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CONSTITUTIVE INEQUALITIES AND EXISTENCE THEOREMS IN NONLINEAR ELASTOSTATICS

1 INTRODUCTION

In these notes I shall describe an approach to the problem of proving the existence of equilibrium solutions in nonlinear elasticity. This problem confronts the analyst with a difficulty typical of nonlinear continuum mechanics, that of choosing hypotheses on the constitutive equations which are both physically reasonable and ensure the existence of solutions with the desired degree of smoothness. The study of constitutive equations in elasticity from the point of view of existence of solutions leads one to consider new constitutive inequalities, and gives insight which it would be hard to gain by other methods.

One-dimensional elasticity

To focus attention on the rôle played by constitutive inequalities in questions of existence, we begin by discussing one-dimensional nonlinear elasticity, which is markedly simpler to treat analytically than the three-dimensional case.

Consider a one-dimensional elastic body (you can think of it as a thin bar) which occupies the unit interval $0 < X < 1$ in a reference configuration (see Figure 1a). In a typical deformed configuration (see Figure 1b) the particle P with position X moves to the point P' having coordinate $x(X)$ with respect to some fixed origin.

(a) reference configuration (b) deformed configuration

Figure 1

Because we are considering elastostatics, x is assumed to be independent of time. We are interested in deformations $x(X)$ satisfying the invertibility condition

$$x'(X) > 0, \quad 0 < X < 1, \tag{1.1}$$

where the prime denotes differentiation with respect to X. Condition (1.1) prevents interpenetration of matter. We assume that the mechanical behaviour of the material is characterized by a stored-energy function $W(X,x')$, in terms of which the total stored-energy is

$$E(x) = \int_0^1 W(X,x'(X))dX. \tag{1.2}$$

Here we have ignored thermal effects. If the body force is conservative with a continuous potential $\psi(x)$ then the equilibrium equation is the Euler-Lagrange equation for the functional

$$I(x) = E(x) + \int_0^1 \psi(x(X))dX, \tag{1.3}$$

namely

$$\frac{\partial\psi}{\partial x} - \frac{d}{dX}\left(\frac{\partial W}{\partial x'}\right) = 0. \tag{1.4}$$

In a displacement boundary-value problem (BVP) we have to solve (1.4) subject to the invertibility condition (1.1) and the boundary conditions

$$x(0) = a, \quad x(1) = b \tag{1.5}$$

where a,b are constants satisfying $a < b$. A 'stable' equilibrium solution will minimize I subject to (1.1) and (1.5).

Whether or not a minimizer exists depends on the form of W. Now it is commonly observed that a rod lengthens when subjected to a tensile force, that is, stress increases with strain. The one-dimensional stress is simply $\sigma = \frac{\partial W}{\partial x'}$, which means that W should be convex in x'. We examine the consequences of assuming this to be so. A typical stress-strain curve for rubber is sketched in Figure 2a, with the corresponding stored-energy function W, assumed independent of X, shown in Figure 2b. The reference configuration is assumed stress-free, so that $\sigma(1) = 0$, and we have taken $W(1) = 0$ without loss of generality. Note that W and σ become large in magnitude for both large and small values of x'; this reflects the fact that a large force is

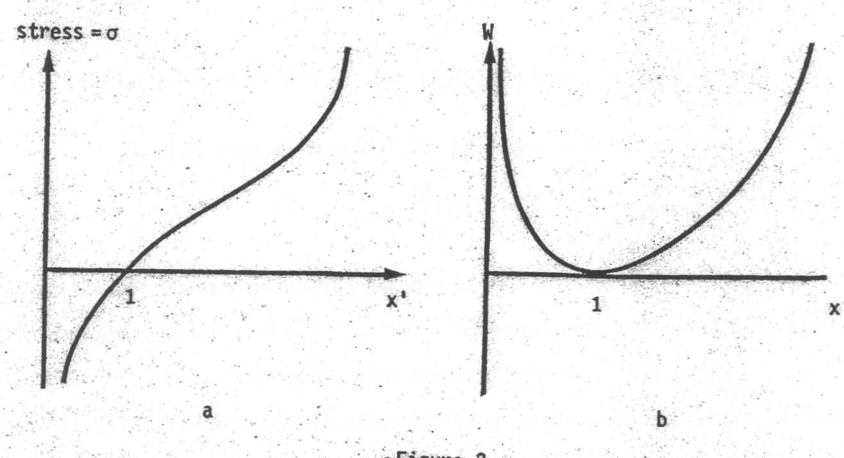


Figure 2

required to effect a large extension or compression. If we take the (somewhat unrealistic) view that any value of $x' > 0$ is possible, then it seems reasonable to assume that for each $X \in (0,1)$

$$W(X,x') \longrightarrow \infty \quad \text{as} \quad x' \longrightarrow 0,\ \infty. \qquad (1.6)$$

Hopefully the first of these requirements will ensure that (1.1) holds.

The convexity of W implies that equation (1.4) is elliptic. Let us sketch a proof of the existence of a minimizer using the 'direct method' of the calculus of variations. We augment (1.6) by assuming that the growth condition

$$W(X,v) \geqslant C + k|v|^p \qquad (1.7)$$

holds, where $p > 1$, $k > 0$ and C are constants. Suppose also, for simplicity, that W is continuous and $\psi \geqslant 0$. Define a class A of admissible functions by

$$A = \{x \in W^{1,p}(0,1) : x(0) = a,\ x(1) = b,\ x' \geqslant 0 \text{ almost everywhere}\}.$$

Here $W^{1,p}(0,1)$ is the Sobolev space of all functions x such that

$$\|x\| \overset{\text{def}}{=} \left(\int_0^1 [|x(X)|^p + |x'(X)|^p]\,dX\right)^{1/p} < \infty,$$

the derivative x' being interpreted in the generalized sense (cf Adams [1]). $W^{1,p}(0,1)$ is a reflexive Banach space. Let $\{x_n\} \subset A$ be a minimizing sequence for I. Using (1.7) and the Poincaré inequality we find that $\|x_n\|$ is bounded, so that a subsequence $\{x_\mu\}$ exists converging weakly in $W^{1,p}(0,1)$ to a function x_0. Since A is weakly closed it follows that $x_0 \in A$. Since W is convex in x' and ψ is continuous we deduce by standard arguments that I is sequentially weakly lower semicontinuous (swlsc), i.e.,

$$I(x_0) \leqslant \liminf_{\mu \to \infty} I(x_\mu).$$

Hence x_0 is a minimizer for I in A. By (1.6), $x_0' > 0$ almost everywhere. To show that x_0 is smooth with $x_0' \geqslant \delta > 0$ in (0,1) is a tricky piece of regularity theory, requiring further hypotheses on W and ψ. The reader is referred to Antman [2,3,4] for details of how this can be done.

The above argument relies crucially on the convexity of W; indeed it has been known since the work of Tonelli that under suitable growth conditions convexity is essentially a necessary and sufficient condition for I to be swlsc. What is more, a simple argument shows that the convexity of W is a necessary condition for the existence of C^1 minimizers for all displacement BVPs of the above type. Consider the case $\psi \equiv 0$, and assume for simplicity that W is C^2 and does not depend explicitly on X. Take $a = 0$, $b > 0$ and suppose that $x_0 \in C^1([0,1])$ minimizes E(x) among all functions $x \in C^1([0,1])$ satisfying $x(0) = 0$, $x(1) = b$, $x'(X) > 0$ for all $X \in [0,1]$. For any $y \in \mathcal{D}(0,1)$, the set of infinitely differentiable functions with compact support in (0,1), we obtain

$$\frac{d^2}{d\varepsilon^2} E(x_0+\varepsilon y)\bigg|_{\varepsilon=0} = \int_0^1 \frac{\partial^2 W}{\partial x'^2}(x_0'(X))y'(X)^2 dX \geqslant 0.$$

Hence, $\frac{\partial^2 W}{\partial x'^2}(x_0'(X)) \geqslant 0$ for all $X \in [0,1]$. But by the mean value theorem there exists $\bar{X} \in (0,1)$ with $x_0'(\bar{X}) = b$. Hence, $\frac{\partial^2 W}{\partial x'^2}(b) \geqslant 0$ and the arbitrariness of b implies that W is convex.

Note that if W is not convex then there may exist minimizers for I that are not C^1. The assumption of convexity of W can be criticized on the grounds that it is not generic, i.e., convexity is not preserved under small perturbations. Indeed, nonconvex W are of physical interest, since they may be associated with materials that undergo phase transitions (Ericksen [12-14]). Nevertheless convexity plays an important rôle even for nonconvex W, since in this case a 'relaxation theorem' of Ekeland and Témam [10] shows that, under certain conditions, from any minimizing sequence for I a subsequence may be extracted converging weakly in a Sobolev space to a minimizer for the functional

$$\bar{I}(x) = \int_0^1 \bar{W}(x'(X))dX + \int_0^1 \psi(x(X))dX,$$

where $\bar{W}$ denotes the lower convex envelope of W (see Figure 3).

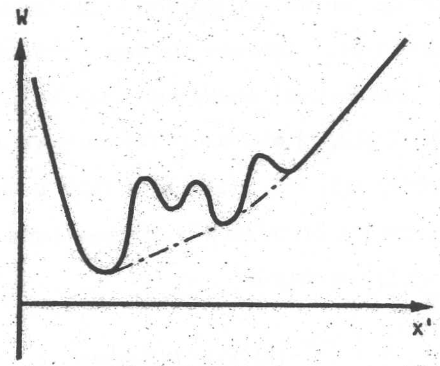


Figure 3

We will not consider this interesting point of view further here.

Three-dimensional elasticity

Consider an elastic body which in a reference configuration occupies the bounded open set $\Omega \subset \mathbb{R}^3$. In a typical deformed configuration the particle P with position vector $\underset{\sim}{X}$ moves to the point P' having position vector $\underset{\sim}{x}(\underset{\sim}{X})$ with respect to fixed Cartesian axes (Figure 4).

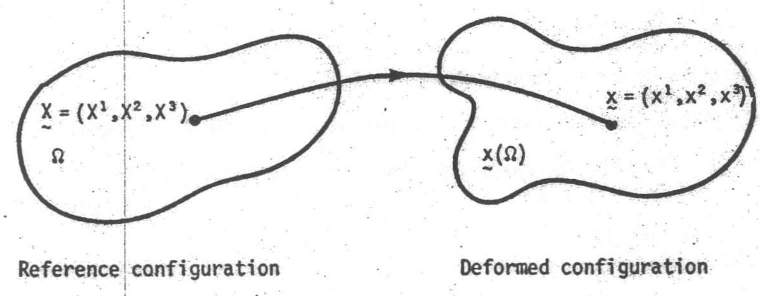


Figure 4

The deformation gradient F is defined by

$$F = \nabla x \; ; \quad F^i_\alpha = x^i_{,\alpha}.$$

We are interested only in deformations which are orientation preserving and globally invertible. In particular we require the local invertibility condition

$$\det F > 0 \quad \text{for all } X \in \Omega \tag{1.8}$$

to hold. By the polar decomposition theorem F=RU for some proper orthogonal matrix R and some positive definite symmetric matrix U. The positive eigenvalues v_i (i = 1,2,3) of U are called the principal stretches of the deformation.

The mechanical properties of the material are characterized by a stored-energy function W(X,F) in terms of which the total stored-energy is

$$E(x) = \int_\Omega W(X, \nabla x(X))\,dX.$$

If the material is isotropic then W has the form

$$W(\underset{\sim}{X},F) = \Phi(\underset{\sim}{X},v_1,v_2,v_3), \tag{1.9}$$

where Φ is symmetric in the v_i for each $\underset{\sim}{X}$.

If the body forces are conservative with potential $\psi(\underset{\sim}{x})$ then the equilibrium equations are the Euler-Lagrange equations for the functional

$$I(\underset{\sim}{x}) = E(\underset{\sim}{x}) + \int_\Omega \psi(\underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X},$$

namely

$$\frac{\partial\psi}{\partial x^i} - \frac{\partial}{\partial X^\alpha}\frac{\partial W}{\partial F^i_\alpha} = 0. \tag{1.10}$$

(Repeated indices are summed from 1 to 3). Let us consider a mixed BVP in which $\underset{\sim}{x}$ is prescribed on a portion $\partial\Omega_1$ of the boundary $\partial\Omega$ of Ω, so that

$$\underset{\sim}{x}(\underset{\sim}{X}) = \bar{\underset{\sim}{x}}(\underset{\sim}{X}) \quad \text{for } \underset{\sim}{X} \in \partial\Omega_1, \tag{1.11}$$

and in which the remainder of the boundary of the body is free of applied surface forces. We seek a minimizer of I subject to (1.8) and (1.11). We do not need to worry about the zero traction condition on $\partial\Omega\backslash\partial\Omega_1$, since this is a natural boundary condition.

What hypotheses should we make on W? Bearing in mind our experience with the one-dimensional case, the natural first try is to assume that $W(\underset{\sim}{X},\cdot)$ is convex; if we make this assumption then it is not hard to generalize our previous analysis from one to three dimensions and to prove various existence theorems. The only trouble is that convexity of $W(\underset{\sim}{X},\cdot)$ is completely unrealistic. There are a number of ways in which this can be seen. Suppose for simplicity that W does not depend explicitly on $\underset{\sim}{X}$, and that $\psi \equiv 0$. If

$W(F)$ were convex and C^1 on some open convex subset S of the space $M^{3\times3}$ of 3×3 matrices, then $E(\underset{\sim}{x})$ would be a Gâteaux differentiable convex function of $\underset{\sim}{x}$ on the relatively open convex set $K = \{\underset{\sim}{x} \in C^1(\bar{\Omega}): \underset{\sim}{x}$ satisfies (1.11) and $\nabla \underset{\sim}{x}(\underset{\sim}{X}) \in S$ for all $\underset{\sim}{X} \in \bar{\Omega}\}$. A result on convex functions (cf Ekeland and Témam [10 Prop. 5.3]) then implies that the set of equilibrium solutions

$$\{\underset{\sim}{x} \in K : E'(\underset{\sim}{x}) = 0\}$$

is convex and consists of absolute minimizers for E on K. In particular, W strictly convex implies the uniqueness of equilibrium solutions in K. Thus convexity rules out the multiple solutions and instabilities commonly observed in buckling. (For examples of nonuniqueness in elastostatics see Rivlin [29,31], John [19,20], Wang and Truesdell [41]). Similar considerations apply in other function spaces. Convexity of W also conflicts with the natural requirement that W be frame-indifferent i.e.,

$$W(\underset{\sim}{X},QF) = W(\underset{\sim}{X},F) \quad \text{for all proper orthogonal matrices } Q. \qquad (1.12)$$

For details see Coleman and Noll [8], Truesdell and Noll [38]. Finally, the inappropriateness of convexity as a constitutive assumption can be seen from the following experiment. Take a homogeneous, isotropic, rubber sheet and subject it to a homogeneous deformation in which $F = \text{diag}\,(v_1, v_2, 1)$. Consider the stored-energy function W as a function of v_1 and v_2. As rubber is almost incompressible it takes a lot of energy to produce a deformation in which v_1v_2 differs greatly from 1. Thus the contours of equal energy are banana-shaped as in Figure 5. This is not consistent with convexity of W. As an illustration, deformations in which $v_1 = 4$, $v_2 = \frac{1}{4}$ or $v_1 = \frac{1}{4}$, $v_2 = 4$ would be easy to produce with one's bare hands, but the case $v_1 = v_2 = \frac{17}{8}$, if possible to achieve at all, would require much more energy. Convexity of W

would imply that $W\left(\frac{17}{8},\frac{17}{8}\right) \leq W(1,4) = W(4,1)$.

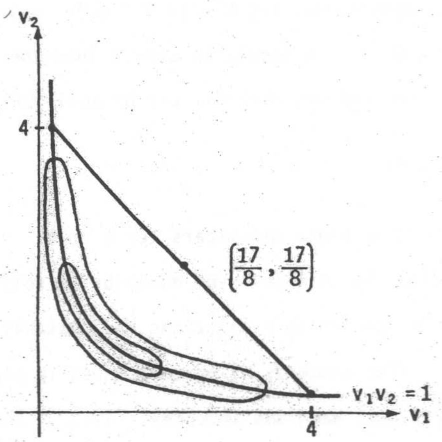


Figure 5

Having disposed of convexity we must search for a suitable substitute. There is a large literature on various alternative convexity hypotheses proposed more or less on ad hoc grounds (see Wang and Truesdell [41] for a summary), and there is no general agreement as to which is the most appropriate†. Whereas in one dimension the intuitive significance of stress increasing with strain is clear, in three dimensions it is not obvious precisely which combinations of surface forces will effect, say, an increase in volume of a unit cube made of <u>any</u> elastic material, and most of the ad hoc inequalities are based on plausibility arguments of this type. An example is the Coleman-Noll condition (see Coleman and Noll [8], Wang and Truesdell [41]). In the case of an isotropic material it implies that $\Phi(\underset{\sim}{X},v_1,v_2,v_3)$ given by (1.9) is convex in

† The general problem of finding suitable constitutive inequalities for nonlinear elasticity was originally posed by Truesdell [37].

the v_i. For rubber this is ruled out by Figure 5 (see also Rivlin [30], Ogden [24]), and for this reason we do not adopt the Coleman-Noll condition.

An older and better motivated inequality is the Legendre-Hadamard or ellipticity condition, which in the case of a smooth stored-energy function requires that

$$\frac{\partial^2 W(\underset{\sim}{X},F)}{\partial F^i_\alpha \partial F^j_\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geqslant 0 \qquad \text{for all } \underset{\sim}{\lambda},\ \underset{\sim}{\mu} \in \mathbb{R}^3. \tag{1.13}$$

Hadamard's theorem [17,18] asserts that (1.13) holds in Ω for any C^1 minimizer of our mixed BVP. If we suppose that (1.13) holds for all $\underset{\sim}{X}$ and F then, as the name suggests, (1.10) becomes an elliptic system. Also $\underset{\sim}{W}(X,\cdot)$ need not be convex, so that multiple equilibrium solutions may be possible. Unfortunately, while there is a well developed existence and regularity theory for linear elliptic systems, no such theory seems to be known for nonlinear elliptic systems. However it can be shown (Theorem 2.1) that if $\underset{\sim}{x}$ is a C^1 minimizer for our BVP then another inequality implying (1.13) holds, namely the quasiconvexity condition of Morrey. This states that

$$\int_D W(\underset{\sim}{X},F+\nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y} \geqslant \int_D W(\underset{\sim}{X},F)d\underset{\sim}{Y} = W(\underset{\sim}{X},F) \times \text{volume of } D \tag{1.14}$$

for all $\underset{\sim}{X} \in \Omega$, $F = \nabla\underset{\sim}{x}(\underset{\sim}{X})$, all bounded open subsets D of $\mathbb{R}^3$ and all $\underset{\sim}{\zeta} \in \mathcal{D}(D)$. If we turn (1.14) into a constitutive hypothesis by requiring it to hold for all $\underset{\sim}{X} \in \Omega$, $F \in M^{3\times3}$ then under certain other growth and continuity conditions $I(\underset{\sim}{x})$ is swlsc and it is possible to prove the existence of equilibrium solutions. This follows from the work of Morrey [22].

The quasiconvexity condition has an interesting physical interpretation that follows immediately from (1.14): For any homogeneous body made from the material found at any point of Ω, and for any displacement BVP with zero body force for such a body that admits as a possible deformation a homogeneous strain, this homogeneous strain must be an absolute minimizer for the total energy. Note that this interpretation would not be in accord with experience if inhomogeneous bodies or mixed BVPs were allowed, as we would expect certain buckled states to have lower energy than the homogeneous strain. Consider for example a displacement BVP for a solid cube consisting of a steel bar imbedded in a rubber matrix. If the homogeneous strain is a compression in the direction of the bar axis then buckling can occur. See Figure 6.

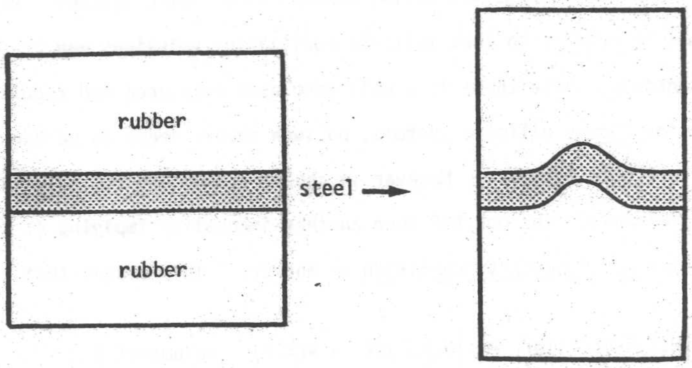


Figure 6

Morrey's existence theorem is very interesting, but one encounters difficulties when applying it to elasticity because of the very strong continuity and growth assumptions used for the proof. In particular $W(\underset{\sim}{X},F)$ is supposed to be defined and continuous for all F. This rules out any singularities of W such as the natural condition

$$W(\underset{\sim}{X},F) \longrightarrow \infty \quad \text{as} \quad \det F \longrightarrow 0. \tag{1.15}$$

(Note that (1.15) is not consistent with convexity of $W(\underset{\sim}{X},\cdot)$). Furthermore the one-dimensional case suggests that we will need to seek a minimum for I on a set of the form

$$A = \{\underset{\sim}{x} \in W^{1,p}(\Omega) : \underset{\sim}{x} = \bar{\underset{\sim}{x}} \text{ on } \partial\Omega_1, \det \nabla\underset{\sim}{x} \geq 0 \text{ almost everywhere}\}$$

for some $p > 1$, and it is not obvious that such a set will be sequentially weakly closed in $W^{1,p}(\Omega)$. Similar problems arise in the important case of incompressible elasticity, where we have to satisfy the constraint

$$\det F = 1. \tag{1.16}$$

A key to the resolution of these difficulties lies in the concept of a null Lagrangian. A continuous function $\phi: M^{3\times 3} \longrightarrow \mathbb{R}$ is a null Lagrangian if the Euler-Lagrange equations for the functional $\int_\Omega \phi(\nabla\underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X}$ reduce to $0 = 0$; i.e., they are identically satisfied for every $\underset{\sim}{x} \in C^1(\bar{\Omega})$. Equivalently

$$\int_\Omega \phi(\nabla\underset{\sim}{x}(\underset{\sim}{X}) + \nabla\underset{\sim}{\zeta}(\underset{\sim}{X}))d\underset{\sim}{X} = \int_\Omega \phi(\nabla\underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X} \tag{1.17}$$

for all $\underset{\sim}{x} \in C^1(\bar{\Omega})$, $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$. Clearly if ϕ is a null Lagrangian then (1.10) is invariant under the transformation $W \longmapsto W + \phi$. In particular, all displacement BVP's for W and $W + \phi$ have the same solutions. We will show in Section 3 that ϕ is a null Lagrangian if and only if

$$\phi(F) = A + A_i^\alpha F_\alpha^i + B_i^\alpha (\operatorname{adj} F)_\alpha^i + C \det F, \tag{1.18}$$

where A, A_i^α, B_i^α, C are constants, and where $\operatorname{adj} F$ denotes the transpose of the matrix of cofactors of F. For our purposes the importance of null Lagrangians lies in various versions of the following result: if $p > 3$ and $|\phi(F)| \leq C(1 + |F|^p)$ for some constant $C > 0$ and all $F \in M^{3\times 3}$, then the map

$\phi(\nabla \underset{\sim}{x}(\cdot)) : W^{1,p}(\Omega) \longrightarrow L^1(\Omega)$ is sequentially weakly continuous (i.e., $\underset{\sim}{x}_n \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$ implies $\phi(\nabla \underset{\sim}{x}_n(\cdot)) \rightharpoonup \phi(\nabla \underset{\sim}{x}(\cdot))$ if and only if ϕ is a null Lagrangian. Related weak continuity results have been studied in so far unpublished work of F. Murat and L. Tartar (1974) and L. Tartar (1976).

A simple constitutive hypothesis which is invariant under the transformation $W \longmapsto W + \phi$ for every null Lagrangian ϕ is the following: there exists a function $g : \Omega \times M^{3\times 3} \times M^{3\times 3} \times (0,\infty) \longrightarrow \mathbb{R}$ with $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$ convex for each $\underset{\sim}{X} \in \Omega$ such that

$$W(\underset{\sim}{X},F) = g(\underset{\sim}{X},F,\operatorname{adj} F, \det F)$$

for all $\underset{\sim}{X} \in \Omega$ and all $F \in M^{3\times 3}$ with $\det F > 0$. We call such functions W polyconvex. It is easily shown that polyconvex functions are quasiconvex, but a polyconvex function need not be convex in F, as the example $W(F) = \det F$ shows. The results on sequential weak continuity mentioned above imply that if W is polyconvex, then under suitable growth and continuity assumptions $I(\underset{\sim}{x})$ is swlsc on $W^{1,p}(\Omega)$. Also, sets of the form A above are sequentially weakly closed. As in one dimension this leads quickly to a proof of the existence of minimizers.

We now turn to the question of what are appropriate growth conditions for W. In the case of compressible elasticity we assume that (1.15) holds. Some condition analogous to (1.7) is also required to restrict the behaviour of W for large $|F|$. Consider a cube of side $\frac{1}{\lambda}$ made from the material found at $\underset{\sim}{X}$. For fixed $F \in M^{3\times 3}$ with $\det F > 0$, imagine deforming the cube by a homogeneous strain with deformation gradient λF. The shape and size of the deformed cube is independent of λ. A plausible requirement is that as $\lambda \longrightarrow \infty$ the total energy of the deformation becomes unbounded, that is

$$\frac{1}{\lambda^3} W(\underset{\sim}{X},\lambda F) \longrightarrow \infty \quad \text{as} \quad \lambda \longrightarrow \infty. \tag{1.19}$$

The stronger condition

$$\frac{W(\underset{\sim}{X},F)}{|F|^3} \longrightarrow \infty \quad \text{as} \quad |F| \longrightarrow \infty \tag{1.20}$$

says that a line segment of positive length cannot be produced from an infinitesimal cube using a finite amount of energy. A sufficient condition for (1.20) to hold is obviously that

$$W(\underset{\sim}{X},F) \geqslant a(\underset{\sim}{X}) + k\,|F|^{3+\varepsilon}, \tag{1.21}$$

for some function $a(\cdot)$, and constants $k > 0$, $\varepsilon > 0$. If (1.21) holds then for certain problems a minimizer $\underset{\sim}{x}$ exists in the space $W^{1,3+\varepsilon}(\Omega)$. The Sobolev imbedding theorem then implies that $\underset{\sim}{x}$ is continuous. This fits in nicely with our motivation for (1.19) and (1.20), since these conditions should prevent holes being formed in the body. In practice we will use a variety of growth conditions; in some cases one can prove existence under conditions that do not imply (1.20).

In the case of incompressible elasticity we assume that $W(\underset{\sim}{X},F)$ is defined for all F with $\det F = 1$. An analogue of (1.20) is that

$$\frac{W(\underset{\sim}{X},F)}{|F|^3} \longrightarrow \infty \quad \text{as} \quad |F| \longrightarrow \infty \quad \text{with } \det F = 1. \tag{1.22}$$

Conditions like (1.20) and (1.22) are especially important for pressure BVP's. For example it is easy to verify that the total energy functional for a spherical shell of Neo-Hookean material ($W(F) = \alpha(\mathrm{tr}(FF^T)-3)$, $\alpha > 0$), under constant internal and external pressures $p > 0$ and zero respectively, is not bounded below, so that no absolute minimizer exists.

As has been indicated above, the existence theorems proved in these notes are of a global type and apply to problems with multiple equilibrium solutions. It is also possible to prove local results, in which the existence and uniqueness of small solutions to BVP's with small body forces and boundary data are established via the inverse function theorem. This has been done by Stoppelli [35] and van Buren [39] (see also Truesdell and Noll [38], Wang and Truesdell [41]). The material response is assumed to be such that existence, uniqueness and regularity theorems hold for the equilibrium equations linearized about the zero data solution. Although these results are limited in scope, they do only need assumptions about the material response for deformation gradients close to those of the zero data solution.

The plan of the remainder of these notes is as follows. In section 2 we discuss in detail the quasiconvexity, ellipticity and polyconvexity conditions. In section 3 we prove the results concerning sequential weak continuity and null Lagrangians. The main existence theorems are given in section 4. In section 5 we comment on the problem of proving that minimizers are smooth and indicate also how the existence theorems in section 4 may be extended to apply to stored-energy functions of 'slow growth' (this involves the use of 'distributional' determinants). In section 6 the existence theorems are applied to various models of rubber. Finally, in section 7 we prove the existence of minimizers for semi-inverse problems of the type recently introduced by Ericksen.

Much of the material presented here appeared first in [A]. Exceptions are the conditions (1.19), (1.20), Theorem 2.6 and the whole of section 7. However the presentation is different (and I hope more readable), and I have tried throughout to use the simplest possible technical assumptions. For example, the existence theorems are proved in Sobolev spaces, rather than in the more general Orlicz-Sobolev spaces used in places in [A].

2 CONSTITUTIVE INEQUALITIES

Let Ω be a bounded open subset of $\mathbb{R}^3$. Let $M^{3\times 3}$ have the induced norm of $\mathbb{R}^9$, and let U be the open subset of $M^{3\times 3}$ consisting of F with $\det F > 0$. Consider a continuous stored-energy function $W : \Omega \times U \longrightarrow \mathbb{R}$.

<u>Definition</u> (cf Morrey [22,23]) W is <u>quasiconvex at a point</u> $(\underset{\sim}{X},F) \in \Omega \times U$ if and only if

$$\int_D W(\underset{\sim}{X}, F+\nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y} \geq W(\underset{\sim}{X},F) \times \text{volume of } D \tag{2.1}$$

for every bounded open subset $D \subset \mathbb{R}^3$ and for every $\underset{\sim}{\zeta} \in \mathcal{D}(D)$ satisfying $F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}) \in U$ for all $\underset{\sim}{Y} \in \Omega$. W is <u>quasiconvex</u> if it is quasiconvex at every $(\underset{\sim}{X},F) \in \Omega \times U$.

If W is quasiconvex at $(\underset{\sim}{X},F)$ then an approximation argument shows that (2.1) holds for any $\underset{\sim}{\zeta} \in C^1(\bar{D})$ with $\underset{\sim}{\zeta} = \underset{\sim}{0}$ on ∂D and such that $F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}) \in D$ for all $\underset{\sim}{Y} \in \bar{D}$.

Let

$$E(\underset{\sim}{x}) = \int_\Omega W(\underset{\sim}{X}, \nabla\underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X},$$

and let $A = \{\underset{\sim}{x} \in C^1(\bar{\Omega}) : \nabla\underset{\sim}{x}(\underset{\sim}{X}) \in U \text{ for all } \underset{\sim}{X} \in \bar{\Omega}\}$. For each $\underset{\sim}{x} \in A$, $\nabla\underset{\sim}{x}(\bar{\Omega})$ is a compact subset of U. Since W is continuous this implies that $E: A \longrightarrow \mathbb{R}$.

The following extension of Hadamard's theorem motivates the quasiconvexity condition by showing that the condition is satisfied at every point of a minimizer for a displacement BVP with zero body force. A similar result is stated by Silverman [34]; see also Busemann and Shephard [7].

Theorem 2.1 Let $\bar{x} \in A$ satisfy

$$E(x) \geq E(\bar{x})$$

for all $x \in A$ with $x-\bar{x} \in \mathcal{D}(\Omega)$ and $\|x-\bar{x}\|_{C(\bar{\Omega})} \overset{\text{def}}{=} \sup_{X \in \Omega} |x(X)-\bar{x}(X)|$ sufficiently small. Then W is quasiconvex at every point $(X, \nabla\bar{x}(X))$, $X \in \Omega$.

Proof. Let D be a bounded open subset of $\mathbb{R}^3$, let $X_o \in \Omega$, and let $\zeta \in \mathcal{D}(D)$ satisfy $\nabla\bar{x}(X_o) + \nabla\zeta(Y) \in U$ for all $Y \in D$. For $\varepsilon > 0$ define $x_\varepsilon : \Omega \longrightarrow \mathbb{R}^3$ by

$$x_\varepsilon(X) = \begin{cases} \bar{x}(X) + \varepsilon\zeta\left(\dfrac{X-X_o}{\varepsilon}\right) & \text{if } \dfrac{X-X_o}{\varepsilon} \in D \\ \bar{x}(X) & \text{otherwise.} \end{cases}$$

For small enough ε the set $X_o + \varepsilon D$ is contained in Ω, so that $x_\varepsilon - \bar{x} \in \mathcal{D}(\Omega)$. Also

$$\nabla x_\varepsilon(X) = \begin{cases} \nabla\bar{x}(X) + \nabla\zeta\left(\dfrac{X-X_o}{\varepsilon}\right) & \text{if } \dfrac{X-X_o}{\varepsilon} \in D \\ \nabla\bar{x}(X) & \text{otherwise.} \end{cases}$$

The continuity assumptions therefore imply that $x_\varepsilon \in A$. Since $\|x_\varepsilon - \bar{x}\|_{C(\bar{\Omega})} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ it follows that $E(x_\varepsilon) \geq E(\bar{x})$ for small enough ε. Making the change of variables $Y = \dfrac{X-X_o}{\varepsilon}$ and dividing by ε^3 we obtain

$$\int_D W(X_o + \varepsilon Y, \nabla\bar{x}(X_o + \varepsilon Y) + \nabla\zeta(Y))dY \geq \int_D W(X_o + \varepsilon Y, \nabla\bar{x}(X_o + \varepsilon Y))dY.$$

Now let $\varepsilon \longrightarrow 0$ to get the result. □

It follows from the theorem that if the quasiconvexity condition (2.1) holds for one bounded open subset $D \subset \mathbb{R}^3$ and all $(\underset{\sim}{X},F) \in \Omega \times U$, then it holds for all such subsets. To see this one simply applies the theorem with $\Omega = D$ and $\bar{\underset{\sim}{x}}(\underset{\sim}{X}) = F\underset{\sim}{X}$.

Note that in one dimension quasiconvexity is equivalent to convexity. To be precise let $W : (0,1) \times (0,\infty) \longrightarrow \mathbb{R}$ satisfy

$$\int_a^b W(X,F+\zeta'(Y))dY \geqslant W(X,F)(b-a) \tag{2.2}$$

for every bounded open interval (a,b), every $\zeta \in \mathcal{D}(a,b)$ satisfying $F + \zeta'(Y) > 0$ for all $Y \in (a,b)$, and every $X \in (0,1)$, $F > 0$. Let $\varepsilon > 0$, $H > 0$, $\lambda \in [0,1]$, $a = 0$, $b = 1$, and define

$$\zeta(Y) = \begin{cases} (1-\lambda)(G-H)Y & \text{for } 0 \leqslant Y \leqslant \lambda \\ \lambda(G-H)(1-Y) & \text{for } \lambda \leqslant Y \leqslant 1 \end{cases}.$$

Setting $F = \lambda G + (1-\lambda)H$ we obtain from (2.2)

$$\lambda W(X,G) + (1-\lambda)W(X,H) \geqslant W(X,\lambda F+(1-\lambda)G), \tag{2.3}$$

so that $W(X,\cdot)$ is convex on $(0,\infty)$. ($\zeta \notin \mathcal{D}(0,1)$ but we can nevertheless deduce (2.3) by approximation). Conversely, if $W(X,\cdot)$ is convex on $(0,\infty)$ and if $\zeta \in \mathcal{D}(a,b)$ satisfies $F + \zeta'(Y) > 0$ for all $Y \in (a,b)$ then

$$W(F+\zeta'(Y)) \geqslant W(F) + A(F)\zeta'(Y)$$

for some $A(F) \in \mathbb{R}$, and (2.2) follows by integration.

In contrast to the situation in one dimension, for homogeneous materials quasiconvexity of W is not necessary for the existence of $C^1(\bar{\Omega})$ minimizers for 'all' displacement BVP's. For simplicity we give a two-dimensional example. Let $\Omega = \{X \in \mathbb{R}^2 : 1 < |X| < 2\}$. Let $W : M^{2\times 2} \longrightarrow \mathbb{R}$ be defined by $W(F) = \rho(r)$, where $r = |F| = [\mathrm{tr}(FF^T)]^{\frac{1}{2}}$, and where $\rho(r) = 0$ for $r \geq 1$, $\rho(r) > 0$ for $0 \leq r < 1$. Consider the map $x^{(n)}(X)$ given in polar coordinates by $(R, \Theta) \longmapsto (R, \Theta + 2n\pi(R-1))$, where $R = |X|$ and $n = 1,2,\ldots$. Clearly $x^{(n)}(X) = X$ for $X \in \partial\Omega$ and $\det \nabla x^{(n)}(X) = 1$ for all $X \in \Omega$. Also it is easily checked that there are numbers a_n, $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that $|\nabla x^{(n)}(X)| \geq a_n$ for all $X \in \bar{\Omega}$. Now let $x_o \in C^1(\bar{\Omega})$ satisfy $\det \nabla x_o(X) > 0$ for all $X \in \bar{\Omega}$. The map $y^{(n)} = x_o \circ x^{(n)}$ satisfies $y^{(n)}(X) = x_o(X)$ for $X \in \partial\Omega$, $\det \nabla y^{(n)}(X) > 0$ for all $X \in \bar{\Omega}$. But

$$a_n \leq |\nabla x^{(n)}(X)| = |\nabla x_o(x^{(n)}(X))^{-1} \nabla y^{(n)}(X)| \leq C|\nabla y^{(n)}(X)|$$

for all $X \in \Omega$ and some constant $C > 0$. Hence

$$\int_\Omega W(\nabla y^{(n)}(X))dX = 0$$

for large enough n, so that any displacement BVP has a $C^1(\bar{\Omega})$ minimizer. However W is not quasiconvex, as can be seen by setting $x_o(X) = \frac{1}{2}X$ in the above argument.

Curiously, if Ω is a cube quasiconvexity of W <u>is</u> a necessary condition for the existence of $C^1(\bar{\Omega})$ minimizers for certain displacement BVP's (see [A Thm 3.2]).

<u>Definition</u> W is <u>rank 1 convex</u> if for each $X \in \Omega$, $W(X,\cdot)$ is convex on all closed line segments in U with end points differing by a matrix of rank 1 i.e., if $X \in \Omega$ then

$$W(X, F + (1-\lambda)\, a\otimes b) \leq \lambda W(X,F) + (1-\lambda)W(X, F + a\otimes b)$$

for all $F \in U$, $\lambda \in [0,1]$, $a, b \in \mathbb{R}^3$, with $F + \mu\, a\otimes b \in U$ for all $\mu \in [0,1]$. Here $(a\otimes b)^i_\alpha \overset{\text{def}}{=} a^i b_\alpha$.

The geometric significance of matrices of rank 1 is the following: if $\underset{\sim}{x}(\underset{\sim}{X})$ is continuous and if $\nabla \underset{\sim}{x}$ takes the constant values F,G on opposite sides of the plane $\underset{\sim}{X}.\underset{\sim}{n} = k$, then $F - G = \underset{\sim}{\lambda} \otimes \underset{\sim}{n}$ for some $\underset{\sim}{\lambda} \in \mathbb{R}^3$.

The next theorem is a consequence of standard results on convex functions. For details see [A].

Theorem 2.2 The following conditions (i), (ii) are equivalent

(i) W is rank 1 convex

(ii) for each $\underset{\sim}{X} \in \Omega$, $F \in U$ there exists $A(\underset{\sim}{X},F) \in M^{3\times 3}$ such that

$$W(\underset{\sim}{X}, F + \underset{\sim}{a} \otimes \underset{\sim}{b}) \geqslant W(\underset{\sim}{X},F) + A_i^\alpha(\underset{\sim}{X},F)a^i b_\alpha$$

whenever $F + \lambda\, \underset{\sim}{a} \otimes \underset{\sim}{b} \in U$ for all $\lambda \in [0,1]$.

If $W(\underset{\sim}{X},\cdot)$ is C^2 for each $\underset{\sim}{X} \in \Omega$ then (i) and (ii) are equivalent to the ellipticity condition

(iii) $$\frac{\partial^2 W(\underset{\sim}{X},F)}{\partial F_\alpha^i \partial F_\beta^j} a^i a^j b_\alpha b_\beta \geqslant 0 \quad \text{for all } \underset{\sim}{X} \in \Omega,\ F \in U,\ \underset{\sim}{a},\underset{\sim}{b} \in \mathbb{R}^3.$$

If $W(\underset{\sim}{X},\cdot)$ is C^2 then the next result is simply Hadamard's theorem. For proofs of Hadamard's theorem see Graves [16] and Morrey [22,23]. A proof of Theorem 2.3 assuming only continuity of W is given in [A].

Theorem 2.3 W quasiconvex implies W rank 1 convex.

Let $E = M^{3\times 3} \times M^{3\times 3} \times (0,\infty)$. We regard E as a subset of $\mathbb{R}^{19}$.

Definition W is polyconvex if there exists a function $g : \Omega \times E \longrightarrow \mathbb{R}$ such that

(i) $W(\underset{\sim}{X},F) = g(\underset{\sim}{X},F,\operatorname{adj} F, \det F)$ for all $\underset{\sim}{X} \in \Omega$, $F \in U$

(ii) $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$ is convex on E.

Theorem 2.4 The following conditions (i) - (iii) are equivalent

(i) W is polyconvex

(ii) for each $\underset{\sim}{X} \in \Omega$, $F \in U$ there exist numbers $a_i^\alpha(\underset{\sim}{X},F)$, $b_i^\alpha(\underset{\sim}{X},F)$, $c(\underset{\sim}{X},F)$ such that

$$W(\underset{\sim}{X},\bar{F}) \geq W(\underset{\sim}{X},F) + a_i^\alpha(\bar{F}_\alpha^i - F_\alpha^i) + b_i^\alpha((\operatorname{adj}\bar{F})_\alpha^i - (\operatorname{adj} F)_\alpha^i) + c(\det \bar{F} - \det F)$$

for all $\bar{F} \in U$.

(iii) for each $\underset{\sim}{X} \in \Omega$, $F \in U$ there exist numbers $A_i^\alpha(\underset{\sim}{X},F)$, $B_i^\alpha(\underset{\sim}{X},F)$, $c(\underset{\sim}{X},F)$ such that

$$W(\underset{\sim}{X},F + \pi) \geq W(\underset{\sim}{X},F) + A_i^\alpha \pi_\alpha^i + B_i^\alpha(\operatorname{adj}\pi)_\alpha^i + c(F)\det\pi$$

for all $F + \pi \in U$.

Proof. That (ii) and (iii) are equivalent follows immediately by setting $\bar{F} = F + \pi$ and rewriting the right hand sides of the inequalities. That (i) implies (ii) is a direct consequence of the convexity of $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$. It remains to show that (ii) implies (i). Define g on $\Omega \times E$ by

$$g(\underset{\sim}{X},G,H,\delta) = \sup_{F \in U} [W(\underset{\sim}{X},F) + a_i^\alpha(\underset{\sim}{X},F)(G_\alpha^i - F_\alpha^i) + b_i^\alpha(\underset{\sim}{X},F)(H_\alpha^i - (\operatorname{adj} F)_\alpha^i) + c(\underset{\sim}{X},F)(\delta - \det F)].$$

Fix $\underset{\sim}{X} \in \Omega$. As $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$ is the supremum of a family of affine functions it is convex. By (ii),

$$g(\underset{\sim}{X},F,\operatorname{adj} F, \det F) = W(\underset{\sim}{X},F), \qquad F \in U.$$

The only thing to check is that $g(\underset{\sim}{X},G,H,\delta) < \infty$ on $\Omega \times E$. Since $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$ is convex it suffices to prove that the convex hull of the set $\{(F,\operatorname{adj} F, \det F): F \in U\}$ is the whole of E. For $k > 0$, define $V_k \subset M^{3\times 3} \times M^{3\times 3}$ by

$$V_k = \{(F,\operatorname{adj} F) : F \in M^{3\times 3}, \det F = k\}.$$

It is enough to show that the convex hull of V_k is $M^{3\times 3}\times M^{3\times 3}$. Suppose not. Then (cf Rockafellar [32 p 99]) there is a closed half-space

$$\pi = \{(F,A)\in M^{3\times 3}\times M^{3\times 3} : F^i_\alpha G^\alpha_i + A^i_\alpha H^\alpha_i \leq \mu\},$$

$(G,H)\neq 0$, with $V_k \subset \pi$. If $R_1, R_2 \in M^{3\times 3}$ are proper orthogonal then

$$F^i_\alpha G^\alpha_i + A^i_\alpha H^\alpha_i = \operatorname{tr}[(R_1FR_2)(R_2{}^T GR_1{}^T) + (R_2{}^T AR_1{}^T)(R_1HR_2)].$$

Since $\operatorname{adj}(R_1FR_2) = R_2{}^T(\operatorname{adj} F)R_1{}^T$, $\det(R_1FR_2) = \det F$, we may without loss of generality suppose that H is diagonal. Suppose that $H\neq 0$ and assume without loss of generality that $H^1_1\neq 0$. Let $F = \operatorname{diag}(kN^{-1}\operatorname{sgn} H^1_1, N^{\frac12}\operatorname{sgn} H^1_1, N^{\frac12})$. Then $\operatorname{adj} F = \operatorname{diag}(N \operatorname{sgn} H^1_1, kN^{-\frac12}\operatorname{sgn} H^1_1, kN^{-\frac12})$ and $\det F = k$. Hence $(F,\operatorname{adj} F)\in V_k$, but for $N>0$ large enough $(F,\operatorname{adj} F)\notin \pi$. If $H=0$ then we can assume that $G^1_1\neq 0$, let $F = \operatorname{diag}(kN\operatorname{sgn} G^1_1, N^{-\frac12}\operatorname{sgn} G^1_1, N^{-\frac12})$ and proceed similarly. Thus $V_k \not\subset \pi$ and this contradiction completes the proof. □

Another equivalent condition for polyconvexity is given in [A] using work of Busemann, Ewald and Shephard [6] on convex functions defined on nonconvex sets.

<u>Open problem</u>

1. Give a physical interpretation of polyconvexity.

The following formulae, which express $\operatorname{adj}\nabla\underset{\sim}{x}$ and $\det\nabla\underset{\sim}{x}$ as divergences, are fundamental to the rest of our work:

$$(\operatorname{adj}\nabla\underset{\sim}{x})^\alpha_i \overset{\text{def}}{=} \tfrac12 \varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} x^j_{,\beta} x^k_{,\gamma} = (\tfrac12 \varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} x^j x^k_{,\gamma})_{,\beta} \tag{2.4}$$

$$\det\nabla\underset{\sim}{x} \overset{\text{def}}{=} \tfrac16 \varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} x^i_{,\alpha} x^j_{,\beta} x^k_{,\gamma} = (\tfrac13 x^i (\operatorname{adj}\nabla\underset{\sim}{x})^\alpha_i)_{,\alpha}. \tag{2.5}$$

Clearly (2.4), (2.5) are valid if $\underset{\sim}{x}$ is C^2.

Theorem 2.5 (Morrey [22]). W polyconvex implies W quasiconvex.

Proof. Fix $\underset{\sim}{X} \in \Omega$. Let D be a bounded open subset of $\mathbb{R}^3$, let $F \in U$, and let $\underset{\sim}{\zeta} \in \mathcal{D}(D)$ satisfy $F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}) \in U$ for all $\underset{\sim}{Y} \in \Omega$. By (2.4) and (2.5),

$$\int_D \zeta^i_{,\alpha} d\underset{\sim}{Y} = \int_D (\text{adj } \nabla\underset{\sim}{\zeta})^i_\alpha d\underset{\sim}{Y} = \int_D \det \nabla\underset{\sim}{\zeta} d\underset{\sim}{Y} = 0.$$

Thus by Theorem 2.4(iii),

$$\int_D W(\underset{\sim}{X}, F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y} \geqslant \int_D W(\underset{\sim}{X}, F)d\underset{\sim}{Y}$$

as required. □

Let us call W convex if $W(\underset{\sim}{X}, \cdot)$ is convex on all closed line segments in U. Then we have the following situation:

$$W \text{ convex} \underset{\not\Leftarrow}{\Rightarrow} W \text{ polyconvex} \underset{\not\Leftarrow}{\Rightarrow} W \text{ quasiconvex} \underset{\overset{\Leftarrow}{?}}{\Rightarrow} W \text{ rank 1 convex}$$

$W(F) = \det F$ is an example for the first nonimplication; for the second see [A].

Open problems (see [A] for discussion)

2. Does W rank 1 convex imply W quasiconvex?
3. Does W quasiconvex imply W polyconvex if W satisfies (1.12)?

My guess is that the answer to both questions is no.

Consider next a pure displacement BVP with boundary condition $\underset{\sim}{x} = \bar{\underset{\sim}{x}}(\underset{\sim}{X})$ for $\underset{\sim}{X} \in \partial\Omega$, where $\bar{\underset{\sim}{x}} : \Omega \longrightarrow \mathbb{R}^3$ is globally one to one. Under assumptions like (1.15), one can hope that any minimizer $\underset{\sim}{x}$ for this problem will be globally one to one. Let $\underset{\sim}{X} = \underset{\sim}{x}^{-1}(\cdot)$, so that $\underset{\sim}{X} : \bar{\underset{\sim}{x}}(\Omega) \longrightarrow \Omega$. It seems reasonable to expect that $\underset{\sim}{X}$ will minimize

$$\hat{I}(\underset{\sim}{X}) = \int_{\bar{\underset{\sim}{x}}(\Omega)} \hat{W}(\underset{\sim}{X}(\underset{\sim}{x}), \nabla \underset{\sim}{X}(\underset{\sim}{x}))d\underset{\sim}{x} + \int_{\bar{\underset{\sim}{x}}(\Omega)} \psi(\underset{\sim}{x}) \det \nabla \underset{\sim}{X}(\underset{\sim}{x})d\underset{\sim}{x}$$

in a suitable function space, subject to the boundary condition $\underset{\sim}{X}(\underset{\sim}{x}) = \bar{\underset{\sim}{x}}^{-1}(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \partial\bar{\underset{\sim}{x}}(\Omega)$, where $\hat{W} : \Omega \times U \longrightarrow \mathbb{R}$ is defined by

$$\hat{W}(\underset{\sim}{X}, F) = W(\underset{\sim}{X}, F^{-1})\det F.$$

Note that ^ is an involution, i.e., $\hat{\hat{W}} = W$. We now ask which constitutive inequalities are invariant under the transformation $W \longmapsto \hat{W}$: it would be disconcerting if a constitutive inequality used as a hypothesis for an existence theorem did not possess this invariance, since then it might be possible to find a minimizer for I but not for $\hat{I}$ (or vice versa). The example $W \equiv 1$ shows that convexity of W is not invariant under ^.

Theorem 2.6 Quasiconvexity, rank 1 convexity, and polyconvexity are all invariant under ^.

Proof. Without loss of generality we take W independent of $\underset{\sim}{X}$. Let W be quasiconvex, D be a bounded open subset of $\mathbb{R}^3$, $\underset{\sim}{\zeta} \in \mathcal{D}(D)$, $F \in U$, and let $F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}) \in U$ for each $\underset{\sim}{Y} \in D$. Define $\underset{\sim}{x}(\underset{\sim}{Y}) = F\underset{\sim}{Y} + \underset{\sim}{\zeta}(\underset{\sim}{Y})$. Clearly $\det \nabla\underset{\sim}{x}(\underset{\sim}{Y}) \geq c > 0$ for all $\underset{\sim}{Y} \in D$. Also $\underset{\sim}{x}(\underset{\sim}{Y}) = F\underset{\sim}{Y}$ for $\underset{\sim}{Y} \in \partial D$, so that $\underset{\sim}{x}$ coincides on ∂D with a globally one to one function. Hence there exists an inverse function $\underset{\sim}{Y} : \underset{\sim}{x}(D) \longrightarrow D$, and $\underset{\sim}{Y}(\underset{\sim}{x}) = F^{-1}\underset{\sim}{x} + \underset{\sim}{\eta}(\underset{\sim}{x}) \in \mathcal{D}(\underset{\sim}{x}(D))$.

(While intuitively obvious, this step is not trivial and requires use of the Brouwer degree. The set $\underset{\sim}{x}(D)$ is open by the invariance of domain theorem.) Thus

$$\int_{D} \hat{W}(F + \nabla \underset{\sim}{\zeta}(\underset{\sim}{Y}))\,d\underset{\sim}{Y} = \int_{\underset{\sim}{x}(D)} W(F^{-1} + \nabla \eta(\underset{\sim}{x}))\,d\underset{\sim}{x} \geqslant \int_{\underset{\sim}{x}(D)} W(F^{-1})\,d\underset{\sim}{x} =$$

$$= W(F^{-1}) \int_{D} \det(F + \nabla \underset{\sim}{\zeta}(\underset{\sim}{Y}))\,d\underset{\sim}{Y} = \int_{D} \hat{W}(F)\,d\underset{\sim}{Y},$$

where at the last step one has to use (2.4), (2.5). This proves that $\hat{W}$ is quasiconvex.

Let W be rank 1 convex. Let $F + \mu\, \underset{\sim}{a} \otimes \underset{\sim}{b} \in U$ for all $\mu \in [0,1]$. Since $(F + \underset{\sim}{a} \otimes \underset{\sim}{b})^{-1} - F^{-1} = -(F + \underset{\sim}{a} \otimes \underset{\sim}{b})^{-1}\, \underset{\sim}{a} \otimes \underset{\sim}{b}\, F^{-1}$ is of rank 1, it follows from Theorem 2.2(iii) that

$$\hat{W}(F + \underset{\sim}{a} \otimes \underset{\sim}{b}) = W(F^{-1} + (F + \underset{\sim}{a} \otimes \underset{\sim}{b})^{-1} - F^{-1}) \det(F + \underset{\sim}{a} \otimes \underset{\sim}{b})$$

$$\geqslant [W(F^{-1}) - A_i^{\alpha}(F)(F + \underset{\sim}{a} \otimes \underset{\sim}{b})^{-1\,i}{}_{\beta}\, a^{\beta} b_j (F^{-1})^j_{\alpha}] \det(F + \underset{\sim}{a} \otimes \underset{\sim}{b})$$

$$= \hat{W}(F) + [(\operatorname{adj} F)(1 - F^{-1}A(F))]^{\alpha}_i (\underset{\sim}{a} \otimes \underset{\sim}{b})^i_{\alpha}.$$

Thus $\hat{W}$ is rank 1 convex by Theorem 2.2.

Let W be polyconvex. Let $F, \bar{F} \in U$. By Theorem 2.4(ii),

$$\hat{W}(\bar{F}) \geqslant (W(F^{-1}) + a_i^{\alpha}(F^{-1})(\bar{F}^{-1} - F^{-1})^i_{\alpha} + b_i^{\alpha}(F^{-1})[\operatorname{adj} \bar{F}^{-1})^i_{\alpha} - (\operatorname{adj} F^{-1})^i_{\alpha}] +$$

$$+ c(F^{-1})[\det \bar{F}^{-1} - \det F^{-1}])\det \bar{F}$$

$$= \hat{W}(F) + b_i^{\alpha}(F^{-1})(\bar{F} - F)^i_{\alpha} + a_i^{\alpha}(F^{-1})(\operatorname{adj} \bar{F} - \operatorname{adj} F)^i_{\alpha} - (\det F)^{-1}[W(F^{-1})\det F +$$

$$+ a_i^{\alpha}(F^{-1})(\operatorname{adj} F)^i_{\alpha} + b_i^{\alpha}(F^{-1})F^i_{\alpha} + c(F^{-1})](\det \bar{F} - \det F).$$

Hence $\hat{W}$ is polyconvex by Theorem 2.4. □

The reader may find it interesting to see what Theorem 2.6 says in one dimension. Most of the results in this section can be carried over to the case of functions $\underset{\sim}{x} : \Omega \longrightarrow \mathbb{R}^n$, $\Omega \subset \mathbb{R}^m$, with little change in the proofs.

3 NULL LAGRANGIANS AND SEQUENTIALLY WEAKLY CONTINUOUS FUNCTIONS

Let Ω be a nonempty bounded open subset of $\mathbb{R}^3$.

<u>Definition</u> Let $\phi : M^{3\times 3} \longrightarrow \mathbb{R}$ be continuous. Then ϕ is said to be a <u>null Lagrangian</u> if

$$\int_\Omega \phi(\nabla \underset{\sim}{x}(\underset{\sim}{X}) + \nabla \underset{\sim}{\zeta}(\underset{\sim}{X}))d\underset{\sim}{X} = \int_\Omega \phi(\nabla \underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X} \tag{3.1}$$

for all $\underset{\sim}{x} \in C^1(\bar{\Omega})$, $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$.

<u>Theorem 3.1</u> The following conditions (i) - (v) are equivalent

(i) ϕ is a null Lagrangian

(ii) $\int_\Omega \phi(F + \nabla \underset{\sim}{\zeta}(\underset{\sim}{X}))d\underset{\sim}{X} = \phi(F) \times \text{volume of } \Omega$

for all $F \in M^{3\times 3}$, $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$. (i.e., ϕ and $-\phi$ are quasiconvex on $M^{3\times 3}$)

(iii) ϕ is rank 1 affine, i.e.,

$$\phi(F + (1-\lambda)\underset{\sim}{a} \otimes \underset{\sim}{b}) = \lambda\phi(F) + (1-\lambda)\phi(F + \underset{\sim}{a} \otimes \underset{\sim}{b}) \tag{3.2}$$

for all $F \in M^{3\times 3}$, $\lambda \in [0,1]$, $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$

(iv)
$$\phi(F) = A + A_i^\alpha F_\alpha^i + B_i^\alpha (\text{adj } F)_\alpha^i + C \det F \tag{3.3}$$

for all $F \in M^{3\times 3}$, where A, A_i^α, B_i^α, C are constants.

(v) ϕ is C^1 and

$$\int_\Omega \frac{\partial \phi}{\partial F_\alpha^i}(\nabla \underset{\sim}{x}(\underset{\sim}{X}))\zeta^i_{,\alpha}(\underset{\sim}{X})d\underset{\sim}{X} = 0 \tag{3.4}$$

for all $\underset{\sim}{x} \in C^1(\bar{\Omega})$, $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$.

Proof. Putting $\underset{\sim}{x}(\underset{\sim}{X}) = F\underset{\sim}{X}$ in (3.1) shows that (i) implies (ii). Let (ii) hold. Then, by an obvious modification of Theorem 2.3, ϕ and $-\phi$ are rank 1 convex on $M^{3\times3}$, i.e., (iii) holds. To show that (iii) implies (iv) assume first that ϕ is C^2, so that (c.p. Theorem 2.2 (iii))

$$\frac{\partial^2\phi(F)}{\partial F^i_\alpha \partial F^j_\beta} a^i a^j b_\alpha b_\beta = 0 \quad \text{for all } F \in M^{3\times3}, \quad \underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3. \tag{3.5}$$

A result of Ericksen [11] shows that (3.5) holds if and only if ϕ has the form (3.3). (See also Edelen [9]). For a general continuous ϕ one can use a mollifier argument to reduce the problem to the case $\phi \in C^2$; for details see [A].

To prove that (iv) implies (v), note first that by approximation we may without loss of generality assume that $\underset{\sim}{x} \in C^2(\Omega)$. Then, if $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$,

$$\int_\Omega \frac{\partial\phi}{\partial F^i_\alpha}(\nabla \underset{\sim}{x}(\underset{\sim}{X}))\zeta^i_{,\alpha}(\underset{\sim}{X})d\underset{\sim}{X} = -\int_\Omega \frac{\partial^2\phi}{\partial F^i_\alpha \partial F^j_\beta}(\nabla \underset{\sim}{x}(\underset{\sim}{X}))x^j_{,\beta\alpha}\zeta^i d\underset{\sim}{X}.$$

But (3.5) implies that $\dfrac{\partial^2\phi}{\partial F^i_\alpha \partial F^j_\beta} = -\dfrac{\partial^2\phi}{\partial F^i_\beta \partial F^j_\alpha}$. This gives (v).

Finally let (v) hold and let $\underset{\sim}{x} \in C^1(\Omega)$, $\underset{\sim}{\zeta} \in \mathcal{D}(\Omega)$. For $t \in [0,1]$ define

$$g(t) = \int_\Omega \phi(\nabla \underset{\sim}{x}(\underset{\sim}{X}) + t\nabla\underset{\sim}{\zeta}(\underset{\sim}{X}))d\underset{\sim}{X}.$$

Then, by (v), $g'(t) \equiv 0$. Hence $g(1) = g(0)$, which is (i). □

Theorem 3.2 Let $\phi : M^{3\times3} \longrightarrow \mathbb{R}$ be continuous. Let $p \geq 1$ and suppose that $|\phi(F)| \leq C(1 + |F|^p)$ for some constant $C \geq 0$ and all $F \in M^{3\times3}$. If the map $\underset{\sim}{x} \longmapsto \phi(\nabla \underset{\sim}{x}(\cdot)) : W^{1,p}(\Omega) \longrightarrow L^1(\Omega)$ is sequentially weakly continuous then ϕ is a null Lagrangian.

Proof We use an argument of Morrey [22]. Let D be the unit cube $0 < X^\alpha < 1$. Let $F \in M^{3\times3}$, $\underset{\sim}{\zeta} \in \mathcal{D}(D)$, $\varepsilon > 0$. Tesselate $\mathbb{R}^3$ by cubes of side ε with faces perpendicular to the X^α axes. Let $D^{(r)} = \underset{\sim}{X}^{(r)} + \varepsilon D$ be a typical cube, and define $\underset{\sim}{x}_\varepsilon : \Omega \longrightarrow \mathbb{R}^3$ by

$$\underset{\sim}{x}_\varepsilon(\underset{\sim}{X}) = \begin{cases} F\underset{\sim}{X} + \varepsilon\underset{\sim}{\zeta}\left(\dfrac{\underset{\sim}{X} - \underset{\sim}{X}^{(r)}}{\varepsilon}\right) & \text{if } \underset{\sim}{X} \in D^{(r)} \subset \Omega \\ F\underset{\sim}{X} & \text{otherwise.} \end{cases}$$

Then

$$\nabla\underset{\sim}{x}_\varepsilon(\underset{\sim}{X}) = \begin{cases} F + \nabla\underset{\sim}{\zeta}\left(\dfrac{\underset{\sim}{X} - \underset{\sim}{X}^{(r)}}{\varepsilon}\right) & \text{if } \underset{\sim}{X} \in D^{(r)} \subset \Omega \\ F & \text{otherwise.} \end{cases}$$

Hence $|\nabla\underset{\sim}{x}_\varepsilon(\underset{\sim}{X})|$ is uniformly bounded for all $\underset{\sim}{X}$ and ε. Since $\underset{\sim}{x}_\varepsilon(\underset{\sim}{X}) \longrightarrow F\underset{\sim}{X}$ uniformly in $\bar{\Omega}$ as $\varepsilon \longrightarrow 0$, it follows that $\underset{\sim}{x}_\varepsilon(\cdot) \longrightarrow F\underset{\sim}{X}$ in $W^{1,p}(\Omega)$ as $\varepsilon \longrightarrow 0$. But if $D^{(r)} \subset \Omega$ then

$$\int_{D^{(r)}} \phi(\nabla\underset{\sim}{x}_\varepsilon(\underset{\sim}{X}))d\underset{\sim}{X} = \varepsilon^3 \int_D \phi(F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y}.$$

As the number of cubes $D^{(r)} \subset \Omega$ is of order $\dfrac{1}{\varepsilon^3}$ X volume of Ω we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \phi(\nabla\underset{\sim}{x}_\varepsilon(\underset{\sim}{X}))d\underset{\sim}{X} = \int_D \phi(F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y} \times \text{volume of } \Omega.$$

Since $\phi(\nabla\underset{\sim}{x}(\cdot))$ is sequentially weakly continuous it follows that

$$\int_D \phi(F + \nabla\underset{\sim}{\zeta}(\underset{\sim}{Y}))d\underset{\sim}{Y} = \int_D \phi(F)d\underset{\sim}{Y}.$$

Hence ϕ is a null Lagrangian by Theorem 3.1. □

The study of sufficient conditions for sequential weak continuity hinges on the identities (2.4), (2.5). First we give a distributional meaning to these identities. (For the definition of the dual space $\mathcal{D}'(\Omega)$ see Schwartz [33].)

Lemma 3.3 (a) If $\underset{\sim}{x} \in W^{1,2}(\Omega)$ then $\operatorname{adj} \nabla \underset{\sim}{x} \in L^1(\Omega)$ and formula (2.4) holds in $\mathcal{D}'(\Omega)$. (b) If $\underset{\sim}{x} \in W^{1,p}(\Omega)$, $p \geq 2$, and $\operatorname{adj} \nabla \underset{\sim}{x} \in L^{p'}(\Omega)$, (where $\frac{1}{p} + \frac{1}{p'} = 1$) then $\det \nabla \underset{\sim}{x} \in L^1(\Omega)$ and formula (2.5) holds in $\mathcal{D}'(\Omega)$.

Proof. (a) Let $\underset{\sim}{x} \in W^{1,2}(\Omega)$. Clearly $\operatorname{adj} \nabla \underset{\sim}{x} \in L^1(\Omega)$. Formula (2.4) holds in $\mathcal{D}'(\Omega)$ if and only if

$$\int_\Omega (\operatorname{adj} \nabla \underset{\sim}{x})^\alpha_i \, \phi \, d\underset{\sim}{X} = - \int_\Omega \tfrac{1}{2} \varepsilon_{ijk} \varepsilon^{\alpha\beta\gamma} x^j x^k_{,\gamma} \phi_{,\beta} \, d\underset{\sim}{X} \quad \text{for all } \phi \in \mathcal{D}(\Omega). \tag{3.6}$$

But (3.6) holds trivially if $\underset{\sim}{x} \in C^\infty(\Omega)$, and $C^\infty(\Omega)$ is norm dense in $W^{1,2}(\Omega)$. Since both sides of (3.6) are continuous functions of $\underset{\sim}{x} \in W^{1,2}(\Omega)$ the result follows.

(b) Let $\underset{\sim}{x} \in W^{1,p}(\Omega)$, $\operatorname{adj} \nabla \underset{\sim}{x} \in L^{p'}(\Omega)$. Then $\det \nabla \underset{\sim}{x} \in L^1(\Omega)$ by Hölder's inequality. For fixed i define $\underset{\sim}{w}_{(i)}$ by $w^\alpha_{(i)} = (\operatorname{adj} \nabla \underset{\sim}{x})^\alpha_i$. If $\underset{\sim}{x} \in C^\infty(\Omega)$ then $\operatorname{Div} \underset{\sim}{w}_{(i)} \overset{\text{def}}{=} w^\alpha_{(i),\alpha} = 0$. Thus

$$\int_\Omega w^\alpha_{(i)} \phi_{,\alpha} \, d\underset{\sim}{X} = 0 \qquad \text{for all } \phi \in \mathcal{D}(\Omega). \tag{3.7}$$

Since $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ and $p \geq 2$ it follows that (3.7) holds for $\underset{\sim}{x} \in W^{1,p}(\Omega)$. Let

$$\rho \in \mathcal{D}(\mathbb{R}^3), \ \rho \geq 0, \int_{\mathbb{R}^3} \rho(X) dX = 1,$$

and define the mollifier ρ_k in the usual way by $\rho_k(\underset{\sim}{X}) = k^3 \rho(k\underset{\sim}{X})$. Extend $\underset{\sim}{w}_{(i)}$ by zero outside Ω. Then the convolution $\rho_k * \underset{\sim}{w}_{(i)} \in C^\infty(\mathbb{R}^3)$ and

$\rho_k * \underset{\sim}{w}_{(i)} \longrightarrow \underset{\sim}{w}_{(i)}$ in $L^{p'}(\mathbb{R}^3)$ as $k \longrightarrow \infty$. Fix $\rho \in \mathcal{D}(\Omega)$. If k is large enough, then by (3.7),

$$\text{Div}\,(\rho_k * \underset{\sim}{w}_{(i)})(\underset{\sim}{X}) = \int_{\mathbb{R}^3} \rho_{k,\alpha}(\underset{\sim}{X}-\underset{\sim}{Y}) w^{\alpha}_{(i)}(\underset{\sim}{Y})\, d\underset{\sim}{Y} = 0 \tag{3.8}$$

for all $\underset{\sim}{X} \in \text{supp}\,\phi$. Let $S \subset \mathbb{R}^3$ be an open ball containing supp ϕ, and let $\underset{\sim}{x}_{(k)} \in C^{\infty}(\Omega)$, $\underset{\sim}{x}_{(k)} \longrightarrow \underset{\sim}{x}$ in $W^{1,p}(\Omega)$. Then, using (3.8),

$$\int_{\Omega} \tfrac{1}{3} x^i_{(k),\alpha} (\rho_k * \underset{\sim}{w}_{(i)})^{\alpha} \phi\, dX = \int_{S} \text{Div}\,[\tfrac{1}{3} x^i_{(k)} (\rho_k * \underset{\sim}{w}_{(i)})\,\phi]\, d\underset{\sim}{X}$$

$$- \int_{\Omega} \tfrac{1}{3} x^i_{(k)} (\rho_k * \underset{\sim}{w}_{(i)})^{\alpha} \phi_{,\alpha}\, d\underset{\sim}{X}$$

$$= - \int_{\Omega} \tfrac{1}{3} x^i_{(k)} (\rho_k * \underset{\sim}{w}_{(i)})^{\alpha} \phi_{,\alpha}\, d\underset{\sim}{X}.$$

Let $k \longrightarrow \infty$. Then

$$\int_{\Omega} \tfrac{1}{3} x^i_{,\alpha} w^{\alpha}_{(i)} \phi\, d\underset{\sim}{X} = - \int_{\Omega} \tfrac{1}{3} x^i w^{\alpha}_{(i)} \phi_{,\alpha}\, d\underset{\sim}{X},$$

which is (2.5). □

The main result of this section is

<u>Theorem 3.4</u> (a) Let $p \geq 2$. If $\underset{\sim}{x}_{(r)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$ then adj $\nabla \underset{\sim}{x}_{(r)} \longrightarrow$ adj $\nabla \underset{\sim}{x}$ in $\mathcal{D}'(\Omega)$. (b) Let $p \geq 2$. If $\underset{\sim}{x}_{(r)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$ and adj $\nabla \underset{\sim}{x}_{(r)} \rightharpoonup$ adj $\nabla \underset{\sim}{x}$ in $L^{p'}(\Omega)$, then det $\nabla \underset{\sim}{x}_{(r)} \longrightarrow$ det $\nabla \underset{\sim}{x}$ in $\mathcal{D}'(\Omega)$.

<u>Proof.</u> (a) Fix $\phi \in \mathcal{D}(\Omega)$. By Lemma 3.3(a), for each r we have

$$\int_{\Omega} (\text{adj}\,\nabla \underset{\sim}{x}_{(r)})^{\alpha}_{i}\, \phi\, d\underset{\sim}{X} = - \int_{\Omega} \tfrac{1}{2} \varepsilon_{ijk} \varepsilon^{\alpha\beta\gamma} x^j_{(r)} x^k_{(r),\gamma} \phi_{,\beta}\, d\underset{\sim}{X}.$$

Let Ω' be an open set with $\Omega \supset \Omega' \supset \operatorname{supp}\phi$ and such that Ω' satisfies the cone condition (cf Adams [1]). Since $\underset{\sim}{x}_{(r)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega')$ it follows by the Rellich-Kondrachov theorem that $\underset{\sim}{x}_{(r)} \longrightarrow \underset{\sim}{x}$ in $L^2(\Omega')$. Hence $x^j_{(r)}\, x^k_{(r),\gamma} \rightharpoonup x^j x^k_{,\gamma}$ in $L^1(\Omega')$, so that

$$\int_\Omega (\operatorname{adj} \nabla \underset{\sim}{x}_{(r)})^\alpha_i \, \phi \, d\underset{\sim}{X} \longrightarrow \int_\Omega (\operatorname{adj} \nabla \underset{\sim}{x})^\alpha_i \, \phi \, d\underset{\sim}{X}.$$

(b) This is proved similarly using Lemma 3.3(b). □

<u>Corollary 3.5</u> (Reshetnyak [27,28]) (a) If $p > 2$ the map $\underset{\sim}{x} \longmapsto L^{p/2}(\Omega)$ is sequentially weakly continuous. (b) If $p > 3$ the map $\underset{\sim}{x} \longmapsto \det \nabla \underset{\sim}{x} : W^{1,p}(\Omega) \longrightarrow L^{p/3}(\Omega)$ is sequentially weakly continuous.

<u>Proof.</u> (a) If $\underset{\sim}{x}_{(r)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$ then $\operatorname{adj} \nabla \underset{\sim}{x}_{(r)}$ is bounded in $L^{p/2}(\Omega)$. Therefore a subsequence $\operatorname{adj} \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup H$ in $L^{p/2}(\Omega)$. By part (a) of the theorem $H = \operatorname{adj} \nabla \underset{\sim}{x}$. Hence the whole sequence converges weakly to $\operatorname{adj} \nabla \underset{\sim}{x}$.
(b) If $\underset{\sim}{x}_{(r)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$ then we have just shown that $\operatorname{adj} \nabla \underset{\sim}{x}_{(r)} \rightharpoonup \operatorname{adj} \nabla \underset{\sim}{x}$ in $L^{p/2}(\Omega)$. Also $\det \nabla \underset{\sim}{x}_{(r)}$ is bounded in $L^{p/3}(\Omega)$, so that a subsequence $\det \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup \delta$ in $L^{p/3}(\Omega)$. By part (b) of the theorem $\delta = \det \nabla \underset{\sim}{x}$. Thus $\det \nabla \underset{\sim}{x}_{(r)} \rightharpoonup \det \nabla \underset{\sim}{x}$ in $L^{p/3}(\Omega)$. □

An amusing illustration of the sequential weak continuity of $\det \nabla \underset{\sim}{x}$ can be given in two dimensions. If Ω is a bounded open subset of $\mathbb{R}^2$, and we consider functions $\underset{\sim}{x} : \Omega \longrightarrow \mathbb{R}^2$, then identical arguments to the above show that $\underset{\sim}{x} \longmapsto \det \nabla \underset{\sim}{x} : W^{1,p}(\Omega) \longrightarrow L^{p/2}(\Omega)$ is sequentially weakly continuous for $p > 2$. Take Ω to be the unit square, and consider the sequence of maps $\underset{\sim}{x}_{(r)} : \Omega \longrightarrow \mathbb{R}^2$ shown in Figure 7 obtained by folding Ω into four along the dotted lines, into four again, and so on, keeping the origin fixed. Clearly $|\nabla \underset{\sim}{x}_{(r)}{}^i_\alpha(\underset{\sim}{X})| = 1$ for all r, i, α and almost all $\underset{\sim}{X} \in \Omega$. Hence some subsequence $\underset{\sim}{x}_{(\mu)} \rightharpoonup \underset{\sim}{x}$ in $W^{1,p}(\Omega)$

for any $p > 1$. Obviously $\mathbf{x}_{(r)} \longrightarrow \mathbf{0}$ in $L^{\infty}(\Omega)$. Hence $\mathbf{x}_{(r)} \longrightarrow \mathbf{0}$ in $W^{1,p}(\Omega)$. The sequence $\det \nabla \mathbf{x}_{(r)}$ must therefore converge weakly to zero in any space $L^{p}(\Omega)$. That this is indeed the case can be seen from Figure 7, where $\det \nabla \mathbf{x}_{(r)}$ takes the values +1, -1 in the unshaded and shaded regions respectively.

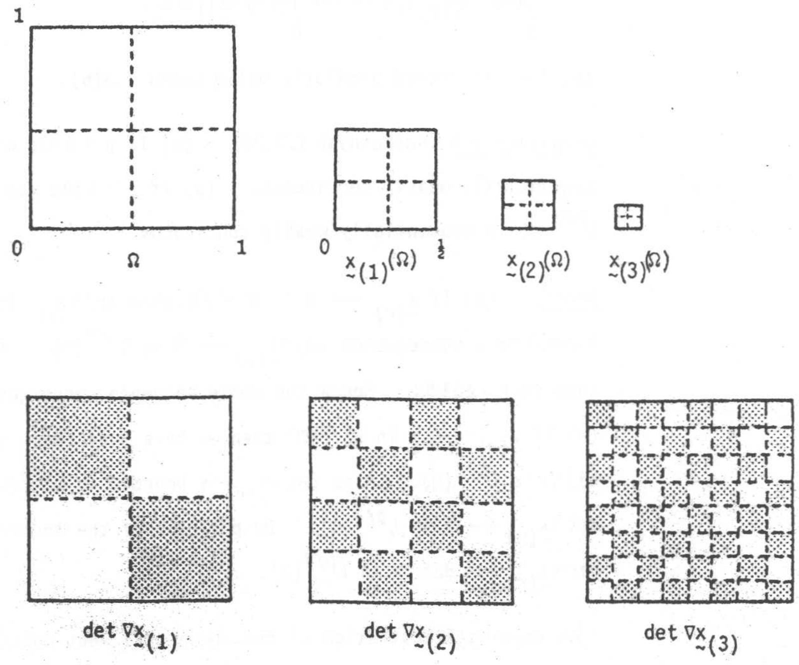


Figure 7

The results of this section carry over in a natural way to the case of maps $\mathbf{x} : \mathbb{R}^{m} \longrightarrow \mathbb{R}^{n}$, $m,n \geq 1$. See Ball [5] for details.

4 EXISTENCE OF MINIMIZERS

We confine attention to mixed displacement zero traction BVP's. A variety of other boundary conditions are treated using similar methods in [A].

We make the following hypotheses on the stored-energy function $W: \Omega \times U \longrightarrow \mathbb{R}$.

(H1) W is polyconvex, and the corresponding function $g: \Omega \times E \longrightarrow \mathbb{R}$ is continuous.

(H2) There exist constants $K > 0$, C, $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, such that

$$g(\underset{\sim}{X}, F, H, \delta) \geq C + K(|F|^p + |H|^q + |\delta|^r)$$

for all $\underset{\sim}{X} \in \Omega$, $(F, H, \delta) \in E$.

(H3) $g(\underset{\sim}{X}, a) \longrightarrow \infty$ as $a \longrightarrow \partial E$.

Remarks: The continuity hypothesis on g may be weakened (see [A]). (H3) corresponds to (1.15).

We define $g(\underset{\sim}{X}, a)$ to be $+\infty$ if $a \in \partial E$, so that by (H3) $g: \Omega \times \bar{E} \longrightarrow \bar{\mathbb{R}}$ is continuous. We suppose that the body force potential $\psi: \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ and is continuous, that the boundary $\partial\Omega$ of Ω satisfies a strong Lipschitz condition, and that $\partial\Omega_1$ is a measurable subset of $\partial\Omega$ with positive measure. We seek a minimizer for

$$I(\underset{\sim}{x}) = \int_\Omega W(\underset{\sim}{X}, \nabla \underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X} + \int_\Omega \psi(\underset{\sim}{x}(\underset{\sim}{X}))d\underset{\sim}{X} \qquad (4.1)$$

subject to

$$\underset{\sim}{x}(\underset{\sim}{X}) = \bar{\underset{\sim}{x}}(\underset{\sim}{X}) \qquad \text{for } X \in \partial\Omega_1, \qquad (4.2)$$

where $\bar{\underset{\sim}{x}}: \partial\Omega_1 \longrightarrow \mathbb{R}^3$ is a given measurable function. We require (4.2) to be satisfied in the sense of trace.

Define the admissibility set A by

$$A = \{\underset{\sim}{x} \in W^{1,p}(\Omega) : \mathrm{adj}\, \nabla \underset{\sim}{x} \in L^q(\Omega),\ \det \nabla \underset{\sim}{x} \in L^r(\Omega),\ \det \nabla \underset{\sim}{x} > 0 \text{ almost everywhere in } \Omega,\ \underset{\sim}{x} = \bar{\underset{\sim}{x}} \text{ almost everywhere in } \partial\Omega_1\}.$$

Theorem 4.1 Suppose there exists $\underset{\sim}{x}_1 \in A$ with $I(\underset{\sim}{x}_1) < \infty$. Then there exists $\underset{\sim}{x}_o \in A$ which minimizes I in A.

Proof. Since $\partial\Omega_1$ has positive measure, a result in Morrey [23 p 82] implies that there exists $k_1 > 0$ such that

$$\int_\Omega |\underset{\sim}{x}|^p\, d\underset{\sim}{X} \leq k_1 \left[\int_\Omega |\nabla \underset{\sim}{x}|^p\, dX + \left(\int_{\partial\Omega_1} |\bar{\underset{\sim}{x}}|\, dS\right)^p\right] \tag{4.3}$$

for all $\underset{\sim}{x} \in W^{1,p}(\Omega)$ with $\underset{\sim}{x} = \bar{\underset{\sim}{x}}$ on $\partial\Omega_1$. Hence by (H2) we have for arbitrary $\underset{\sim}{x} \in A$,

$$I(\underset{\sim}{x}) \geq C\,(\text{volume of } \Omega) + K\left[\int_\Omega |\nabla \underset{\sim}{x}|^p\, dX + \int_\Omega |\mathrm{adj}\, \nabla \underset{\sim}{x}|^q\, d\underset{\sim}{X} + \int_\Omega (\det \nabla \underset{\sim}{x})^r\, d\underset{\sim}{X}\right]$$

$$\geq \text{const.} + K_1 \|\underset{\sim}{x}\|^p_{W^{1,p}(\Omega)} + K\left(\int_\Omega |\mathrm{adj}\, \nabla \underset{\sim}{x}|^q\, d\underset{\sim}{X} + \int_\Omega (\det \nabla \underset{\sim}{x})^r\, d\underset{\sim}{X}\right), \tag{4.4}$$

where $K_1 > 0$ is a constant.

Let $\underset{\sim}{x}_{(n)}$ be a minimizing sequence for I in A. It follows from (4.4) that some subsequence $\underset{\sim}{x}_{(\mu)}$ satisfies

$$\underset{\sim}{x}_{(\mu)} \rightharpoonup \underset{\sim}{x}_o \text{ in } W^{1,p}(\Omega),\quad \underset{\sim}{x}_{(\mu)} \rightarrow \underset{\sim}{x}_o \text{ almost everywhere in } \Omega \text{ and } \partial\Omega_1,$$

$$\mathrm{adj}\, \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup H \text{ in } L^q(\Omega),\quad \det \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup \delta \text{ in } L^r(\Omega).$$

By Theorem 3.4, $H = \operatorname{adj} \nabla \underset{\sim}{x}_o$ and $\delta = \det \nabla \underset{\sim}{x}_o$. Hence,

$$(\nabla \underset{\sim}{x}_{(\mu)}, \operatorname{adj} \nabla \underset{\sim}{x}_{(\mu)}, \det \nabla \underset{\sim}{x}_{(\mu)}) \longrightarrow (\nabla \underset{\sim}{x}_o, \operatorname{adj} \nabla \underset{\sim}{x}_o, \det \nabla \underset{\sim}{x}_o) \text{ in } L^1(\Omega).$$

Since $g(\underset{\sim}{X},\cdot,\cdot,\cdot)$ is convex, it follows by a lower semicontinuity theorem of Ekeland and Témam [10 Thm 2.1 p 226] that

$$I(\underset{\sim}{x}_o) \leqslant \varliminf_{\mu\to\infty} I(\underset{\sim}{x}_{(\mu)}).$$

But clearly $\underset{\sim}{x}_o = \bar{\underset{\sim}{x}}_o$ on $\partial\Omega_1$. Also, as $I(\underset{\sim}{x}_o) < \infty$ we must have $\det \nabla \underset{\sim}{x}_o > 0$ almost everywhere. Hence $\underset{\sim}{x}_o \in A$. This completes the proof. □

Incompressible elasticity

We retain the same hypotheses on Ω, $\partial\Omega_1$, ψ and $\bar{\underset{\sim}{x}}$, but replace (H1) - (H3) by hypotheses (H1)', (H2)' below.

Let $V = \{F \in M^{3\times3} : \det F = 1\}$.

(H1)' $W : \Omega \times V \longrightarrow \mathbb{R}$ and there exists a continuous function $g : \Omega \times M^{3\times3} \times M^{3\times3} \longrightarrow \mathbb{R}$, with $g(\underset{\sim}{X},\cdot,\cdot)$ convex, such that

$$W(\underset{\sim}{X},F) = g(\underset{\sim}{X},F, \operatorname{adj} F) \quad \text{for all } \underset{\sim}{X} \in \Omega, F \in V.$$

(H2)' There exist constants $K > 0$, C, $p \geqslant 2$, $q \geqslant p/p-1$, such that $g(\underset{\sim}{X},F,H) \geqslant C + K(|F|^p + |H|^q)$ for all $\underset{\sim}{X} \in \Omega$, $F,H \in M^{3\times3}$.

Let $A' = \{\underset{\sim}{x} \in W^{1,p}(\Omega) : \operatorname{adj} \nabla \underset{\sim}{x} \in L^q(\Omega), \det \nabla \underset{\sim}{x} = 1 \text{ almost everywhere in } \Omega, \underset{\sim}{x} = \bar{\underset{\sim}{x}}$ almost everywhere in $\partial\Omega_1\}$.

Theorem 4.2 Suppose there exists $\underset{\sim}{x}_1 \in A'$ with $I(\underset{\sim}{x}_1) < \infty$. Then there exists $\underset{\sim}{x}_o \in A'$ which minimizes I on A'.

Proof. Let $\underset{\sim}{x}_{(n)}$ be a minimizing sequence for I from A'. Since $\det \nabla \underset{\sim}{x}_{(n)} = 1$ almost everywhere in Ω, we have in particular that $\det \nabla \underset{\sim}{x}_{(n)}$ is bounded in

$L^2(\Omega)$, say. Proceeding as in the proof of Theorem 4.1, we obtain a minimizing sequence $\underset{\sim}{x}_{(\mu)} \subset A'$ with the properties

$$\underset{\sim}{x}_{(\mu)} \rightharpoonup \underset{\sim}{x}_o \text{ in } W^{1,p}(\Omega), \; \underset{\sim}{x}_{(\mu)} \longrightarrow \underset{\sim}{x}_o \text{ almost everywhere in } \Omega \text{ and } \partial\Omega_1,$$

$$\operatorname{adj} \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup \operatorname{adj} \nabla \underset{\sim}{x}_o \text{ in } L^q(\Omega), \; \det \nabla \underset{\sim}{x}_{(\mu)} \rightharpoonup \det \nabla \underset{\sim}{x}_o \text{ in } L^2(\Omega).$$

Thus $\det \nabla \underset{\sim}{x}_o = 1$ almost everywhere in Ω. Hence $\underset{\sim}{x}_o \in A'$ and we obtain the theorem as before. □

Note that in the proof of Theorem 4.2 we made essential use of the fact that the pointwise constraint $\det F = 1$ was weakly closed. The only other homogeneous constraints of this type, as we have seen in Theorem 3.2, are of the form

$$\phi(F) = A + A_i^\alpha F_\alpha^i + B_i^\alpha (\operatorname{adj} F)_\alpha^i + D \det F = 0,$$

where A, A_i^α, B_i^α and D are constants. One can easily show that the only <u>frame-indifferent</u> constraints of this form (i.e., ϕ satisfying $\phi(QF) = \phi(F)$ for all proper orthogonal Q) are those with $A_i^\alpha = B_i^\alpha = 0$, so that $\det F$ is specified. Note that the constraint of inextensibility (Truesdell and Noll [38 p 72]) is not included. This makes one wonder about the mathematical status of inextensible elasticity.

Note also that if $\partial\Omega_1 = \partial\Omega$ and $\bar{\underset{\sim}{x}} \in C^1(\bar{\Omega})$ then a necessary condition for A' to be nonempty is that

$$\int_\Omega \det \nabla \bar{\underset{\sim}{x}}(\underset{\sim}{X})\, d\underset{\sim}{X} = \text{volume of } \Omega. \tag{4.5}$$

<u>Open problem</u>

4. Let $\bar{\underset{\sim}{x}}$ be a diffeomorphism satisfying (4.5). Does there exist a volume-preserving diffeomorphism $\underset{\sim}{x}$ with $\underset{\sim}{x} = \bar{\underset{\sim}{x}}$ on $\partial\Omega$?

Open problem

5. Prove that under suitable hypotheses the minimizers in Theorems 4.1 and 4.2 are smooth.

A necessary prerequisite for solving the regularity problem is presumably to show that the minimizers are weak solutions of the equilibrium equations. In the case of Theorem 4.1 the problem is very delicate due to the pointwise constraint $\det F > 0$ and the associated growth condition (H3). Even the simpler case of Theorem 4.2 presents serious difficulties; formally one could regard $\det \nabla \underset{\sim}{x} = 1$ as a Banach space valued constraint and apply an appropriate Lagrange multiplier theorem, identifying the Lagrange multiplier with the familiar hydrostatic pressure of incompressible elasticity. However, satisfying the hypotheses of standard Lagrange multiplier theorems is not straightforward because it is not a priori obvious that the minimizer $\underset{\sim}{x}_0$ is invertible, even in the case of a pure displacement BVP. One could minimize in a class of invertible functions, but then other difficulties arise. The only result on weak solutions I know of avoids these problems by assuming that they have already been overcome.

In addition to the hypotheses of Theorem 4.1 we suppose that $p \geq q \geq r$ and

(H4) $W(\underset{\sim}{X}, \cdot) \in C^1(U)$ for each $\underset{\sim}{X} \in \Omega$, and for each $d > 0$ there exists a constant $C(d)$ such that

$$\left|\frac{\partial W(\underset{\sim}{X}, F)}{\partial F}\right| \leq C(d)\,(1 + |F|^p + |\mathrm{adj}\, F|^q + (\det F)^r)$$

for all $\underset{\sim}{X} \in \Omega$, $F \in M^{3\times 3}$ with $\det F \geq d$.

Let ψ be C^1, and if $p < 3$ assume that there exist constants C_1 and γ, $1 < \gamma < \frac{3p}{3-p}$, $\gamma > 1$ arbitrary if $p = 3$, such that

$$\left|\frac{\partial\psi}{\partial \underset{\sim}{x}}\right| \leq C_1 (1 + |\underset{\sim}{x}|^{\gamma})$$

for all $\underset{\sim}{x} \in \mathbb{R}^3$.

Theorem 5.1 Let $\underset{\sim}{x}_o$ be the minimizer of Theorem 4.1. Suppose that $\det \nabla \underset{\sim}{x}_o(\underset{\sim}{X}) \geq d_1 > 0$ almost everywhere in some open subset E of Ω. Then $\underset{\sim}{x} = \underset{\sim}{x}_o$ satisfies the Euler-Lagrange equation

$$\int_E \left[\frac{\partial\psi}{\partial x^i} v^i + \frac{\partial W}{\partial F^i_\alpha} v^i_{,\alpha}\right] d\underset{\sim}{X} = 0 \quad \text{for all } \underset{\sim}{v} \in \mathcal{D}(E). \tag{5.1}$$

Proof. Let $\underset{\sim}{v} \in \mathcal{D}(E)$. Since $p \geq q \geq r$ it follows that $\underset{\sim}{x}_o + \varepsilon\underset{\sim}{v} \in A$ for small enough $|\varepsilon|$. Also there exists a constant d such that $\det \nabla(\underset{\sim}{x}_o + \varepsilon\underset{\sim}{v})(\underset{\sim}{X}) \geq d > 0$ for almost all $\underset{\sim}{X} \in E$ and all small enough $|\varepsilon|$.

We must show that $\frac{d}{d\varepsilon} I(\underset{\sim}{x}_o + \varepsilon\underset{\sim}{v})\Big|_{\varepsilon=0}$ exists and is given by the left hand side of (5.1). But

$$\frac{I(\underset{\sim}{x}_o + \varepsilon\underset{\sim}{v}) - I(\underset{\sim}{x}_o)}{\varepsilon} = \int_E \frac{\psi(\underset{\sim}{x}_o(\underset{\sim}{X}) + \varepsilon\underset{\sim}{v}(\underset{\sim}{X})) - \psi(\underset{\sim}{x}_o(\underset{\sim}{X}))}{\varepsilon} d\underset{\sim}{X}$$

$$+ \int_E \frac{W(\underset{\sim}{X}, \nabla\underset{\sim}{x}_o(\underset{\sim}{X}) + \varepsilon\nabla\underset{\sim}{v}(\underset{\sim}{X})) - W(\underset{\sim}{X}, \nabla\underset{\sim}{x}_o(\underset{\sim}{X}))}{\varepsilon} d\underset{\sim}{X}.$$

Using the mean value theorem, hypothesis (H4) and the dominated convergence theorem, it is easily proved that the second integral tends to

$$\int_E \frac{\partial W}{\partial F^i_\alpha}(\underset{\sim}{X}, \nabla\underset{\sim}{x}_o(\underset{\sim}{X}))\, v^i_{,\alpha}(\underset{\sim}{X})\, d\underset{\sim}{X} \text{ as } \varepsilon \longrightarrow 0.$$

The first integral is treated similarly, using the facts that $\underset{\sim}{x}_o \in C(\text{supp}\, \underset{\sim}{v})$ if $p > 3$, $\underset{\sim}{x}_o \in L^{3p/(3-p)}(\text{supp}\, \underset{\sim}{v})$ if $p < 3$, $\underset{\sim}{x}_o \in L^{\gamma}(\text{supp}\, \underset{\sim}{v})$ for any $\gamma > 1$ if $p = 3$.

□

No regularity theory seems to be known for nonlinear elliptic systems of the type encountered in elasticity. Such a theory would probably require strong growth conditions on W such as (1.20) to prevent the formation of 'holes'. This is certainly what is indicated by an example due to Giusti and Miranda [15] of an analytic integrand $f(\underset{\sim}{X},\underset{\sim}{x},\nabla\underset{\sim}{x})$ with $f(\underset{\sim}{X},\underset{\sim}{x},\cdot)$ convex, such that the function

$$\underset{\sim}{x}_o(\underset{\sim}{X}) = \frac{\underset{\sim}{X}}{|\underset{\sim}{X}|}, \quad \underset{\sim}{X} \in \Omega,$$

Ω a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, is a solution of the Euler-Lagrange equations for

$$J(\underset{\sim}{x}) = \int_{\Omega} f(\underset{\sim}{X},\underset{\sim}{x},\nabla\underset{\sim}{x})d\underset{\sim}{X}.$$

For large enough n, $\underset{\sim}{x}_o(\underset{\sim}{X})$ is in fact the unique minimizer for J subject to $\underset{\sim}{x} = \underset{\sim}{x}_o$ on $\partial\Omega$. In Giusti and Miranda's example f is quadratic in $\nabla\underset{\sim}{x}$, so that (1.20) is not satisfied.

Let us say that a stored-energy function $W(\underset{\sim}{X},F)$ is of <u>slow growth</u> if (1.20) is not satisfied. W may be of slow growth and still satisfy the hypotheses of Theorem 4.1. However, it is possible to treat other such stored-energy functions by refining the sequential weak continuity results of Section 3. We first observe that the right hand sides of (2.4), (2.5) can have meaning as distributions when the conditions of Lemma 3.3 are not satisfied. To be precise, let $\underset{\sim}{x} \in W^{1,\frac{3}{2}}(\Omega)$; then $\underset{\sim}{x} \in L^3_{loc}(\Omega)$ by the imbedding theorems, and hence $x^j x^k_{,\gamma} \in L^1_{loc}(\Omega)$. Thus (note the capital letter)

$$(\text{Adj}\,\nabla\underset{\sim}{x})^{\alpha}_i \overset{\text{def}}{=} (\tfrac{1}{2}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma}x^j x^k_{,\gamma})_{,\beta}$$

is a well defined distribution. Similarly, if $\underset{\sim}{x} \in W^{1,p}(\Omega)$ and $\text{Adj}\,\nabla\underset{\sim}{x} \in L^q(\Omega)$ for $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < \frac{4}{3}$, then

$$\text{Det}\,\nabla\underset{\sim}{x} \overset{\text{def}}{=} \left(\tfrac{1}{3} x^i (\text{Adj}\,\nabla\underset{\sim}{x})^\alpha_i\right)_{,\alpha}$$

is a well defined distribution.

Lemma 3.3 says that if $\underset{\sim}{x} \in W^{1,2}(\Omega)$ then $\text{adj}\,\nabla\underset{\sim}{x} = \text{Adj}\,\nabla\underset{\sim}{x}$, and if $\underset{\sim}{x} \in W^{1,p}(\Omega)$, $\text{adj}\,\nabla\underset{\sim}{x} \in L^{p'}(\Omega)$, $p \geq 2$, then $\det\nabla\underset{\sim}{x} = \text{Det}\,\nabla\underset{\sim}{x}$. It is not always true that $\text{adj}\,\nabla\underset{\sim}{x} = \text{Adj}\,\nabla\underset{\sim}{x}$, $\det\nabla\underset{\sim}{x} = \text{Det}\,\nabla\underset{\sim}{x}$. As an example, consider the map

$$\underset{\sim}{x}(\underset{\sim}{X}) = (1 + |\underset{\sim}{X}|)\frac{\underset{\sim}{X}}{|\underset{\sim}{X}|} \quad \text{for } |\underset{\sim}{X}| < 1.$$

This map produces a spherical hole of radius 1 at $\underset{\sim}{X} = \underset{\sim}{0}$. One can check (see [A]) that $\underset{\sim}{x} \in W^{1,p}(\Omega)$ for $1 \leq p < 3$, $\text{adj}\,\nabla\underset{\sim}{x} \in L^q(\Omega)$ for $1 \leq q < \frac{3}{2}$. But $\text{Det}\,\nabla\underset{\sim}{x} \neq \det\nabla\underset{\sim}{x}$ since $\text{Det}\,\nabla\underset{\sim}{x}$ has an atom of measure $\frac{4\pi}{3}$ at $\underset{\sim}{X} = \underset{\sim}{0}$.

<u>Open problem</u>

6. Need $\det\nabla\underset{\sim}{x} = \text{Det}\,\nabla\underset{\sim}{x}$ if $\text{Det}\,\nabla\underset{\sim}{x}$ is a <u>function</u> (and a similar question for $\text{Adj}\,\nabla\underset{\sim}{x}$)?

It is obvious that the arguments of Theorem 3.4 carry over to the distributions $\text{Adj}\,\nabla\underset{\sim}{x}$, $\text{Det}\,\nabla\underset{\sim}{x}$ under weaker conditions on p. Thus one can prove the existence of minimizers for functionals of the form

$$\bar{I}(\underset{\sim}{x}) = \int_\Omega \psi(\underset{\sim}{x})d\underset{\sim}{X} + \int_\Omega G(\underset{\sim}{X}, \nabla\underset{\sim}{x}, \text{Adj}\,\nabla\underset{\sim}{x}, \text{Det}\,\nabla\underset{\sim}{x})d\underset{\sim}{X}$$

with $G(\underset{\sim}{X},\cdot,\cdot,\cdot)$ convex, under coercivity conditions on G weaker than the corresponding conditions on g in (H2). The reader is referred to [A] for details of these results. The minimization is carried out in a class of functions $\underset{\sim}{x}$ such that $\text{Adj}\,\nabla\underset{\sim}{x}$ and $\text{Det}\,\nabla\underset{\sim}{x}$ are functions, but the relationship of the integrand $G(\underset{\sim}{X}, \nabla\underset{\sim}{x}, \text{Adj}\,\nabla\underset{\sim}{x}, \text{Det}\,\nabla\underset{\sim}{x})$ to $W(\underset{\sim}{X},F) = G(\underset{\sim}{X},F,\text{adj}\,F,\det F)$ is unclear because the open problem 6 is unsolved.

6 APPLICATIONS TO SPECIFIC ELASTIC MATERIALS

We now investigate to what extent the hypotheses of our existence theorems are satisfied by accurate models of real elastic materials. We confine attention to isotropic materials. For an isotropic material W has the form

$$W(\underset{\sim}{X},F) = \Phi(\underset{\sim}{X},v_1,v_2,v_3), \tag{6.1}$$

where v_i are the eigenvalues of $\sqrt{FF^T}$ and where Φ is symmetric in the v_i. Because the transformation $F \longmapsto (v_1,v_2,v_3)$ is nonlinear, it is by no means obvious under what conditions on Φ the stored-energy function W is quasiconvex or polyconvex. The corollary to the following result gives some sufficient conditions. For the proof of the theorem see Thompson and Freede [36] and [A].

Theorem 6.1 Let $\psi(v_1,v_2,v_3)$ be a symmetric real valued function defined on $\mathbb{R}_3^+ = \{v_i \geqslant 0\}$. For $F \in M^{3\times 3}$ define

$$\sigma(F) = \psi(v_1,v_2,v_3),$$

where the v_i are the eigenvalues of $\sqrt{FF^T}$. Then σ is convex on $M^{3\times 3}$ if and only if ψ is convex and nondecreasing in each variable v_i.

Corollary 6.2 For $j = 1,2$ let $\Phi_j : \Omega \times \mathbb{R}_3^+ \longrightarrow \mathbb{R}$ be continuous and such that $\Phi_j(\underset{\sim}{X},\cdot,\cdot,\cdot)$ is symmetric, convex and nondecreasing for each $\underset{\sim}{X} \in \Omega$. Let $\Phi_3 : \Omega \times (0,\infty) \longrightarrow \mathbb{R}$ be continuous and such that $\Phi(\underset{\sim}{X},\cdot)$ is convex for each $\underset{\sim}{X} \in \Omega$.

Let

$$\Phi(\underset{\sim}{X}, v_1, v_2, v_3) = \Phi_1(\underset{\sim}{X}, v_1, v_2, v_3) + \Phi_2(\underset{\sim}{X}, v_2v_3, v_3v_1, v_1v_2) + \Phi_3(\underset{\sim}{X}, v_1v_2v_3).$$

Then W, defined by (6.1), is polyconvex.

<u>Proof.</u> By the theorem $\Phi_1(\underset{\sim}{X}, v_1, v_2, v_3)$ is convex in F for each $\underset{\sim}{X}$. Also, since th eigenvalues of $\sqrt{(\text{adj } F)(\text{adj } F)^T}$ are v_2v_3, v_3v_1, v_1v_2 it follows from the theorem that $\Phi_2(\underset{\sim}{X}, v_2v_3, v_3v_1, v_1v_2)$ is a convex function of adj F. Since $\det F = v_1v_2v_3$ we obtain the result. □

We consider a slight modification of a class of stored-energy functions introduced by Ogden [25]. For $\alpha \geq 1, \beta \geq 1$, let

$$\rho(\alpha) = v_1^\alpha + v_2^\alpha + v_3^\alpha - 3, \quad \chi(\beta) = (v_2v_3)^\beta + (v_3v_1)^\beta + (v_1v_2)^\beta - 3.$$

Consider first incompressible materials, and let

$$W(\underset{\sim}{X}, F) = \sum_{i=1}^{M} a_i(\underset{\sim}{X})\rho(\alpha_i) + \sum_{j=1}^{M} b_j(\underset{\sim}{X})\chi(\beta_j), \tag{6.2}$$

where $\alpha_1 \geq \ldots \geq \alpha_M \geq 1$, $\beta_1 \geq \ldots \geq \beta_N \geq 1$, and where a_i, b_j are continuous functions on $\bar{\Omega}$ satisfying

$$a_i(\underset{\sim}{X}) \geq 0,\ b_j(\underset{\sim}{X}) \geq 0, \quad \text{for } 1 \leq i \leq M,\ 1 \leq j \leq N,\ \underset{\sim}{X} \in \bar{\Omega},$$
$$a_1(\underset{\sim}{X}) \geq k > 0,\ b_1(\underset{\sim}{X}) \geq k > 0, \quad \text{for } \underset{\sim}{X} \in \bar{\Omega},$$

where k is some constant.

By Corollary 6.2, W is polyconvex, the continuity of W following from the convexity of $\rho(\alpha_i)$, $\chi(\beta_j)$ as functions of F, adj F respectively. (The special case of Theorem 6.1 used here is due to von Neumann [40]). The continuity of $\rho(\alpha_1)$, $\chi(\beta_1)$ also implies the existence of positive constants a,b such that

$$\rho(\alpha_1) \geqslant a|F|^{\alpha_1}, \ \chi(\beta_1) \geqslant b|\text{adj } F|^{\beta_1}$$

for all $F \in M^{3\times3}$. Therefore hypotheses (H1)', (H2)' of Theorem 4.2 are satisfied, provided

$$\alpha_1 \geqslant 2, \ \beta_1 \geqslant \frac{\alpha_1}{\alpha_1 - 1} \ . \tag{6.3}$$

As a special case, consider the inhomogeneous Mooney-Rivlin material, for which $M = N = 1$, $\alpha_1 = \beta_1 = 2$, so that

$$W(\underset{\sim}{X},F) = a_1(\underset{\sim}{X})(I_B - 3) + b_1(\underset{\sim}{X})(II_B - 3),$$

where I_B and II_B are the first two principal invariants of $B = FF^T$. Clearly (6.3) is satisfied, so that the Mooney-Rivlin material is included in the existence theory. An application to buckling of a rod made of Mooney-Rivlin material is described in [A]. The incompressible Neo-Hookean material

$$W(\underset{\sim}{X},F) = a_1(\underset{\sim}{X})(I_B - 3)$$

is not covered by Theorem 4.2. To illustrate this, consider the single-term stored-energy function

$$W(F) = \rho(\alpha_1).$$

It is not hard to show that $(H_2)'$ is satisfied if and only if $\alpha_1 \geqslant 3$. By use of Adj and Det one can reduce α_1 to $2\frac{1}{2}$, but this still does not cover the Neo-Hookean material.

Ogden curve-fitted a stored-energy function of the form (6.2) with three terms ($M = 2$, $N = 1$) to data of Treloar for homogeneous vulcanized rubber. The values of the various constants obtained by him were

$$\alpha_1 = 5.0, \quad \alpha_2 = 1.3, \quad \beta_1 = 2,$$

$$a_1 = 2.4 \times 10^{-3}, \quad a_2 = 4.8, \quad b_1 = 0.05 \text{ kg cm}^{-2}.$$

Similar values are given by Jones and Treloar [21]. Clearly (6.3) is satisfied. Furthermore, since $\alpha_1 > 3$, condition (1.20) holds. If $\partial\Omega$ satisfies a strong Lipschitz condition then by the imbedding theorem of Morrey [23] the minimizer in Theorem 4.2 belongs to $C^{0,0.4}(\bar{\Omega})$.

In the compressible case Ogden [26] considered the effect of adding a term $\Gamma(\det F)$ to (6.2). (Actually he replaced the term $\chi(2)$ by $v_1^{-2} + v_2^{-2} + v_3^{-2} - 3$, but the difference is negligible experimentally since for rubber $v_1 v_2 v_3 \approx 1$). Suppose that

$$\Gamma(t) \geq c + d\, t^r \quad \text{for all } t > 0,$$

where $d > 0$, $r > 1$ and c are constants. Assume that Γ is convex on $(0,\infty)$, $\Gamma(t) \longrightarrow \infty$ as $t \longrightarrow 0+$. Then the modified stored-energy function satisfies hypotheses (H1) - (H3) of Theorem 4.1, provided (6.3) holds.

<u>Open problem</u>

7. Find necessary and sufficient conditions on Φ for $W(\underset{\sim}{X},F)$ to be polyconvex.

A generalization of Corollary 6.2 is given in [A], but it does not solve the above problem.

7 EXISTENCE OF SEMI-INVERSE SOLUTIONS

In this section we prove the existence of minimizers for semi-inverse problems of the type discussed by Ericksen in [14] and in his article in this volume.†

Let D be a bounded open subset of $\mathbb{R}^2$. We denote coordinates in $\mathbb{R}^2$ by X^Γ, $\Gamma = 1,2$. Consider an elastic body which in a reference configuration

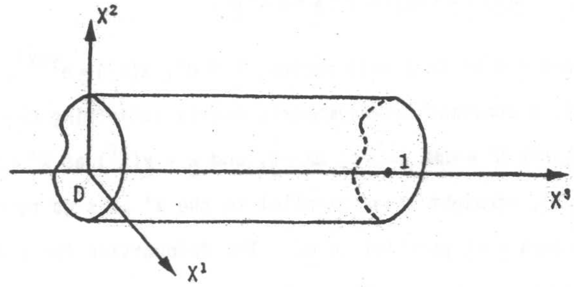


Figure 8

We make the following assumptions on W.

(A1) W is independent of X^3, so that $W : D \times U \longrightarrow \mathbb{R}$.

(A2) W is frame-indifferent (see (1.12)).

(A3) W is polyconvex and the corresponding function $g : D \times E \longrightarrow \mathbb{R}$ is continuous.

†I would like to thank Professor Ericksen for some useful discussions concerning the material in this section.

(A4) There exist constants $K > 0$, C, $p \geq 2$, $q > 1$, such that

$$g(X^\Gamma, F, H, \delta) \geq C + K(|F|^p + |H|^q)$$

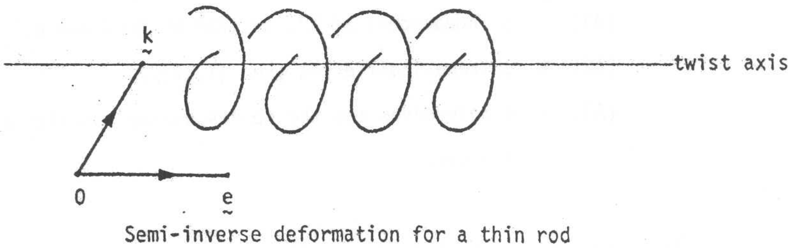
for all $(X^1, X^2) \in D$, $(F, H, \delta) \in E$.

(A5) $g(X^\Gamma, a) \longrightarrow \infty$ as $a \longrightarrow \partial E$.

For simplicity suppose that there is no body force. The semi-inverse solutions have the form

$$\underset{\sim}{x}(\underset{\sim}{X}) = \underset{\sim}{k} + R[\underset{\sim}{y}(X^\Gamma) - \underset{\sim}{k} + \alpha X^3 \underset{\sim}{e}], \qquad (7.1)$$

where $\underset{\sim}{e} \in \mathbf{R}^3$ is a unit vector, $\underset{\sim}{k} \in \mathbf{R}^3$, $R(X^3) = e^{\gamma\Omega X^3}$, α, γ are constants, and Ω is a constant skew-symmetric matrix satisfying $\Omega\underset{\sim}{v} = \underset{\sim}{e} \wedge \underset{\sim}{v}$ for all $\underset{\sim}{v} \in \mathbf{R}^3$. Clearly $R' = \gamma\Omega R$, $R\underset{\sim}{e} = \underset{\sim}{e}$, $\Omega\underset{\sim}{e} = \underset{\sim}{0}$, and $\underset{\sim}{x} = \underset{\sim}{y}(X^\Gamma)$ at $X^3 = 0$. In the deformation (7.1), straight lines parallel to the X^3 axis go to helices about the line through $\underset{\sim}{x} = \underset{\sim}{k}$ parallel to $\underset{\sim}{e}$. The deformation for a thin rod thus looks roughly as shown in Figure 9.



Semi-inverse deformation for a thin rod

Figure 9

If $\underset{\sim}{x}$ has the form (7.1) then $\nabla \underset{\sim}{x} = R(\nabla \underset{\sim}{y}, \underset{\sim}{z})$, where $\underset{\sim}{z} = \alpha \underset{\sim}{e} + \gamma \underset{\sim}{e} \wedge (\underset{\sim}{y} - \underset{\sim}{k})$. Hence by (A2)

$$W(X^\Gamma, \nabla \underset{\sim}{x}) = W(X^\Gamma, \nabla \underset{\sim}{y}, \underset{\sim}{z}). \tag{7.2}$$

Note that

$$\det \nabla \underset{\sim}{x} = \underset{\sim}{z} \cdot (\underset{\sim}{y}_{,1} \wedge \underset{\sim}{y}_{,2}). \tag{7.3}$$

Consider the functional

$$J(\underset{\sim}{y}) = \int_D W(X^\Gamma, \nabla \underset{\sim}{y}, \underset{\sim}{z})\, dS, \qquad dS = dX^1 dX^2. \tag{7.4}$$

The Euler-Lagrange equations for (7.4) are

$$\frac{\partial W}{\partial \underset{\sim}{y}} - \left(\frac{\partial W}{\partial \underset{\sim}{y}_{,\Gamma}}\right)_{,\Gamma} = \underset{\sim}{0}. \tag{7.5}$$

It is easily verified using (A1), (A2) that if (7.5) holds then, formally, so do the equilibrium equations

$$\frac{\partial}{\partial X^\alpha}\left(\frac{\partial W}{\partial x^i_{,\alpha}}\right) = 0, \tag{7.6}$$

where $\underset{\sim}{x}$ is given by (7.1). We therefore consider the problem of minimizing $J(\underset{\sim}{y})$. We will only consider the case when the curved surface of the cylinder is traction free; this is a natural boundary condition, so that for the purposes of minimization it can be ignored. Let $A = \{\underset{\sim}{y} \in W^{1,p}(D)$: $\operatorname{adj}(\nabla \underset{\sim}{y}, \underset{\sim}{0}) \in L^q(D)$, $\underset{\sim}{z} \cdot (\underset{\sim}{y}_{,1} \wedge \underset{\sim}{y}_{,2}) > 0$ for almost all $(X^1, X^2) \in D\}$.

Theorem 7.1 Let D satisfy the cone condition. Let $\underset{\sim}{e}, \underset{\sim}{k}, \alpha$ and γ be fixed, and assume $\alpha^2 + \gamma^2 \neq 0$. Suppose there exists $\underset{\sim}{y}_1 \in A$ with $J(\underset{\sim}{y}_1) < \infty$. Then there exists $\underset{\sim}{y}_o \in A$ which minimizes J on A.

To prove the theorem we need the following version of the Poincaré inequality:

Lemma 7.2 Let D satisfy the cone condition, let $p > 1$, and let $\underset{\sim}{e} \in \mathbf{R}^3$ be a unit vector. Then the inequality

$$\int_D |\underset{\sim}{y}|^p \, dS \leq \text{const.}[(\int_D \underset{\sim}{y}.\underset{\sim}{e}\, dS)^p + \int_D |\underset{\sim}{e} \wedge \underset{\sim}{y}|^p \, dS + \int_D |\nabla \underset{\sim}{y}|^p \, dS] \tag{7.7}$$

holds for all $\underset{\sim}{y} : D \longrightarrow \mathbf{R}^3$ belonging to $W^{1,p}(D)$.

Proof of lemma. We use a similar argument to Morrey [23 p 82]. Since D satisfies the cone condition and is bounded, it has only finitely many connected components. Hence without loss of generality we may assume D to be connected. Suppose the lemma is false. Then there exists a sequence $\{\underset{\sim}{y}_N\} \subset W^{1,p}(D)$ such that $\int_D |\underset{\sim}{y}_N|^p \, dS = 1$ and

$$\int_D |\nabla \underset{\sim}{y}_N|^p \, dS + \int_D |\underset{\sim}{e} \wedge \underset{\sim}{y}_N|^p \, dS + (\int_D \underset{\sim}{y}_N \cdot \underset{\sim}{e}\, dS)^p < \frac{1}{N}. \tag{7.8}$$

Thus $\underset{\sim}{y}_N$ is bounded in $W^{1,p}(D)$, so that a subsequence $\underset{\sim}{y}_\mu$ converges weakly in $W^{1,p}(D)$ to some $\underset{\sim}{y}$. Since the imbedding of $W^{1,p}(D)$ into $L^p(D)$ is compact, it follows that $\underset{\sim}{y}_\mu \longrightarrow \underset{\sim}{y}$ in $L^p(D)$. Hence

$$\int_D |\underset{\sim}{y}|^p \, dS = 1, \tag{7.9}$$

and by (7.8) and the fact that $\int_D |\nabla \underset{\sim}{y}|^p \, dS \leq \varliminf_{\mu\to\infty} \int_D |\nabla \underset{\sim}{y}_\mu|^p \, dS$, we have that

$$\nabla y = 0,\ e \wedge y = 0 \text{ for almost all } (X^1, X^2) \in D,$$

and

$$\int_D e.y\, dS = 0.$$

Since D is connected, y is constant, and hence $y = 0$. This contradicts (7.9).

Proof of theorem. Suppose first that $\gamma \neq 0$. Then $W(X^r, \nabla y, z)$ is invariant under the transformation $y \longmapsto y + \lambda e$. Hence we may without loss of generality seek a minimum for J on the set $\bar{A} = \{y \in A : \int_D y.e\, dS = 0\}$. Let $y_{(n)}$ be a minimizing sequence for J in $\bar{A}$. Then by (A4)

$$\int_D |\nabla y_{(n)}|^p\, dS \leq \text{constant}, \quad \int_D |e \wedge y_{(n)}|^p\, dS \leq \text{constant},$$

$$\int_D |\mathrm{adj}(\nabla y_{(n)}, z_{(n)})|^q\, dS \leq \text{constant}, \quad \text{where } z_{(n)} = \alpha e + \gamma e \wedge (y_{(n)} - k).$$

Hence by the lemma $\|y_{(n)}\|_{W^{1,p}(D)} \leq$ constant, and so there exists a subsequence $y_{(\mu)}$ of $y_{(n)}$ satisfying

$$y_{(\mu)} \rightharpoonup y_o \text{ in } W^{1,p}(D), \quad y_\mu \longrightarrow y_o \text{ in } L^r(D) \text{ for any } r > 1,$$

$$\mathrm{adj}(\nabla y_{(\mu)}, z_{(\mu)}) \rightharpoonup H \text{ in } L^q(D).$$

Arguments similar to those in Section 3 show that $H = \mathrm{adj}(\nabla y_o, z_o)$, where $z_o = \alpha e + \gamma e \wedge (y_o - k)$. Hence also

$$z_{(\mu)} \cdot (y_{(\mu),1} \wedge y_{(\mu),2}) = z^i_{(\mu)}\, \mathrm{adj}(\nabla y_{(\mu)}, 0)^3_i$$

converges weakly to $\underset{\sim}{z}\cdot(\underset{\sim}{y}_{0,1}\wedge\underset{\sim}{y}_{0,2})$ in $L^1(D)$. Hence by the same argument as in Theorem 4.1 we find that $\underset{\sim}{y}_0 \in \bar{A}$ and $J(\underset{\sim}{y}_0) = \inf_{\bar{y}\in\bar{A}} J(\bar{\underset{\sim}{y}})$.

If $\gamma = 0$, $\alpha \neq 0$, then $W(X^\Gamma, \nabla\underset{\sim}{y}, \underset{\sim}{z})$ is invariant under the transformation $\underset{\sim}{y} \longmapsto \underset{\sim}{y} + \underset{\sim}{a}$. Hence it is sufficient to minimize J on the set $\bar{\bar{A}} = \{\underset{\sim}{y} \in A : \int_D \underset{\sim}{y}\, dS = \underset{\sim}{0}\}$. This is done in the same way as for $\gamma \neq 0$ by using the Poincaré inequality

$$\int_D |\underset{\sim}{y}|^p\, dS < \text{const.}[|\int_D \underset{\sim}{y}\, dS|^p + \int_D |\nabla\underset{\sim}{y}|^p\, dS]$$

for all $\underset{\sim}{y} \in W^{1,p}(D)$. □

Note that because D is two-dimensional we get existence under the growth condition (A4), which is weaker than the corresponding hypothesis (H2) for the full three-dimensional problem.

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