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Asymptotic Behaviour and Changes of Phase in One-Dimensional Nonlinear Viscoelasticity*

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1. INTRODUCTION

In this paper we study the asymptotic behaviour as $t \to \infty$ of solutions $u(x, t)$ to the nonlinear partial differential equation

$$u_{tt} = (\sigma(u_x) + u_{xt})_x, \qquad 0 < x < 1, \ t > 0, \tag{1.1}$$

with initial conditions

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad 0 < x < 1, \tag{1.2}$$

and boundary conditions either

$$u(0, t) = u(1, t) = 0, \qquad t > 0, \tag{1.3a}$$

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or

$$u(0, t) = 0, \qquad \sigma(u_x(1, t)) + u_{xt}(1, t) = P, \tag{1.3b}$$

where P is a given constant.

Equation (1.1) governs the one-dimensional motion under zero body forces of a homogeneous nonlinear viscoelastic material of rate type, $u(x, t)$ denoting the displacement at time t of a particle having position x in a given reference configuration. For this material the stress $S(x, t)$ is given by the constitutive equation

$$S = \sigma(u_x) + u_{xt}, \tag{1.4}$$

and the density in the reference configuration is assumed to be unity. The case in which the density is a positive constant and (1.4) is replaced by

$$S = \sigma(u_x) + \mu u_{xt},$$

where $\mu > 0$ is a constant, can be reduced to (1.1) by a suitable scaling of t and σ. While, strictly speaking, (1.1) applies only to one-dimensional motion of an infinite slab of material with faces normal to the x-axis, it is also a useful approximate model for purely longitudinal motion of a homogeneous thin bar of uniform cross-section and unit length. The boundary conditions (1.3a) say that the displacements of the ends of the rod are zero; constant nonzero displacements can be reduced to (1.3a) by a change of variables. The boundary conditions (1.3b) correspond to the situation in which the end $x = 0$ of the rod is fixed, while the end $x = 1$ is subjected to a given force P. The elastic part σ of the stress is assumed to be a locally Lipschitz real valued function defined on all of $\mathbb{R}$; since, however, a priori estimates will show that the values of $u_x(x, t)$ for all $x \in (0, 1)$ and $t \geqslant 0$ are confined to a bounded interval, our analysis applies also to certain cases in which σ is not everywhere defined.

Our principal objective is to study (1.1)–(1.3) in the case when σ is not a monotone increasing function, so that the *stored-energy function*

$$W(u_x) \stackrel{\text{def}}{=} \int_0^{u_x} \sigma(z)\, dz \tag{1.5}$$

is not convex. This implies that the corresponding equilibrium problem, namely, to solve

$$\sigma(u'(x)) = \text{const} \tag{1.6}$$

subject to the boundary conditions either

$$u(0) = u(1) = 0 \tag{1.7a}$$

or

$$u(0)=0, \qquad \sigma(u'(1))=P, \tag{1.7b}$$

has in general infinitely many solutions. For example, in the case of the boundary conditions (1.7b) any piecewise affine function $u(x)$ passing through the origin and having slopes which are roots of the equation $\sigma(z)=P$ is an equilibrium solution. Solutions of problems (1.6), (1.7a) and (1.6), (1.7b) are stationary points, in suitable function classes, of the functionals

$$I(u) \overset{\text{def}}{=} \int_0^1 W(u'(x))\,dx \tag{1.8a}$$

and

$$I_p(u) \overset{\text{def}}{=} \int_0^1 W(u'(x))\,dx - Pu(1), \tag{1.8b}$$

respectively.

The equilibrium problem (1.6)–(1.7) has been studied recently by Ericksen (1975), one of whose aims was to clarify the extent to which elasticity theory can model materials which change phase. Different phases of a material can in this context be identified with appropriate ranges of values of the deformation gradient. For example, in one dimension a particular phase might be identified with a maximal interval of values of u_x in which σ is monotone; with this interpretation a piecewise affine equilibrium solution comprises homogeneous strains of different material phases separated by points representing phase boundaries. When considering what type of stored-energy function gives rise to equilibrium solutions possessing sharply defined phase boundaries, the following result (Ball, 1980) is of some relevance: whatever be the spatial dimension, a homogeneous nonlinear elastic material can possess nontrivial piecewise affine equilibrium solutions under zero body forces if and only if the stored-energy function W fails to be *strictly rank* 1 *convex*. (Strict rank 1 convexity of W is essentially equivalent to strong ellipticity, and in one dimension is the same as strict convexity of W.) It should be noted that strong ellipticity may be lost in the reduction from three-dimensional elasticity to a one-dimensional rod theory of the type considered here; for example, a piecewise affine solution of (1.6) can represent necking of a rod, but the same phenomenon can also be modelled using a more refined rod theory in which strong ellipticity is assumed (Antman, 1973; Antman and Carbone, 1977). Ericksen's analysis has been extended by James (1979), who also gives a summary of some experimental literature on changes of phase in polymers and metals. The propagation of phase boundaries in an elastic bar (with no viscoelastic damping) is analyzed in James (1980). Also relevant are the papers of James (1981) and Fosdick and James (1981) which treat aspects of the equilibrium theory of elastic rods in the case when the stored-energy function is not strongly elliptic.

In the case when σ is monotone increasing the asymptotic behaviour of solutions has been studied by Greenberg *et al.* (1968), Greenberg (1969) and Greenberg and MacCamy (1970). The solution to (1.1)–(1.3a) then tends exponentially to zero as $t \to \infty$. When σ is not monotone, the multiplicity of equilibrium solutions makes the problem of asymptotic behaviour much more complicated, a point emphasized by Dafermos (1969). In this case, on account of the viscoelastic term in (1.4), it is at first sight natural to conjecture that each solution $u(x, t)$ of (1.1)–(1.3) converges to a particular equilibrium solution as $t \to \infty$. Furthermore, one could expect that most solutions would converge to (at least local) minimizers of I or I_p. However, when σ is not monotone it is well known that a minimizing sequence $u_{(j)}(x)$ of I or I_p may converge uniformly to a function which is not an equilibrium solution. In this case the corresponding sequence of derivatives $u'_{(j)}(x)$ converges weakly but not strongly. The resulting limit can be viewed as an ordinary curve with a superimposed "infinitesimal zigzag," or, in the terminology of Young (1969), a "generalized curve." The ordinary curve minimizes the lower convex envelope of I or I_p (cf. Ekeland and Témam, 1974; Dacorogna, 1981). Since the total energy is nonincreasing along solutions of (1.1)–(1.3) the possibility arises that such a minimizing sequence could be given by $u_{(j)}(x) = u(x, t_j)$ for some sequence $t_j \to \infty$. For such a solution the deformations $u(\cdot, t_j)$ would consist of progressively finer phase mixtures. The behaviour of solutions established in this paper is consistent with this possibility. Indeed we give conditions under which $u(\cdot, t)$ converges in the sense of generalized curves as $t \to \infty$, so that in particular $u(\cdot, t)$ converges uniformly to a function $v(\cdot)$ as $t \to \infty$. We have not been able to determine whether this result is optimal, in the sense that there exists some solution u of (1.1)–(1.3) converging to a function v which is not an equilibrium solution, or if so whether this is a common or rare phenomenon. On the one hand it seems to be far from obvious how to construct such an example, while on the other hand we have made numerous unsuccessful attempts to apply the various versions currently available of the LaSalle invariance principle (for references, see Ball, 1978) so as to conclude that u converges to an equilibrium. A careful numerical investigation might throw light on this question.

The plan of the paper is as follows. We begin in Section 2 by reviewing and extending slightly the existence theorems of Andrews (1979, 1980) for problem (1.1)–(1.3), laying particular emphasis on the boundary conditions (1.3b) which were treated in Andrews (1979, 1980) only for the special case when $\sigma(u_0'(1)) = p$. Under natural hypotheses on σ, appropriate for solids and not implying monotonicity, a unique weak solution u exists for all time $t \geqslant 0$. An important step in the proof of global existence is an a priori estimate of Andrews stating that $u(\cdot, t)$ is bounded in $W^{1,\infty}(0, 1)$ for all $t \geqslant 0$. This estimate is crucial also for our study of asymptotic behaviour. In Section 4 we study the asymptotic behaviour of u in the case of the boundary conditions (1.3b). In Theorem 4.1 we show that as $t \to \infty$, $u_t(\cdot, t) \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(0, 1)$, that $\sigma(u_x(\cdot, t)) \to P$ in $L^2(0, 1)$, and that $u(\cdot, t)$ converges in the sense of generalized curves; i.e., there exists a family of probability measures $\{\nu_x\}_{x \in (0,1)}$ on $\mathbb{R}$ such that $\Phi(u_x(\cdot, t)) \overset{*}{\rightharpoonup} \langle \nu_x, \Phi \rangle$ in $L^\infty(0, 1)$ for each continuous function Φ. Furthermore $\operatorname{supp} \nu_x \subset K_P = \{z : \sigma(z) = P\}$ a.e. In Corollaries 4.2, 4.3 the case when $K_P = \{z_1, ..., z_k\}$ is finite is discussed. If $k = 1$ then $u(x, t) \to z_1 x$ strongly in $W^{1,p}(0, 1)$ for all p, $1 \leqslant p < \infty$, while if $k > 1$ then the local phase fractions converge. In Corollary 4.4 we prove that $u(\cdot, t)$ tends to the *set* of equilibrium solutions strongly in $W^{1,p}(0, 1)$ for all p, $1 \leqslant p < \infty$. The main idea of the proof of Theorem 4.1 is to use various "energy" estimates to prove that $\lim_{t \to \infty} \int_0^1 \psi(x)\, \Phi(u_x(x, t))\, dx$ exists for *certain* ψ, Φ and then to use an approximation lemma proved in Section 3 to show that the limit exists in general. Analogous results are proved in Section 5 for the boundary conditions (1.3a), but the argument is more delicate because the limiting value P of $\sigma(u_x(\cdot\ t))$ is not known a priori, and we give proofs only for the case when σ satisfies an extra (possibly unnecessary) nondegeneracy condition. In Section 6 we discuss further whether $u(\cdot, t)$ converges to a unique equilibrium solution, and show that convergence to equilibrium *does* hold for solutions to the modified equation

$$u_{tt} = (\sigma(u_x) + u_{xt} - \varepsilon u_{xxx})_x, \qquad 0 < x < 1, \quad t > 0, \tag{1.9}$$

with appropriate boundary conditions, where $\varepsilon > 0$ is a constant. We also discuss briefly the relationship between equilibrium solutions for (1.1) and those for (1.9) as $\varepsilon \to 0$.

It would be interesting to extend the analysis of this paper to the equation of one-dimensional isothermal motion of a linear viscous fluid (in Lagrangian coordinates),

$$u_{tt} = (-p(u_x) + u_{xt}/u_x)_x, \tag{1.10}$$

with a non-monotone pressure function $p(\cdot)$ (such as that for a van der Waal's fluid). Note that not only does the dissipative term differ from that in (1.1), but that our hypotheses (Ha), (Hb) in Section 2 are not appropriate for gases. An existence theorem for (1.10) has been announced by Kazhikov and Nikolaev (1979) (see also Solonnikov & Kazhikov, 1981).

2. Existence of Solutions

We summarize and extend slightly results of Andrews (1979, 1980) concerning the existence of solutions to (1.1)–(1.3).

Notation

$\|\cdot\|_p$ denotes the norm in $L^p(0,1)$, $1 \leqslant p \leqslant \infty$. For $k = 1, 2, \ldots$, the norm in the Sobolev space $W^{k,p}(0,1)$, $1 \leqslant p \leqslant \infty$, is denoted by $\|\cdot\|_{k,p}$. $W_0^{1,p}(0,1) \overset{\text{def}}{=} \{v \in W^{1,p}(0,1): v(0) = v(1) = 0\}$. (For information concerning Sobolev spaces see Adams (1975).)

Local Existence

We begin with the boundary conditions (1.3a). If $u(x,t)$ is a classical solution of (1.1)–(1.3a) then u satisfies the integral equation

$$u(x,t) = u_0(x) + \int_0^t \int_0^1 G(x,y,s)\, u_1(y)\, dy\, ds$$
$$- \int_0^t \int_0^s \int_0^1 G_y(x,y,s-\tau)\, \sigma(u_y(y,\tau))\, dy\, d\tau\, ds \tag{2.1}$$

for $(x,t) \in [0,1] \times [0,\infty)$, where $G(x,y,t)$ denotes the Green's function for the linear heat equation

$$w_t(x,t) = w_{xx}(x,t), \qquad x \in (0,1),\ t > 0,$$

with boundary conditions

$$w(0,t) = w(1,t) = 0, \qquad t > 0.$$

For $T > 0$ let $X(T)$ denote the Banach space $C([0,T]; W_0^{1,\infty}(0,1))$ with norm $\|v\|_{X(T)} \overset{\text{def}}{=} \sup_{t\in[0,T]} \|v(\cdot,t)\|_{1,\infty}$. For $0 < \gamma < 1$ let

$$X_\gamma(T) = \{v \in X(T): \|v(\cdot,t_2) - v(\cdot,t_1)\|_{1,\infty} \leqslant K((t_2-t_1)^\gamma / t_2^{1/2})$$
$$\text{for all } t_1, t_2 \in [0,T] \text{ with } t_1 < t_2 \text{ and some constant } K\}.$$

Theorem 2.1. *Let* $u_0 \in W^{1,\infty}(0,1)$, $u_1 \in L^2(0,1)$ *and let* $\frac{1}{2} < \gamma < \frac{3}{4}$. *Then for sufficiently small* $T > 0$ *there exists a unique solution u of* (2.1) *in* $X_\gamma(T)$. *Furthermore, u is a weak solution of* (1.1)–(1.3a) *in the sense that* $u(x,t)$ *satisfies* $u(x,0) = u_0(x)$ *and*

$$-\int_0^t \int_0^1 u_s(x, s)\, \phi_s(x, s)\, dx\, ds + \int_0^t \int_0^1 u_{xs}(x, s)\, \phi_x(x, s)\, dx\, ds$$

$$+ \int_0^t \int_0^1 \sigma(u_x(x, s))\, \phi_x(x, s)\, dx\, ds$$

$$= \int_0^1 u_1(x)\, \phi(x, 0)\, dx - \int_0^1 u_t(x, t)\, \phi(x, t)\, dx \tag{2.2}$$

whenever $\phi \in C([0, T]; W_0^{1,1}(0, 1))$, $\phi_t \in C([0, T]; L^2(0, 1))$.

Theorem 2.1 is proved in Andrews (1980) for the case when $u_1 \in W_0^{1,2}(0, 1)$; the extension to the case when $u_1 \in L^2(0, 1)$ is carried out in Andrews (1979).

For the boundary conditions (1.3b) we suppose that $u_0 \in W^{1,\infty}(0, 1)$, that $u_0(0) = 0$, and that $u_0'(1) \stackrel{\text{def}}{=} \text{ess lim}_{x \to 1-} u_0'(x)$ exists. Let $g(t)$ denote the solution of the initial value problem.

$$\dot{g}(t) + \sigma(g(t)) = P,$$
$$g(0) = u_0'(1). \tag{2.3}$$

Since σ is locally Lipschitz the solution of (2.3) exists for sufficiently small $t > 0$ and is unique. Let $T > 0$ be sufficiently small. The integral equation corresponding to (2.1) is

$$u(x, t) = U(x, t) - \int_0^t \int_0^s \int_0^1 \tilde{G}_y(x, y, s - \tau)$$
$$\times (\sigma(u_y(y, \tau)) - \sigma(g(\tau)))\, dy\, d\tau\, ds, \tag{2.4}$$

where $\tilde{G}(x, y, t)$ denotes the Green's function for the linear heat equation

$$w_t(x, t) = w_{xx}(x, t), \qquad x \in (0, 1),\ t > 0,$$

with boundary conditions

$$w(0, t) = w_x(1, t) = 0, \qquad t > 0,$$

and where U denotes the solution of the initial boundary value problem

$$U_{tt} = U_{xxt}, \qquad x \in (0, 1),\ t \in (0, T],$$
$$U(x, 0) = u_0(x), \qquad U_t(x, 0) = u_1(x), \qquad x \in (0, 1),$$
$$U(0, t) = 0, \qquad U_x(1, t) = g(t), \qquad t \in [0, T]. \tag{2.5}$$

By substituting for $U - g(t)\,x$ it is easily seen that

$$U(x, t) = u_0(x) + (g(t) - g(0))\,x + \int_0^t \int_0^1 \tilde{G}(x, y, s)(u_1(y) - \dot{g}(0)\,y)\,dy\,ds - \int_0^t \int_0^s \int_0^1 \tilde{G}(x, y, s-\tau)\,\ddot{g}(\tau)\,y\,dy\,d\tau\,ds. \tag{2.6}$$

For $T > 0$ and $0 < \gamma < 1$ let $\tilde{X}(T) = \{v \in C([0, T]; W^{1,\infty}(0, 1))\colon v(0, t) = 0$ for all $t \in [0, T]\}$ with the norm induced by $C([0, T]; W^{1,\infty}(0, 1))$, and

$$\tilde{X}_\gamma(T) = \{v \in \tilde{X}(T)\colon \|v(\cdot, t_2) - v(\cdot, t_1)\|_{1,\infty} \leqslant K((t_2 - t_1)^\gamma / t_2^{1/2}) \text{ for all } t_1, t_2 \in [0, T] \text{ with } t_1 < t_2 \text{ for some constant } K\}.$$

THEOREM 2.2. *Let* $u_0 \in W^{1,\infty}(0, 1)$, $u_0(0) = 0$, *let* $u_0'(1)$ *exist, and suppose* $u_1 \in L^2(0, 1)$. *If* $\frac{1}{2} < \gamma < \frac{3}{4}$ *then for sufficiently small* $T > 0$ *there exists a unique solution* u *of* (2.4) *in* $\tilde{X}_\gamma(T)$. *Furthermore,* u *is a weak solution of* (1.1), (1.2), (1.3b) *in the sense that* $u(x, t)$ *satisfies* $u(x, 0) = u_0(x)$ *and*

$$-\int_0^t \int_0^1 u_s(x, s)\,\phi_s(x, s)\,dx\,ds + \int_0^t \int_0^1 u_{xs}(x, s)\,\phi_x(x, s)\,dx\,ds + \int_0^t \int_0^1 \sigma(u_x(x, s))\,\phi_x(x, s)\,dx\,ds = P\int_0^t \phi(1, s)\,ds + \int_0^1 u_1(x)\,\phi(x, 0)\,dx - \int_0^1 u_t(x, t)\,\phi(x, t)\,dx \tag{2.7}$$

for every ϕ *in the set*

$$\{\phi \in C([0, T]; W^{1,1}(0, 1)); \phi(0, t) = 0, \phi_t \in C([0, 1]; L^2(0, 1))\}.$$

The boundary condition at $x = 1$ *holds in the sense that for all* $t \in [0, T]$

$$\operatorname*{ess\,lim}_{x \to 1-} u_x(x, t) = g(t). \tag{2.8}$$

Sketch of proof. Theorem 2.2 was stated in Andrews (1979, 1980) for the case when $u_0'(1)=0$ and $\sigma(0)=0$ (or, equivalently, when $\sigma(u_0'(1))=P$), and the pattern of the proof in the general case is the same.

We first note that $\tilde{G}(x,y,t)$ can be given explicitly by the formula

$$\begin{aligned}\tilde{G}(x,y,t)=\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty}\Big\{&\exp\left[\frac{-(x-y-4n)^2}{4t}\right]\\ &+\exp\left[\frac{-(x+y-4n-2)^2}{4t}\right]-\exp\left[\frac{-(x-y-4n-2)^2}{4t}\right]\\ &-\exp\left[\frac{-(x+y-4n)^2}{4t}\right]\Big\}.\end{aligned}\tag{2.9}$$

For positive constants R and K the set

$$\begin{aligned}A(R,K)=\{v\in\tilde{X}(T)\colon \|v\|_{\tilde{X}(T)}\leqslant R,\ &\|v(\cdot,t_2)-v(\cdot,t_1)\|_{1,\infty}\\ &\leqslant K((t_2-t_1)^{\gamma}/t_2^{1/2})\text{ for all }t_1,t_2\in[0,T]\text{ with }t_1<t_2\}\end{aligned}$$

is a closed, bounded and convex subset of $\tilde{X}(T)$. For $v\in A(R,K)$ define

$$\begin{aligned}(\mathscr{F}v)(x,t)&=U(x,t)-\int_0^t\int_0^s\int_0^1\tilde{G}_y(x,y,s-\tau)\\ &\qquad\times(\sigma(v_y(y,\tau))-\sigma(g(\tau)))\,dy\,d\tau\,ds,\\ (\mathscr{G}v)(x,t)&=\int_0^t\int_0^s\int_0^1\tilde{G}_y(x,y,s-\tau)\\ &\qquad\times(\sigma(v_y(y,s))-\sigma(v_y(y,\tau)))\,dy\,d\tau\,ds,\\ (\mathscr{C}v)(x,t)&=U(x,t)-\int_0^t\int_0^s\int_0^1\tilde{G}_y(x,y,s-\tau)\\ &\qquad\times(\sigma(v_y(y,s))-\sigma(g(\tau)))\,dy\,d\tau\,ds,\end{aligned}$$

so that $\mathscr{F}v=\mathscr{G}v+\mathscr{C}v$.

By estimating $\tilde{G}(x,y,t)$ using (2.9), and applying the techniques of Andrews (1979, 1980) one can prove that R, K can be chosen such that, for a sufficiently small $T>0$,

(i) $\mathscr{G}v_1+\mathscr{C}v_2\in A(R,K)$ for all $v_1,v_2\in A(R,K)$,

(ii) $\mathscr{G}\colon A(R,K)\to\tilde{X}(T)$ is compact and continuous, and

(iii) $\mathscr{C}\colon A(R,K)\to A(R,K)$ is a contraction.

Hence $\mathscr{F}$ has a fixed point u, which is a solution of (2.4).

We next show that (2.8) holds. For $0 < \delta < 1$ let

$$h(\delta, T) = \operatorname*{ess\,sup}_{x \in (1-\delta, 1)} |u_x(x, t) - g(t)|. \tag{2.10}$$

We estimate $h(\delta, t)$ using (2.4), which we write in the form

$$\begin{aligned} u(x, t) &= U(x, t) - \int_0^t \int_0^s \int_0^1 \tilde{G}_y(x, y, s-\tau) f(y, s)\, dy\, d\tau\, ds \\ &\quad + \int_0^t \int_0^s \int_0^1 \tilde{G}_y(x, y, s-\tau)(f(y, s) - f(y, \tau))\, dy\, d\tau\, ds \\ &= U(x, t) - I(x, t) + J(x, t), \end{aligned} \tag{2.11}$$

where

$$f(x, t) = \sigma(u_x(x, t)) - \sigma(g(t)).$$

Using estimates for $\tilde{G}(x, y, t)$, the boundedness of $\ddot{g}(t)$, the fact that $u_1 \in L^2(0, 1)$, the relation $\tilde{G}_x(1, y, t) = 0$ and the dominated convergence theorem, it is easily proved that

$$U_x(x, t) - g(t) = u_0'(x) - g(0) + a(x, t), \tag{2.12}$$

where

$$\lim_{\delta \to 0} \operatorname*{ess\,sup}_{\substack{x \in (1-\delta, 1) \\ t \in [0, T]}} |a(x, t)| = 0. \tag{2.13}$$

Similarly, using the Hölder continuity of $f(\cdot, t)$,

$$\lim_{\delta \to 0} \operatorname*{ess\,sup}_{\substack{x \in (1-\delta, 1) \\ t \in [0, T]}} |J_x(x, t)| = 0. \tag{2.14}$$

Let $H(x, y, t)$ denote the Green's function for the linear heat equation

$$w_t(x, t) = w_{xx}(x, t), \qquad x \in (0, 1),\ t > 0,$$

with boundary conditions

$$w_x(0, t) = w(1, t) = 0.$$

Then $H_x(x, y, t) = -\tilde{G}_y(x, y, t)$, and so

$$\begin{aligned} I_x(x, t) &= -\int_0^t \int_0^s \int_0^1 H_t(x, y, s-\tau) f(y, s)\, dy\, d\tau\, ds \\ &= -\int_0^t \left[\int_0^1 H(x, y, s) f(y, s)\, dy - f(x, s) \right] ds. \end{aligned}$$

It follows that

$$I_x(x, t) = \int_0^t [\sigma(u_x(x, s)) - \sigma(g(s))]\, ds + b(x, t), \tag{2.15}$$

where

$$\lim_{\delta \to 0} \operatorname*{ess\,sup}_{\substack{x \in (1-\delta, 1) \\ t \in [0, T]}} |b(x, t)| = 0. \tag{2.16}$$

Combining (2.10)–(2.16) we deduce that

$$h(\delta, t) \leqslant h(\delta, 0) + \nu(\delta) + k \int_0^t h(\delta, s)\, ds \tag{2.17}$$

for all $t \in [0, T]$, where k is a local Lipschitz constant for σ and where $\lim_{\delta \to 0} \nu(\delta) = 0$. Applying Gronwall's inequality and letting $\delta \to 0$ we obtain (2.8).

The weak form (2.7) of the equation and the uniqueness assertion now follows as in Andrews (1979, 1980). ∎

Global Existence

To obtain global existence we make the further hypothesis that, for the boundary conditions (1.3a),

W is bounded below, and there exists $h > 0$ such that

$$(\sigma(z_1) - \sigma(z_2))(z_1 - z_2) > 0 \qquad \text{whenever } |z_1 - z_2| \geqslant h. \tag{Ha}$$

Then we have the following result.

THEOREM 2.3. *Let the hypotheses of Theorem 2.1 hold. Then there exists a unique solution u of (2.1) in $X_\gamma(T)$ for any $T > 0$. The energy equation*

$$\tfrac{1}{2} \|u_t(\cdot, t_2)\|_2^2 + \int_0^1 W(u_x(x, t_2))\, dx + \int_{t_1}^{t_2} \|u_{xs}(\cdot, s)\|_2^2\, ds$$
$$= \tfrac{1}{2} \|u_t(\cdot, t_1)\|_2^2 + \int_0^1 W(u_x(x, t_1))\, dx \tag{2.18}$$

holds for all $t_1, t_2 \geqslant 0$. Moreover there exists a constant M depending only on $\|u_0\|_{1,\infty}$, $\|u_1\|_2$ and σ such that

$$\|u(\cdot, t)\|_{1,\infty} \leqslant M \qquad \textit{for all } t > 0, \tag{2.19}$$

and

$$\|u_t(\cdot, t)\|_2 \leqslant M \qquad \textit{for all } t \geqslant 0, \tag{2.20}$$

and, for any $\delta > 0$, there exists a constant $M_1 > 0$ depending only on $\|u_0\|_{1,\infty}$, $\|u_1\|_2$, σ and δ such that

$$\|u_t(\cdot, t)\|_{1,\infty} \leqslant M_1 \qquad \textit{for all } t \geqslant \delta. \tag{2.21}$$

Theorem 2.3 is proved in Andrews (1979, 1980) for sufficiently smooth σ; the result for general locally Lipschitz σ can be obtained by approximating in the way indicated in the proof of Theorem 2.4 below.

For the boundary conditions (1.3b) we make the hypothesis that there exists $h > 0$ such that

$$(\sigma(z) - P)\, z > 0 \qquad \text{for all } |z| \geqslant h. \tag{Hb}$$

THEOREM 2.4. *Let the hypotheses of Theorem 2.2 hold. Then there exists a unique solution u of (2.4) in $\tilde{X}_\gamma(T)$ for any $T > 0$. The energy equation*

$$\begin{aligned} &\tfrac{1}{2}\|u_t(\cdot, t_2)\|_2^2 + \int_0^1 W(u_x(x, t_2))\, dx - Pu(1, t_2) \\ &\quad + \int_{t_1}^{t_2} \|u_{xs}(\cdot, s)\|_2^2\, ds \\ &= \tfrac{1}{2}\|u_t(\cdot, t_1)\|_2^2 + \int_2^1 W(u_x(x, t_1))\, dx - Pu(1, t_1) \end{aligned} \tag{2.22}$$

holds for all $t_1, t_2 \geqslant 0$. Moreover, the estimates (2.19)–(2.21) *hold, where M, M_1 may depend also on P.*

Sketch of proof. The theorem is proved in Andrews (1979, 1980) for the special case when σ is sufficiently smooth and $\sigma(u_0'(1)) = 0$, and the proof in the general case follows the same pattern. The crucial point is to prove the estimate (2.19). We do this first for a sufficiently regular solution u of (2.4). Fix any $x_0 \in [0, 1]$ and let

$$q(t) \overset{\text{def}}{=} \int_1^{x_0} u_t(y, t)\, dy - u_x(x_0, t). \tag{2.23}$$

Then

$$\dot{q}(t) = \sigma(u_x(x_0, t)) - P. \tag{2.24}$$

Since u is smooth it satisfies the energy equation (2.22), and since by (Hb), $W(z) - Pz$ is bounded below it follows that (2.20) holds and hence that

$$\left| \int_0^x u_t(y, t)\, dy \right| \leqslant k$$

for all $t \geqslant 0$ and all $x \in [0, 1]$. The bound (2.19) now follows immediately from an easily proved lemma.

LEMMA 2.5. *Let $T > 0$ and suppose that $|a(t)| \leqslant k$ for all $t \in [0, T]$. If $q \in C^1([0, T])$ satisfies*

$$\dot{q}(t) = \sigma(a(t) - q(t)) - P$$

then

$$|q(t)| \leqslant \max\{|q(0)|, h + k\} \qquad \text{for all } t \in [0, T].$$

The estimate (2.21) follows from (2.19) and the integral equation (2.4). Suppose now that σ is smooth. For sufficiently regular u_0, u_1 there exists a sufficiently regular solution u of (2.4) defined on an interval $[0, T]$ whose length depends only on $\|u_0\|_{1,\infty}$, $\|u_1\|_2$ and σ (compare Andrews, 1979, Proposition 3.5). By approximating u_0, u_1 and σ by smooth functions and using continuous dependence one can establish (2.19)–(2.22) in the general case. ∎

A physically reasonable solution u should satisfy the invertibility condition $u_x(x, t) > -1$. Suppose that $\sigma: (-1, \infty) \to \mathbb{R}$ is locally Lipschitz and that there exist constants $h > 0$, $\gamma \in (-1, 0)$ such that $(\sigma(z) - P)\, z > 0$ if $z > h$ or $-1 < z < \gamma$. Let $\omega_{\min} \stackrel{\text{def}}{=} \inf_{z \in (-1,\infty)} W(z) - Pz$. Suppose that for some $\varepsilon > 0$,

$$u_0'(x) \geqslant -1 + \varepsilon \qquad \text{a.e.} \quad x \in [0, 1],$$

and that the initial energy is small, so that

$$E(0) \stackrel{\text{def}}{=} \tfrac{1}{2} \int_0^1 u_1^2(x)\, dx + \int_0^1 [W(u_0'(x)) - Pu_0'(x)]\, dx < \omega_{\min} + \delta,$$

where $2\sqrt{2\delta} < \min(\gamma + 1, \varepsilon)$. Defining $q(t)$ by (2.23), noting that by (2.22), $|\int_0^x u_t\, dx| \leqslant \sqrt{2\delta}$, and applying the same argument as in the above proof, we deduce that

$$-1 + \nu \leqslant u_x(x, t) \leqslant M \tag{2.25}$$

for all $t \geqslant 0$ and a.e. $x \in [0, 1]$, where ν, M are positive constants. In this case one can therefore redefine $\sigma(z)$ for $z < -1 + \nu$ so that $\sigma: \mathbb{R} \to \mathbb{R}$, and apply our analysis. (The same argument works in one-dimensional thermoviscoelasticity of rate type; see Dafermos and Hsiao (1981) and Dafermos (1981).)

For equations of the type

$$u_{tt} = (\sigma(u_x) + \lambda(u_x)\, u_{xt})_x,$$

where $\lambda: (-1, \infty) \to (0, \infty)$, $\Lambda(z) \overset{\text{def}}{=} \int_0^z \lambda(\xi)\, d\xi \to -\infty$ as $z \to -1$, and $\sigma: (-1, \infty) \to \mathbb{R}$ is as above, one can prove (cf. Andrews, 1979) that $u_x(x, t) > -1$ for all $t > 0$, a.e. $x \in [0, 1]$ without a small data assumption. In this case we use the function

$$q(t) \overset{\text{def}}{=} \int_0^{x_0} u_t(x, t)\, dx - \Lambda(u_x(x_0, t))$$

in place of (2.23).

3. Approximation Lemmas

The following approximation lemmas are a key ingredient in studying the asymptotic behaviour of solutions to (1.1)–(1.3).

Lemma 3.1. *Let* $\sigma \in C(\mathbb{R})$, and let $M > 0, P$ be given constants. Then the set

$$S = \operatorname{span}\{\Phi \in C^2([-M, M]) : \Phi'(z)(\sigma(z) - P) \geqslant 0 \text{ if } |z| \leqslant M\}$$

is dense in $C([-M, M])$.

Proof. Let $A = \{z \in \mathbb{R} : \sigma(z) \neq P\}$; then A is open, and ∂A is a closed, nowhere dense set. We suppose without loss of generality that $\pm M \notin \partial A$. Consider first a function $f \in C([-M, M])$ for which

$$\begin{aligned} f(z) &= f(\alpha), \qquad z \leqslant \alpha, \\ &= f(\beta), \qquad z \geqslant \beta, \end{aligned} \tag{3.1}$$

where $[\alpha, \beta] \subset [-M, M] \backslash \partial A$. If f is smooth and monotone then we can choose $\Phi = \pm f$ and so $f \in S$. Since mollifying f preserves monotonicity, and since A is open, it follows that if f is merely continuous and monotone then $f \in \bar{S}$. Next suppose that f is smooth but not necessarily monotone; then we can write

$$f(z) = f(\alpha) + \int_\alpha^z f'(y)_+\, dy - \int_\alpha^z f'(y)_-\, dy,$$

which is the difference of two continuous monotone functions of the type (3.1). Hence $f \in \bar{S}$. By mollifying we deduce that any continuous f of the type (3.1) belongs to $\bar{S}$.

Now let $f \in C([-M, M])$ be arbitrary, and let $\varepsilon > 0$. Then the compact set $\partial A \cap [-M, M]$ can be covered by a finite number of open intervals $I_i = (z_i - \gamma_i, z_i + \delta_i)$, $1 \leqslant i \leqslant k$, such that

$$|f(z) - f(z_i)| < \varepsilon \qquad \text{if} \quad z \in \bar{I}_i.$$

Since ∂A is nowhere dense we can suppose without loss of generality that the $\bar{I}_i$ are disjoint subintervals of $(-M, M)$. Let $g(z) = f(z_i)$ for $z \in \bar{I}_i$, $1 \leqslant i \leqslant k$; then we can define g on the remainder of $[-M, M]$ in such a way that $g \in C([-M, M])$ and $\| g - f\|_{C([-M,M])} < \varepsilon$. Clearly g can be written in the form

$$g(z) = \sum_{i=1}^{k} g_i(z),$$

where each $g_i \in C([-M, M])$ has the form (3.1). Thus g belongs to $\bar{S}$, and since ε was arbitrary so does f. ∎

LEMMA 3.2. *Let $\sigma \in C(\mathbb{R})$ be constant on no interval, and let $M > 0$, P be given constants. For $\varepsilon > 0$ let*

$$S_\varepsilon = \text{span}\{\Phi \in C^2([-M, M]) \colon \Phi'(z)(\sigma(z) - \tau) \geqslant 0$$
$$\textit{whenever} \quad |z| \leqslant M \textit{ and } |\tau - P| < \varepsilon\}.$$

Then $S \stackrel{\text{def}}{=} \bigcup_{\varepsilon > 0} S_\varepsilon$ is dense in $C([-M, M])$.

Proof. Define A as in the proof of Lemma 3.1, and consider a function $f \in C([-M, M])$ of the type (3.1). Since σ is not constant on any interval there exists $\varepsilon > 0$ such that $\sigma(z) - \tau$ is of one sign whenever $z \in [\alpha, \beta]$ and $|\tau - P| < \varepsilon$. Hence $\pm f \in S_\varepsilon$ and so $f \in S$. The remainder of the proof is the same. ∎

4. Asymptotic Behaviour for Mixed Boundary Conditions

Let P be given, and suppose σ satisfies (Hb). Let $u_0 \in W^{1,\infty}(0, 1)$ with $u_0(0) = 0$, suppose $u_0'(1)$ exists, and let $u_1 \in L^2(0, 1)$. Then Theorem 2.4 guarantees the global existence of a suitably defined unique weak solution $u(x, t)$ to (1.1), (1.2), (1.3b). We study the asymptotic behaviour of this solution as $t \to \infty$.

THEOREM 4.1. (i) *$u_t(\cdot, t) \stackrel{*}{\rightharpoonup} 0$ in $W^{1,\infty}(0, 1)$ as $t \to \infty$.*

(ii) *$u(\cdot, t) \stackrel{*}{\rightharpoonup} v(\cdot)$ in $W^{1,\infty}(0, 1)$ as $t \to \infty$ for some v with $v(0) = 0$.*

(iii) *$\sigma(u_x(\cdot, t)) \to P$ in $L^2(0, 1)$ as $t \to \infty$.*

(iv) *$m \stackrel{\text{def}}{=} \lim_{t \to \infty} u_x(1, t)$ exists, and $\sigma(m) = P$.*

(v) *There exists a family of probability measures* $\{\nu_x\}_{x\in(0,1)}$ *on* $\mathbb{R}$ *(depending measurably on* x*) with* $\operatorname{supp}\nu_x \subset K_P = \{z\colon \sigma(z)=P\}$ *such that if* $\Phi \in C(\mathbb{R})$ *and*

$$f_\Phi(x) = \langle \nu_x, \Phi\rangle \qquad a.e.,$$

then

$$\Phi(u_x(\cdot, t)) \overset{*}{\rightharpoonup} f_\Phi(\cdot) \qquad in \quad L^\infty(0,1) \ as \ t\to\infty.$$

Proof. Since $W(z)-Pz$ is bounded below, it follows from the energy equation (2.22) that

$$\int_0^\infty \|u_{xt}(\cdot, t)\|_2^2\, dt < \infty, \tag{4.1}$$

and hence by the Poincaré inequality that

$$\int_0^\infty \|u_t(\cdot, t)\|_2^2\, dt < \infty. \tag{4.2}$$

From (2.22) we also have that for $t, \tau > 0$

$$\begin{aligned} &\big|\, \|u_t(\cdot, t+\tau)\|_2^2 - \|u_t(\cdot, t)\|_2^2 \,\big| \\ &\quad\leqslant 2\int_t^{t+\tau} \|u_{xs}(\cdot, s)\|_2^2\, ds + 2\left| \int_t^{t+\tau}\int_0^1 (\sigma(u_x(x,s))-P)\, u_{xs}(x,s)\, dx\, ds \right| \\ &\quad\leqslant 2\tau\left[\sup_{s\geqslant t} \|u_{xs}(\cdot, s)\|_\infty^2 + \sup_{s\geqslant t}\int_0^1 |\sigma(u_x(x,s))-P|\, |u_{xs}(x,s)|\, dx\, ds\right]. \end{aligned}$$

It follows from (2.19), (2.21) that $\|u_t(\cdot, t)\|_2^2$ is uniformly continuous on $[\delta, \infty)$ for any $\delta > 0$. By (4.2) this implies that $\lim_{t\to\infty} \|u_t(\cdot, t)\|_2 = 0$, and (i) follows by (2.21).

Part (iv) is an immediate consequence of (2.3) and the fact that, by (2.19), $g(t) = \operatorname{ess\,lim}_{x\to 1-} u_x(x,t)$ is bounded.

Suppose $\psi \in L^2(0,1)$ with $\psi \geqslant 0$, and that $\Phi \in C^2([-M, M])$ satisfies

$$\Phi'(z)(\sigma(z)-P) \geqslant 0 \qquad \text{for } |z| \leqslant M,$$

where M is the upper bound in (2.19).

Inserting

$$\phi(x,t) = \int_0^x \psi(y)\, \Phi'(u_y(y,t))\, dy$$

in (2.7) we obtain

$$\int_0^1 u_t(x,t)\left(\int_0^x \psi(y)\,\Phi'(u_y(y,t))\,dy\right)dx$$
$$-\int_0^1 u_1(x)\int_0^x \psi(y)\,\Phi'(u_0'(y))\,dy\,dx$$
$$-\int_0^t\int_0^1 u_s(x,s)\left(\int_0^x \psi(y)\,\Phi''(u_y(y,s))\,u_{ys}(y,s)\,dy\right)dx\,ds$$
$$+\int_0^1 \psi(x)\,\Phi(u_x(x,t))\,dx-\int_0^1 \psi(x)\,\Phi(u_0'(x))\,dx$$
$$+\int_0^t\int_0^1 \psi(x)\,\Phi'(u_x(x,s))(\sigma(u_x(x,s))-P)\,dx\,ds=0. \tag{4.3}$$

Now, for the first term in (4.3),

$$\left|\int_0^1 u_t(x,t)\left(\int_0^x \psi(y)\,\Phi'(u_y(y,t))\,dy\right)dx\right|$$
$$\leqslant \|u_t(\cdot,t)\|_2\int_0^1 |\psi(x)\,\Phi'(u_x(x,t))|\,dx$$
$$\leqslant \|u_t(\cdot,t)\|_2\,\|\psi\|_2\,\|\Phi'(u_x(\cdot,t))\|_2$$
$$\leqslant C\,\|u_t(\cdot,t)\|_2,$$

which tends to zero as $t\to\infty$ by (i). (Here and below C denotes a generic constant.)

For the third term in (4.3) we note that

$$\left|\int_0^1 u_t(x,t)\left(\int_0^x \psi(y)\,\Phi''(u_y(y,t))u_{yt}(y,t)\,dy\right)dx\right|$$
$$\leqslant \|u_t(\cdot,t)\|_2\,\|\psi\|_2\,\|\Phi''(u_x(\cdot,t))\|_\infty\,\|u_{xt}(\cdot,t)\|_2$$
$$\leqslant C\,\|u_{xt}(\cdot,t)\|_2^2 \tag{4.4}$$

for all $t\geqslant 0$. Hence, by (4.1) and the dominated convergence theorem,

$$\lim_{t\to\infty}\int_0^t\int_0^1 u_s(x,s)\left(\int_0^x \psi(y)\,\Phi''(u_y(y,s))\,u_{ys}(y,s)\,dy\right)dx\,ds$$

exists.

Since

$$\left| \int_0^1 \psi(x)\, \Phi(u_x(x,t))\, dx \right| \leqslant \|\psi\|_2 \, \|\Phi(u_x(\cdot,t))\|_2,$$

the fourth term in (4.3) is bounded.

The second and fifth terms in (4.3) are constants. Therefore the sixth and last term is bounded, and since the integrand in this term is by assumption nonnegative it follows that

$$\lim_{t\to\infty} \int_0^t \int_0^1 \psi(x)\, \Phi'(u_x(x,s))(\sigma(u_x(x,s)) - P)\, dx\, ds \tag{4.5}$$

exists.

In particular, taking $\psi \equiv 1$ and $\Phi(z) = W(z) - Pz$, we see from (4.5) that

$$\int_0^\infty \|\sigma(u_x(\cdot,t)) - P\|_2^2\, dt < \infty. \tag{4.6}$$

Since

$$\left| \frac{d}{dt} \|\sigma(u_x(\cdot,t)) - P\|_2^2 \right|$$

$$\leqslant 2\, \|\sigma(u_x(\cdot,t)) - P\|_\infty \, \|\sigma'(u_x(\cdot,t))\|_\infty \int_0^1 |u_{xt}(x,t)|\, dx$$

is bounded as $t \to \infty$, it follows from (4.6) that (iii) holds.

Returning to (4.3), we have shown that every term apart from $\int_0^1 \psi(x)\, \Phi(u_x(x,t))\, dx$ is either independent of t or tends to a limit as $t \to \infty$. Hence

$$\lim_{t\to\infty} \int_0^1 \psi(x)\, \Phi(u_x(x,t))\, dx$$

exists for all $\psi \in L^2(0,1)$ with $\psi \geqslant 0$, and thus for all $\psi \in L^2(0,1)$. Therefore

$$\Phi(u_x(\cdot,t)) \rightharpoonup f_\Phi(\cdot) \qquad \text{in} \quad L^2(0,1)$$

as $t \to \infty$ for some $f_\Phi \in L^2(0,1)$. Since $\|\Phi(u_x(\cdot,t))\|_\infty \leqslant C$ it follows that $f_\Phi \in L^\infty(0,1)$ and

$$\Phi(u_x(\cdot,t)) \overset{*}{\rightharpoonup} f_\Phi(\cdot) \qquad \text{in} \quad L^\infty(0,1). \tag{4.7}$$

Next let $\Phi \in C([-M, M])$ be arbitrary, and let $\psi \in L^1(0, 1)$. We show that $\{\int_0^1 \psi(x)\, \Phi(u_x(x, t))\, dx; t \geqslant 0\}$ is Cauchy. Given $\varepsilon > 0$, there exists by Lemma 3.1 a function $\Phi_\varepsilon \in S$ such that

$$2\, \|\psi\|_1\, \|\Phi - \Phi_\varepsilon\|_{C([-M,M])} \leqslant \frac{\varepsilon}{2}.$$

By (4.7)

$$\left| \int_0^1 \psi(x)\, \Phi_\varepsilon(u_x(x, t))\, dx - \int_0^1 \psi(x)\, \Phi_\varepsilon(u_x(x, s))\, dx \right| < \frac{\varepsilon}{2}$$

for sufficiently large s, t. Therefore

$$\begin{aligned} &\left| \int_0^1 \psi(x)\, \Phi(u_x(x, t))\, dx - \int_0^1 \psi(x)\, \Phi(u_x(x, s))\, dx \right| \\ &\qquad < 2\, \|\psi\|_1\, \|\Phi - \Phi_\varepsilon\|_{C([-M,M])} + \frac{\varepsilon}{2} \\ &\qquad < \varepsilon \end{aligned}$$

for sufficiently large s and t, as required. Hence

$$\lim_{t \to \infty} \int_0^1 \psi(x)\, \Phi(u_x(x, t))\, dx$$

exists for all $\psi \in L^1(0, 1)$, and it follows easily that (4.7) holds for an arbitrary $\Phi \in C([-M, M])$. Choosing $\Phi(z) \equiv z$ we immediately obtain (ii).

The existence of the probability measures ν_x follows at once from (4.7) and Tartar (1979, Theorem 5) (see also Balakrishnan, 1976, p. 31). To prove that $\operatorname{supp} \nu_x \subset K_P$ a.e. it suffices to show that if Φ is zero on K_P then $\langle \nu_x, \Phi \rangle = 0$ a.e. But if Φ is zero on K_P it follows from (iii) that $\Phi(u_x(\cdot, t)) \to 0$ in measure as $t \to \infty$. Therefore $\Phi(u_x(\cdot, t)) \overset{*}{\rightharpoonup} 0$ in $L^\infty(0, 1)$ as $t \to \infty$, and hence $\langle \nu_x, \Phi \rangle = 0$ a.e. as required. ∎

COROLLARY 4.2. *Suppose the equation $\sigma(z) = P$ has only one root z_1. Then as $t \to \infty$*

$$u(x, t) \to z_1 x \qquad \textit{strongly in} \quad W^{1,p}(0, 1)$$

for all p, $1 \leqslant p < \infty$.

Proof. Since $\operatorname{supp} \nu_x \subset \{z_1\}$ we have $\nu_x = \delta_{z_1}$ a.e., and therefore

$$\Phi(u_x(\cdot, t)) \overset{*}{\rightharpoonup} \Phi(z_1) \qquad \text{in} \quad L^\infty(0, 1)$$

for all $\Phi \in C(\mathbb{R})$. In particular $u_x(\cdot, t) \overset{*}{\rightharpoonup} z_1$ and $|u_x(\cdot, t)|^p \overset{*}{\rightharpoonup} |z_1|^p$ in $L^\infty(0, 1)$. Hence $u_x(\cdot, t) \to z_1$ strongly in $L^p(0, 1)$, and the result follows. ∎

COROLLARY 4.3. *Suppose the equation $\sigma(z) = P$ has exactly $k > 1$ roots $z_1, ..., z_k$. Then there exist nonnegative functions $\mu_i \in L^\infty(0, 1)$, $1 \leqslant i \leqslant k$, such that if $\Phi \in C(\mathbb{R})$ then*

$$\Phi(u_x(x, t)) \overset{*}{\rightharpoonup} \sum_{i=1}^{k} \Phi(z_i)\, \mu_i(x) \qquad in \quad L^\infty(0, 1) \tag{4.8}$$

as $t \to \infty$. Furthermore

$$\sum_{i=1}^{k} \mu_i(x) = 1 \qquad a.e., \tag{4.9}$$

$$u(x, t) \overset{*}{\rightharpoonup} \sum_{i=1}^{k} z_i \int_0^x \mu_i(y)\, dy \qquad in \quad W^{1,\infty}(0, 1) as \ t \to \infty, \tag{4.10}$$

and if $\varepsilon < \min_{1 \leqslant i < j \leqslant k} |z_i - z_j|$ then for $1 \leqslant i \leqslant k$

$$\lim_{t \to \infty} \operatorname{meas}\{x \in A : |u_x(x, t) - z_i| < \varepsilon\} = \int_A \mu_i(x)\, dx, \tag{4.11}$$

for any measurable subset $A \subset [0, 1]$.

Proof. Since supp $\nu_x \subset \{z_1, ..., z_k\}$ we have that

$$\nu_x = \sum_{i=1}^{k} \mu_i(x)\, \delta_{z_i},$$

where the $\mu_i \in L^\infty(0, 1)$ are nonnegative and satisfy (4.9). Hence (4.8) and (4.9) hold. To prove (4.11) let $1 \leqslant i \leqslant k$ and choose $\Phi(z)$ to be 1 if $|z - z_i| < \varepsilon$ and 0 if $|z - z_i| > \frac{1}{2}(\varepsilon + \min_{j \neq i} |z_i - z_j|)$. By (4.8)

$$\lim_{t \to \infty} \int_A \Phi(u_x(x, t))\, dx = \int_A \mu_i(x)\, dx. \tag{4.12}$$

But

$$\int_A \Phi(u_x(x, t))\, dx \geqslant \operatorname{meas}\{x \in A : |u_x(x, t) - z_i| < \varepsilon\} \overset{\text{def}}{=} A_i(t). \tag{4.13}$$

Since, by (iii), $\lim_{t \to \infty} \sum_{j=1}^{k} A_j(t) = \operatorname{meas} A$, it follows from (4.9), (4.12) and (4.13) that (4.11) holds. ∎

COROLLARY 4.4. *Let the hypotheses of Theorem 4.1 hold. Then for each $t \geqslant 0$ there exists an equilibrium solution $w(x; t) \in W^{1,\infty}(0, 1)$, that is, a solution of*

$$\begin{aligned} \sigma(w_x(x; t)) &= P \qquad a.e. \quad x \in (0, 1), \\ w(0; t) &= 0, \end{aligned} \tag{4.14}$$

such that for any p, $1 \leqslant p < \infty$,

$$\lim_{t \to \infty} \|u(\cdot, t) - w(\cdot; t)\|_{1,p} = 0.$$

Proof. We define $w(\cdot; t)$ successively on the intervals $[T_i, T_{i+1})$, where $0 = T_0 < T_1 < \cdots$ and $\lim_{i \to \infty} T_i = \infty$. For each i we choose $T_i \geqslant T_{i-1}$ sufficiently large so that

$$\text{meas}\{x: \text{dist}(u_x(x, t), K_P) > 1/i\} < 1/i$$

for $t \geqslant T_i$; this is possible because by (iii), $\sigma(u_x(\cdot, t)) \to P$ in measure. For $t \in [T_i, T_{i+1})$ we define $w(0; t) = 0$ and

$$\begin{aligned} w_x(x; t) &= \max\{z: \sigma(z) = P \text{ and } |z - u_x(x, t)| \leqslant 1/i\} \\ &\qquad\qquad \text{if} \quad \text{dist}(u_x(x, t), K_P) \leqslant 1/i \\ &= z_0 \qquad \text{if} \quad \text{dist}(u_x(x, t), K_P) > 1/i, \end{aligned}$$

where $z_0 \in K_P$ is fixed. Then

$$\int_0^1 |u_x(x, t) - w_x(x; t)|^p \, dx \leqslant \frac{1}{i} (|z_0| + M)^p + \frac{1}{i^p}$$

for $t \in [T_i, T_{i+1})$, giving the result. ∎

Note that by Theorem 4.1(ii) and Corollary 4.4, $w(x; t) \overset{*}{\rightharpoonup} v(x)$ in $W^{1,\infty}(0, 1)$ as $t \to \infty$. Hence $v(x)$ is in the $W^{1,\infty}(0, 1)$ weak* closure of the set of equilibrium solutions. Unfortunately this information is very weak, since any function $w(x) \in W^{1,\infty}(0, 1)$ which is such that $w(0) = 0$ and

$$\min\{z: z \in K_p\} \leqslant w'(x) \leqslant \max\{z: z \in K_p\} \qquad \text{a.e.}$$

can be expressed as the weak* limit in $W^{1,\infty}(0, 1)$ of a sequence of equilibrium solutions (see Tartar, 1979, Theorem 3). We have *not* proved that v itself is an equilibrium solution.

We remark that a result analogous to Theorem 4.1(iii) was proved in

Dafermos (1969) for a more general model of one-dimensional nonlinear viscoelasticity in the case when stress boundary conditions are imposed at $x = 0, 1$.

5. Asymptotic Behaviour for Displacement Boundary Conditions

Suppose σ satisfies (Ha) and that $u_0 \in W_0^{1,\infty}(0, 1)$, $u_1 \in L^2(0, 1)$. Then Theorem 2.3 guarantees the global existence of a suitably defined unique weak solution $u(x, t)$ to (1.1)–(1.3a). We study the asymptotic behaviour of this solution as $t \to \infty$. We make the following extra nondegeneracy assumptions on σ:

(a) $\text{meas}\{z : \sigma'(z) = 0\} = 0$,

(b) the set N of local maximum and local minimum values of σ (not necessarily strict) is nowhere dense, and

(c) if $[p, q] \subset N^c$ with $p < q$, and if $\zeta_i \in W^{1,1}([p, q])$, $1 \leqslant i \leqslant 2k + 1$, $k = k(p, q)$, are the distinct inverse functions to σ on $[p, q]$ (odd in number by (Ha)), then $\{\zeta_i'\}_{1 \leqslant i \leqslant 2k+1}$ are linearly independent in $L^1([p, q])$.

THEOREM 5.1. (i) *$u_t(\cdot, t) \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(0, 1)$ as $t \to \infty$.*

(ii) *$u(\cdot, t) \overset{*}{\rightharpoonup} v(\cdot)$ in $W_0^{1,\infty}(0, 1)$ as $t \to \infty$ for some v.*

(iii) *$\sigma(u_x(\cdot, t)) \to P$ in $L^2(0, 1)$ as $t \to \infty$ for some constant P.*

(iv) *There exists a family of probability measures $\{\nu_x\}_{x \in (0,1)}$ on $\mathbb{R}$ (depending measurably on x) with* $\text{supp}\, \nu_x \subset K_P = \{z : \sigma(z) = P\}$ *such that if $\Phi \in C(\mathbb{R})$ and*

$$f_\Phi(x) = \langle \nu_x, \Phi \rangle \qquad a.e.,$$

then

$$\Phi(u_x(\cdot, t)) \overset{*}{\rightharpoonup} f_\Phi(\cdot) \qquad in \quad L^\infty(0, 1) \;\; as \;\; t \to \infty.$$

Proof. Part (i) is proved as in Theorem 4.1 using the energy equation (2.18).

Let $\psi \in L^2(0, 1)$ with $\psi \geqslant 0$, and suppose $\Phi \in W^{2,\infty}([-M, M])$, where M is the upper bound in (2.19).

Inserting

$$\phi(x, t) = \int_0^x \left(\psi(\xi)\, \Phi'(u_\xi(\xi, t)) - \int_0^1 \psi(y)\, \Phi'(u_y(y, t))\, dy\right) d\xi$$

in (2.2) we obtain

$$\int_0^1 u_t(x,t)\,\phi(x,t)\,dx - \int_0^1 u_1(x)\,\phi(x,0)\,dx - \int_0^t\int_0^1 u_s(x,s)\,\phi_s(x,s)\,dx\,ds$$
$$+ \int_0^1 \psi(x)\,\Phi(u_x(x,t))\,dx - \int_0^1 \psi(x)\,\Phi(u_0'(x))\,dx$$
$$+ \tfrac{1}{2}\int_0^t\int_0^1\int_0^1 (\psi(x)\,\Phi'(u_x(x,s)) - \psi(y)\,\Phi'(u_y(y,s)))$$
$$\times\,(\sigma(u_x(x,s)) - \sigma(u_y(y,s)))\,dx\,dy\,ds = 0. \tag{5.1}$$

The argument used in the proof of Theorem 4.1 shows that, provided

$$\int_0^1\int_0^1 (\psi(x)\,\Phi'(u_x(x,t)) - \psi(y)\,\Phi'(u_y(y,t)))$$
$$\times\,(\sigma(u_x(x,t)) - \sigma(u_y(y,t)))\,dx\,dy \geqslant 0 \tag{5.2}$$

for all sufficiently large $t \geqslant 0$, then

$$\lim_{t\to\infty}\int_0^1 \psi(x)\,\Phi(u_x(x,t))\,dx \text{ exists,} \tag{5.3}$$

and

$$\int_0^\infty\int_0^1\int_0^1 (\psi(x)\,\Phi'(u_x(x,t)) - \psi(y)\,\Phi'(u_y(y,t)))$$
$$\times\,(\sigma(u_x(x,t)) - \sigma(u_y(y,t)))\,dx\,dy\,dt < \infty. \tag{5.4}$$

We first make the choice $\psi \equiv 1$, $\Phi = W$. Then (5.2) holds, and so by (5.4) we have

$$\tfrac{1}{2}\int_0^\infty\int_0^1\int_0^1 [\sigma(u_x(x,t)) - \sigma(u_y(y,t))]^2\,dx\,dy\,dt$$
$$= \int_0^\infty\int_0^1 \left(\sigma(u_x(x,t)) - \int_0^1 \sigma(u_y(y,t))\,dy\right)^2 dx\,dt < \infty. \tag{5.5}$$

Since

$$\left|\frac{d}{dt}\int_0^1 \left(\sigma(u_x(x,t)) - \int_0^1 \sigma(u_y(y,t))\,dy\right)^2 dx\right|$$

is bounded as $t \to \infty$, it follows from (5.5) that

$$\sigma(u_x(\cdot, t)) - \int_0^1 \sigma(u_y(y, t))\, dy \to 0 \qquad \text{strongly in } L^2(0, 1) \tag{5.6}$$

as $t \to \infty$.

Our next goal is to show that if

$$\theta(t) \overset{\text{def}}{=} \int_0^1 \sigma(u_x(x, t))\, dx \tag{5.7}$$

then

$$P \overset{\text{def}}{=} \lim_{t \to \infty} \theta(t) \qquad \text{exists.} \tag{5.8}$$

The proof of (5.8) is somewhat lengthy. We first choose $\psi \equiv 1$ and

$$\Phi'(z) = \sigma(z) + \varepsilon F(\sigma(z)),$$

where F is an arbitrary smooth function. For sufficiently small $|\varepsilon|$, $r + \varepsilon F(r)$ is monotone increasing for r in a bounded set. Hence (5.2) holds, and so by (5.3)

$$\lim_{t \to \infty} \int_0^1 \int_{-M}^{u_x(x,t)} [\sigma(z) + \varepsilon F(\sigma(z))]\, dz\, dx \qquad \text{exists.}$$

Hence

$$\lim_{t \to \infty} \int_0^1 \int_{-M}^{u_x(x,t)} F(\sigma(z))\, dz\, dx \qquad \text{exists.} \tag{5.9}$$

Let χ be the characteristic function of a closed interval. Let F_j be a sequence of smooth functions converging monotonically to χ. Using the monotone convergence theorem it is easily proved that

$$\int_{-M}^{z} F_j(\sigma(z))\, dz \to \int_{-M}^{z} \chi(\sigma(z))\, dz$$

uniformly for $z \in [-M, M]$. Hence by the same argument as that used in the proof of Theorem 4.1 (after (4.7)) we deduce from (5.9) that

$$\lim_{t \to \infty} \int_0^1 \int_{-M}^{u_x(x,t)} \chi(\sigma(z))\, dz\, dx \qquad \text{exists.} \tag{5.10}$$

Suppose for contradiction that $\theta(t)$ does not tend to a limit as $t \to \infty$. Then there exist numbers p, q with $p < q$ such that the bounded continuous function $\theta(t)$ takes the values p and q for arbitrarily large values of t. Since the set N is nowhere dense we can suppose that $[p, q] \subset N^c$. The graph of the function $r = \sigma(z)$ intersects the strip $p \leqslant r \leqslant q$ in an odd number $2k+1$ of alternately strictly monotonic increasing and strictly monotonic decreasing segments of curves C_i, $1 \leqslant i \leqslant 2k+1$. If t is such that $\theta(t) \in [p, q]$ we denote the z-coordinate of the intersection of C_i with the line $r = \theta(t)$ by $\alpha_i(t)$ (see Fig. 1). We denote by $\zeta_i = \zeta_i(r)$, $1 \leqslant i \leqslant 2k+1$, the inverse function to σ on $[p, q]$ whose graph is C_i. The absolute continuity of ζ_i follows from assumption (a) and a standard change of variables formula (cf. Federer, 1969, pp. 244–245) which we use without further comment below.

For $\varepsilon > 0$ sufficiently small and t such that $\theta(t) \in [p, q]$ define

$$S_i(t) = \{x \in [0, 1] : |u_x(x, t) - \alpha_i| < \varepsilon\},$$

and

$$\mu_i(t) = \text{meas } S_i(t).$$

Then the sets $S_i(t)$ are disjoint, and by (5.6), (5.7)

$$\lim_{\substack{t \to \infty \\ \theta(t) \in [p,q]}} \sum_{i=1}^{2k+1} \mu_i(t) = 1. \tag{5.11}$$

It follows from (5.6), (5.7), (5.10) and (5.11) that

$$\lim_{\substack{t \to \infty \\ \theta(t) \in [p,q]}} \sum_{i=1}^{2k+1} \mu_i(t) \int_{-M}^{\alpha_i(t)} \chi(\sigma(z))\, dz \qquad \text{exists} \tag{5.12}$$

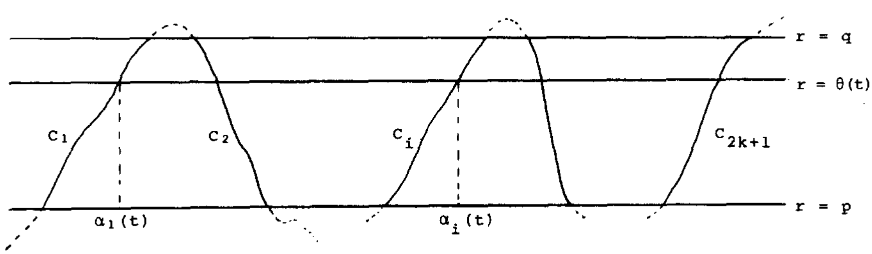

FIGURE 1

for any characteristic function χ of a closed interval. Let $p < s < s+\delta < p_1 < q_1 < q$, and apply (5.12) with χ the characteristic function of $[s, s+\delta]$. Dividing by δ we deduce that

$$\lim_{\substack{t\to\infty \\ \theta(t)\in[p_1,q_1]}} \left\{ \sum_{j=1}^{k} \left([\mu_{2j-1}(t) + \mu_{2j}(t)] \sum_{i=1}^{2j-1} \frac{1}{\delta} \int_s^{s+\delta} \zeta_i'(r)\, dr \right) + \mu_{2k+1}(t) \sum_{i=1}^{2k+1} \frac{1}{\delta} \int_s^{s+\delta} \zeta_i'(r)\, dr \right\} \text{ exists.}$$

We now choose $s_l \in (p, p_l)$, $1 \leqslant l \leqslant 2k+1$, to be Lebesgue points of each of the functions ζ_i' and such that the $(\zeta_i'(s_l))$, $l = 1,..., 2k+1$, are linearly independent vectors in $\mathbb{R}^{2k+1}$; this is possible by part (c) of the nondegeneracy assumptions. Letting $\delta \to 0+$ and using the boundedness of the $\mu_j(t)$ we obtain

$$\lim_{\substack{t\to\infty \\ \theta(t)\in[p_1,q_1]}} \left\{ \sum_{j=1}^{k} \left([\mu_{2j-1}(t) + \mu_{2j}(t)] \sum_{i=1}^{2j-1} \zeta_i'(s_l) \right) + \mu_{2k+1}(t) \sum_{i=1}^{2k+1} \zeta_i'(s_l) \right\} = a_l \tag{5.13}$$

for some a_l, $1 \leqslant l \leqslant 2k+1$. Writing (5.13) in the form

$$\lim_{\substack{t\to\infty \\ \theta(t)\in[p_1,q_1]}} \left\{ \zeta_1'(s_l) \sum_{i=1}^{2k+1} \mu_i(t) + \sum_{j=1}^{k} (\zeta_{2j}'(s_l) + \zeta_{2j+1}'(s_l)) \sum_{i=2j+1}^{2k+1} \mu_i(t) \right\} = a_l,$$

and using the linear independence of the $(\zeta_i'(s_l))$ we deduce that the sums $\sum_{i=2j-1}^{2k+1} \mu_i(t)$, $i = 1,..., k+1$, tend to limits as $t \to \infty$ with $\theta(t) \in [p_1, q_1]$. By repeating the above argument with $q_1 < s < s+\delta < q$ we find also that the sums $\sum_{i=2j}^{2k} \mu_i(t)$, $i = 1,..., k$, tend to limits as $t \to \infty$ with $\theta(t) \in [p_1, q_1]$. Thus the limits

$$\mu_i \stackrel{\text{def}}{=} \lim_{\substack{t\to\infty \\ \theta(t)\in[p_1,q_1]}} \mu_i(t) \tag{5.14}$$

exist.

We next choose χ to be the characteristic function of $[p, q]$. Then from (5.12), (5.14) it follows easily that

$$\lim_{\substack{t\to\infty \\ \theta(t)\in[p_1,q_1]}} \int_p^{\theta(t)} \left(\sum_{i=1}^{2k+1} \mu_i \zeta_i'(r) \right) dr \qquad \text{exists.}$$

Therefore

$$\sum_{i=1}^{2k+1} \mu_i \zeta_i'(r) = 0 \quad \text{a.e.} \quad r \in [p_1, q_1].$$

Since $\sum_{i=1}^{2k+1} \mu_i = 1$ by (5.11), (5.14) the ζ_i' are linearly dependent in $L^1([p_1, q_1])$, contradicting the nondegeneracy assumption. This contradiction proves (5.8), and thus part (ii) of the theorem.

Returning to (5.2), (5.3), we choose $\psi \in L^2(0, 1)$ with $\psi \geqslant 0$ and $\Phi \in C^2([-M, M])$ such that for some $\varepsilon > 0$,

$$\Phi'(z)(\sigma(z) - \tau) \geqslant 0 \qquad \text{if} \quad |z| \leqslant M, \; |\tau - P| < \varepsilon. \tag{5.15}$$

Rewriting the integral in (5.2) in the form

$$2 \int_0^1 \psi(x)\, \Phi'(u_x(x, t))(\sigma(u_x(x, t)) - \theta(t))\, dx$$

we deduce from (5.8), (5.15) that (5.2) holds for sufficiently large $t \geqslant 0$. We thus get (5.3). Note that, by (a), σ is constant on no interval. Thus by Lemma 3.2 and the argument used in Theorem 4.1 we deduce that

$$\lim_{t \to \infty} \int_0^1 \psi(x)\, \Phi(u_x(x, t))\, dx$$

exists for all $\psi \in L^1(0, 1)$, $\Phi \in C([-M, M])$. The remainder of the proof is also the same as that for Theorem 4.1. ∎

Corollary 5.2. *Suppose σ is strictly increasing. Then as $t \to \infty$*

$$u(x, t) \to 0 \qquad \textit{strongly in} \quad W_0^{1,p}(0, 1)$$

for all p, $1 \leqslant p < \infty$.

Proof. This is the same as the proof of Corollary 4.2. Examination of the proof of Theorem 5.1 shows that the nondegeneracy assumption (a) (which need not hold for strictly increasing functions) is unnecessary if σ is strictly increasing. ∎

The statement and proof of Corollary 4.3 carry over to displacement boundary conditions without change.

Corollary 5.3. *Let the hypotheses of Theorem 5.1 hold. Then for each $t \geqslant 0$ there exists an equilibrium solution $w(x; t) \in W_0^{1,\infty}(0, 1)$ satisfying*

$$\sigma(w_x(x; t)) = P \qquad \textit{a.e.} \quad x \in (0, 1), \tag{5.16}$$

such that for any p, $1 \leqslant p < \infty$,

$$\lim_{t\to\infty} \|u(\cdot, t) - w(\cdot; t)\|_{1,p} = 0.$$

In particular, $v(x)$ *is in the* $W_0^{1,\infty}(0, 1)$ *weak* closure of the set of equilibrium solutions satisfying* (5.16).

Proof. Since ν_x is a probability measure with $\operatorname{supp} \nu_x \subset K_P$, it follows that

$$\langle \nu_x, \text{identity} \rangle \in \operatorname{conv} K_P \qquad \text{a.e.}$$

By part (iv) of the theorem

$$\int_0^1 \langle \nu_x, \text{identity} \rangle \, dx = \lim_{t\to\infty} \int_0^1 u_x(x, t) \, dx = 0.$$

Hence $0 \in \operatorname{conv} K_P$.

Case 1. $\sigma(z) = P$ *has no negative root.* Then K_P is a closed bounded subset of $[0, \infty]$, and hence $0 \in K_P$. Also $\langle \nu_x$, identity$\rangle \geqslant 0$ a.e., and therefore

$$\langle \nu_x, \text{identity} \rangle = 0 \qquad \text{a.e.}$$

Hence $\operatorname{supp} \nu_x \subset \{0\}$ a.e., and thus $\nu_x = \delta_0$ a.e. This implies that

$$u(\cdot, t) \to 0 \qquad \text{strongly in} \quad W^{1,p}(0, 1), \;\; 1 \leqslant p < \infty,$$

as $t \to \infty$. By part (iii) of Theorem 5.1, $\sigma(0) = P = 0$. Therefore 0 is an equilibrium solution satisfying (5.16), and we can take $w(\cdot; t) \equiv 0$.

Case 2. $\sigma(z) = P$ *has no positive root.* This is handled as in Case 1.

Case 3. $\sigma(z) = P$ *has at least one negative root* z_- *and one positive root* z_+. Define $v(x; t)$ exactly as was $w(x; t)$ in Corollary 4.4. If $v(1; t) = 0$ we set $w(x; t) = v(x; t)$; then w is an equilibrium solution satisfying (5.16).

Suppose $v(1; t) < 0$. For $y \in [-\infty, 0]$ define

$$A(y, t) = \{x \in [0, 1]: v_x(x; t) \leqslant y\},$$

and let

$$g(y, t) = z_+ \operatorname{meas}(A(y, t)) + \int_{A(y,t)^c} v_x(x; t) \, dx.$$

Then $g(\cdot, t)$ is nondecreasing, $\lim_{y\to-\infty} g(y, t) = v(1, t) < 0$ and $\lim_{y\to 0-} g(y, t) > 0$. Let $y_0 = \sup\{y: g(y, t) \leqslant 0\}$, so that $y_0 < 0$. If

$g(y_0, t) = 0$ define $B(t) = A(y_0, t)$. If $g(y_0, t) > 0$, $s \in [0, 1]$, let $B(t, s) = \bigcup_{y < y_0} A(y, t) \cup (A(y_0, t) \cap [0, s])$ and

$$h(s, t) = z_+ \text{ meas}(B(t, s)) + \int_{B(t,s)^c} v_x(x; t)\, dx.$$

Then $h(\cdot, t)$ is continuous and nondecreasing, $h(0, t) = \sup_{y < y_0} g(y, t)$ and $h(1, t) = g(y_0, t)$. Therefore $h(s_0, t) = 0$ for some $s_0 \in [0, 1]$; in this case we define $B(t) = B(t, s_0)$.

Now let $w(0; t) = 0$ and

$$\begin{aligned} w_x(x; t) &= z_+ \qquad &&\text{if} \quad x \in B(t) \\ &= v_x(x; t) &&\text{otherwise.} \end{aligned}$$

Then

$$w(1; t) = \int_0^1 w_x(x; t)\, dx = 0,$$

so that $w(x; t)$ is an equilibrium solution satisfying (5.16). But

$$z_+ \text{ meas } B(t) = \int_{B(t)} v_x(x; t)\, dx - v(1, t),$$

and so

$$\text{meas } B(t) \leqslant -v(1; t)/z_+.$$

Hence

$$\int_0^1 |v_x(x; t) - w_x(x; t)|^p\, dx \leqslant \text{const. meas } B(t) \to 0$$

as $t \to \infty$. Applying a similar argument to the case $v(1; t) > 0$ we deduce finally that

$$\lim_{t \to \infty} \|u(\cdot, t) - w(\cdot; t)\|_{1,p} = 0,$$

as required. ∎

6. Concluding Remarks

A central question left open by our analysis is whether the limiting displacements $v(x)$ in Theorem 4.1, 5.1 need be equilibrium solutions. If $v(x)$

were not an equilibrium solution then $u(\cdot, t)$ would tend to an "infinitesimal zigzag" as $t \to \infty$. Is there a mechanism which might produce such increasing oscillations as $t \to \infty$ even in the presence of dissipation? A possible such mechanism is suggested by linearizing (1.1)–(1.3a) around the trivial equilibrium solution $u \equiv 0$ in the case when $\sigma(0) = 0$ and $\sigma'(0) < 0$. The resulting initial boundary value problem is

$$w_{tt} = \sigma'(0)\, w_{xx} + w_{xxt}, \tag{6.1}$$

$$w(x, 0) = w_0(x), \qquad w_t(x, 0) = w_1(x), \tag{6.2}$$

$$w(0, t) = w(1, t) = 0. \tag{6.3}$$

The solution of (6.1)–(6.3) has the form

$$w(x, t) = \sum_{n=1}^{\infty} (a_n e^{\lambda_n^+ t} + b_n e^{\lambda_n^- t}) \sin n\pi x, \tag{6.4}$$

where

$$\lambda_n^\pm = \tfrac{1}{2}[-n^2\pi^2 \pm \sqrt{n^4\pi^4 - 4\sigma'(0)\, n^2\pi^2}]. \tag{6.5}$$

As $n \to \infty$,

$$\lambda_+^n \sim -\sigma'(0),$$
$$\lambda_n^- \sim \sigma'(0) - n^2\pi^2.$$

Thus for large n the amplitudes of all modes are increased by roughly the same factor $e^{-\sigma'(0)t}$. Suppose that a similar phenomenon occurs for solutions to the nonlinear system (1.1)–(1.3a) in the neighbourhood of points x where $\sigma'(u_x(x, t)) < 0$. If u is a smooth solution of (1.1)–(1.3a) and $u_x(x_1, t)$, $u_x(x_2, t)$ lie on adjacent increasing portions of the σ-curve then $\sigma'(u_x(x_3, t)) < 0$ for some $x_3 \in (x_1, x_2)$. Increasing oscillations in the neighbourhood of x_3 might then produce further points near x_3 where u_x lies on different increasing portions of the σ-curve, thus further intermediate points where $\sigma' < 0$, and so on. This argument is not wholly convincing, of course, because of the possibility of stabilizing nonlinear interactions and nonlocal effects.

We now show that by modifying the constitutive equation for the stress one can establish that solutions converge to equilibria as $t \to \infty$. In place of (1.4) we suppose that

$$S = \sigma(u_x) + u_{xt} - \varepsilon u_{xxx}, \tag{6.6}$$

where $\varepsilon > 0$ is a constant. This constitutive equation corresponds to a special viscoelastic material of second grade. It is also a special case of a constitutive equation proposed by Korteweg (1901) in his theory of interfacial capillarity in fluids; for fluids of constant viscosity the term u_{xt} should, however, be replaced by u_{xt}/u_x. For information concerning Korteweg's theory the reader is referred to Truesdell and Noll (1965), Serrin (1981), and Slemrod (1981). A typical initial boundary value problem for the material (6.6), corresponding to (1.1)–(1.3a), is

$$u_{tt} = (\sigma(u_x) + u_{xt} - \varepsilon u_{xxx})_x, \qquad 0 < x < 1,\ t > 0, \tag{6.7}$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad 0 < x < 1, \tag{6.8}$$

$$u(0, t) = u_{xx}(0, t) = u(1, t) = u_{xx}(1, t) = 0, \qquad t > 0. \tag{6.9}$$

(The choice of boundary conditions is physically somewhat artificial, but will enable us to give a relatively simple theory.) The corresponding energy equation is

$$E(t) + \int_0^t \int_0^1 u_{xs}^2 \, dx \, ds = E(0), \tag{6.10}$$

where

$$E(t) = \int_0^1 \left[\frac{1}{2} u_t(x, t)^2 + W(u_x(x, t)) + \frac{\varepsilon}{2} u_{xx}(x, t)^2 \right] dx. \tag{6.11}$$

The form of the extra term $(\varepsilon/2)\, u_{xx}^2$ in the energy density suggests that in this model increasing oscillations in u_x will not occur and that solutions may converge to equilibria. This expectation is confirmed by the following theorem.

THEOREM 6.1. *Let* $W \in C^3(\mathbb{R})$ *with* W *bounded below. Let* $Y = W^{2,2}(0, 1) \cap W_0^{1,2}(0, 1)$, $X = Y \times L^2(0, 1)$ *and suppose that* $\{u_0, u_1\} \in X$. *Then given any* $T > 0$ *there exists a unique weak solution* u *of* (6.7)–(6.9) *with* $\{u, u_t\} \in C([0, T]; X)$.

Let $Z = \{v \in C^4([0, 1])\colon \varepsilon v''''(x) = \sigma(v'(x))'$ *for* $x \in [0, 1]$, *and* $v(0) = v''(0) = v(1) = v''(1) = 0\}$ *denote the set of equilibrium solutions of* (6.7), (6.9). *Then as* $t \to \infty$, $u_t(\cdot, t) \to 0$ *strongly in* $L^2(0, 1)$, *and*

$$\operatorname{dist}_Y(u(\cdot, t), Z) \to 0.$$

If, further, the elements of Z *are isolated in* Y *then there exists a unique* $v \in Y$ *such that* $u(\cdot, t) \to v$ *strongly in* Y *as* $t \to \infty$.

Remark. An appropriate definition of a weak solution, together with information concerning the regularity of the solution for $t > 0$, is given in the proof.

Proof. We write (6.7)–(6.9) in the form

$$\dot{w} = Aw + f(w),\qquad w(0) = w_0, \tag{6.12}$$

where

$$w = \begin{pmatrix} u \\ u_t \end{pmatrix},\qquad A = \begin{pmatrix} 0 & I \\ -\varepsilon \dfrac{d^4}{dx^4} & \dfrac{d^2}{dx^2} \end{pmatrix},$$

$$D(A) = (W^{4,2}(0,1) \cap W_0^{1,2}(0,1)) \times X,$$

$$f(w) = \begin{pmatrix} 0 \\ \sigma'(u_x)\,u_{xx} \end{pmatrix} \quad \text{and} \quad w_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

We note that A generates a C^0 semigroup $\{e^{At}\}$ of bounded linear operators on X given explicitly by

$$e^{At}w_0 = \sum_{n=1}^{\infty} w_n(t) \sin n\pi x, \tag{6.13}$$

where

$$w_0 = \sum_{n=1}^{\infty} \begin{pmatrix} \dfrac{1}{n^2\pi^2}\, u_{0n} \\ u_{1n} \end{pmatrix} \sin n\pi x, \qquad w_n(t) = \begin{pmatrix} a_n e^{\lambda_n^+ t} + b_n e^{\lambda_n^- t} \\ a_n \lambda_n^+ e^{\lambda_n^+ t} + b_n \lambda_n^- e^{\lambda_n^- t} \end{pmatrix},$$

$$a_n = \frac{[1 + (1-4\varepsilon)^{1/2}](u_{0n}/2) + u_{1n}}{n^2\pi^2(1-4\varepsilon)^{1/2}},$$

$$b_n = \frac{[-1 + (1-4\varepsilon)^{1/2}](u_{0n}/2) - u_{1n}}{n^2\pi^2(1-4\varepsilon)^{1/2}},$$

and

$$\lambda_n^{\pm} = (n^2\pi^2/2)(-1 \pm (1-4\varepsilon)^{1/2}).$$

It follows from (6.13) that $\{e^{At}\}$ is an analytic semigroup and, by the Arzela–Ascoli theorem, that e^{At} is compact for $t > 0$. Since $\sigma \in C^2(\mathbb{R})$ and since the imbedding of $W^{2,2}(0,1)$ in $C^1([0,1])$ is continuous, the mapping $f\colon X \to X$ is locally Lipschitz. Hence (cf. Segal, 1962) there is a unique solution $w \in C([0,T];X)$ of the integral equation

$$w(t) = e^{At}w_0 + \int_0^t e^{A(t-s)} f(w(s))\,ds, \tag{6.14}$$

provided $T > 0$ is sufficiently small, and w depends continuously on w_0. Equivalently, there exists a unique weak solution of (6.12) in the sense defined in Balakrishnan (1976) and Ball (1977). Since $\{e^{AT}\}$ is analytic, it follows from Pazy (1975, Theorem 5.2) that $w \in C^1((0, T]; X)$, that $w(t) \in D(A)$ for $t \in (0, T]$, and that (6.12) holds for all $t \in (0, T]$. In particular, the energy equation (6.10) holds for all $t \in [0, T]$. Since W is bounded below it follows that $\|w(t)\|_X \leqslant C$ for all $t \in [0, T_1]$ for any solution $w \in C([0, T_1]; X)$ of (6.14), $T_1 > 0$, where C is a constant independent of T_1. This bound implies that the solution $w(t)$ of (6.14) exists for all $t \geqslant 0$. Since e^{At} is compact for $t > 0$, it follows from Pazy (1975, Theorem 4.1) that the positive orbit $\mathcal{O}^+(w_0) \stackrel{\text{def}}{=} \bigcup_{t \geqslant 0} w(t)$ is precompact in X. The total energy $E(t)$ is nonincreasing and is a continuous functional on X. Furthermore, by (6.10) the only solutions along which $E(t)$ is constant are equilibria. By the version in Hale (1969) of the LaSalle invariance principle the ω-limit set of $w(\cdot)$ is contained in $Z \times \{0\}$, and therefore $\operatorname{dist}_X(w(t), Z \times \{0\}) \to 0$ as $t \to \infty$. Since the ω-limit set is connected, if Z consists only of isolated points then $w(t) \to v$ as $t \to \infty$ for a unique $v \in Z$. ∎

We now briefly investigate the relationship between equilibrium solutions of (1.1) and those of (6.7) in the limit $\varepsilon \to 0$. Consider an equilibrium solution u of (1.1) (for any boundary conditions) which for some $\delta > 0$, $x_0 \in (0, 1)$ has the form

$$\begin{aligned} u'(x) &= p_-, \qquad x_0 - \delta < x < x_0, \\ &= p_+, \qquad x_0 < x < x_0 + \delta, \end{aligned}$$

where $p_- \neq p_+$. Necessarily we must have

$$\sigma(p_-) = \sigma(p_+). \tag{6.15}$$

We examine whether there can be a sequence of equilibria u_ε of (6.7) converging to u in $(x_0 - \delta, x_0 + \delta)$ as $\varepsilon \to 0$. As is customary we in fact look for travelling wave solutions

$$u_\varepsilon(x) = \varepsilon^{1/2} f\left(\frac{x - x_0}{\varepsilon}\right). \tag{6.16}$$

Since

$$\varepsilon u_\varepsilon''''(x) = \sigma(u_\varepsilon'(x))' \tag{6.17}$$

we obtain

$$f''''(\eta) = \sigma(f'(\eta))', \tag{6.18}$$

where

$$\eta = \frac{x - x_0}{\varepsilon^{1/2}}, \qquad (\ )' = \frac{d}{d\eta}, \tag{6.19}$$

and we look for solutions to (6.18) satisfying

$$f'(\pm\infty) = p_\pm. \tag{6.20}$$

THEOREM 6.2. *There exists a solution of* (6.18) *and* (6.20) *if and only if*

$$W(p_-) - p_-\sigma(p_-) = W(p_+) - p_+\sigma(p_+) \tag{6.21}$$

and

$$W(p) - W(p_-) - (p - p_-)\sigma(p_-) > 0 \tag{6.22}$$

for all $p \in (p_-, p_+)$.

Proof. The argument follows the usual pattern (see, for example, Wendroff, 1972). Let $g = f'$. Integrating (6.18), any solution satisfies

$$g'' = \sigma(g) + c, \tag{6.23}$$

where clearly we must have $c = -\sigma(p_+) = -\sigma(p_-)$. Integrating again we obtain

$$\tfrac{1}{2} g'^2 = W(g) - g\sigma(p_-) + d,$$

so that by (6.20)

$$W(p_-) - p_-\sigma(p_-) = W(p_+) - p_+\sigma(p_+) = -d.$$

Then

$$\tfrac{1}{2} g'^2 = W(g) - W(p_-) - (g - p_-)\sigma(p_-). \tag{6.24}$$

Thus the expression in (6.22) is nonnegative for all $p \in [p_-, p_+]$. But if $g'(\eta^*) = 0$ for some η^* with $g(\eta^*) \in (p_-, p_+)$ then $g(\eta^*)$ minimizes the right-hand side of (6.24), and thus $\sigma(g(\eta^*)) = \sigma(p_-)$. By the uniqueness of solutions to (6.23), $g(\eta) = g(\eta^*)$ for all η, which is impossible. We have thus proved that conditions (6.21), (6.22) are necessary.

Conversely, suppose (6.21), (6.22) hold. Then we may solve the initial value problem

$$g' = \pm[2(W(g) - W(p_-) - (g - p_-)\sigma(p_-))]^{1/2},$$
$$g(0) = \tfrac{1}{2}(p_- + p_+)$$

locally, where the + (resp. −) sign is taken if $p_+ > p_-$ (resp. $p_+ < p_-$). Then the uniqueness of solutions to (6.23) implies that $g(\eta) \in (p_-, p_+)$ as long as the solution exists. Thus $g(\eta)$ exists for all $\eta \in \mathbb{R}$, and obviously

$$g(\pm\infty) = p_\pm.$$

Thus there exists a solution of (6.18), (6.20). ∎

Equations (6.15), (6.21) are the Weierstrass–Erdmann corner conditions (cf. Bolza, 1973) which say that the chord joining the points $(p_-, W(p_-))$ and $(p_+, W(p_+))$ is a common tangent to the graph of W at $p_\pm$. Condition (6.22) says that this chord lies strictly below the graph of W. The theorem shows that not all equilibria for (1.1) are limits of equilibrium solutions to (6.7) of the type (6.16).

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