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MEASURABILITY AND CONTINUITY CONDITIONS FOR NONLINEAR EVOLUTIONARY PROCESSES

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ABSTRACT. This paper generalizes to nonlinear evolutionary processes on a metric space the well-known results connecting measurability and continuity properties with respect to time of linear semigroups of continuous operators on a Banach space.

1. **Introduction.** Let X be a topological space. By definition, an *evolutionary process* on X is a family of operators $U(t,s)$: $X \to X$, defined for $t \in \mathbf{R}^+$, $s \in \mathbf{R}$ and satisfying (i) $U(0,s) =$ identity; (ii) $U(t+\tau,s) = U(t,s+\tau)U(\tau,s)$ for $t, \tau \in \mathbf{R}^+$, $s \in \mathbf{R}$. Such processes arise in the mathematical modelling of nonautonomous systems, when $U(t,s)x$ represents the position (or state) at time $t+s$ of the point which at time s was at x. In the special case when the operators $U(t,s) \overset{\text{def}}{=} T(t)$ are independent of s, the evolutionary process defines a *semigroup* $\{T(t)\}$, $t \in \mathbf{R}^+$. In the above $\mathbf{R}^+$ denotes the nonnegative reals.

In [3] it was shown that in certain situations measurability and continuity properties known to be satisfied for a semigroup could be strengthened using the semigroup properties. We extend this work to evolutionary processes. Our Theorem 1 is, however, new even for semigroups for which it takes the following form.

THEOREM 1′. *Let $\{T(t)\}$, $t \geqslant 0$, be a semigroup on a metric space X. If $T(t)$ is continuous for each $t \geqslant 0$, and if the map $t \mapsto T(t)x$ is strongly measurable on $(0,\infty)$ for each $x \in X$, then the map $(t,x) \mapsto T(t)x$ is continuous on $(0,\infty) \times X$.*

Theorem 1′ generalizes classical results due to von Neumann [17], Dunford [12] and Phillips [20] for the case when X is a Banach space and each $T(t)$ is linear, and improves the result of Phillips (see Crandall and Pazy [8]) for X Banach and $\{T(t)\}$, $t \geqslant 0$, a semigroup of nonexpansions. It is a Lebesgue measure counterpart for its category version due to Chernoff and Marsden [6], [7] (see also [3, Theorem 5.1]).

The other theorems of the paper follow in a straightforward way from those in [3]. Included are some counterexamples indicating directions in which the results cannot be improved.

2. **Preliminaries.** Throughout this section let X be a metric space with metric d and denote Lebesgue measure in $\mathbf{R}$ by m. A function f: $(0,\infty) \to X$ is said

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to be *strongly measurable* if there exists a sequence $\{f_n\}$ of measurable countably-valued functions which converges almost everywhere to f on $(0, \infty)$, and *almost separably-valued* if there exists a subset $E \subseteq (0, \infty)$ of zero measure such that $f((0, \infty)\backslash E)$ has a countable dense subset. It is easily shown (see Dunford and Schwartz [13, p. 147] for an analogous proof) that f is strongly measurable if and only if (a) f is almost separably-valued and (b) $f^{-1}U$ is Lebesgue measurable for every open $U \subseteq X$.

We need the following version of Lusin's theorem, our proof of which is adapted from that in Oxtoby [19].

LEMMA 1. *A function f: $(0, \infty) \to X$ is strongly measurable if and only if given any $\varepsilon > 0$ there exists a closed set F with $m((0, \infty)\backslash F) < \varepsilon$ such that f is continuous when restricted to F.*

PROOF. Let f be strongly measurable. Then there exists $E \subseteq (0, \infty)$ of zero measure such that $Z \overset{\text{def}}{=} f((0, \infty)\backslash E)$ has a countable base of open sets $U_i \cap Z$ $(i = 1, 2, \ldots)$ with U_i open in X. For each i there exists an open set $G_i \supseteq f^{-1}U_i$ such that $m(G_i\backslash f^{-1}U_i) < \varepsilon/2^{i+1}$. Let $S = \bigcup_{i=1}^{\infty}(G_i\backslash f^{-1}U_i)$ so that $m(S) < \varepsilon/2$. Let $U \subseteq X$ be open. Then $U \cap Z = \bigcup_k(U_{i_k} \cap Z)$ and

$$f^{-1}(U \cap Z) = \bigcup_k (G_{i_k}\backslash S) \cap f^{-1}Z = \bigcup_k G_{i_k}\backslash(S \cup E)$$

which is open in $(0, \infty)\backslash(S \cup E)$. There exists a closed set

$$F \subseteq (0, \infty)\backslash(S \cup E)$$

with $m((0, \infty)\backslash F) < \varepsilon$, and clearly f restricted to F is continuous.

Conversely let F_i be closed sets with $m((0, \infty)\backslash F_i) < 1/i$ and such that f is continuous when restricted to each F_i. Let $F = \bigcup_{i=1}^{\infty} F_i$. Then $m((0, \infty)\backslash F) = 0$ and $f(F) = \bigcup_{i=1}^{\infty} f(F_i)$ has a countable dense subset. Thus f is almost separably-valued. Let $U \subseteq X$ be open. For each i there exists an open set $G_i \subseteq (0, \infty)$ with $(f^{-1}U) \cap F_i = G_i \cap F_i$. Hence

$$f^{-1}U = ((f^{-1}U)\backslash F) \cup \bigcup_{i=1}^{\infty}(G_i \cap F_i),$$

which is clearly Lebesgue measurable. $\square$

3. **Main results.** Throughout this section we suppose that the evolutionary process $\{U(t, s)\}$ defined on the topological space X satisfies the hypothesis:

(A) For each $t \in \mathbf{R}^+$ the map $(s, x) \mapsto U(t, s)x$ is (jointly) sequentially continuous from $\mathbf{R} \times X \to X$.

THEOREM 1. *Let X be a metric space. Suppose that for each $s \in \mathbf{R}$, $x \in X$ the map $t \mapsto U(t, s)x$ is strongly measurable on $(0, \infty)$. Then the map $(t, s, x) \mapsto U(t, s)x$ is continuous on $(0, \infty) \times \mathbf{R} \times X$.*

PROOF. We first prove Theorem 1′. Let $x \in X$. We show that the map $f(t) \equiv T(t)x$ is continuous on $(0, \infty)$. The result then follows from a theorem of Chernoff and Marsden [6] (see also [3]). Let $0 < a < a + \delta < \infty$ and denote by I and J the open intervals $(a, a + \delta)$ and $(a + \delta/3, a + 2\delta/3)$ respectively. Since f is strongly measurable, by Lemma 1 there exists in I a closed set F_r of measure greater than $\delta - 1/r^2$ on which the restriction of f is continuous. The continuity being uniform, there exists $\delta/3 > \eta_r > 0$ such that $t, t+h \in F_r$ and $|h| < \eta_r$ imply that $d(f(t+h), f(t)) < 1/r$. Fix h_r with $|h_r| < \eta_r$. The set $\{t \in F_r \cap J: t + h_r \notin F_r\}$ has measure less than $1/r^2$. Therefore $d(f(t+h_r), f(t)) < 1/r$ holds for all t in a subset $E_r \subseteq J$ of measure greater than $\delta/3 - 2/r^2$. Clearly $J \backslash \underline{\lim}_{r\to\infty} E_r$ has measure zero. Therefore $T(t+h_r)x \to T(t)x$ almost everywhere in J. (This argument is due to Auerbach [2].) Let $t \in J$. There exists $t_1 < t$ belonging to J such that $T(t_1 + h_r)x \to T(t_1)x$. Then

$$T(t+h_r)x = T(t-t_1)T(t_1+h_r)x \to T(t-t_1)T(t_1)x = T(t)x$$

by the assumed continuity of $T(t-t_1)$. Thus $T(t+h_r)x \to T(t)x$ everywhere in J. Since from *any* sequence $\{h_k\}$ tending to zero we may extract a subsequence $\{h_{k_r}\}$ with $|h_{k_r}| < \eta_r$, it follows that $T(t+h_k)x \to T(t)x$ everywhere in J. This completes the proof in the semigroup case.

The proof in the general case follows immediately by applying Theorem 1′ to the semigroup $\{S(t)\}$, $t \in \mathbf{R}^+$, which is defined on the space $\mathbf{R} \times X$ by

$$S(t)\begin{pmatrix} s \\ x \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} s+t \\ U(t,s)x \end{pmatrix}. \quad \square \tag{1}$$

COROLLARY. *Let X be a subset of a Banach space. Suppose that for each $s \in \mathbf{R}$, $x \in X$, the map $t \mapsto U(t,s)x$ is weakly continuous from the right on $(0,\infty)$. Then the map $(t,s,x) \mapsto U(t,s)x$ is continuous on $(0,\infty) \times \mathbf{R} \times X$ with respect to the norm topology on X.*

PROOF. See [3, Theorem 5.2]. $\square$

REMARKS. 1. The proof of Theorem 1 bears some resemblance to that of Banach [4] of the result of Fréchet that every Lebesgue measurable real-valued solution g of the functional equation

$$g(s) + g(t) = g(s+t) \tag{2}$$

is continuous, and thus of the form $g(t) = At$ for some constant A. (This is a special case of Theorem 1, since any f satisfying (2) generates a semigroup on $\mathbf{R}$ given by $T(t)\tau = e^{g(t)}\tau$.)

2. In this case $X = \mathbf{R}$, Theorem 1 may also be proved by an argument used by Alexiewicz and Orlicz [1] in their proof of Fréchet's result. With the notation of the above proof, the function f is measurable and thus approximately continuous almost everywhere in $(0,\infty)$; by the semigroup property and the continuity of $T(t)$ for $t \geqslant 0$ it follows that f is approximately continuous everywhere in $(0,\infty)$, and hence f is continuous (see Denjoy [11] and Looman [16]). It might be possible to extend this argument to arbitrary metric X.

3. There is an obvious modification of Theorem 1 to the case when the evolutionary process is defined only locally in t.

4. It is not in general possible to deduce that $(t,s,x) \mapsto U(t,s)x$ is continuous on $[0,\infty) \times \mathbf{R} \times X$, even if $t \mapsto U(t,s)x$ is continuous on $[0,\infty)$ for every (s,x). (See Chernoff [5].)

All the other results in [3] may be generalised in a straightforward way to evolutionary processes using the transformation (1). We give as a sample two such generalisations and take the opportunity to weaken slightly the hypotheses of the corresponding results in [3, Theorems 5.1, 5.3] (the proofs are very similar).

THEOREM 2. *Let X be arbitrary. Suppose that for each $s \in \mathbf{R}$, $x \in X$, the map $t \mapsto U(t,s)x$ is Baire continuous on* $(0, \infty)$ *and, when restricted to the complement of some first category set, has second countable range. Then the map* $(t,s,x) \mapsto U(t,s)x$ *is sequentially continuous on* $(0,\infty) \times \mathbf{R} \times X$.

THEOREM 3. *Let X be a subset of a uniformly convex Banach space. Suppose that*

(a) *for each $s_1, s_2 \in \mathbf{R}$, $x_1, x_2 \in X$, $t_n \to 0+$ implies*

$$\liminf_{n\to\infty} \|U(t_n,s_1)x_1 - U(t_n,s_2)x_2\| \leqslant \|x_1 - x_2\|,$$

(b) *for each $s \in \mathbf{R}$, $x \in X$, the map $t \mapsto U(t,s)x$ is weakly continuous from the right at $t = 0$.*

Then for each $s \in \mathbf{R}$, $x \in X$, the map $t \mapsto U(t,s)x$ is continuous on $[0,\infty)$ *with respect to the norm topology on X.*

We remark that condition (i) in the definition of an evolutionary process is not needed for the validity of Theorems 1 and 2. We remark also that there are useful methods of generating a semigroup from a given process other than by (1). (See Dafermos [9] and the references therein.) However, these methods, while having definite advantages over (1) for stability theory, do not improve our results.

4. **Some counterexamples.** Perhaps the simplest example of an evolutionary process is when $X = \mathbf{R}$ and each operator $U(t,s)$ is linear and defined on $\mathbf{R} \times \mathbf{R}$. Let $\{U(t,s)\}$ have the form

$$U(t,s)r = e^{g(t,s)}r \tag{3}$$

for some function $g\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. g satisfies the functional equation

$$g(t+\tau,s) = g(t,\tau+s) + g(\tau,s), \quad \text{for all } t, \tau, s \in \mathbf{R}. \tag{4}$$

The general solution of (4) is

$$g(t,s) = h(t+s) - h(s), \tag{5}$$

where $h\colon \mathbf{R} \to \mathbf{R}$ is arbitrary. It is therefore clear that, for example, neither strong measurability nor Baire continuity of $(t,s) \mapsto U(t,s)x$, $x \in X$, suffices to prove continuity of this map when (A) is replaced by an assumption of continuity of $U(t,s)x$ with respect to x alone.

When $\{U(t,s)\} = \{T(t)\}$ is a semigroup then $g(t,s) \equiv g(t)$, where f satisfies Cauchy's equation (2). In 1905 Hamel [14] showed using the axiom of choice that there are discontinuous solutions of (2). Thus even for semigroups of continuous linear operators on a Banach space Theorems 1 and 2 are false without the hypothesis of strong measurability or Baire continuity on the map $t \mapsto T(t)x$. Can this hypothesis be weakened to the requirement of precompactness of $T((\alpha,\beta))x$ for all $\alpha, \beta \in \mathbf{R}^+$? This question is motivated by the result of Ostrowski [18], who, extending work of Darboux [10] and Sierpiński [21], showed that any solution f of (2) which is bounded above on a set of positive measure is necessarily continuous (for an alternative proof see Kestelman [15]). The answer is no. For example, define $\{T(t)\}$, $t \in \mathbf{R}$, by

$$(6) \qquad \begin{aligned} T(t)(\pm\pi/2) &= \pm\pi/2, \\ T(t)\tau &= \tan^{-1}[g(t) + \tan\tau], \qquad \tau \in (-\pi/2, \pi/2), \end{aligned}$$

where g is any discontinuous solution of (2). It is easily checked that (6) defines a group of continuous (nonlinear) operators on $[-\pi/2, \pi/2]$ such that each nontrivial orbit is discontinuous (in fact, there will be just one orbit in $(-\pi/2, \pi/2)$ if and only if g is bijective–such solutions g to (2) are easy to construct using a Hamel basis of $\mathbf{R}$ over the rationals). $\{T(t)\}$, $t \in \mathbf{R}$, can be trivially extended to $\mathbf{R}$. Finally, we remark that $(S(t)\theta)(\tau) = \theta(T(t)\tau)$ for $\theta \in C([-\pi/2, \pi/2])$ defines a group $\{S(t)\}$, $t \in \mathbf{R}$, of linear isometries on $C([-\pi/2, \pi/2])$ with the maximum norm such that each nontrivial orbit is discontinuous.

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