DOES RANK-ONE CONVEXITY IMPLY QUASICONVEXITY ?

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1. Background

Let $\Omega \subset R^m$ be a bounded open set. Let $M^{n\times m}$ denote the set of real $n \times m$ matrices and suppose that $W: M^{n\times m} \to \overline{R}$ is Borel measurable and bounded below. (Here $\overline{R}$ denotes the extended real line with its usual topology.) We are interested in the problem of minimizing

$$I(u) = \int_\Omega W(Du(x))dx \tag{1.1}$$

among functions $u \in W^{1,1}(\Omega;R^n)$ satisfying appropriate boundary conditions. An important application is to nonlinear elasticity, when $W = W(A)$ is the stored-energy function of a homogeneous material and $u(x)$ is the deformed position of the particle at $x \in \Omega$ in a reference configuration; in this case we usually take $m = n = 3$, but the cases $1 \leq m \leq n \leq 3$ are also of interest and cover certain string and membrane problems. It is convenient to allow W to take the value $+\infty$ so as to include various constraints. In compressible nonlinear elasticity ($m = n = 3$), for example, we may set $W(A) = +\infty$ for $\det A \leq b$, where $b > 0$ is a constant, to reflect the fact that infinite energy is required to make a reflection of the body or to homogeneously compress it to b times its original volume. Similarly, for an incompressible material it is convenient to set $W(A) = +\infty$ if and only if $\det A \neq 1$.

Connected with the existence and properties of minimizers for (1.1) are certain convexity conditions on W. Two of these conditions are <u>rank-one convexity</u> and <u>quasiconvexity</u>, and as we shall see the question raised in the title amounts roughly to asking whether they are the same. This has been an open problem since quasiconvexity was introduced by Morrey [21] over 30 years ago.

Most of the material in the paper is drawn from the existing literature, though the remarks in §6, §8(b), (c) are perhaps new.

2. Definitions

Let $A \in M^{n\times m}$. We say that W is rank-one convex at A if

$$W(A) \leq tW(A_1) + (1-t)W(A_2) \qquad (2.1)$$

whenever $t \in [0,1]$, $A = tA_1 + (1-t)A_2$ and $A_1, A_2 \in M^{n\times m}$ with $A_2 - A_1 = \lambda \otimes \mu$ for some vectors $\lambda \in R^n$, $\mu \in R^m$. We say that W is rank-one convex if it is rank-one convex at every $A \in M^{n\times m}$; equivalently, W is convex along all line segments in $M^{n\times m}$ whose end-points differ by a matrix of rank one.

Replacing λ by $\frac{-\lambda}{1-t}$ in (2.1) we see that W is rank-one convex at A if and only if

$$W(A) \leq tW(A + \lambda \otimes \mu) + (1-t)W(A - \frac{t}{1-t}\lambda \otimes \mu) \qquad (2.2)$$

for all $t \in (0,1)$, $\lambda \in R^n$, $\mu \in R^m$. If W is finite in a neighbourhood of A and differentiable at A it follows easily that W is rank-one convex at A if and only if

$$W(A + \lambda \otimes \mu) \geq W(A) + DW(A)(\lambda \otimes \mu) \qquad (2.3)$$

for all $\lambda \in R^n$, $\mu \in R^m$. If in addition W is twice differentiable at A then (2.3) implies that the Legendre-Hadamard condition

$$D^2W(A)(\lambda \otimes \mu, \lambda \otimes \mu) = \frac{\partial^2 W(A)}{\partial A^i_\alpha \partial A^j_\beta} \lambda^i \mu_\alpha \lambda^j \mu_\beta \geq 0 \qquad (2.4)$$

$$\text{for all } \lambda \in R^n, \mu \in R^m$$

holds. Conversely, suppose that

$$\text{dom } W := \{A \in M^{n\times m} : W(A) < \infty\}$$

is a rank-one convex set (i.e. $tA_1 + (1-t)A_2 \in \text{dom } W$ whenever $A_1, A_2 \in \text{dom } W$, $A_2 - A_1 = \lambda \otimes \mu$, $t \in [0,1]$) and open, that $W \in C^2(\text{dom } W)$ and that (2.4) holds for all $A \in \text{dom } W$. Then by integrating $\frac{d^2}{dt^2} W(A + t\lambda \otimes \mu)$ twice we see that W is rank-one convex.

For $1 \leq p \leq \infty$ and $E \subset R^m$ a bounded open set we denote by $W^{1,p}(E; R^n)$ the Sobolev space of all weakly differentiable mappings $u: E \to R^n$ such that $\|u\|_{L^p(E;R^n)} + \|Du\|_{L^p(E;M^{n\times m})} < \infty$, and by $W_0^{1,p}(E;R^n)$ the subset of $W^{1,p}(E;R^n)$ consisting of those u vanishing in the usual sense (cf. [9, p. 227]) on the boundary ∂E of E.

Let $A \in M^{n\times m}$. We say that W is $W^{1,p}$-quasiconvex at A if

$$\int_E W(A + D\phi(x))dx \geq \int_E W(A)dx = (\text{meas } E)W(A) \qquad (2.5)$$

for every bounded open set $E \subset R^m$ with meas $\partial E = 0$ and all $\phi \in W_0^{1,p}(E;R^n)$, and that W is $W^{1,p}$-quasiconvex if it is $W^{1,p}$-quasiconvex at every $A \in M^{n\times m}$. If $p = \infty$ we abbreviate $W^{1,\infty}$-quasiconvex to quasiconvex.

The definition of quasiconvexity was introduced by Morrey [21]; the generalization to $W^{1,p}$-quasiconvexity was made in [9]. The definition is independent of E in the following sense; if (2.5) holds for one nonempty bounded open set $E \subset R^m$, some $A \in M^{n\times m}$ and all $\phi \in W_0^{1,p}(E;R^n)$ then W is $W^{1,p}$-quasiconvex at A (see [20, 9 Prop. 2.3]). Clearly if $1 \leq p \leq q \leq \infty$ and W is $W^{1,p}$-quasiconvex at A then W is $W^{1,q}$-quasiconvex at A.

The open question posed in the title can now be stated precisely: is every rank-one convex function W quasiconvex? Alternatively one can modify the question by adding the hypothesis that W be continuous (or finite and continuous). Whether adding such regularity hypotheses could affect the answer is not obvious.

3. Quasiconvexity Implies Rank-One Convexity

Suppose that in (2.5) we take the function ϕ to be piecewise affine, so that E is the disjoint union of a finite number N of open simplices E_i and a set of measure zero and $D\phi(x) = A_i - A$ for a.e. $x \in E_i$, where $A_i \in M^{n\times m}$

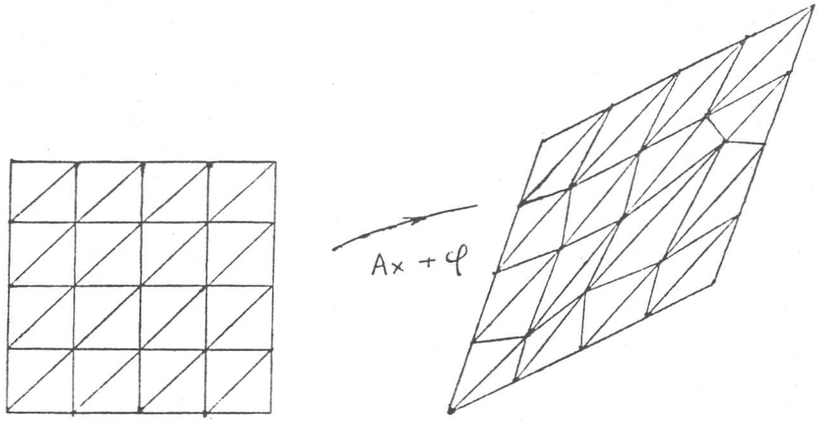

Figure 3.1

is constant. An example with $m = n = 2$ is illustrated in Figure 3.1 above. Let $\lambda_i = \frac{\text{meas } E_i}{\text{meas } E}$, so that $\sum_{i=1}^{N} \lambda_i = 1$. Then

$$A = \frac{1}{\text{meas } E} \int_E D(Ax + \phi(x))dx = \sum_{i=1}^{N} \lambda_i A_i, \tag{3.1}$$

and (2.5) becomes

$$W(\sum_{i=1}^{N} \lambda_i A_i) \le \sum_{i=1}^{N} \lambda_i W(A_i). \tag{3.2}$$

Conversely, for a polyhedral domain E and any $\phi \in W_0^{1,\infty}(E;R^n)$ there exists a bounded sequence $\phi^{(j)} \in W_0^{1,\infty}(E;R^n)$ of piecewise affine functions such that $\phi^{(j)} \to \phi$ uniformly and $D\phi^{(j)}(x) \to D\phi(x)$ a.e. in E (cf. [14, Chap. X]). Therefore, for W finite and continuous, quasiconvexity at A is equivalent to the convexity condition (3.2) holding for all piecewise affine functions. If the matrices A_i were independent then (3.2) would be equivalent to convexity of W at A. They are not independent, however, because together they form the gradient of a mapping; to understand the resulting compatibility conditions we recall an observation of Hadamard.

Let S be a smooth $(m - 1)$-dimensional surface with normal μ at the point $x \in S$. Let N be a neighbourhood of x in R^m and suppose that $u:N \to R^n$ is continuous across S and C^1 on either side of S. Let A, B denote the limits at x of Du from either side of S. Equating the tangential derivatives at x we find that

$$B - A = \lambda \otimes \mu \tag{3.3}$$

for some $\lambda \in R^n$. Thus for a piecewise affine function the gradient jumps by a matrix of rank ≤ 1 across the faces of adjoining simplices.

By choosing E to be a rectangular parallelpiped, considering piecewise affine functions with just one interior node, and using the argument of Morrey [21, p. 45], we obtain the following result.

<u>Theorem 3.1</u>

Let $A \in M^{n \times m}$. Suppose that W is quasiconvex at A, that $W(A) < \infty$ and that W is continuous at A. Then W is rank-one convex at A.

As observed in [9, p. 232], it follows that if W is continuous (with values in $\overline{R}$) and quasiconvex then W is rank-one convex. Thus for continuous W quasiconvexity is equivalent to convexity when $m = 1$ or $n = 1$. Without some continuity assumption Theorem 3.1 is false, as shown by the example (see [9, p. 232])

$$W(0) = W(a \otimes b) = 0, \quad W(A) = +\infty \text{ otherwise},$$

where $a \in R^n$, $b \in R^m$ are given nonzero vectors and $m > 1$. As discussed in [9] the moral of this example is perhaps that for general W taking infinite values some other version of the quasiconvexity condition (for example, one based on weak lower semicontinuity or that in [7]) should be taken as the basic definition. However this issue is not crucial for the main problem discussed in this paper which is unresolved even for smooth integrands.

4. Quasiconvexity as a Necessary Condition Satisfied by a Minimizer

Let $x_0 \in \Omega$. We say that $u \in W^{1,1}(\Omega;R^n)$ is a <u>strong local minimizer of I at x_0</u> if there are numbers $\rho > 0$, $\varepsilon > 0$ such that $I(v) \ge I(u)$ whenever $v \in W^{1,1}(\Omega;R^n)$ with $v(x) = u(x)$ for a.e. $x \in \Omega$ satisfying $|x - x_0| > \rho$ and $|v(x) - u(x)| \le \varepsilon$ for a.e. $x \in \Omega$.

A version of the following result was first proved by Meyers [20] (see also

Busemann & Shephard [11]).

Theorem 4.1

Assume W is continuous on dom W. Let $x_0 \in \Omega$ and let u be a strong local minimizer of I at x_0. Suppose further that u is C^1 in a neighbourhood of x_0 with $Du(x_0) = A$ and $W(A) < \infty$. Then

$$\int_E W(A + D\phi(x))dx \geq \int_E W(A)dx \tag{4.1}$$

for all bounded open subsets $E \subset R^m$ and all $\phi \in W_0^{1,\infty}(E;R^n)$ such that $\operatorname{ess\,sup}_{x \in E} W(A + D\phi(x)) < \infty$.

Idea of proof

We 'blow up' the minimization problem in a neighbourhood of x_0, so that u becomes linear. This is done by defining, for $\varepsilon > 0$ sufficiently small,

$$u_\varepsilon(x) = u(x) + \varepsilon\tilde{\phi}\left(\frac{x - x_0}{\varepsilon}\right),$$

where $\tilde{\phi}$ is ϕ extended by zero outside E, making the change of variables $x - x_0 = \varepsilon y$, and letting $\varepsilon \to 0$ in the inequality $I(u_\varepsilon) \geq I(u)$. □

Refinements of Theorem 4.1, including treatment of the case when $x_0 \in \partial\Omega$, are given in [8]. The condition (4.1) says roughly that W is quasiconvex at A, and this follows if W does not take the value $+\infty$. The proof of Theorem 3.1 still applies and gives the following result.

Corollary 4.2 (Graves [16])

Let the hypotheses of Theorem 4.1 hold. Then W is rank-one convex at A.

In view of the above discussion it would be very interesting (i) to give useful necessary and/or sufficient conditions for W to be quasiconvex at A, and (ii) to identify quasiconvexity at the values of $Du(x)$ as one of a set of sufficient conditions for u to be a local minimizer of I in, say, $W^{1,p}(\Omega;R^n)$.

5. Other Rôles Played by Rank-One Convexity and Quasiconvexity

Quasiconvexity was introduced by Morrey in connection with the direct method of the calculus of variations. In [21] he showed that if W is finite and continuous then quasiconvexity of W is a necessary condition for I to be sequentially weak*lower semicontinuous on $W^{1,\infty}(\Omega;R^n)$. (The same argument shows in general [9, p. 230] that $W^{1,p}$-quasiconvexity of W is necessary for I to be sequentially weakly (weak * if $p = \infty$) lower semicontinuous on $W^{1,p}(\Omega,R^n)$.) He then showed that quasiconvexity is sufficient for I to be sequentially weakly lower semicontinuous on $W^{1,1}(\Omega;R^n)$ provided W also satisfies certain growth conditions. Extensions of this result can be found in [20,1,19], but unfortunately they cannot be used to prove the existence of minimizers for I in nonlinear elasticity since they assume that W is everywhere finite. At present the only existence theorems applying to elasticity [3,7,9] allow W to be singular at the expense of assuming that W is polyconvex, i.e. W can be written as a convex function of minors of A of all orders r, $1 \leq r \leq \min(m,n)$. Polyconvexity implies quasiconvexity, but the converse is false [21,25,23,6].

Quasiconvexity is necessary for the existence of minimizers to certain perturbations of I. In fact the following result is proved in [9, Thm 5.1].

Theorem 5.1

Let $1 \leq p \leq \infty$, $A \in M^{n\times m}$ and $X_A = \{u \in W^{1,p}(\Omega;R^n): u - Ax \in W_0^{1,p}(\Omega;R^n)\}$. Let $\phi:[0,\infty) \to R$ be bounded and continuous with $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and set $\psi(x,u) = \phi(|u - Ax|^2)$. Assume meas $\partial\Omega = 0$. If

$$J(u) := \int_\Omega [W(Du) + \phi(x,u)]dx \tag{5.1}$$

attains an absolute minimum on X_A then W is $W^{1,p}$-quasiconvex at A.

For a given W and boundary conditions I may or may not attain a minimum. In either case it is of interest to study the behaviour of minimizing sequences of I, and it has been shown by Acerbi & Fusco [1], Dacorogna [12] (see also [2,17,13]) that they possess subsequences converging weakly in $W^{1,1}(\Omega:R^n)$ to minimizers of the relaxed functional

$$\bar{I}(u) := \int_\Omega QW(Du(x))dx, \qquad (5.2)$$

where QW denotes the supremum of all quasiconvex functions less than W. Again these results do not apply to elasticity on account of the strong growth hypotheses made.

The Euler-Lagrange equations for I are given by

$$\frac{\partial}{\partial x^\alpha}\frac{\partial W}{\partial A^i_\alpha}(Du) = 0, \quad i = 1,\ldots,n. \qquad (5.3)$$

By definition, these equations are strongly elliptic if (2.4) holds for all A with equality only if $\lambda \otimes \mu = 0$. The slightly weaker condition of strict rank-one convexity (i.e. rank-one convexity with equality in (2.1) only if $\lambda \otimes \mu = 0$ or $t = 0,1$) is necessary, and nearly sufficient, for there to be no piecewise C^1 weak solution of (5.3) whose gradient jumps across a smooth (m-1)-dimensional surface (for the details see [4]). Neither strong ellipticity nor strict quasiconvexity are sufficient to prevent weak solutions having other types of singularities, such as that occurring in cavitation [5]. However, recently Evans [15] has proved a partial regularity result for absolute minimizers of I under a strict quasiconvexity hypothesis. Although he assumes W is everywhere finite, his theorem offers the first hope of a regularity theorem applying to nonlinear elasticity.

6. Rank-One Convexity at A Does not Imply Quasiconvexity at A

Let $m > 1$, $n > 1$. Then the closed cone

$$\Lambda := \{\lambda \otimes \mu : \lambda \in R^n, \ \mu \in R^m\}$$

is a proper subset of $M^{n\times m}$. Let $A \in M^{n\times m}$ and let B be an open ball contained in $M^{n\times m} \setminus (A + \Lambda)$. Let $W \in C^\infty(M^{n\times m})$ be negative in B and zero otherwise. Since W is zero on $A + \Lambda$ it follows that W is rank-one convex at A. However, by choosing $\phi \in W_0^{1,\infty}(E;R^n)$ such that $A + D\phi(x) \in B$ on a set of positive measure we can violate (2.5), so that W is not quasiconvex at A. This simple remark shows that in general Theorem 4.1 provides a stronger necessary condition than that of Graves.

The above example is easily adapted so as to apply to isotropic nonlinear elasticity with $m = n > 1$. The isotropy is expressed by the requirement that

$$W(A) = \Phi(v_1,\ldots,v_n), \quad A \in M^{n\times m} \qquad (6.1)$$

for some symmetric function Φ of the singular values $v_i = v_i(A)$ of A (the eigenvalues of $(A^TA)^{1/2}$). Let $A = 1$, $e = (1,1,\ldots,1) \in R^n$, $\gamma > 1$, $\varepsilon > 0$, and let $\Phi \in C^\infty(R^n)$ be such that $\Phi(v) < 0$ for $|v - \gamma e| < \varepsilon$, $\Phi(v) = 0$ otherwise. We claim that for ε sufficiently small, $W(1 + \lambda \otimes \mu) = 0$ for all $\lambda,\mu \in R^n$. If not, there would exist a sequence $v^{(r)}$ converging to γe in R^n, orthogonal matrices $Q^{(r)}$, $R^{(r)}$, and vectors $\lambda^{(r)},\mu^{(r)} \in R^n$ such that

$$1 + \lambda^{(r)} \otimes \mu^{(r)} = Q^{(r)}(\mathrm{diag}\, v^{(r)})R^{(r)}. \qquad (6.2)$$

Extracting convergent subsequences and passing to the limit we find that

$$1 + \lambda \otimes \mu = \gamma Q \qquad (6.3)$$

for some $\lambda,\mu \in R^n$ and orthogonal matrix Q. This is easily seen to be impossible (for example, by evaluating QQ^T and Q^TQ). We have thus shown that for $\varepsilon > 0$ sufficiently small W is rank-one convex at 1, and the same arguments as before shows that W is not quasiconvex at 1. Note that by adding to Φ a term Φ_0, where $\delta > 0$ is sufficiently small, we can arrange that W satisfies any desired growth conditions as $|A| \to \infty$, $\det A \to 0+$, and that W be strictly rank-one convex, but not quasiconvex, at 1.

7. The Evidence Against

We collect together some remarks which might suggest that rank-one convexity does not imply quasiconvexity.

(a) The inequalities (3.2) arising from writing down the quasiconvexity condition for piecewise affine functions ϕ do not obviously follow from rank-one convexity (for example, in the case discussed in [3, p. 355], where there are 3 interior nodes). A possible riposte to this, suggested by the results of Tartar [24] on separately convex functions, is that to derive (3.2) from rank-one convexity it may be necessary to use values of the deformation gradient other than those taken by $A + D\phi(x)$.

(b) The analogous statement to 'rank-one convexity implies quasiconvexity' for integrands depending on higher derivatives of u is false. It is shown in [7, p. 146] that if $m = 2$, $n = 3$ and

$$W(D^2u) = \varepsilon_{ijk}u^i_{,11}u^j_{,12}u^k_{,22}\,, \tag{7.1}$$

where ε_{ijk} is the usual permutation symbol, then the map

$$t \to W(A + t\lambda \otimes \mu \otimes \mu)$$

is affine for every $A = (A^i_{\alpha\beta})$, $\lambda \in R^3$, $\mu \in R^2$, where $(\lambda \otimes \mu \otimes \mu)^i_{\alpha\beta} := \lambda^i\mu_\alpha\mu_\beta$, but W does not satisfy the quasiconvexity condition

$$\int_E W(A + D^2\phi(x))dx \geq \int_E W(A)dx \tag{7.2}$$

for all $\phi \in W_0^{2,\infty}(E;R^3)$, for any A.

(c) Rank-one convexity does not imply $W^{1,p}$-quasiconvexity in general. For example, if $m = n \geq 3$ and

$$W(A) = \mathrm{tr}(A^TA) + (\det A)^2, \tag{7.3}$$

Then (see [3,9]) W is polyconvex, and thus quasiconvex, but is not $W^{1,p}$-quasiconvex if $1 \leq p < n$.

8. The Evidence in Favour

The following remarks might suggest that rank-one convexity does imply quasiconvexity.

(a) If W is quadratic, that is

$$W(A) = c^{\alpha\beta}_{ij}A^i_\alpha A^j_\beta \tag{8.1}$$

for constants $c^{\alpha\beta}_{ij}$, and rank-one convex then W is quasiconvex. The only proof that seems to be known for this fact (see [26,22]) is to show that

$$I(\phi) = \int_E c^{\alpha\beta}_{ij}\,\phi^i_{,\alpha}\,\phi^j_{,\beta}\,dx \geq 0$$

for all $\phi \in W_0^{1,2}(E;R^n)$ by extending ϕ by zero outside E, taking Fourier transforms and using Plancherel's formula. Functions W formed by combining polyconvex and quadratic functions seem to be the only known examples of quasiconvex functions.

(b) For isotropic nonlinear elasticity, rank-one convexity implies that the quasiconvexity inequality holds for radial deformations.

Taking $m = n = 3$, for example, with $B = \{x \in R^3\colon\ |x| < 1\}$, a radial deformation $u\colon B \to R^3$ is one having the form

$$u(x) = \frac{r(R)}{R}x,\ R = |x|, \tag{8.2}$$

where $r\colon [0,1] \to [0,\infty)$. If $u \in W^{1,\infty}(B;R^3)$ with $\det Du(x) > 0$ a.e. $x \in B$ then $r \in W^{1,\infty}(0,1)$ with $r(0)= 0$, $r'(R) > 0$, a.e. $R \in (0,1)$ and $\sup_{R\in(0,1)} \frac{r(R)}{R} < \infty$ (cf. [5, p. 566]). If the stored-energy function $W \in C^1(M^{3\times3}_+)$, where $M^{3\times3}_+ := \{A \in M^{3\times3}\colon \det A > 0\}$, then there exists a symmetric function $\Phi = \Phi(v_1,v_2,v_3)$, defined and continuously differentiable for positive arguments, of the singular values v_1,v_2,v_3 of A. For a radial deformation (8.2) these singular values are given by $r'(R)$, $\frac{r(R)}{R}$ and $\frac{r(R)}{R}$. Hence

$$\int_B W(Du(x))dx = 4\pi \int_0^1 R^2\phi(r',\frac{r}{R},\frac{r}{R})dR. \qquad (8.3)$$

Taking $A = \text{diag}(v_1,v_2,v_3)$, λ, μ parallel to the x^i-axis $(i = 1,2,3)$, it follows from (2.3), as is well known, that W rank-one convex implies that ϕ is convex in each v_i separately. Hence

$$\phi(r',\frac{r}{R},\frac{r}{R}) \geq \phi(\frac{r}{R},\frac{r}{R},\frac{r}{R}) + (r' - \frac{r}{R})\phi_1(\frac{r}{R},\frac{r}{R},\frac{r}{R}),$$

$$\text{a.e. } R \in (0,1). \qquad (8.4)$$

We now note that

$$\frac{d}{dR}\left[\frac{R^3}{z}\phi(\frac{r}{R},\frac{r}{R},\frac{r}{R})\right] = R^2\left[\phi(\frac{r}{R},\frac{r}{R},\frac{r}{R}) + (r' - \frac{r}{R})\phi_1(\frac{r}{R},\frac{r}{R},\frac{r}{R})\right],$$

$$\text{a.e. } R \in (0,1). \qquad (8.5)$$

Combining (8.3)-(8.5) we deduce that for a radial deformation $u \in W^{1,\infty}(B;R^3)$ with $\det Du(x) > 0$ a.e. $x \in B$ and satisfying

$$u(x) = \lambda x, \quad x \in \partial B, \qquad (8.6)$$

we have

$$\int_B W(Du(x))dx \geq 4\pi\int_0^1 R^2\phi(\lambda,\lambda,\lambda)dR = \int_B W(\lambda 1)dx, \qquad (8.7)$$

which is the required quasiconvexity inequality. For related results see [5, §6.3, 18].

The above argument can be thought of as an application of the field theory of the calculus of variations [10], the extremal $\overline{r}(R) = \lambda R$ being regarded as embedded in the global field of extremals $r(R) = \mu R$, $\mu > 0$. The slope function of this field is given by $p(R,r) = r/R$ and (8.4) expresses the positivity of the corresponding Weierstrass excess function.

It is instructive to note that in (8.4) rank-one convexity is applied at matrices whose choice is not at all evident a priori, and which do not form the gradient of a deformation.

(c) We present a plausibility argument that rank-one convexity implies quasiconvexity in general. For simplicity we suppose that $W \in C^2(M^{n\times m})$. The argument is based on the following interesting result of Knops & Stuart [18].

<u>Theorem 8.1</u>

Let $\Omega \subset R^m$ be a bounded, star-shaped domain with smooth boundary. Let W be rank-one convex, let $A \in M^{n\times m}$ and let $u \in C^2(\overline{\Omega};R^n) \cap C^1(\Omega;R^n)$ be a solution to the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\alpha}\frac{\partial W}{\partial A^i_\alpha}(Du) = 0, \quad x \in \Omega, \qquad (8.8)$$

satisfying

$$u(x) = Ax, \quad x \in \partial\Omega. \qquad (8.9)$$

Then

$$I(u) \leq I(Ax). \qquad (8.10)$$

Suppose that W is rank-one convex but not quasiconvex. Then there exist $A \in M^{n\times m}$ and $\overline{\phi} \in W_0^{1,\infty}(\Omega;R^n)$ such that

$$I(Ax + \overline{\phi}) < I(Ax). \qquad (8.11)$$

Define, for $\varepsilon > 0$,

$$W_\varepsilon(A) = W(A) + \varepsilon\, \text{tr}\,(A^TA), \qquad (8.12)$$

$$I_\varepsilon(u) = \int_\Omega W_\varepsilon(Du(x))dx. \qquad (8.13)$$

Then

$$I_\varepsilon(Ax + \overline{\phi}) < I_\varepsilon(Ax) \qquad (8.14)$$

provided ε is sufficiently small. With ε so chosen, and assuming as we may that Ω is star-shaped with smooth boundary, we note that by the argument in (a) the second variation

$$\delta^2 I_\varepsilon(Ax)(\phi,\phi) := \frac{d^2}{dt^2} I_\varepsilon(Ax + t\phi)|_{t=0}$$

$$\geq 2\varepsilon \int_\Omega |D\phi|^2 dx.$$

Hence [26,8] the linear map Ax minimizes I_ε locally in $W^{1,\infty}(\Omega;R^n)$ subject to the boundary condition (8.9). If we could apply an approximate 'mountain-pass' lemma we could conclude from (8.14) that there exists a critical point u of I_ε with $I_\varepsilon(u) > I_\varepsilon(Ax)$. If we could also assert that u had sufficient regularity for Theorem 8.1 to hold then we would have a contradiction to Theorem 8.1, applied to the rank-one convex function W_ε.

Examination of the proof of Theorem 8.1 shows that, as in (b), the rank-one convexity of W is applied in the above argument at some matrices, A and $Du(x)$, whose choice was not evident a priori.

9. Concluding Quotations

We end by quoting two passages from the work of Morrey concerning the problem discussed in this paper; the terminology has been altered to conform with ours.

(From Morrey [21]) 'It would seem that there is still a wide gap in the general case between the necessary and sufficient conditions for quasi-convexity which the writer has obtained. In fact, after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves W and only a finite number of its derivatives, and which is both necessary and sufficient for quasi-convexity in the general case'.

(From Morrey [22]) 'It is an unsolved problem to prove or disprove the theorem that every rank-one convex fucntion of Du is quasi-convex'.

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