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TWO EXAMPLES IN NONLINEAR ELASTICITY

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This note is concerned with extremals for the integral

$$J(u) = \int_0^1 W(u_x)\,dx$$

with W a given smooth function of $u_x = \frac{du}{dx}$ and with u prescribed at x = 0 and x = 1; say

$$u(0) = 0,\ u(1) = p_0.$$

In applications to one dimensional elasticity, W is the stored energy function. We will call $u_0(x) = p_0 x$ the <u>trivial solution</u>.

Our examples point out the care needed in choosing function spaces when discussing the existence and stability of equilibrium solutions in elasticity, and they are indicative of difficulties for realistic models of nonlinear elastic materials in one and higher dimensions.

The purpose of these examples, more specifically, is as follows.

1. The trivial solution need <u>not</u> be isolated in any Sobolev space $W^{1,p} = W^{1,p}(0,1)$, $1 \leq p < \infty$ even though
   (a) the second variation of J is positive definite, and
   (b) it <u>is</u> isolated in $W^{2,p}$.

   In particular, an implicit function theorem cannot be used to prove local existence and uniqueness in $W^{1,p}$ under assumption (a) alone.

2. Positivity of the second variation at the trivial solution implies u_0 locally minimizes J in a topology as strong as $W^{1,\infty}$ although
   (a) it <u>need not</u> imply u_0 locally minimizes J in $W^{1,p}$ for any p, $1 \leq p < \infty$.

(b) in any topology as strong as $W^{1,\infty}$ we <u>always have</u> for $\varepsilon > 0$ sufficiently small,

$$\inf_{\|u-u_0\|=\varepsilon} J(u) = J(u_0).$$

Before proceeding to these examples, we make some remarks.

(i) The space $W^{1,p}$ plays a basic role in the existence theory for minimizers in elasticity (Ball [1]). In example 1, however, W is not convex.

(ii) The second example shows that <u>in general</u> potential wells (the standard sufficient conditions for stability; cf. references [5], [6]) are impossible in topologies as strong as $W^{1,\infty}$. The above conclusions in example 2 were given by Knops [3] for the case $W(u_x) = \frac{1}{2}(u_x^2 - u_x^4)$ and by Knops and Payne [4] in some related three dimensional examples.

(iii) If convexity and polynomial growth conditions are imposed, conditions for a potential well may be met in $W^{1,p}$ by inspection. However it is unknown whether the equations of nonlinear elastodynamics are well posed for suitable weak solutions in $W^{1,p}$ (for any nontrivial choice of stored energy function).

(iv) Koiter [6] has remarked that in practice the energy criterion is very successful. However this is consistent with the possibility that the energy criterion may fail for hyperelastodynamics. Indeed "in practice" one usually does not observe the very high frequency motions. Masking them may amount to replacing the quasilinear equations of elastodynamics by semilinear approximations. For the latter the proof of the validity of the energy criterion is basically trivial (cf. [7], [8]).

(v) The second example illustrates that the Morse lemma for the function J will fail in $W^{1,p}$, $1 \leqslant p < \infty$, but be valid in $W^{s,p}$, $s \geqslant 2$, $1 < p < \infty$. See Tromba [9].

<u>The First Example</u> Let W be a smooth function of $\mathbb{R}$ to $\mathbb{R}$ and let $p_- < p_0 < p_+$ be such that

$$W'(p_-) = W'(p_0) = W'(p_+)$$

and

$$W''(p_0) > 0 \quad .$$

See figure 1.

In $W^{2,p}$ (with the boundary conditions $u(0) = 0$, $u(1) = p_0$ as before), the trivial solution is isolated because the map

$$u \longmapsto W(u_x)_x$$

from $W^{2,p}$ to L^p is smooth and its derivative at u_0 is the linear operator

$$v \longmapsto W''(p_0)v_{xx},$$

which is an isomorphism. Therefore, by the inverse function theorem, u_0 is an isolated zero of $W(u_x)_x$.

The second variation of J is positive definite (relative to the $W^{1,2}$ topology) at u_0 because if v is in $W^{1,2}$ and vanishes at $x = 0,1$,

$$\frac{d^2}{d\varepsilon^2} J(u_0 + \varepsilon v)\Big|_{\varepsilon=0} = W''(p_0)\int_0^1 v_x^2 \, dx$$

$$\geqslant c\|v\|^2_{W^{1,2}} \; .$$

Now we show that u_0 is not isolated in $W^{1,p}$. Given $\varepsilon > 0$, let

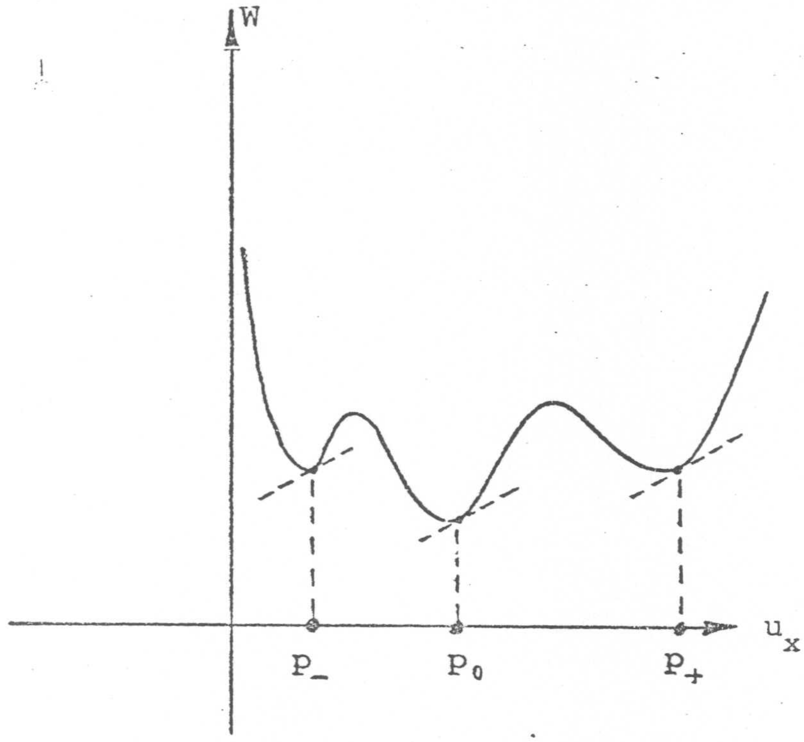

Figure 1

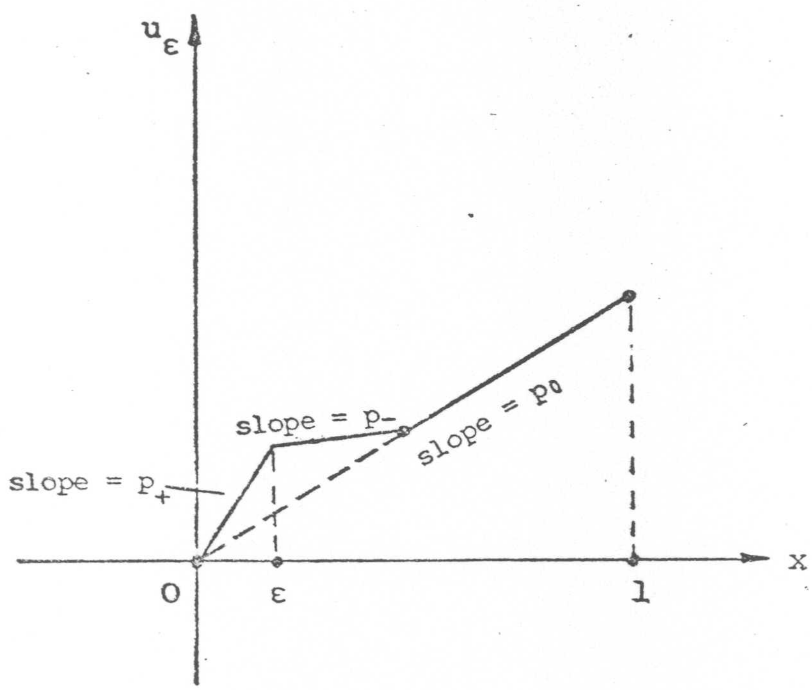

Figure 2

$$u_\varepsilon(x) = \begin{cases} p_+x & \text{for } 0 \leqslant x \leqslant \varepsilon \\ p_+\varepsilon + p_-(x-\varepsilon) & \text{for } \varepsilon \leqslant x \leqslant (p_+ - p_-)\varepsilon/(p_0 - p_-) \\ p_0 x & \text{for } (p_+ - p_-)\varepsilon/(p_0 - p_-) \leqslant x \leqslant 1 \end{cases}$$

See fig. 2. Since $W'(u_{\varepsilon x})$ is constant each u_ε is an extremal. Also

$$\int_0^1 |u_{\varepsilon x} - u_{0x}|^p \, dx = \varepsilon |p_+ - p_0|^p + \left(\frac{p_+ - p_0}{p_0 - p_-}\right) \varepsilon |p_- - p_0|^p$$

which tends to zero as $\varepsilon \to 0$. Thus u_0 is not isolated in $W^{1,p}$.

Remarks. 1. If $W(p_-) = W(p_+) = W(p_0)$ and if $W(p) \geqslant W(p_0)$ for all p, the same argument shows that there are absolute minima of J arbitrarily close to u_0 in $W^{1,p}$.

2. Phenomena like this seem to have first been noticed by Weierstrass. See Bolza [2], footnote 1, p.40.

The Second Example. Let $W : \mathbb{R} \to \mathbb{R}$ be a smooth function with $W'(p_0) = 0$ and $W''(p_0) > 0$. As in the first example, $u_0(x) = p_0 x$ is an extremal and the second variation of J at u_0 is positive definite. Let X be a Banach space continuously included in $W^{1,\infty}$. Then there is an $\varepsilon > 0$ such that

$$\text{if } 0 < \|u - u_0\|_X < \varepsilon \text{ then } J(u) > J(u_0)$$

i.e. u_0 is a strict local minimum for J. This follows trivially from the fact that p_0 is a local minimum of W and that the topology on X is as strong as that of $W^{1,\infty}$.

In $W^{1,p}$ one cannot conclude that u_0 is a local minimum. Indeed the example $W(u_x) = \frac{1}{2}(u_x^2 - u_x^4)$ with $p_0 = 0$ shows that in any $W^{1,p}$ neighbourhood, $J(u)$ can be unbounded below, even though its second variation at u_0 is positive definite.

Finally we show that

$$\inf_{\|u-u_0\|_X=\varepsilon} J(u) = J(u_0)$$

Indeed, by Taylor's theorem,

$$J(u) - J(u_0) = \int_0^1 (W(u_x) - W(p_0))dx$$

$$= \int_0^1\int_0^1 (1-s)W''(su_x + (1-s)p_0)(u_x - p_0)^2 dsdx$$

$$\leqslant C\int_0^1 (u_x - p_0)^2 dx$$

where $C > 0$, since $su_x + (1-s)p_0$ is essentially uniformly bounded (by the assumption $X \subset W^{1,\infty}$) and W'' is continuous. However, the topology on X is strictly stronger than the $W^{1,2}$ topology, and so

$$\inf_{\|u-u_0\|_X=\varepsilon} \int_0^1 (u_x - p_0)^2 dx = 0.$$

This proves our claim.

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