb{R}$ be continuous, and suppose that for each fixed $x_0 \in \Omega$, $u_0 \in \mathbb{R}^n$, $\psi(x_0,u_0,\cdot)$ is C^1. Then the map $u \longmapsto \psi(\cdot,u(\cdot),\ \nabla u(\cdot))$ is sequentially continuous from $(W_n^{1,\infty}(\Omega)$, weak *$) \longrightarrow (L^1(\Omega)$, weak$)$ if and only if for each fixed $x_0 \in \Omega, u_0 \in \mathbb{R}^n$, $\psi(x_0,u_0,\cdot) \in N$.

Proof

Let $u \longmapsto \psi(\cdot,u(\cdot),\ \nabla u(\cdot))$ have the stated continuity property. Let $x_0 \in \Omega, u_0 \in \mathbb{R}^n$ and define $\phi(F) = \psi(x_0,u_0,F)$. By Corollary 2.1 we have that

$$\int_D \phi(F_0 + \nabla\zeta(x))dx = \mu(D)\ \phi(F_0) \qquad (5)$$

for all bounded open subsets $D \subseteq \mathbb{R}^m$, $F_0 \in M^{n\times m}$, $\zeta \in C_0^\infty(D)$. Let $\rho \in C_0^\infty(M^{n\times m})$ satisfy $\rho \geq 0$, $\rho(F) = 0$ if $|F| \geq 1$, $\int_{M^{n\times m}} \rho(F)dF = 1$. For $\varepsilon > 0$ let $\rho_\varepsilon(F) = \varepsilon^{-mn}\rho(F/\varepsilon)$. Then $\phi_\varepsilon \overset{def}{=} \rho_\varepsilon * \phi$ is C^∞ and satisfies (5). Hence (cf for example Morrey [7 p 11])

$$\frac{\partial^2\phi_\varepsilon(F)}{\partial F^i_\alpha \partial F^j_\beta}\lambda^i\lambda^j\mu_\alpha\mu_\beta = 0$$

for all $F \in M^{n\times m}$, $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. Thus $\dfrac{\partial^2\phi_\varepsilon(F)}{\partial F^i_\alpha \partial F^j_\beta} = -\dfrac{\partial^2\phi_\varepsilon(F)}{\partial F^i_\beta \partial F^j_\alpha}$, so that (4) holds for ϕ_ε. Letting $\varepsilon \longrightarrow 0$ we see that $\phi \in N$.

The converse follows by noting that if $\phi \in N$ then (5) holds for all F_0, ζ. □

The null-space N has been characterized for arbitrary m,n by Edelen [3] (see also Ericksen [4]). Edelen assumes that the functions u in (4) are C^2, but his results hold for u that are C^1 by approximation. By Theorem 3 we are interested only in elements $\phi(F)$ of N which do not depend on x,u. These are given by linear combinations of 1 and all $r \times r$ subdeterminants of F for $1 \leq r \leq \min(m,n)$. Thus, for example, if $m = n = 1$, 2 or 3 then $\phi(F) \in N$ if and only if ϕ has the form

(n = 1) $\quad \phi(F) = a + bF$

(n = 2) $\quad \phi(F) = a + A^\alpha_i F^i_\alpha + B\det F$

(n = 3) $\quad \phi(F) = a + A^\alpha_i F^i_\alpha + B^\alpha_i(\operatorname{adj}F)^i_\alpha + C\det F$,

where a, b, A^α_i, B^α_i, B, C are constants.

5. Sequentially weakly continuous functionals on $W^{1,p}(\Omega)$

Corollary 2.1 and Theorem 3 show in particular that if ψ is continuous and such that for some $1 \leqslant p < \infty$ the map $\theta : \underset{\sim}{u} \longmapsto \psi(\cdot, \underset{\sim}{u}(\cdot), \nabla \underset{\sim}{u}(\cdot))$ is sequentially continuous from $(W_n^{1,p}(\Omega), \text{weak}) \longrightarrow (L^1(\Omega), \text{weak})$, then $\psi(\underset{\sim}{x}_0, \underset{\sim}{u}_0, \cdot) \in N$ for all $\underset{\sim}{x}_0 \in \mathbb{R}^m$, $\underset{\sim}{u}_0 \in \mathbb{R}^n$. In this section we investigate to what extent the converse holds.

Lemma 1

Let $K \geqslant 2$, $m \geqslant 2$, $n \geqslant 2$, and suppose that $y^i \in W^{1,p}(\Omega)$ for $1 \leqslant i \leqslant K$, where $p \geqslant p_0 = \min(m,n)$. Then the formula

$$\frac{\partial(y^1,\ldots,y^K)}{\partial(x_1,\ldots,x_K)} = \sum_{s=1}^{K} (-1)^{s+1} \frac{\partial}{\partial x_s}\left\{ y^1 \frac{\partial(y^2,\ldots\ldots\ldots\ldots,y^K)}{\partial(x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_K)}\right\} \quad (6)$$

holds in the sense of distributions, where

$$\frac{\partial(y^1,\ldots,y^K)}{\partial(x_1,\ldots,x_K)} \overset{\text{def}}{=} \det\left(\frac{\partial y^i}{\partial x_j}\right).$$

Proof

First suppose that each $y^i \in C^2(\Omega)$. Then the right hand side of (6) equals

$$\sum_{s=1}^{K} (-1)^{s+1}\, y^1_{,s}\, \frac{\partial(y^2,\ldots\ldots\ldots\ldots,y^K)}{\partial(x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_K)} +$$

$$+\, y^1 \sum_{s=1}^{K} (-1)^{s+1} \frac{\partial}{\partial x_s} \frac{\partial(u^2,\ldots\ldots\ldots\ldots,u^K)}{\partial(x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_K)}.$$

The second term is zero by Morrey [7 Lemma 4.4.6], while the first equals $\frac{\partial(y^1,\ldots,y^K)}{\partial(x_1,\ldots,x_K)}$ as required.

Now suppose $y^i \in W_n^{1,p}(\Omega)$ for $1 \leqslant i \leqslant K$ and let $\alpha \in C_0^\infty(\Omega)$. There exists a sequence y_r^i of $C^2(\Omega)$ functions such that $y_r^i \longrightarrow y^i$ in $W^{1,p}(\Omega')$ for some $\Omega' \supset \operatorname{supp} \alpha$.

Then

$$\int_\Omega \frac{\partial(y_r^1,\ldots,y_r^K)}{\partial(x_1,\ldots,x_K)}\, \alpha(\underset{\sim}{x})\, d\underset{\sim}{x} = \sum_{s=1}^{K} (-1)^s \int_\Omega y_r^1 \frac{\partial(y_r^2,\ldots\ldots\ldots\ldots,y_r^K)}{\partial(x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_K)} \frac{\partial\alpha}{\partial x_s}\, d\underset{\sim}{x}$$

Note that $y_r^1 \longrightarrow y^1$ in $L^q(\Omega')$ for $q \geqslant 1$, $\frac{1}{q} > \frac{1}{p_0} - \frac{1}{m}$, and that $\frac{2}{p_0} - \frac{1}{m} < 1$. Using the Hölder inequality we obtain (6). □

Theorem 4

Let $1 \leqslant K \leqslant p_0 = \min(m,n)$, $1 \leqslant i_1 < i_2 < \ldots < i_K \leqslant n$, $1 \leqslant j_1 < j_2 < \ldots < j_K \leqslant m$, and define

$$\phi(\nabla \underset{\sim}{u}) = \frac{\partial(u^{i_1},\ldots,u^{i_K})}{\partial(x_{j_1},\ldots,x_{j_K})}.$$

Let $p \geqslant p_0$ and let $\underset{\sim}{u}_r \longrightarrow \underset{\sim}{u}$ in $W_n^{1,p}(\Omega)$. Then $\phi(\nabla \underset{\sim}{u}_r) \longrightarrow \phi(\nabla \underset{\sim}{u})$ in the sense of distributions.

If $p > p_0$ then $\phi(\nabla \underset{\sim}{u}_r) \longrightarrow \phi(\nabla \underset{\sim}{u})$ in $L^{p/p_0}(\Omega)$.

Proof

Suppose $K \geqslant 2$, $m \geqslant 2$, $n \geqslant 2$, since the other cases are trivial. Let $\alpha \in C_0^\infty(\Omega)$. Then $\underset{\sim}{u}_r \longrightarrow \underset{\sim}{u}$ in $L_n^q(\Omega')$ for some $\Omega' \supset \operatorname{supp} \alpha$ and for $q \geqslant 1$, $\frac{1}{q} > \frac{1}{p_0} - \frac{1}{n}$. Hence

$$u_r^1 \frac{\partial(u_r^{i_2},\ldots\ldots\ldots\ldots,u_r^{i_K})}{\partial(x_{j_1},\ldots,x_{j_{s-1}},x_{j_{s+1}},\ldots,x_{j_K})} \longrightarrow u^1 \frac{\partial(u^{i_2},\ldots\ldots\ldots\ldots,u^{i_K})}{\partial(x_{j_1},\ldots,x_{j_{s-1}},x_{j_{s+1}},\ldots,x_{j_K})}$$

in $L^1(\Omega')$ as $r \to \infty$. The result follows from Lemma 1.

If $p > p_0$ then $\phi(\nabla \underset{\sim}{u}_r)$ is bounded in $L^{p/p_0}(\Omega)$, so that a subsequence $\phi(\nabla \underset{\sim}{u}_\mu) \longrightarrow \theta$ in $L^{p/p_0}(\Omega)$. By the first part

$0 = \phi(\nabla u)$ and thus the whole sequence converges to $\phi(\nabla u)$. □

Note that the right hand side of (6) may have meaning as a distribution when $p < p_0$. In fact we just need that the products

$$y^1 \frac{\partial(y^2,\ldots\ldots\ldots\ldots,y^K)}{\partial(x_1,\ldots,x_{s-1},x_{s+1},\ldots,x_K)} \tag{7}$$

are in $L^1(\Omega)$ when $y^i \in W^{1,p}(\Omega)$ for $1 \leq i \leq K$, and conditions on p,m for this to hold are easily derivable from the imbedding theorems. In fact, one may go further and define the Jacobians in (7) under correspondingly weaker conditions. Rather than give a complete inductive definition of these generalized Jacobians we here restrict ourselves to an illustrative example.

Let $m = n = 3$. Define the distributions

$$(\text{Adj } \nabla u)^\alpha_i = (u^{i+2}u^{i+1}_{,\alpha+1})_{,\alpha+2} - (u^{i+2}\, u^{i+1}_{,\alpha+2})_{,\alpha+1}\,,$$

where the indices are taken modulo 3, and

$$\text{Det } \nabla u = [u^1(\text{Adj } \nabla u)^j_1]_{,j}\,.$$

Note that if $u \in W^{1,p}_3(\Omega)$, $p \geq 2$ then $\text{Adj } \nabla u = \text{adj } \nabla u$, where $(\text{adj } \nabla u)^T$ is the matrix of cofactors of ∇u, and that if $u \in W^{1,p}_3(\Omega)$, $p \geq 2$, and $\text{Adj } \nabla u \in L^{p'}_q(\Omega)$ then $\text{Det } \nabla u = \det \nabla u$. In general, however, $\text{Adj } \nabla u \neq \text{adj } \nabla u$, $\text{Det } \nabla u \neq \det \nabla u$. The following theorem may be proved by similar methods to Theorem 4 (cf [2] for details).

Theorem 5

(i) Let $p > 3/2$. If $u_r \longrightarrow u$ in $W^{1,p}_3(\Omega)$ then $\text{Adj } \nabla u_r \longrightarrow \text{Adj } \nabla u$ in the sense of distributions.

(ii) Let $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} < \frac{4}{3}$. If $u_r \longrightarrow u$ in $W^{1,p}_3(\Omega)$ and if $\text{Adj } \nabla u_r \longrightarrow \text{Adj } \nabla u$ in $L^q_9(\Omega)$ then $\text{Det } \nabla u_r \longrightarrow \text{Det } \nabla u$ in the sense of distributions.

Remark: Results analogous to Theorems 4,5 can be proved in an Orlicz-Sobolev space setting (see [2]).

6. Lower semicontinuity theorems

Let $\phi_1(F),\ldots,\phi_K(F)$ belong to N and let $g: \Omega \times \mathbb{R}^n \times \mathbb{R}^K \to \overline{\mathbb{R}}$ satisfy the conditions

(a) for almost all $x \in \Omega$, $g(x,\cdot,\cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^K$,

(b) for all $u \in \mathbb{R}^m, a \in \mathbb{R}^K, g(\cdot,u,a)$ is measurable,

(c) for almost all $x \in \Omega$ and for all $u \in \mathbb{R}^n, g(x,u,\cdot)$ is convex,

(d) $g(x,u,a) \geq \alpha(x) + \eta(|a|)$,

where $\alpha \in L^1(\Omega)$ and $\eta(t)$ is a real-valued, continuous, even, convex function of $t \in \mathbb{R}$ satisfying $\eta(t) > 0$ for $t > 0$, $\eta(t)/t \longrightarrow 0$ as $t \longrightarrow 0$, $\eta(t)/t \longrightarrow \infty$ as $t \longrightarrow \infty$.

Define $f: \Omega \times \mathbb{R}^n \times M^{n\times m} \longrightarrow \overline{\mathbb{R}}$ by

$$f(x,u,F) = g(x,u,\phi_1(F),\ldots,\phi_K(F)) \tag{8}$$

and let $I(u,\Omega)$ be given by (1).

Theorem 6

Let $u_r \longrightarrow u$ in $W^{1,p}_n(\Omega)$, where $p > p_0 = \min(m,n)$. Then

$$I(u,\Omega) \leq \liminf_{r\to\infty} I(u_r,\Omega).$$

Proof

For $i = 1,2,\ldots$ let Ω_i be the union of all open balls contained in Ω of radius less than $1/i$. Each Ω_i satisfies the cone condition, so that by the imbedding theorems a subsequence $u_\mu \longrightarrow u$ almost everywhere on Ω_i. A standard diagonal argument shows that we may assume that $u_\mu \longrightarrow u$ almost everywhere on each Ω_i and thus almost everywhere on Ω. Since each $\phi_i(F)$ is a finite linear combination of subdeterminants of F of order less than or equal to p_0, we may suppose without loss of generality that $\phi_i(\nabla u_\mu) \rightharpoonup \theta_i$ in $L^1(\Omega)$, and hence, by Theorem 4, $\theta_i = \phi_i(\nabla u)$. By a known theorem [5 p 226]

$$I(u,\Omega) \leq \liminf_{\mu \to \infty} I(u_\mu,\Omega),$$

and the result follows. □

Remarks: 1. If $g: \Omega \times \mathbb{R}^n \times \mathbb{R}^K \longrightarrow \mathbb{R}$ is continuous, then by Theorems 2 and 6 it follows that f is quasiconvex.

2. Other lower semicontinuity theorems can be proved using Theorem 5 and analogous results.

Integrands of the form (8) occur in nonlinear elasticity. An example is the Mooney-Rivlin strain-energy function.

$$W(F) = A(I-3) + B(II-3),$$

where $A > 0$, $B > 0$ are constants, and where $I = \mathrm{tr}(FF^T)$, $II = \mathrm{tr}[(\mathrm{adj}F)(\mathrm{adj}F)^T]$. Theorems 4 and 6 can be applied simply to prove the existence of equilibrium solutions for various boundary value problems for the Mooney-Rivlin material subject to the pointwise constraint of incompressibility

$$\det \nabla u = 1 \quad \text{almost everywhere in } \Omega. \tag{9}$$

More general existence theorems are proved in [2].

7. Conclusion

The method in this paper would seem to have the following advantages.

(i) It enables the existence of minimizers for integrands of the form (8) to be established under significantly weaker continuity and growth conditions than those of Morrey [5 Thm 4.4.5], and the proofs are much simpler.

(ii) It can treat 'weakly continuous' pointwise constraints such as (9).

(iii) It can be extended to equations which do not arise from the calculus of variations.

(iv) It can be extended to higher order equations.

On the other hand, there are examples of quasiconvex integrands which cannot be written in the form (8), so that Morrey's theorem applies but Theorem 6 does not.

We end with a few examples illustrating (iv)†. We consider integrands depending only on second derivatives of u. In the case m = 2, n = 1, the only nonlinear element of the null-space of the corresponding Euler-Lagrange operator is

$$u_{,11}u_{,22} - u_{,12}^2 = (u_{,1}u_{,22})_{,1} - (u_{,1}u_{,12})_{,2} \tag{10}$$

while if m = n = 2 the basic nonlinear elements of the null-space are

$$u^1_{,11}u^2_{,12} - u^1_{,12}u^2_{,11},\quad u^1_{,22}u^2_{,21} - u^1_{,12}u^2_{,22},\quad u^1_{,11}u^1_{,22} - (u^1_{,12})^2$$

$$u^1_{,11}u^2_{,22} - u^1_{,12}u^2_{,12},\quad u^2_{,11}u^1_{,22} - u^2_{,12}u^1_{,12},\quad u^2_{,11}u^2_{,22} - (u^2_{,12})^2$$

Results analogous to Theorems 4, 5 and 6 may be proved. We remark that expressions of the form (9) occur, for example, in connection with the Monge-Ampère equation and the von Karman plate equations.

REFERENCES

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