MINIMIZERS AND THE EULER-LAGRANGE EQUATIONS

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Consider the problem of minimizing an integral of the form

$$I(u) = \int_\Omega f(x,u(x),\nabla u(x))dx$$

subject to given boundary conditions, where $\Omega \subset \mathbb{R}^m$ is a bounded open set and the competing functions $u : \Omega \to \mathbb{R}^n$. Frequently it is possible to use the direct method of the calculus of variations to establish the existence of a minimizer u in an appropriate Sobolev space $W^{1,p}(\Omega;\mathbb{R}^n)$. Then formally we expect that u satisfies the weak form of the Euler-Lagrange equations

$$\int_\Omega \left[\frac{\partial f}{\partial u^i_{,\alpha}} \varphi^i_{,\alpha} + \frac{\partial f}{\partial u^i} \varphi^i\right]dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega;\mathbb{R}^n), \tag{1}$$

but a search of the literature reveals that in general the theorems guaranteeing this make stronger growth assumptions on f than are necessary to prove existence. That this is not just a technical difficulty can be seen from one-dimensional examples due to Mizel and myself that are announced in [6]. One of these examples concerns the problem of minimizing

$$I(u) = \int_{-1}^{1} [(x^4-u^6)^2 (u')^{2r} + \varepsilon(u')^2]dx \tag{2}$$

subject to $u(-1)=-k$, $u(1)=k$, where $r \geq 14$ is an integer, $\varepsilon > 0$ and $0 < k \leq 1$. (Here $m=n=1$ and the prime denotes $\frac{d}{dx}$.) Note that the integrand $f(x,u,u')$ in (2) is smooth and <u>regular</u> (i.e., $f_{u'u'} > 0$) so that the Euler-Lagrange equation can be reduced to the form $u'' = g(x,u,u')$. Given k, let $\varepsilon > 0$ be sufficiently small. Then I attains an absolute minimum on the set $\mathcal{A} = \{v \in W^{1,1}(-1,1) : v(\pm 1) = \pm k\}$ and any minimizer u satisfies $u(0)=0$, $u'(0)=+\infty$. Furthermore $f_u \notin L^1_{loc}(-1,1)$ and hence (1) does not hold. Also, we have that

$$\inf_{\substack{v \in W^{1,\infty}(-1,1) \\ v(\pm 1) = \pm k}} I(v) > I(u) \quad \text{(the Lavrentiev phenomenon)}. \tag{3}$$

I will now sketch the most important part of the proof, which establishes (3), that $u(0)=0$, and that if $0 \leq \mu < 1$ then $|u(x)| \geq \mu k|x|^{2/3}$

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for all $x \in [-1,1]$, provided $\varepsilon > 0$ is sufficiently small. The argument is an adaptation of Mania [9] (cf. Cesari [8, p.514]). Further details can be found in Ball and Mizel [7]. Let v be any element of $\mathcal{A}$. Then $v(x_0) = 0$ for some $x_0 \in (-1,1)$ and by symmetry we can suppose that $x_0 \geq 0$. Suppose further either that $x_0 \neq 0$ or $x_0 = 0$ and $0 < v(\bar{x}) < \mu k \bar{x}^{2/3}$ for some $\bar{x} \in (0,1)$. Let $\mu < \nu < 1$. In either case there exists an interval (x_1,x_2), $0 < x_1 < x_2 < 1$, on which $\mu k x^{2/3} \leq v(x) \leq \nu k x^{2/3}$ and such that $v(x_1) = \mu k x_1^{2/3}$, $v(x_2) = \nu k x_2^{2/3}$. On this interval $(x^4 - v^6)^2 \geq x^8(1-(\nu k)^3)^2$, and hence

$$I(v) \geq (1-(\nu k)^3)^2 \int_{x_1}^{x_2} x^8 (v')^{2r} dx.$$

Putting $y = x^\theta$, where $\theta = \frac{2r-9}{2r-1}$, we get, using Jensen's inequality

$$\int_{x_1}^{x_2} x^8 (v')^{2r} dx = \theta^{2r-1} \int_{x_1^\theta}^{x_2^\theta} \left(\frac{dv}{dy}\right)^{2r} dy$$

$$\geq \theta^{2r-1} k^{2r} \frac{(\nu x_2^{2/3} - \mu x_1^{2/3})^{2r}}{(x_2^\theta - x_1^\theta)^{2r-1}} \stackrel{\text{def}}{=} h(x_1,x_2).$$

It is easily verified that if $r \geq 14$ then $\inf_{0<x_1<x_2<1} h(x_1,x_2) > 0$, and it follows that $I(v) \geq \eta > 0$ for all v as above, η being independent of ε. Now let $\bar{v}(x) = |x|^{2/3} \operatorname{sign} x$ for $|x| \leq k^{3/2}$, $\bar{v}(x) = k$ for $x > k^{3/2}$, $\bar{v}(x) = -k$ for $x < -k^{3/2}$. Then $\bar{v} \in \mathcal{A}$ and

$$I(\bar{v}) = 2\varepsilon \int_0^{k^{3/2}} (\tfrac{2}{3} x^{-1/3})^2 dx,$$

which is less than η if ε is sufficiently small. Thus $u(0) = 0$, $|u(x)| \geq \mu k |x|^{2/3}$ for any minimizer u, and (3) holds. As far as we are aware the examples in [6,7] are the first in which the singular set in Tonelli's partial regularity theorem [10, p. 359] has been shown to be nonempty.

I now turn to nonlinear elastostatics, which in fact motivated the work in [6,7]. Consider a simple mixed boundary value problem in which it is required to minimize

$$I(u) = \int_\Omega W(\nabla u(x)) dx$$

on the set $\mathcal{A} = \{u \in W^{1,1}(\Omega;\mathbb{R}^n) : I(u) < \infty, \ u|_{\partial\Omega_1} = u_0$ in the sense of trace$\}$. Here $\Omega \subset \mathbb{R}^n$ is a strongly Lipschitz bounded open set, $\partial\Omega_1 \subset \partial\Omega$ has positive $n-1$ dimensional measure, and $W : M^{n\times n} \to \mathbb{R}^+ \cup \{+\infty\}$ is the stored-energy function of a homogeneous material. We suppose that $W \in C^1(M_+^{n\times n})$, where $M_+^{n\times n} = \{A \in M^{n\times n} : \det A > 0\}$, that $W(A) = +\infty$ if $\det A \leq 0$, $W(A) \to +\infty$ as $\det A \to 0+$, and that for some $\varepsilon_0 > 0$

$$\left|\frac{\partial W}{\partial A}(CA)A^T\right| \leq \text{const.}\ (W(A) + 1) \tag{4}$$

for all $A, C \in M_+^{n\times n}$ with $|C - 1| < \varepsilon_0$. Let u minimize I in $\mathcal{A}$; extra hypotheses guaranteeing the existence of a minimizer can be found in [1,5]. Let $v : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 with ∇v uniformly bounded and $v \circ u_0|_{\partial\Omega_1} = 0$. Define for $\varepsilon > 0$

$$u_\varepsilon(x) = u(x) + \varepsilon v(u(x)).$$

Then it is not hard to show that $u_\varepsilon \in \mathcal{A}$ and that

$$\frac{d}{d\varepsilon} I(u_\varepsilon)\Big|_{\varepsilon=0} = \int_\Omega \frac{\partial W}{\partial u^i_{,\alpha}}(\nabla u) u^j_{,\alpha} v^i_{,j}(u(x)) dx = 0. \tag{5}$$

Under further hypotheses (c.f. [2]) u is invertible and (5) can then be recognized as a weak form of the Cauchy equilibrium equations

$$\frac{\partial}{\partial u^j} T^j_i = 0,$$

where T^j_i is the Cauchy stress tensor. If instead we define for $v|_{\partial\Omega} = 0$,

$$u_\varepsilon(x) = u(z), \quad x = z + \varepsilon v(z),$$

and make an analogous hypothesis to (4), we obtain the weak form of the equation

$$\frac{\partial}{\partial x^\beta}\left(W\delta^\alpha_\beta - u^i_{,\beta} \frac{\partial W}{\partial u^i_{,\alpha}}\right) = 0. \tag{6}$$

Details of these results will appear in [3]. To obtain the weak form

$$\int_\Omega \frac{\partial W}{\partial u^i_{,\alpha}} \varphi^i_{,\alpha} dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega;\mathbb{R}^n) \tag{7}$$

of the equilibrium equations one would need to show that $I(u_\varepsilon)$ is differentiable with respect to ε, with the obvious derivative, for a large class of variations $u_\varepsilon(x) = u(x) + \varepsilon\varphi(x)$, and it is not clear how to do this under any realistic hypotheses on W. The one-dimensional examples suggest that infinite values of $\nabla u(x)$ or $\nabla u(x)^{-1}$ may occur in minimizers; this could be the source of the difficulty, and may also be relevant to the onset of fracture.

Finally I remark that the Lavrentiev phenomenon severely restricts the class of numerical methods capable of detecting singular minimizers; see [4].

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