Dynamics and Minimizing Sequences

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1 Introduction

What is the relationship between the Second Law of Thermodynamics and the approach to equilibrium of mechanical systems? This deep question has permeated science for over a century, yet is still poorly understood. Particularly obscure is the connection between the way the question is traditionally analysed at different levels of mathematical modelling, for example those of classical and quantum particle mechanics, statistical physics and continuum mechanics.

In this article I make some remarks, and discuss examples, concerning one part of the picture, the justification of variational principles for dynamical systems (especially in infinite dimensions) endowed with a Lyapunov function. For dynamical systems arising from physics the Lyapunov function will typically have a thermodynamic interpretation (entropy, free energy, availability), but its origin will not concern us here. Modern continuum thermomechanics provides such Lyapunov functions for general deforming materials as a consequence of assumed statements of the Second Law such as the Clausius-Duhem inequality (c.f. Coleman & Dill [18], Duhem [20], Ericksen [21], Ball & Knowles [12]). By contrast, statistical physics provides Lyapunov functions only for very special materials (the paradigm being the H-functional for the Boltzmann equation, which models a moderately rarified monatomic gas).

Let $T(t)_{t\geq 0}$ be a dynamical system on some (say, topological) space X. Thus (*i*) $T(0) = identity$, (*ii*) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$, and (*iii*) the mapping $(t,\varphi) \mapsto T(t)\varphi$ is continuous. We suppose that $T(t)_{t\geq o}$ is endowed with a continuous Lyapunov function $V : X \to \mathbf{R}$, that is $V(T(t)\varphi)$ is nonincreasing on $[0,\infty)$ for each $\varphi \in X$. (In some situations variations on these assumptions would be appropriate; for example, solutions may not be unique or always globally defined.) The central conjecture is that *if* $t_j \to \infty$ *then* $T(t_j)\varphi$ *will be a minimizing sequence for* V. If true, this would give a dynamical justification for the variational principle:

$$\text{Minimize } V. \tag{1.1}$$

What are the obstacles to making this more precise? First, there may exist constants of motion that force the solution $T(t)\varphi$ to remain on some submanifold. These constants of motion must be incorporated as constraints in the variational principle. For example, if the constants of motion are $c_i : X \to \mathbf{R}$, $1 = 1, \dots, N$, so that

$$c_i(T(t)\varphi) = c_i(\varphi) \qquad \text{for all } t \geq 0,\ i = 1, \dots, N, \tag{1.2}$$

then the modified variational principle would be

$$\operatorname*{Min}_{\substack{c_i(\psi)=\alpha_i \\ i=1,\dots,N}} V(\psi), \tag{1.3}$$

where the α_i are constants. Second, there may be points $\psi \in X$ which are local minimizers (in some sense) but not absolute minimizers of V, so that an appropriate definition of a 'local minimizing sequence' is needed. Third, the conjecture is false for initial data φ belonging to the region of attraction of a rest point that is not a local minimizer of V; such exceptional initial data must somehow be excluded. Fourth, the minimum of V may not be attained, rendering even more problematical a good definition of a local minimizing sequence (c.f. Ball [4]). We are thus searching for a result (applying to a general class of dynamical systems, or to interesting examples) of the type:

Prototheorem *For most initial data φ, and any sequence $t_j \to \infty$, $T(t_j)\varphi$ is a local minimizing sequence for V subject to appropriate constraints.*

The trivial one-dimensional example in Figure 1 illustrates a further difficulty. In the example there are three critical points A,B,C. The Lyapunov

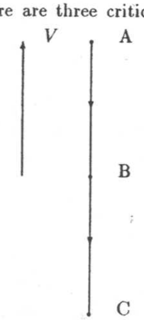

Figure 1:

function V is the vertical coordinate. There is clearly no nontrivial constant of motion, since such a function would have to be constant on the closed intervals

[A,B] and [B,C]. Yet for any $\varphi \in$ [A,B] the solution tends as $t \to \infty$ to a rest point which is not a local minimizer of V. One could have at least three reactions to this example (*i*) that staying in the invariant region [A,B] should be incorporated as a constraint in the variational principle, (*ii*) that the example is not generic, because the rest point B is not hyperbolic, or (*iii*) that stochastic effects should be introduced so that the upper orbit can get through the barrier at B. For, example, taking the point of view (*ii*), a version of the prototheorem can be proved for an ordinary differential equation in $\mathbf{R}^n$.

Theorem 1 *Consider the equation*

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n, \tag{1.4}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is C^1. Suppose that there exists a continuous Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$ for (1.4) satisfying

$$\lim_{|x|\to\infty} V(x) = \infty, \tag{1.5}$$

and such that if x is a solution of (1.4) with $V(x(t)) =$const. for all $t \geq 0$ then x is a rest point. Suppose further that there are just a finite number of rest points a_i, $i = 1, \dots, N$ of (1.4), and that they are each hyperbolic. Then the union of the regions of attraction of the local minimizers of V in $\mathbf{R}^n$ is open and dense.

Proof. I sketch the standard argument. By (1.5) each solution $x(t)$ is bounded for $t \geq 0$, so that by the invariance principle (Barbashin & Krasovskii [14], LaSalle [25]) $x(t) \to a_i$ as $t \to \infty$ for some i. Thus

$$\mathbf{R}^n = \bigcup_{i=1}^{N} A(a_i), \tag{1.6}$$

where $A(a_i)$ denotes the region of attraction of a_i. But a hyperbolic rest point a_i is stable if and only if it is a local minimizer of V, while if a_i is unstable then $A(a_i)$ is closed and nowhere dense. □

Note that from (1.6) it follows that under the hypotheses of Theorem 1 there is no nontrivial continuous constant of motion $c : \mathbf{R}^n \to \mathbf{R}$.

Similar results to Theorem 1 can be proved for some classes of (especially semilinear) partial differential equations by combining the invariance principle with linearization (c.f. Hale [23], Henry [24], Dafermos [19], Ball [6,5]), provided the set of rest points is, in an appropriate sense, hyperbolic. However, many interesting examples lie well outside the scope of these results, and no version of the prototheorem of wide applicability is known to me.

The work of Carr & Pego [16] on the Ginzburg-Landau equation with small diffusion shows that, even when the prototheorem holds, solutions may in practice take an extremely long time to approach their asymptotic state, getting stuck along the way in metastable states that are not close to local minimizers.

2 Two variational problems of elasticity

The examples in this section illustrate some of the features described in Section 1. In the first there are nontrivial constants of motion, while in the second the minimum is not attained.

Example 2.1. (*The pure traction problem of thermoelasticity*)
Consider a thermoelastic body in free space, occupying in a reference configuration a bounded domain $\Omega \subset \mathbf{R}^3$. It is assumed that the external body force and volumetric heat supply are zero, that there are no applied surface forces, and that the boundary of the body is insulated. Let $y = y(x,t) \in \mathbf{R}^3$ denote the position at time t of the particle at $x \in \Omega$ in the reference configuration, $v \overset{\text{def}}{=} \dot{y}(x,t)$ the velocity, $\epsilon = \epsilon(x,t)$ the internal energy density, and $\rho_R = \rho_R(x)$ the given density in the reference configuration. Then the balance laws of linear momentum, angular momentum and energy imply that

$$\frac{d}{dt}\int_\Omega \rho_R v \, dx = 0, \tag{2.1}$$

$$\frac{d}{dt}\int_\Omega \rho_R y \wedge v \, dx = 0, \tag{2.2}$$

$$\frac{d}{dt}\int_\Omega \rho_R(\epsilon + \frac{1}{2} \mid v \mid^2) \, dx = 0, \tag{2.3}$$

respectively, while as a consequence of the Clausius-Duhem inequality we have that

$$\frac{d}{dt}\int_\Omega -\rho_R \eta(x, Dy, \epsilon) \, dx \leq 0, \tag{2.4}$$

where η denotes the entropy density and Dy the gradient of y. It is assumed that η is frame indifferent, that is

$$\eta(x, RA, \epsilon) = \eta(x, A, \epsilon) \tag{2.5}$$

for all x, A, ϵ and all $R \in SO(3)$. By changing to centre of mass coordinates we may assume that

$$\int_\Omega \rho_R y \, dx = 0, \qquad \int_\Omega \rho_R v \, dx = 0. \tag{2.6}$$

This motivates the variational principle

$$\text{Minimize} \quad \int_\Omega -\rho_R \eta(x, Dy, \epsilon) \, dx \tag{2.7}$$

subject to the constraints

$$\int_\Omega \rho_R(\epsilon + \frac{1}{2} \mid v \mid^2) \, dx = \alpha, \tag{2.8}$$

$$\int_\Omega \rho_R y \, dx = 0, \qquad \int_\Omega \rho_R v \, dx = 0, \tag{2.9}$$

$$\int_\Omega \rho_R y \wedge v \, dx = b, \tag{2.10}$$

where $\alpha \in \mathbf{R}$ and $b \in \mathbf{R}^3$ are constant.

The minimization problem (2.7)-(2.10) has recently been studied by Lin [26], who proved that under reasonable polyconvexity and growth conditions on η the minimum is attained at some state $(\overline{y}, \overline{v}, \overline{\epsilon})$. Of course $\overline{y}, \overline{v}, \overline{\epsilon}$ are functions of x alone. As a consequence of (2.5), the minimization problem is invariant to the transformation $(y, v, \epsilon) \mapsto (Ry, Rv, \epsilon)$ for any $R \in SO(3)$ satisfying $Rb = b$. Hence, for any such R, $(R\overline{y}, R\overline{v}, \epsilon)$ is also a minimizer. In fact it is proved in [26] that for any minimizer $(\overline{y}, \overline{v}, \overline{\epsilon})$ there exists a skew matrix Λ such that $\Lambda b = b$, $\overline{v} = \Lambda \overline{y}$, and such that

$$y(x,t) = e^{\Lambda t}\overline{y}(x) \tag{2.11}$$

$$\epsilon(x,t) = \overline{\epsilon}(x) \tag{2.12}$$

is a weak solution of the equations of motion. Furthermore

$$\frac{\partial \eta}{\partial \epsilon}(x, Dy(x,t), \epsilon(x,t)) = \theta^{-1} \tag{2.13}$$

for all t, where θ is a constant. The motion (2.11), (2.12) corresponds to a rigid rotation at constant temperature θ. Note that in this example the Lyapunov function V is constant along nontrivial orbits, such as that given by (2.13). In particular, solutions to the dynamic equations need not tend to a rest point as time $t \to \infty$.

Example 2.2. (*A theory of crystal microstructure*)
Consider an elastic crystal, occupying in a reference configuration a bounded domain $\Omega \subset \mathbf{R}^3$ with sufficiently smooth boundary $\partial\Omega$. Assume that part of the boundary $\partial\Omega_1$ is maintained at a constant temperature θ_0 and at a given deformed position

$$y|_{\partial\Omega} = \overline{y}, \tag{2.14}$$

where $\overline{y} = \overline{y}(\cdot)$, while the remainder of the boundary is insulated and traction free. Then an argument similar to that in Example 2.1, but using a different Lyapunov function, the availability, motivates the variational principle

$$\text{Minimize} \quad \int_\Omega W(Dy(x)) \, dx \tag{2.15}$$

subject to

$$y|_{\partial\Omega} = \bar{y}, \quad (2.16)$$

where W is the Helmholtz free energy at temperature θ_0 (see Ericksen [21], Ball [3].) It is supposed that W is frame indifferent, i.e.

$$W(RA) = W(A) \quad (2.17)$$

for all A in the domain of W and all $R \in SO(3)$. In addition to (2.17), W has other symmetries arising from the crystal lattice structure, as a consequence of which W is nonelliptic. This lack of ellipticity implies in turn that the minimum in (2.15),(2.16) is in general *not attained* in the natural spaces of admissible mappings. In this case, in order to get closer and closer to the infimum of the energy it is necessary to introduce more and more microstructure. Such microstructure is frequently observed in optical and electron micrographs, where one may see multiple interfaces (occurring, for example, in the form of very fine parallel bands), each corresponding to a jump in Dy. The observed microstructure is not, of course, infinitely fine, as would be predicted by the model here. The conventional explanation for this is that one should incorporate in the energy functional contributions due to interfacial energy; this should predict a limited fineness and impose additional geometric structure (c.f.Parry [29], Fonseca [22]). Since the interfacial energy is very small (witness the large amount of surface observed) it is a reasonable expedient to ignore it, and in fact this successfully predicts many features of the observed microstructure (see Ball & James [10], Chipot & Kinderlehrer [17]). An example in which the nonattainment of a minimum can be rigorously established is the following (a special case of a result of Ball & James [11]). Let $W \geq 0$ with $W(A) = 0$ if and only if $A \in M$, where

$$M = SO(3)S^+ \cup SO(3)S^-, \quad (2.18)$$

where

$$S^{\pm} = \mathbf{1} \pm \delta e_3 \otimes e_1, \quad (2.19)$$

and where $\delta > 0$ and $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbf{R}^3$. Suppose that $\partial\Omega_1 = \partial\Omega$ and that

$$\bar{y}(x) = (\lambda S^+ + (1-\lambda)S^-)\,x, \qquad \lambda \in (0,1). \quad (2.20)$$

Then under some technical hypotheses it is proved in [11] that the infimum of (2.15) subject to (2.16) is zero, and that if $y^{(j)}$ is a minimizing sequence then the Young measure corresponding to $Dy^{(j)}$ is unique and given by

$$\nu_x = \lambda\delta_{S^+} + (1-\lambda)\delta_{S^-}, \qquad \text{for a.e. } x \in \Omega. \quad (2.21)$$

In particular, because ν_x is not a Dirac mass a.e., it follows that the minimum is not attained. The minimizing set M in (2.18) occurs, for example, in the case of an orthorhombic to monoclinic transformation.

It would be very interesting to carry out a dynamical analysis corresponding to the above variational problem, to see if the dynamics produces minimizing sequences with microstructure after the fashion of the prototheorem. This could lead to important insight into a controversial area of metallurgy, that of martensitic nucleation.

3 Some dynamical examples

In this section some infinite-dimensional problems are discussed for which the prototheorem can either be proved or, in the case of Example 3.2, related information obtained.

Example 3.1. (*Stabilization of a rod using the axial force as a control*) The problem of feedback stabilization of an elastic rod using the axial force as a control leads to the initial-boundary value problem

$$u_{tt} + u_{xxxx} + \left(\int_0^1 u_{xx}u_t dx\right) u_{xx} = 0, \qquad 0 < x < 1, \quad (3.1)$$

$$u = u_{xx} = 0, \qquad x = 0, 1, \quad (3.2)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad 0 < x < 1. \quad (3.3)$$

Here $u(x,t)$ denotes the transverse displacement of the rod, while the boundary conditions (3.2) correspond to the case of simply supported ends. This and similar problems were formulated and analyzed in Ball & Slemrod [13]. Using the Lyapunov function

$$V(t) = \int_0^1 \frac{1}{2}(u_t^2 + u_{xx}^2)\,dx, \quad (3.4)$$

which has time derivative

$$\dot{V}(t) = -\left(\int_0^1 u_{xx}u_t\,dx\right)^2, \quad (3.5)$$

it was proved that if $\{u_0, u_1\} \in X \stackrel{\text{def}}{=} (H^2(0,1) \cap H_0^1(0,1)) \times L^2(0,1)$ then the unique weak solution $\{u, u_t\}$ of (3.1)-(3.3) satisfies

$$\{u, u_t\} \rightharpoonup \{0, 0\} \qquad \text{weakly in } X \text{ as } t \to \infty. \quad (3.6)$$

Considered as a functional on X, V has only one critical point $\{0,0\}$, which is an absolute minimizer. The conclusion of the prototheorem therefore holds if and only if

$$\{u, u_t\} \to \{0, 0\} \qquad \text{strongly in } X \text{ as } t \to \infty. \quad (3.7)$$

This has recently been proved by Müller [28] by means of a delicate analysis of the infinite system of ordinary differential equations satisfied by the coefficients $u_j(t)$ of the Fourier expansion

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\sin(j\pi x) \tag{3.8}$$

of a solution. Müller also established the interesting result that given any continuous function $g : [0,\infty) \to (0,\infty)$ with $lim_{t\to\infty} g(t) = 0$ there exists initial data $\{u_0, u_1\} \in X$ such that the solution of (3.1)-(3.3) satisfies

$$V(t) \geq Cg(t) \tag{3.9}$$

for all $t \geq 0$ and some constant $C > 0$. Thus solutions may have an arbitrary slow rate of decay as $t \to \infty$. It is an open question whether strong convergence holds for the case of clamped ends

$$u = u_x = 0 \qquad \text{at } x = 0, 1, \tag{3.10}$$

or for various other feedback stabilization problems for which the analogue of (3.6) was established in [13].

Example 3.2. *(Phase transitions in one-dimensional viscoelasticity)*
Consider one-dimensional motion of a viscoelastic rod. The equation of motion is taken to be

$$u_{tt} = (\sigma(u_x) + u_{xt})_x, \qquad 0 < x < 1, \tag{3.11}$$

with boundary conditions

$$u = 0 \text{ at } x = 0, \qquad \sigma(u_x) + u_{xt} = 0 \text{ at } x = 1, \tag{3.12}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad 0 < x < 1. \tag{3.13}$$

For simplicity, assume that

$$\sigma(u_x) = W'(u_x), \qquad W(u_x) = (u_x^2 - 1)^2. \tag{3.14}$$

Let

$$V(u,p) = \int_0^1 [\frac{1}{2}p^2 + W(u_x)]\,dx. \tag{3.15}$$

Then $V(u,u_t)$ is a Lyapunov function for (3.11)-(3.13) with time derivative

$$\dot{V}(u,u_t) = -\int_0^1 u_{xt}^2\,dx \leq 0. \tag{3.16}$$

The corresponding variational problem

$$\underset{X}{\text{Min}}\, V, \tag{3.17}$$

where $X = \{\{u,p\} : u \in W^{1,\infty}(0,1),\ u(0) = 0,\ p \in L^2(0,1)\}$ has uncountably many absolute minimizers, given by any pair $\{u,0\} \in X$ with $u_x = \pm 1$ a.e.. In particular it is easily proved that given any smooth function v on $[0,1]$ with $v(0) = 0$ and $| v' | \leq 1$, there exists a sequence $\{u^{(j)}, 0\}$ of absolute minimizers such that $u^{(j)} \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,1)$. This raises the interesting question as to whether a solution $\{u, u_t\}$ to (3.11)-(3.13) could exhibit similar behaviour, converging weakly but not strongly to a pair $\{v, 0\}$ which is not a rest point. This question was resolved by Pego [30], following earlier work of Andrews & Ball [1]. Pego showed that for any solution $\{u, u_t\}$, as $t \to \infty$,

$$u(\cdot,t) \to v(\cdot) \qquad \text{strongly in } W^{1,p}(0,1), \tag{3.18}$$

$$u_t(\cdot,t) \to 0 \qquad \text{strongly in } W^{1,2}(0,1), \tag{3.19}$$

for all $p > 1$, where $\{v, 0\}$ is a rest point of (3.11)-(3.13). Thus solutions to the dynamical equations do not mimic the typical behaviour of minimizing sequences. The results of Pego do not seem, however, to be sufficient to establish whether or not a version of the prototheorem holds.

Example 3.3. *(The Becker-Döring cluster equations)*
These are the infinite set of ordinary differential equations

$$\begin{aligned} \dot{c}_r &= J_{r-1}(c(t)) - J_r(c(t)), \qquad r \geq 2, \\ \dot{c}_1 &= -J_1(c(t)) - \sum_{r=1}^{\infty} J_r(c(t)), \end{aligned} \tag{3.20}$$

where $c(t)$ denotes the infinite vector $(c_r(t))$,

$$J_r(c) = a_r c_1 c_r - b_{r+1} c_{r+1}, \tag{3.21}$$

and the coefficients $a_r > 0$, $b_r > 0$ are constant. The physical significance of (3.20) is discussed in the article in this volume by Carr [15].

Let $X = \{y = (y_r) : \| y \| \overset{\text{def}}{=} \sum_{r=1}^{\infty} r \mid y_r \mid < \infty\}$. X is a Banach space with the indicated norm. Solutions of (3.20) are sought as continuous functions $c : [0,\infty) \to X^+$, where

$$X^+ = \{y \in X : y_r \geq 0, \qquad r = 1, 2, \ldots\}. \tag{3.22}$$

The system (3.20) possesses the Lyapunov function

$$V(c) = \sum_{r=1}^{\infty} c_r \left(\ln\left(\frac{c_r}{Q_r}\right) - 1\right), \tag{3.23}$$

where $Q_1 = 1$, $Q_{r+1}/Q_r = a_r/b_{r+1}$, and there is a constant of motion, the density

$$\rho = \sum_{r=1}^{\infty} r c_r . \tag{3.24}$$

For suitable coefficients a_r, b_r there exists $\rho_s > 0$ such that there is a unique rest point $c^{(\rho)}$ of (3.20) with density ρ for $\rho \in [0, \rho_s]$, and no rest point with any density $\rho > \rho_s$. Furthermore $c^{(\rho)}$ is the unique absolute minimizer of the problem

$$\underset{c \in X^+,\ \sum_{r=1}^{\infty} r c_r = \rho}{\text{Minimize}} \quad V(c) . \tag{3.25}$$

The equations (3.20) were analyzed in Ball, Carr & Penrose [8], Ball & Carr [7]; see also Ball [2] for remarks on the variational problem (3.25). It follows from [8],[7] that under suitable hypotheses on the a_r, b_r the conclusion of the prototheorem holds. That is, given $c(0) \in X^+$ with $\sum_{r=1}^{\infty} r c_r(0) = \rho$, and any sequence $t_j \to \infty$, $c(t_j)$ is a minimizing sequence for (3.25). Note that this conclusion holds even in the case $\rho > \rho_s$, when the minimum in (3.25) is not attained.

Example 3.4. *(Model equations related to phase transitions in solids)* In Example 3.2, the Lyapunov function V given by (3.15) has minimizing sequences that oscillate more and more finely, converging weakly to a state that is not a minimizer. On the other hand there are minimizing sequences which do not behave like this, consisting, for example, of a single minimizer. The results of Pego show that the dynamics chooses to imitate the latter kind of minimizing sequence rather than the former. In the crystal problem described in Example 2.2 minimizing sequences are forced to oscillate more and more finely, leading to interesting possibilities for a corresponding dynamical model. Does the dynamics imitate the minimizing sequences, or is it still the case that all solutions tend to equilibria? This is a formidable problem, so it makes sense to first try out some one-dimensional examples. The most obvious candidate is the problem

$$u_{tt} = (\sigma(u_x) + u_{xt})_x - 2u, \qquad 0 < x < 1, \tag{3.26}$$

with boundary conditions

$$u = 0 \text{ at } x = 0, 1, \tag{3.27}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad 0 < x < 1. \tag{3.28}$$

As before, assume that

$$\sigma(u_x) = W'(u_x), \qquad W(u_x) = (u_x^2 - 1)^2 . \tag{3.29}$$

Then $V(u, u_t)$ is a Lyapunov function for (3.26)-(3.28), where

$$V(u,p) = \int_0^1 [\frac{1}{2}p^2 + W(u_x) + u^2]\, dx . \tag{3.30}$$

The minimizing sequences of V subject to (3.27) all oscillate faster and faster, converging weakly but not strongly to $\{0,0\}$ in $W_0^{1,4}(0,1) \times L^2(0,1)$. (See the paper in this volume by Müller [27] for a study of this variational problem with surface energy added.)

The problem (3.26)-(3.28) has been studied in joint work of P.J.Holmes, R.D.James, R.L.Pego, P.Swart and the author [9], together with the much more tractable problem consisting of the equation

$$u_{tt} = \left(\int_0^1 u_x^2\, dx - 1\right) u_{xx} + u_{xxt} - 2u, \qquad 0 < x < 1, \tag{3.31}$$

with boundary and initial conditions (3.27),(3.28). This problem has the Lyapunov function $\overline{V}(u, u_t)$, where

$$\overline{V}(u,p) = \int_0^1 [\frac{1}{2}(p^2 - u_x^2) + u^2]\, dx + \frac{1}{4}\left(\int_0^1 u_x^2\, dx\right)^2 . \tag{3.32}$$

There are countably many rest points of (3.31),(3.27) given by

$$u_k = a_k \sin k\pi x, \qquad k \text{ an integer}, \tag{3.33}$$

for suitable coefficients a_k. It can easily be proved that

$$\inf_X \overline{V} = -\frac{1}{4}, \tag{3.34}$$

where $X = H_0^1(0,1) \times L^2(0,1)$. Then we have the result

Theorem 2 *Let u be any weak solution of (3.31),(3.27). As $t \to \infty$ either*
(i) $\{u, u_t\} \to \{u_k, 0\}$ *strongly in X for some k, or*
(ii) $\{u, u_t\} \rightharpoonup \{0, 0\}$ *weakly in X, but not strongly, and*

$$\lim_{t\to\infty} \overline{V}(t) = -\frac{1}{4}. \tag{3.35}$$

The alternatives (i),(ii) both occur for dense sets of initial data in X, the set corresponding to (ii) being of second category.

By contrast, for the problem (3.26)-(3.28) it is shown in [9] that there is no solution $\{u, u_t\}$ for which

$$\lim_{t\to\infty} V(t) = 0, \tag{3.36}$$

i.e. no solution which realizes an absolute minimizing sequence.

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