Finite Time Blow-Up in Nonlinear Problems

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1. Introduction.

It is well known that solutions of ordinary differential equations may blow up in finite time; for example, $x(t) = \frac{1}{1-t}$ is a solution of the equation $\dot{x} = x^2$. Furthermore, for ordinary differential equations of the form

$$\dot{x} = f(x,t), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R} \ , \tag{1.1}$$

with $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ continuous, a standard existence and continuation theorem (cf Hartman [11].) asserts that finite time blow-up is equivalent to global nonexistence. More precisely, if $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ there exists a solution $x(t)$ of (1.1) with $x(t_0) = x_0$, defined on a maximal interval of existence $[t_0, t_{max})$, where $t_0 < t_{max} \leq \infty$, and if $t_{max} < \infty$ then

$$\lim_{t \nearrow t_{max}} |x(t)| = \infty \ .$$

The situation for infinite-dimensional initial value problems, such as those arising from partial differential equations, is more complicated and it is not possible to make any general statement relating blow-up and nonexistence. In part this is due to the coexistence of nonequivalent norms each of which may serve as a measure for the size of a solution.

In recent years, much experience has been gained in the use of differential inequalities for the study of global nonexistence for infinite-dimensional problems. The reader is

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referred to the papers [6], [8-9], [12-15], [18-21], [24], [29-30] for some of this work. The idea is to derive a differential inequality for a real valued functional $F(u(t))$ of the solution u of the problem under consideration. The inequality is then solved, subject to appropriate initial conditions at $t = t_0$, so as to obtain a lower bound for $F(u(t))$ that blows up at some finite time $t_1 > t_0$. If the definition of solution requires F to be finite for all time then global nonexistence has been established. It cannot in general be concluded, however, that $F(u(t))$ itself blows up at some finite time, since the maximal half-open interval of existence of the solution may be $[t_0, t_{max})$, where $t_{max} < t_1$. (See Figure 1.) An example of this phenomenon for a backwards nonlinear heat equation is described in Ball [1].

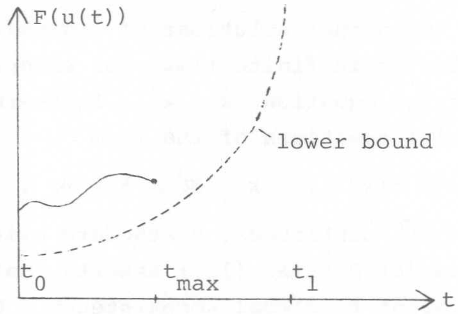


Figure 1.

Once it is known that $t_{max} < \infty$ then blow-up of u follows if a continuation theorem analogous to that described above for ordinary differential equations holds. Such a theorem will imply that <u>some</u> norm of u blows up, though not necessarily that $F(u)$ does. To obtain blow-up of $F(u)$ or other measures of u arguments based on more detailed structure of the problem may be needed.

The interpretation of blow-up theorems in physical problems often poses difficulties; blow-up may indicate either a real phenomenon or a failure of the physical model. In continuum mechanics, for example, hypotheses concerning the behaviour of constitutive functions for unbounded arguments may be necessary to establish blow-up, yet such values of the arguments may not be physically realistic. In such cases careful quantitative estimates are needed to decide on a valid interpretation. In some problems a solution may blow up in finite time with respect to one norm, yet be continuable as a solution in an appropriately weakened sense; this situation occurs for nonlinear hyperbolic equations, for which spatial derivatives of a globally defined weak solution can blow up in finite time due to shock formation (Lax [17,18]).

The preceding remarks are illustrated in this paper through the discussion of two examples. In Section 2 we consider an initial boundary value problem for a nonlinear elastic body subjected to constant pressures applied to its surface. The boundary conditions are, for example, appropriate to the case of inflation of a hollow shell under a maintained internal pressure. The stored-energy function of the elastic material is assumed to satisfy a special case of the 'concavity inequality' of Knops, Levine and Payne [13]. This assumption is interpreted as a precise statement about the weakness of the material for large strains. The concavity method [13] is adapted to show that for suitable initial conditions and pressures no weak solution can exist for all time, and that the L^2 norm of the solution possesses a lower bound which blows up in finite time. It is possible that for certain materials this result is connected with the onset of rupture. The lack of a suitable existence theory for nonlinear hyperbolic systems unfortunately prevents us from making any definite assertion concerning blow-up of the solution.

In Section 3 a model problem is considered for which an existence and continuation theorem leads to a proof of blow-up for certain solutions. The problem consists of the semilinear wave equation

$$u_{tt} = \Delta u + |u|^{\gamma-1}u, \quad t > 0,\ x \in \Omega \ ,$$

with boundary conditions

$$u\big|_{\partial\Omega} = 0 \ , \ t > 0 \ ,$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $\gamma > 1$. Blow-up of weak solutions in various norms is established for suitable initial conditions and under stated hypotheses on γ and n. The results improve those in Ball [1].

Other blow-up theorems for semilinear equations have been proved by Ball [1] for the parabolic problem

$$u_t = \Delta u + |u|^{\gamma-1}u, \quad t > 0, \ x \in \Omega \ ,$$
$$u|_{\partial\Omega} = 0, \quad t > 0 \ ,$$

(see also Weissler [31] for some relevant continuation results), and by Glassey [9,10] for the nonlinear Schrödinger equation

$$iw_t = \Delta w + |w|^{\gamma-1}w \ , \ t > 0, \ x \in \mathbb{R}^n \ .$$

In both cases strong hypotheses are made concerning the size of γ relative to n.

2. Dynamic behaviour of an elastic body under pressure.

Consider a nonlinear elastic body which occupies the bounded open set $\Omega \subset \mathbb{R}^3$ in a reference configuration. We suppose that the boundary $\partial\Omega$ of Ω is the disjoint union of piecewise smooth closed surfaces $\partial\Omega_r$ (r=1,...,M).

In a typical motion the particle occupying the point $\underset{\sim}{x} \in \Omega$ in the reference configuration is displaced to $\underset{\sim}{u}(\underset{\sim}{x},t)$ at time t. (See Figure 2.)

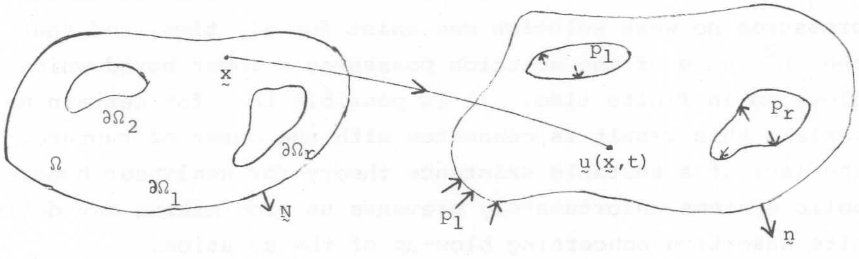


Figure 2.

Let $M_+^{3\times 3}$ denote the set of real 3×3 matrices with positive determinant. The material properties of the body are characterized by a smooth stored-energy function $W:\Omega \times M_+^{3\times 3} \to \mathbb{R}$, in terms of which the total stored energy of the body is given by

$$V = \int_\Omega W(\underset{\sim}{x},\nabla\underset{\sim}{u}(\underset{\sim}{x},t))d\underset{\sim}{x} \ .$$

Consider the initial boundary value problem

$$(2.1)\quad \rho_0(\underset{\sim}{x})\underset{\sim}{u}_{tt} = \operatorname{div}\frac{\partial W(\underset{\sim}{x},\nabla\underset{\sim}{u}(\underset{\sim}{x},t))}{\partial\nabla\underset{\sim}{u}}\ ,\ t>0,\ \underset{\sim}{x}\in\Omega\ ,$$

$$(2.2)\quad \underset{\sim}{t}(\underset{\sim}{x},t) = -p_r\underset{\sim}{n}(\underset{\sim}{x},t)\ ,\ t>0,\ \underset{\sim}{x}\in\partial\Omega_r,\ (r=1,\dots,M)\ ,$$

$$(2.3)\quad \underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}_0(\underset{\sim}{x}),\ \underset{\sim}{u}_t(\underset{\sim}{x},0) = \underset{\sim}{u}_1(\underset{\sim}{x}),\quad \underset{\sim}{x}\in\Omega\ ,$$

where $\rho_0 \in L^\infty(\Omega)$ is the density in the reference configuration, $\underset{\sim}{t}$ is the Cauchy stress vector, $\underset{\sim}{n}$ is the unit outward normal to the deformed surface, $p_r(r=1,\dots,M)$ are constant pressures, and $\underset{\sim}{u}_0,\underset{\sim}{u}_1$ are sufficiently smooth given initial functions. We assume that $\operatorname{ess\,inf}_{\underset{\sim}{x}\in\Omega} \rho_0(\underset{\sim}{x}) > 0$. The hypotheses on $\partial\Omega$ imply the existence of a function $\underset{\sim}{p} \in C_0^\infty(\mathbb{R}^3)$ which takes the values p_r on $\partial\Omega_r$.

Let $\tau > 0$, and suppose that $\underset{\sim}{u}$ is a smooth solution of (2.1)- (2.3) on the time interval $[0,\tau]$. Suppose further that $\underset{\sim}{u}(\cdot,t)$ is invertible on $\bar{\Omega}$ with smooth inverse for each $t \in [0,\tau]$. Multiplying (2.1) by a smooth function $\underset{\sim}{v}(\underset{\sim}{x})$ and integrating over Ω we obtain

$$\frac{d}{dt}\int_\Omega \rho_0\underset{\sim}{u}_t\cdot\underset{\sim}{v}\,d\underset{\sim}{x} = \int_{\partial\Omega}\frac{\partial W}{\partial u^i_{,\alpha}}v^iN_\alpha dS - \int_\Omega\frac{\partial W}{\partial u^i_{,\alpha}}v^i_{,\alpha}\,d\underset{\sim}{x}\ ,$$

where $\underset{\sim}{N}$ denotes the unit outward normal to $\partial\Omega$ and where we are using the summation convention for repeated suffices. Applying the definition of $\underset{\sim}{t}$, the boundary conditions (2.2), and the divergence theorem, we obtain

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial W}{\partial u^i_{,\alpha}}v^iN_\alpha dS &= \int_{\partial\underset{\sim}{u}(\Omega)} t_iv^i ds\\ &= -\int_{\partial\underset{\sim}{u}(\Omega)} pv^in_i ds\\ &= -\int_{\underset{\sim}{u}(\Omega)}\frac{\partial(pv^i)}{\partial u^i}d\underset{\sim}{u}\\ &= -\int_\Omega (pv^i)_{,\alpha}\frac{\partial x^\alpha}{\partial u^i}\det\nabla\underset{\sim}{u}\,d\underset{\sim}{x}\ .\end{aligned}$$

Hence

$$(2.4)\quad \frac{d}{dt}\int_\Omega \rho_0\underset{\sim}{u}_t\cdot\underset{\sim}{v}\,d\underset{\sim}{x} = -\int_\Omega (pv^i)_{,\alpha}(\operatorname{adj}\nabla\underset{\sim}{u})^\alpha_i\,d\underset{\sim}{x} - \int_\Omega\frac{\partial W}{\partial u^i_{,\alpha}}v^i_{,\alpha}d\underset{\sim}{x}\ ,$$

where $\operatorname{adj}\nabla\underset{\sim}{u}$ is the transpose of the matrix of cofactors of $\nabla\underset{\sim}{u}$.

Using the facts that $\partial\Omega_r$ is closed and p_r is constant for each r one can prove (cf Sewell[28], Ball [3]) the following

Transport Lemma

$$\frac{d}{dt}\int_{\partial\underset{\sim}{u}(\Omega)} p\,\underset{\sim}{u}\cdot\underset{\sim}{n}\,ds = 3\int_{\partial\underset{\sim}{u}(\Omega)} p\,\underset{\sim}{u}_t\cdot\underset{\sim}{n}\,ds \quad .$$

(The coefficient 3 is not a misprint; $\underset{\sim}{u}(\Omega)$ and $\underset{\sim}{n}$ depend on t.)

Multiplying (2.1) by $\underset{\sim}{u}_t$, it follows from the lemma that the energy identity

$$(2.5) \qquad E(t) = E(0), \quad t \in [0,\tau] \quad ,$$

holds, where

$$(2.6)\quad E(t) \overset{\text{def}}{=} \int_\Omega [\tfrac{1}{2}\rho_0|\underset{\sim}{u}_t|^2 + W(\underset{\sim}{x},\nabla\underset{\sim}{u}) + \tfrac{1}{3}(pu^i)_{,\alpha}(\operatorname{adj}\nabla\underset{\sim}{u})^\alpha_i]d\underset{\sim}{x} \ .$$

With the above as motivation we make the following

Definition

Let $D \subset (W^{1,1}(\Omega))^3$. A function $\underset{\sim}{u}:[0,\tau] \to D$ is a weak solution of (2.1) - (2.3) if

(i) $\underset{\sim}{u} \in C^1([0,\tau];\ (L^2(\Omega))^3)$ and satisfies (2.3);

(ii) For any $\underset{\sim}{v} \in D$ the integrals on the right-hand side of (2.4) exist and belong to $C([0,\tau])$,

$\int_\Omega \rho_0\underset{\sim}{u}_t\cdot\underset{\sim}{v}\,d\underset{\sim}{x} \in C^1([0,\tau])$, and (2.4) holds;

(iii) $E(t)$ is well defined for all $t \in [0,\tau]$ and satisfies the energy inequality

$$E(t) \le E(0), \quad t \in [0,\tau] \quad ;$$

(iv) For each $t \in [0,\tau]$

$$\det \nabla\underset{\sim}{u}(\underset{\sim}{x},t) > 0 \quad \text{for almost all } \underset{\sim}{x} \in \Omega \quad .$$

Remarks: Property (iii) is consistent with the use of entropy conditions in the theory of nonlinear hyperbolic conservation laws; energy may be dissipated by shock waves. Property (iv) could be strengthened by requiring that $\underset{\sim}{u}$ be invertible, but we do not assume this.

We suppose that for each $\underset{\sim}{x} \in \Omega$, $F \in M^{3\times3}_+$, W satisfies

$$3W(\underset{\sim}{x},F) \ge \frac{\partial W}{\partial F^i_\alpha}(\underset{\sim}{x},F)F^i_\alpha \ .$$

We write this constitutive inequality in the abbreviated form

$$3W(F) \geq \frac{\partial W}{\partial F}(F)\cdot F \quad . \tag{C}$$

Condition (C) is a special case of the 'concavity inequality' of Knops, Levine and Payne [13]. It is also in a certain sense the opposite of a condition studied in Ball [4]. Consider a homogeneous cube of material of side $\frac{1}{\lambda}$. Fix $F \in M_+^{3\times 3}$ and consider a uniform deformation of the cube with deformation gradient $\nabla \underset{\sim}{u} = \lambda F$. The shape and size of the deformed cube is independent of λ. The total stored energy of the deformation is given by

(2.7) $$g(\lambda) = \frac{W(\lambda F)}{\lambda^3} \quad .$$

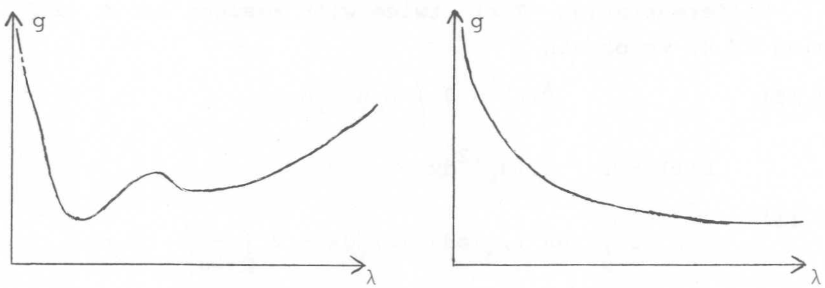
Thus the property

(2.8) $$g(\lambda) \to \infty \quad \text{as} \quad \lambda \to \infty$$

can be viewed as characterizing a material which is 'strong' for large strains. This was the condition proposed in Ball [4]. If we also suppose that

(2.9) $$W(F) \to \infty \quad \text{as} \quad \det F \to 0$$

(i.e. that infinite energy is required to effect a compression to zero volume) then clearly $g(\lambda)$ tends to infinity as $\lambda \to 0+$. Hence the graph of g for a strong material has the general form shown in Figure 3(a).



(a) (b)

Figure 3

A condition which might be satisfied by a 'weak' material is that $g'(\lambda) \leq 0$ (see Figure 3b). Differentiating (2.7) it is clear that this condition is equivalent to (C). Condition (C) is satisfied, for example, by stored energy functions of the form

$$W(F) = \operatorname{tr}(FF^T) + h(\det F) \quad ,$$

where $sh'(s) \leq h(s)$ for all $s > 0$; such W can satisfy (2.9).

Theorem 2.1

Let W satisfy condition (C). Let $E(0) < 0$, or $E(0) = 0$ and $\int_\Omega \rho_0 \underset{\sim}{u}_0 \cdot \underset{\sim}{u}_1 d\underset{\sim}{x} > 0$. Let $\underset{\sim}{u}$ be a weak solution of (2.1) - (2.3) on $[0,\tau]$, and define

$$F(t) = \int_\Omega \rho_0 |\underset{\sim}{u}(\underset{\sim}{x},t)|^2 d\underset{\sim}{x} \quad .$$

Then

$$F(t) \geq \frac{a}{(1-kt)^4} \quad ,$$

where a and k are positive constants depending on $\int_\Omega \rho_0 |\underset{\sim}{u}_0|^2 d\underset{\sim}{x}$, $\int_\Omega \rho_0 \underset{\sim}{u}_0 \cdot \underset{\sim}{u}_1 d\underset{\sim}{x}$ and $E(0)$. In particular, if $[0,t_{max})$ is the maximal half-open interval of existence of a weak solution $\underset{\sim}{u}$, then

$$t_{max} \leq k^{-1} < \infty \quad .$$

Proof

Differentiating $F(t)$ twice with respect to t and using (2.4) we obtain

$$\dot{F}(t) = 2 \int_\Omega \rho_0 \underset{\sim}{u} \cdot \underset{\sim}{u}_t d\underset{\sim}{x} \quad , \tag{2.10}$$

$$\begin{aligned} \ddot{F}(t) = {} & 2 \int_\Omega \rho_0 |\underset{\sim}{u}_t|^2 d\underset{\sim}{x} - \\ & 2 \int_\Omega (pu^i)_{,\alpha} (\operatorname{adj} \nabla \underset{\sim}{u})_i^\alpha d\underset{\sim}{x} - 2 \int_\Omega \frac{\partial W}{\partial u^i_{,\alpha}} u^i_{,\alpha} d\underset{\sim}{x} \quad . \end{aligned} \tag{2.11}$$

Substituting for the second integral in (2.11) from the energy inequality gives

$$\begin{aligned} \ddot{F}(t) \geq {} & 5 \int_\Omega \rho_0 |\underset{\sim}{u}_t|^2 d\underset{\sim}{x} + \\ & 2 \int_\Omega [3W(\underset{\sim}{x}, \nabla \underset{\sim}{u}) - \frac{\partial W}{\partial u^i_{,\alpha}} (\underset{\sim}{x}, \nabla \underset{\sim}{u}) u^i_{,\alpha}] d\underset{\sim}{x} - 6E(0) \quad . \end{aligned} \tag{2.12}$$

Suppose that $E(0) \leq 0$ and $\int_\Omega \rho_0 \underset{\sim}{u}_0 \cdot \underset{\sim}{u}_1 d\underset{\sim}{x} > 0$. By condition (C) and (2.12),

$$\ddot{F}(t) \geq 5 \int_\Omega \rho_0 |\underset{\sim}{u}_t|^2 d\underset{\sim}{x} \quad .$$

Multiplying by $F(t)$ and using (2.10) and Schwarz's inequality we get

$$F(t)\ddot{F}(t) - \frac{5 \cdot 2}{4} F^2(t) \geq 0 \quad . \tag{2.13}$$

Let $g(t) = F^{-\frac{1}{4}}(t)$. Then (2.13) becomes

$$\ddot{g}(t) \leq 0 \quad .$$

But $\dot{g}(0) = -\frac{1}{4} F^{-\frac{5}{4}}(0)\dot{F}(0) < 0$. Since

$$g(t) \leq g(0) + \dot{g}(0)t$$

the result follows with

$$a = \int_\Omega \rho_0 |\underset{\sim}{u}_0|^2 d\underset{\sim}{x}, \; k = \int_\Omega \rho_0 \underset{\sim}{u}_0 \cdot \underset{\sim}{u}_1 d\underset{\sim}{x} / 2\int_\Omega \rho_0 |\underset{\sim}{u}_0|^2 d\underset{\sim}{x} \quad .$$

In the case $E(0) < 0$ it follows from (2.12) that $\dot{F}(t_1) > 0$ if $t_1 > \dot{F}(0)/6E(0)$. Then $\dot{g}(t_1) < 0$, $E(t_1) < 0$ so that the previous argument applies. □

Remarks

1. The pressure term in (2.6) may be written in the form

$$\sum_{r=1}^{M} p_r \int_{\underset{\sim}{u}(\partial\Omega_r)} \underset{\sim}{u} \cdot \underset{\sim}{n} \, ds \quad .$$

   Thus if $\int_{\underset{\sim}{u}(\partial\Omega_r)} \underset{\sim}{u}_0 \cdot \underset{\sim}{n} \, ds \neq 0$ for some r it is possible to satisfy the condition $E(0) \leq 0$ by a suitable choice of pressures.
2. Some results in the case $E(0) > 0$ can be obtained using methods of Knops, Levine and Payne [13].
3. If W satisfies

$$3W - \frac{\partial W}{\partial F} \cdot F \geq k$$

   for some constant k, then $W - \frac{k}{3}$ satisfies condition (C), so that Theorem 2.1 may be applied.

4. Some information concerning the stability of equilibrium solutions for certain pressure boundary value problems is contained in Coleman and Dill [5].

3. Blow-up for a semilinear wave equation.

Consider the problem

$$(3.1)\qquad \begin{aligned} &u_{tt} = \Delta u + |u|^{\gamma-1}u, \quad t > 0,\ x \in \Omega,\\ &u\big|_{\partial\Omega} = 0,\ t \geq 0;\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and where $\gamma > 1$ is a constant satisfying $\gamma \leq \frac{n}{n-2}$ if $n \geq 3$.

Let $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$ and define $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ with $D(A) = (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \times W_0^{1,2}(\Omega)$. It is well known that A generates a strongly continuous group $T(\cdot)$ of bounded linear operators on X. Let $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ |u|^{\gamma-1}u \end{pmatrix}$. We write (3.1) in the form

$$(3.2)\qquad \dot{w} = Aw + f(w),\ w(0) = \varphi,$$

where $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$. We assume that $\varphi = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ belongs to X. A weak solution w of (3.1) is by definition a solution of the integral equation

$$(3.3)\qquad w(t) = T(t)\varphi + \int_0^t T(t-s)f(w(s))ds \quad .$$

The corresponding function u satisfies (3.1) in the sense of distributions. The hypotheses on γ imply that $f:X \to X$ and is locally Lipschitz. Therefore a standard argument (cf Segal [27]) gives the following existence and continuation theorem.

Proposition 3.1

There exists a unique maximally defined weak solution $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$, $w \in C([0,t_{max});X)$, $t_{max} > 0$, of (3.1). If $t_{max} < \infty$ then

$$\lim_{t \nearrow t_{max}} \|w(t)\|_X = \infty \quad .$$

It can be shown (cf Reed [26], Ball [2]) that the solution u in Proposition 3.1 satisfies the energy equation

$$(3.4)\quad E(u(\cdot,t),u_t(\cdot,t)) = E(u_0,u_1), \quad t \in [0,t_{max}) \quad ,$$

where $E:X \to \mathbb{R}$ is defined by

$$E(v,y) = \int_\Omega [\tfrac{1}{2}|y|^2 + \tfrac{1}{2}|\nabla v|^2 - \tfrac{1}{\gamma+1}|v|^{\gamma+1}]dx \quad .$$

<u>Notation</u>: $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote the norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$ respectively. The inner product in $L^2(\Omega)$ is written $(\ ,\ )$.

Global nonexistence results for (3.1) have been proved by Glassey [8], Levine [20] and Tsutsumi [30]. Following the work of these authors we prove

<u>Theorem 3.2</u>.

If $E_0 \overset{def}{=} E(u_0(\cdot),u_1(\cdot)) < 0$, or if $E_0 = 0$ and $(u_0,u_1) > 0$, then $t_{max} < \infty$ and

$$(3.5)\quad \lim_{t\nearrow t_{max}} \|u(t)\|_{\gamma+1} = \lim_{t\nearrow t_{max}} [\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2]^{\frac{1}{2}} = \infty \quad .$$

<u>Proof</u>

In view of (3.4) and Proposition 3.1, it suffices to show that $t_{max} < \infty$. Suppose $t_{max} = \infty$. Let $F(t) = \|u(t)\|_2^2$. Using (3.1) and (3.4) we obtain the differential inequality

$$(3.6)\quad \ddot{F}(t) \geq \frac{2(\gamma-1)}{\gamma+1} \int_\Omega |u|^{\gamma+1}dx - 4E_0 \geq kF^{(\gamma+1)/2}(t) - 4E_0 \ ,$$

where $k > 0$ is a constant. Solving this inequality under the given initial conditions leads to a contradiction. For the details the reader is referred to Ball [1]. □

In order to sharpen the blow-up result (3.5) we will make stronger hypotheses on γ.

<u>Theorem 3.3</u>

Let the hypotheses of Theorem 3.2 hold.

(i) Let

$$1 < \gamma \leq 1 + \frac{4}{n} \quad \text{if} \quad n = 2,3 \ ,$$

$$1 < \gamma \leq \frac{n}{n-2} \quad \text{if} \quad n \geq 4 \quad .$$

Then

$$\lim_{t \nearrow t_{max}} (u,u_t)(t) = \infty \quad .$$

(ii) Let $1 \leq p < \gamma + 1$, and

$$1 < \gamma < 1 + \frac{2p}{n} \quad .$$

Suppose also that $\gamma \leq \frac{n}{n-2}$ if $n \geq 3$.

Then

$$\lim_{t \nearrow t_{max}} \|u(t)\|_p = \infty \quad .$$

Proof

(ii) Since $E_0 \leq 0$, (3.4) implies that

$$(3.7) \qquad \int_\Omega |\nabla u|^2 dx \leq \frac{2}{\gamma+1} \int_\Omega |u|^{\gamma+1} dx$$

on the interval $[0,t_{max})$.

An interpolation inequality of Gagliardo [7] and Nirenberg [23] (see also Ladyzhenskaya, Solonnikov and Ural'ceva [16]) implies that

$$(3.8) \qquad \|v\|_q \leq K\|v\|_p^{1-a}\|\nabla v\|_2^a \; , \qquad \text{for all } v \in W_0^{1,2}(\Omega) \quad ,$$

where

$$\frac{1}{q} = (1-a)\frac{1}{p} + a\left(\frac{1}{2} - \frac{1}{n}\right) ,$$

$$\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{2} - \frac{1}{n} ,$$

and where the constant K depends only on p, q and n.

Applying (3.8) with $q = \gamma + 1$, noting that by hypothesis

$$\frac{1}{\gamma+1} \geq \frac{1}{2} - \frac{1}{n} ,$$

and using (3.7), we obtain

$$\|u\|_{\gamma+1} \leq K\|u\|_p^{1-a}\|\nabla u\|_2^a$$

$$\leq C\|u\|_p^{1-a}\|u\|_{\gamma+1}^{(\gamma+1)a/2} \quad ,$$

where here and below C denotes a generic constant. But

$(\gamma+1)a/2 < 1$ if and only if $\gamma < 1 + \frac{2p}{n}$. The result follows from (3.5).

(i) We first suppose that $n > 1$. Note that if $1 < \gamma < 1 + \frac{4}{n}$ then the result follows from (ii) with $p = 2$, and from the fact that $F(t)$ is convex by (3.6). Thus for $n > 1$ we need only consider the case $\gamma = 1 + \frac{4}{n}$. However, to show how the number $1 + \frac{4}{n}$ arises we shall consider the case of general γ.

Since $\|T(t)\| \le M$ it follows from (3.3) and Proposition 3.1 that

$$\int_0^{t_{max}} \|f(w(t))\|_X dt = \infty \quad .$$

Thus

$$\int_0^{t_{max}} \left(\int_\Omega |u|^{2\gamma} dx\right)^{\frac{1}{2}} dt = \infty \quad . \tag{3.9}$$

By (3.6) it suffices to prove that

$$\int_0^{t_{max}} \int_\Omega |u|^{\gamma+1} dx\, dt = \infty \quad . \tag{3.10}$$

Clearly (3.10) will follow from (3.9) if we show that u satisfies the estimate

$$\|u\|_{2\gamma}^{\gamma} \le C\|u\|_{\gamma+1}^{b(\gamma+1)} \tag{3.11}$$

with $0 < b \le 1$. Applying (3.8) with $q = 2\gamma$ and $p = \gamma+1$, noting that

$$\frac{1}{\gamma+1} > \frac{1}{2\gamma} \ge \frac{1}{2} - \frac{1}{n} \quad ,$$

and using (3.7), we obtain (3.11) with

$$b = \frac{1}{\gamma+1}\,[(1-a)\gamma + a\gamma(\gamma+1)/2] \quad .$$

But $b \le 1$ if and only if $a \le \frac{2}{\gamma(\gamma-1)}$, and a short calculation shows that this holds provided

$$\gamma^2 - \frac{4\gamma}{n} - (1 + \frac{4}{n}) \le 0 \quad .$$

The result for $n > 1$ follows.

Finally we consider the case $n = 1$. Let $Y = W_0^{1,1}(\Omega) \times L^1(\Omega)$. It is well known that A generates a strongly continuous group $T(t)$ on Y. (This follows immediately from D'Alembert's solution of the wave equation; it is false for $n > 1$ (cf Littman [22]).) It is easily verified that $f:Y \to Y$ and is locally Lipschitz. Hence there exists a unique maximally defined solution $w = \binom{u}{u_t}$, $w \in C([0,t_1);Y), t_1 > 0$, of (3.1). Clearly $t_1 \geq t_{max}$. But by (3.4)

$$\lim_{t \nearrow t_{max}} \|u(t)\|_{\gamma+1} = \infty \quad .$$

Hence

$$\lim_{t \nearrow t_{max}} \|u(t)\|_{1,1} = \infty \quad ,$$

so that $t_1 = t_{max}$.

As before we deduce that

$$(3.12) \qquad \int_0^{t_{max}} \|f(w(t))\|_Y dt = \int_0^{t_{max}} \int_\Omega |u|^\gamma dx\, dt = \infty \quad .$$

It follows immediately that

$$\int_0^{t_{max}} \int_\Omega |u|^{\gamma+1} dx\, dt = \infty \quad ,$$

so that (i) holds. □

<u>Remarks</u>

1. Equations (3.9), (3.10) and (3.12) may be interpreted as statements about rate of blow-up. By integrating the inequality in (3.6) one may also obtain the upper bound

$$\|u(t)\|_2 \leq C(t_{max} - t)^{2/(1-\gamma)} \quad .$$

2. Similar results to Theorems 3.2 and 3.3 can be obtained by exactly the same methods for the problem

$$u_{tt} = \Delta u + f(u)$$

$$u|_{\partial\Omega} = 0 \; ; \; u(x,0) = u_0(x), \; u_t(x,0) = u_1(x) \quad ,$$

under suitable polynomial growth hypotheses on f.

3. Little seems to be known about the global behaviour of all solutions of (3.1), although some results have been obtained by Payne and Sattinger [25] based on the study of potential wells. For example, is it true that every solution either blows up in finite time or remains bounded for all time?

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