WEAK CONVERGENCE THEOREMS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST AND SECOND ORDER

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ABSTRACT

We prove under various fairly weak assumptions that if a sequence of functions u^n converges to a function u, and if each u^n solves some appropriate fully nonlinear partial differential equation, then u solves the limit equation.

1. *Introduction*

In this paper we establish some convergence theorems for solutions of fully nonlinear partial differential equations of first and second order, having the general forms

$$F\big(Du(x), u(x), x\big) = 0 \qquad \text{a.e. } x \in \Omega\,, \tag{1.1}$$

and

$$F\big(D^2u(x), Du(x), u(x), x\big) = 0 \qquad \text{a.e. } x \in \Omega\,, \tag{1.2}$$

respectively, where Ω is a bounded open subset of $\mathbb{R}^m$ $(m \geqslant 1)$.

Our most substantial results concern elliptic equations of the form (1.2), and a simple case exhibiting the essential difficulties occurs when (1.2) has the form

$$F\big(D^2u(x)\big) = f(x)\,, \tag{1.3}$$

where $F : S^{m\times m} \to \mathbb{R}$, $S^{m\times m}$ denotes the space of real symmetric $m\times m$ matrices, and f is measurable. In this situation we say that F is *elliptic* if F is nondecreasing on $S^{m\times m}$; that is, if $A \geqslant B$ implies $F(A) \geqslant F(B)$ for $A, B \in S^{m\times m}$. It is easily shown (Proposition 1) that should F be C^1 this definition is equivalent to the more familiar condition

$$\frac{\partial F}{\partial r_{ij}}(A)\xi_i\xi_j \geqslant 0 \qquad \text{for all } A \in S^{m\times m},\ \xi \in \mathbb{R}^m\,.$$

One typical convergence theorem proved by our techniques is the following.

THEOREM 1. *Let $F : S^{m\times m} \to \mathbb{R}$ be continuous and elliptic. Suppose $u^n \in W^{2,p}(\Omega)$ (for some $p > m$) solves*

$$F\big(D^2u^n(x)\big) = f^n(x) \qquad \textit{a.e. } x \in \Omega\ (n = 1, 2, \ldots) \tag{1.4}$$

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and

(i) $u^n \to u$ *weakly in* $W^{2,p}(\Omega)$;

(ii) $f^n \to f$ *a.e. in* Ω.

Then

$$F(D^2u(x)) = f(x) \qquad \text{a.e. } x \in \Omega .$$

Theorem 1 is a special case of a more general convergence theorem, Theorem 3, stated and proved in Section 2. A weaker version of Theorem 1 (requiring that $u^n \to u$ weakly in $W^{2,p}(\Omega)$, $p > m$, and $f^n \to f$ uniformly on $\bar{\Omega}$) was proved by completely different methods in Evans [**6**]; some additional applications of the techniques from [**6**] are presented in Evans and Friedman [**8**], Evans [7], and P. L. Lions [**15**]. N. V. Krylov in [**11**] has used techniques from stochastic differential game theory to prove theorems like ours in the case where $f^n \to f$ in $L^m(\Omega)$, $u^n \to u$ weakly in $W^{2,m}(\Omega)$ and $F(D^2u)$ is expressible as the min-max of affine elliptic operators (see also [7; §2] concerning this last idea). Also relevant are Krylov's papers [**12–14**]. Although Theorem 3 partially generalizes the above results, the main interest of it lies in the method of proof, which is in the spirit of the maximum principle for elliptic equations and is thus independent of stochastic differential game theory.

Theorem 1 is a local result, in that the functions u^n are not required to satisfy any boundary conditions. Note also that for a nonlinear function F, the fact that $u^n \to u$ weakly in $W^{2,p}(\Omega)$ does not in general imply that $F(D^2u^n) \to F(D^2u)$ in any sense. In particular, the *only* functions F having the property that (for suitable p) $u^n \to u$ weakly in $W^{2,p}(\Omega)$ implies that $F(D^2u^n) \to F(D^2u)$ in the sense of distributions are given by

$$F(A) = \text{affine combination of minors of } A\,, \tag{1.5}$$

(so that, for example, if $m = 2$ the only nonlinear sequentially weakly continuous function $F(D^2u)\colon W^{2,2}(\Omega) \to \mathscr{D}'(\Omega)$ is $u_{x_1x_1}u_{x_2x_2} - (u_{x_1x_2})^2$). This result is proved in Ball, Currie and Olver [3]. Thus hypothesis (ii) in Theorem 1 cannot be replaced by $f^n \to f$ weakly in $L^q(\Omega)$. On the other hand, if F has the form (1.5), if $u^n \to u$ weakly in $W^{2,p}(\Omega)$ (p sufficiently large), and if $f^n \to f$ weakly in $L^1(\Omega)$, then we can pass to the limit in (1.4) using the results of [3]; since such F are not elliptic this raises the question of finding necessary and sufficient conditions on F for Theorem 1 to be valid.

A variant of the above results in which it is assumed that u^n, u are C^2, that $f^n \to f$ uniformly, but only that $u^n \to u$ uniformly, is also presented as Theorem 2 in Section 2. The reader is advised to read the proof of this first, since it illustrates our method in its simplest and most instructive form.

In Section 3 we consider first order equations of the form (1.1). Again the essential difficulties are exhibited by the special case of equations having the form

$$F(Du(x)) = f(x)\,, \tag{1.6}$$

where $F\colon \mathbb{R}^m \to \mathbb{R}$ is continuous and f is measurable. Here our methods give a surprising result; namely, if $u^n, u \in C^1(\Omega)$, $u^n \to u$ in $C(\bar{\Omega})$, $f^n \to f$ in $C(\bar{\Omega})$, and

$$F(Du^n(x)) = f^n(x) \qquad \text{for all } x \in \Omega\,,$$

then u satisfies (1.6), there being no extra hypothesis on F. If u^n, u are supposed to be Lipschitz, rather than C^1, this result is true if and only if

$$F(y) = g(a \cdot y), \qquad y \in \mathbb{R}^n,$$

for some $a \in \mathbb{R}^n$ and some nonincreasing or nondecreasing function g.

Notation. We write $x = (x_1, \dots, x_m)$, $Du = (u_{x_1}, \dots, u_{x_m})$, $D^2u = ((u_{x_i x_j}))$, $B(x,r) = \{y \in \mathbb{R}^m : |y-x| \leqslant r\}$; and $\fint_A u\,dx = \frac{1}{\text{meas } A} \int_A u\,dx$ = average of u over A. The letter "C" denotes various constants depending only on known quantities. We employ the summation convention throughout.

2. *Second order elliptic equations*

We first show that our definition of ellipticity coincides with the usual one if F is C^1.

PROPOSITION 1. *Let $F : S^{m \times m} \to \mathbb{R}$ be C^1. Then F is elliptic if and only if*

$$\frac{\partial F}{\partial r_{ij}}(A)\xi_i\xi_j \geqslant 0 \qquad \textit{for all } A \in S^{m \times m},\ \xi \in \mathbb{R}^m. \tag{2.1}$$

Proof. Let F be elliptic. Then $A + t\xi \otimes \xi \geqslant A$ for $t \geqslant 0$, and so

$$\frac{d}{dt} F(A + t\xi \otimes \xi)\Big|_{t=0} = \frac{\partial F}{\partial r_{ij}}(A)\xi_i\xi_j \geqslant 0.$$

Conversely, let (2.1) hold and suppose that $A \geqslant B$. Then

$$F(A) - F(B) = \int_0^1 \frac{\partial F}{\partial r_{ij}}\big(sA + (1-s)B\big)ds(A_{ij} - B_{ij}) = \text{trace}\big(C(A-B)\big),$$

where

$$C_{ij} \equiv \int_0^1 \frac{\partial F}{\partial r_{ij}}\big(sA + (1-s)B\big)ds$$

and

$$A = ((A_{ij})), \qquad B = ((B_{ij})), \qquad C = ((C_{ij})).$$

Changing bases so that $A - B$ is diagonal, say $A' - B' = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_i \geqslant 0$ $(i = 1, \dots, m)$, we calculate

$$F(A) - F(B) = \text{trace}\big(C'(A' - B')\big) = \sum_{i=1}^m C'_{ii}\lambda_i \geqslant 0,$$

since $C' = ((C'_{ij}))$ is similar to C and therefore nonnegative definite.

The following theorem and its corollary exhibit our method in its simplest form.

THEOREM 2. *Let $F^n : S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}\times\Omega\to\mathbb{R}$ be continuous functions satisfying these hypotheses for $n = 1, 2, \ldots$:*

(a) *$F^n(\cdot, p, z, x)$ is elliptic for all p, z, x;*

(b) *$F^n \to F$ uniformly on compact subsets of $S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}\times\Omega$.*

Let $u^n, u\in C^2(\Omega)$, assume $u^n\to u$ uniformly in Ω, and suppose that

$$F^n\big(D^2u^n(x), Du^n(x), u^n(x), x\big) = 0 \qquad \text{for all } x\in\Omega,\ n = 1, 2, \ldots .$$

Then

$$F\big(D^2u(x), Du(x), u(x), x\big) = 0 \qquad \text{for all } x\in\Omega .$$

COROLLARY. *Let $F : S^{m\times m}\to\mathbb{R}$ be continuous and elliptic, let $u^n, u\in C^2(\Omega)$, $f^n, f\in C(\Omega)$, and suppose that*

$$F\left(D^2u^n(x)\right) = f^n(x) \qquad \text{for all } x\in\Omega,\ n = 1, 2, \ldots .$$

If (i) *$u^n\to u$ uniformly in Ω, and* (ii) *$f^n\to f$ uniformly in Ω, then*

$$F\big(D^2u(x)\big) = f(x) \qquad \text{for all } x\in\Omega .$$

Note carefully that we do not assume here that $D^2u^n\to D^2u$ in any sense.

Proof of Theorem 2. Choose any point $x_0\in\Omega$. Fix $\varepsilon > 0$ and then choose $N(\varepsilon)$ so large that

$$\sup_{x\in\Omega}|u^n(x)-u(x)| \leqslant \varepsilon \qquad \text{for } n\geqslant N(\varepsilon). \tag{2.2}$$

Set $r_\varepsilon \equiv 2\varepsilon^{1/4}$ and define, for $n\geqslant N(\varepsilon)$ fixed, the auxiliary functions

$$v^\pm(x) \equiv -\varepsilon^{1/2}|x-x_0|^2 \pm \big(u^n(x)-u(x)\big)+2\varepsilon .$$

Then

$$v^+(x_0) = u^n(x_0)-u(x_0)+2\varepsilon \geqslant \varepsilon ,$$

while

$$v^+(x) = -2\varepsilon+\big(u^n(x)-u(x)\big) \leqslant -\varepsilon$$

if $x\in\partial B(x_0, r_\varepsilon)$. Thus v^+ attains its maximum on $B(x_0, r_\varepsilon)$ at some interior point x_ε, where

$$2\varepsilon^{1/2}(x_\varepsilon-x_0)+Du(x_\varepsilon)-Du^n(x_\varepsilon) = 0 ,$$

$$2\varepsilon^{1/2}I+D^2u(x_\varepsilon)-D^2u^n(x_\varepsilon) \geqslant 0 .$$

By (a),

$$F^n\big(D^2u(x_\varepsilon)+2\varepsilon^{1/2}I,\ Du(x_\varepsilon)+2\varepsilon^{1/2}(x_\varepsilon-x_0),\ u^n(x_\varepsilon), x_\varepsilon\big)$$
$$\geqslant F^n\big(D^2u^n(x_\varepsilon), Du^n(x_\varepsilon), u^n(x_\varepsilon), x_\varepsilon\big) = 0 .$$

Now let $\varepsilon\searrow 0$, $n\to\infty$, $n\geqslant N(\varepsilon)$. Since $u\in C^2$, $x_\varepsilon\to x_0$, and $u^n(x_\varepsilon)\to u(x_0)$, we obtain using (b) the inequality

$$F\big(D^2u(x_0), Du(x_0), u(x_0), x_0\big) \geqslant 0 .$$

The opposite inequality follows by considering v^-.

Remark 1. If each u^n is only a subsolution for F^n, that is,

$$F^n(D^2u^n(x), Du^n(x), u^n(x), x) \geqslant 0 \qquad \text{for all } x \in \Omega,$$

then the proof shows that u is a subsolution for F. An analogous remark holds for supersolutions and applies also to Theorem 3 below.

Remark 2. Suppose in the corollary that F is C^1, that

$$\frac{\partial F}{\partial r_{ij}}(A)\xi_i\xi_j \geqslant \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^m$$

for some constant $\lambda > 0$, and that $u^n|_{\partial\Omega} = u|_{\partial\Omega}$ for each n. Then hypothesis (i) in fact follows from (ii). To see this note that $a_{ij}(u-u^n)_{x_ix_j} = f-f^n$ for

$$a_{ij}(x) \equiv \int_0^1 \frac{\partial F}{\partial r_{ij}}(sD^2u(x)+(1-s)D^2u^n(x))ds$$

and recall the estimate $\|u-u^n\|_{L^\infty(\Omega)} \leqslant C\|f-f^n\|_{L^\infty(\Omega)}$ (cf. Gilbarg and Trudinger [9; p. 35]).

Remark 3. The corollary is false if (ii) is weakened to read $f^n \to f$ a.e. in Ω. We give a one-dimensional example. Let

$$u^n(x) = \int_0^x\int_0^y f^n(s)dsdy,$$

where f^n is smooth, periodic on $\mathbb{R}$ with period $1/n$, $\int_0^{1/n} f^n(x)dx = 0$, $f^n \leqslant 1$, and meas$\{x \in (0, 1/n): f^n(x) = 1\} > (1/n)-(1/n^2)$. It is clear that $f^n \to 1$ in measure, so that a subsequence, again denoted f^n, converges to 1 a.e. Also

$$\int_0^1 |f^n(x)|dx = \int_0^1 [|f^n(x)|+f^n(x)]dx \leqslant 2.$$

But

$$\left|\int_0^y f^n(s)ds\right| \leqslant \int_0^{1/n} |f^n(s)|ds \leqslant \frac{2}{n}$$

(in fact, $f^n \to 0$ in $\mathscr{D}'(0, 1)$). Taking F as the identity, we thus have that $F(u^n_{xx}) = u^n_{xx} = f^n \to 1$ a.e., $u^n \to 0$ uniformly, but $F(0) \neq 1$. Note also that u^n is uniformly bounded in $W^{2,1}(0, 1)$.

Our main result is the following.

THEOREM 3. *Let* $F, F^n : S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}\times\Omega\to\mathbb{R}$ $(n = 1, 2, \ldots)$ *satisfy these hypotheses:*

(a) *F, F^n are Carathéodory functions, that is, for each $(r, p, z)\in S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}$ the functions $F(r, p, z, \cdot)$, $F^n(r, p, z, \cdot)$ are measurable, and for almost all $x\in\Omega$ the functions $F(\cdot, \cdot, \cdot, x)$, $F^n(\cdot, \cdot, \cdot, x)$ are continuous;*

(b) *for almost all $x\in\Omega$, as $n\to\infty$*

$$F^n(r, p, z, x)\to F(r, p, z, x),$$

uniformly on compact subsets of $S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}$;

(c) *for almost all $x\in\Omega$ and every $(p, z)\in\mathbb{R}^m\times\mathbb{R}$ the function $F^n(\cdot, p, z, x)$ is elliptic.*

Let $u\in W^{2,1}(\Omega)$ and suppose that $u^n - u\to 0$ weakly in $W^{2,p}(\Omega)$ (for some $p > m$) as $n\to\infty$, where u^n satisfies

$$F^n\big(D^2u^n(x), Du^n(x), u^n(x), x\big) = 0 \qquad a.e.\ x\in\Omega,\ n = 1, 2, \ldots. \tag{2.3}$$

Then

$$F\big(D^2u(x), Du(x), u(x), x\big) = 0 \qquad a.e.\ x\in\Omega. \tag{2.4}$$

Remark. Theorem 1 is clearly a special case of Theorem 3.

Our proof of Theorem 3 requires two lemmas. For $h\in L^1(\Omega)$ we extend h to be zero in Ω^c, and then define

$$\mathscr{M}[h](x) \equiv \sup_{r>0} \fint_{B(x,r)} |h(y)|dy, \qquad x\in\mathbb{R}^m, \tag{2.5}$$

the *maximal function* of h.

LEMMA 1. *Let $\{h^n\}$ be a bounded sequence in $L^1(\Omega)$. Then for each $\delta > 0$ there exists a constant K_δ and a measurable subset $\Omega_\delta\subset\Omega$ such that*

(i) $\operatorname{meas}\Omega_\delta\leqslant\delta$;

(ii) *for each $x_0\in\Omega\setminus\Omega_\delta$ there exists a subsequence $n_k = n_k(x_0)\to\infty$ with*

$$\mathscr{M}[h^{n_k}](x_0)\leqslant K_\delta \qquad (k = 1, 2, \ldots). \tag{2.6}$$

Proof. We make use of the classical estimate (cf. Stein [2; p. 5]):

$$\operatorname{meas}\{x : \mathscr{M}[h^n](x)\geqslant\varepsilon\}\leqslant\frac{C_1}{\varepsilon}\int_{\mathbb{R}^m}|h^n|dx, \qquad \varepsilon > 0. \tag{2.7}$$

For $\delta > 0$ we set

$$K_\delta\equiv\frac{C_1}{\delta}\max_n\int_\Omega|h^n|dx.$$

Define $S^n \equiv \{x \in \Omega : \mathscr{M}[h^n](x) \geqslant K_\delta\}$. By (2.7) meas $S^n \leqslant \delta$ $(n = 1, 2, \ldots)$. Now define

$$\Omega_\delta \equiv \liminf_{n\to\infty} S^n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} S^k ;$$

we have meas $\Omega_\delta \leqslant \delta$. If $x_0 \in \Omega \setminus \Omega_\delta$, then $x_0 \in (S^n)^c$ for infinitely many values of n. This proves (2.6).

LEMMA 2. *Let F, F^n satisfy hypotheses* (a), (b) *of Theorem* 3. *Then for every $\delta > 0$ there exists a measurable subset $E_\delta \subset \Omega$, with* meas $(\Omega \setminus E_\delta) \leqslant \delta$, *such that F, F^n are continuous on $S_\delta \equiv S^{m\times m} \times \mathbb{R}^m \times \mathbb{R} \times E_\delta$ and $F^n \to F$ uniformly on compact subsets of S_δ.*

Proof. We identify $S^{m\times m} \times \mathbb{R}^m \times \mathbb{R}$ with $\mathbb{R}^s$, $s = \frac{1}{2}m(m+1)+m+1$, so that $F, F^n : \mathbb{R}^s \times \Omega \to \mathbb{R}$. Define $g(a, t, x)$ on $(\mathbb{R}^s \times \mathbb{R}) \times \Omega$ by

$$g(a,t,x) = \begin{cases} F(a,x), & t \leqslant 0, \\ n(n+1)\left[F^n(a,x)\left(t - \dfrac{1}{n+1}\right) + F^{n+1}(a,x)\left(\dfrac{1}{n} - t\right)\right], & t \in \left[\dfrac{1}{n+1}, \dfrac{1}{n}\right], \\ F^1(a,x), & t \geqslant 1 . \end{cases}$$

It is easily checked that as a consequence of (a), (b) g is a Carathéodory function on $(\mathbb{R}^s \times \mathbb{R}) \times \Omega$. By the Scorzà–Dragoni theorem (cf. Ekeland and Témam [5; p. 218]) there exists for every $\delta > 0$ a measurable subset $E_\delta \subset \Omega$, with meas $(\Omega \setminus E_\delta) \leqslant \delta$, such that g is continuous on $(\mathbb{R}^s \times \mathbb{R}) \times E_\delta$. The result follows.

Proof of Theorem 3. The principal idea in the proof is first to choose a typical point $x_0 \in \Omega$, consider then some collection of balls $B(x_0, r_\varepsilon)$ ($r_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$), and next to show by a comparison argument that u 'almost solves' (2.4) at some point $x_\varepsilon \in B(x_0, r_\varepsilon)$. We finally let $\varepsilon \searrow 0$ to prove that (2.4) holds at x_0. This was the technique used in the proof of Theorem 2; but in the present case, since the various hypotheses hold only a.e. and $|D^2(u^n - u)|$ is only in $L^p(\Omega)$ and not necessarily bounded, we need to arrange things more carefully to ensure that the points x_0 and x_ε mentioned above lie in the 'good' subset of Ω where the assumptions hold. For this define

$$v^n \equiv u^n - u . \tag{2.8}$$

Fix $\delta > 0$. Now there exists a measurable subset $G = G_\delta \subset \Omega$ with

$$\text{meas}\,(\Omega \setminus G) \leqslant \delta , \tag{2.9}$$

such that

$$\left.\begin{array}{ll}\text{(i)} & D^2u^n(x),\ Du^n(x),\ u^n(x),\ D^2u(x),\ Du(x),\ u(x) \text{ are defined and } \\ & F^n(D^2u^n(x), Du^n(x), u^n(x), x) = 0\ (n = 1,2,\dots) \text{ for all } x \in G, \\ \text{(ii)} & F^n, F \text{ are continuous on } S_\delta \equiv S^{m\times m}\times\mathbb{R}^m\times\mathbb{R}\times G \text{ and } F^n \to F \\ & \text{uniformly on compact subsets of } S_\delta \text{ (Lemma 2)}, \\ \text{(iii)} & D^2u,\ Du,\ u \text{ are continuous on } G \text{ (Lusin's theorem)}, \\ \text{(iv)} & \text{there is a sequence } n_k \to \infty \text{ for which } v^{n_k} \to 0,\ Dv^{n_k} \to 0, \text{ uniformly} \\ & \text{on } G \text{ (Sobolev embedding theorem)}, \\ \text{(v)} & \text{for each } \eta > 0,\ x_0 \in G \text{ there is a further subsequence} \\ & n_{k,l} = n_{k_l}(\eta, x_0) \to \infty \text{ such that} \\ & \qquad \mathscr{M}[|-\eta I \pm D^2 v^{n_{k,l}}|^p](x_0) \leqslant K_\delta \qquad (l = 1,2,\dots) \\ & \text{(Lemma 1).}\end{array}\right\} \tag{2.10}$$

Now select $x_0 \in G$ to be some point of density of G; that is.

$$\lim_{R\to 0} \frac{\operatorname{meas}(G \cap B(x_0, R))}{\operatorname{meas} B(x_0, R)} = 1\,. \tag{2.11}$$

Fix $\eta > 0$, and for ease of notation reindex so that (2.10) (iv), (v) hold for the whole sequence $n = 1, 2, \dots$.

These preliminaries aside, we describe now our principal observation:

$$\left.\begin{array}{ll} & \text{there exist } \beta > 0,\ r_\varepsilon > 0 \text{ such that} \\ \text{(a)} & r_\varepsilon \to 0 \text{ as } \varepsilon \to 0\,, \\ \text{(b)} & \dfrac{\operatorname{meas}\{x \in B(x_0, r_\varepsilon) : D^2 v^n(x) \leqslant \eta I\}}{\operatorname{meas} B(x_0, r_\varepsilon)} \geqslant \beta \\ & \text{for all } \varepsilon > 0 \text{ and } n \geqslant N(\varepsilon).\end{array}\right\} \tag{2.12}$$

Our proof of (2.12) incorporates an idea of Bony [4] (cf. also Pucci [**18**]). For $\varepsilon > 0$, let $N(\varepsilon)$ be so large that

$$\|v^n\|_{L^\infty(\Omega)} \leqslant \varepsilon \qquad \text{for } n \geqslant N(\varepsilon)\,.$$

Fix $n \geqslant N(\varepsilon)$ and set

$$v \equiv -\frac{\eta}{2}|x - x_0|^2 + v^n + 2\varepsilon, \qquad r_\varepsilon \equiv (8\varepsilon/\eta)^{1/2}\,.$$

Then

$$v(x_0) \geqslant \varepsilon,\ v(x) \leqslant -\varepsilon \qquad \text{if } x \in \partial B(x_0, r_\varepsilon)\,. \tag{2.13}$$

Set

$A \equiv \{x_1 \in B(x_0, r_\varepsilon)$: there exists a supporting hyperplane $\rho_{x_1}(x) \equiv Dv(x_1)\cdot(x-x_1)+v(x_1)$ such that $v(x) \leqslant \rho_{x_1}(x)$ for all $x \in B(x_0, r_\varepsilon)\}$,

$B \equiv \{Dv(x_1) : x_1 \in A\} \subset \mathbb{R}^m$.

We first show that

$$B \supset B\left(0, \frac{\eta}{8} r_\varepsilon\right). \tag{2.14}$$

Let $z \in B\left(0, \frac{\eta}{8} r_\varepsilon\right)$ and define

$$q(x) = z\cdot(x-x_0)+v(x_0).$$

Then for $x \in \partial B(x_0, r_\varepsilon)$,

$$q(x) \geqslant -|z||x-x_0|+\varepsilon \geqslant -\frac{\eta}{8} r_\varepsilon^2+\varepsilon = 0 > -\varepsilon \geqslant v(x),$$

where we have used (2.13). Thus $q(x)+c$ for some $c \geqslant 0$ is a supporting hyperplane from above, at some point $x_1 \in B(x_0, r_\varepsilon)$. So $x_1 \in A$, and hence $z = Dv(x_1) \in B$; this proves (2.14). It follows immediately that

$$\operatorname{meas} B \geqslant C_1 r_\varepsilon^m, \qquad C_1 > 0. \tag{2.15}$$

Now since $v \in W^{2,p}(\Omega)$ $(p > m)$ we have

$$\operatorname{meas} B = \operatorname{meas} Dv(A) \leqslant \int_A |\det D^2 v(x)| dx$$

$$\leqslant C \int_A |D^2 v(x)|^m dx;$$

hence (2.15) implies that

$$0 < C_1 \leqslant \frac{C}{r_\varepsilon^m} \int_A |D^2 v(x)|^m dx$$

$$\leqslant C\mathscr{M}[|D^2 v|^p]^{m/p} \left(\frac{\operatorname{meas} A}{r_\varepsilon^m}\right)^{1-(m/p)}$$

$$\leqslant C K_\delta^{m/p} \left(\frac{\operatorname{meas} A}{r_\varepsilon^m}\right)^{1-(m/p)} \qquad \text{by (2.10)(v)}.$$

Therefore $\operatorname{meas} A \geqslant \beta \operatorname{meas} B(x_0, r_\varepsilon)$, for some $\beta > 0$. If $x_1 \in A$, then $v(x) \leqslant \rho_{x_1}(x)$ for all $x \in B(x_0, r_\varepsilon)$, and hence $D^2 v(x_1) = D^2 v^n(x_1) - \eta I \leqslant 0$. This proves (2.12).

In view of (2.11), (2.12) there must exist some point

$$x_\varepsilon \in G \cap \{x \in B(x_0, r_\varepsilon) : D^2 v^n(x) \leqslant \eta I\},$$

provided that ε is small enough and $n \geqslant N(\varepsilon)$. At this point we have by (2.10) (i), (ii) and the definition of ellipticity

$$F^n(D^2 u(x_\varepsilon)+\eta I, Du^n(x_\varepsilon), u^n(x_\varepsilon), x_\varepsilon) \geqslant F^n(D^2 u^n(x_\varepsilon), Du^n(x_\varepsilon), u^n(x_\varepsilon), x_\varepsilon) = 0$$

Now let $\varepsilon \searrow 0$, $n \to \infty$, $n \geqslant N(\varepsilon)$. By (2.10)(ii), (iii), (iv),

$$F(D^2 u(x_0)+\eta I, Du(x_0), u(x_0), x_0) \geqslant 0,$$

and so, letting $\eta \to 0$,

$$F(D^2 u(x_0), Du(x_0), u(x_0), x_0) \geqslant 0.$$

Applying the above argument with the new auxiliary function

$$\bar{v} \equiv -\frac{\eta}{2}|x-x_0|^2 - v^n + 2\varepsilon$$

gives the opposite inequality. Since x_0 was any point of density of G, we have therefore proved that

$$F(D^2 u(x_0), Du(x_0), u(x_0), x_0) = 0 \qquad \text{a.e. } x_0 \in G.$$

Since δ was arbitrary it follows from (2.9) that (2.4) holds.

3. *First order equations*

THEOREM 4. *Let $F^n : \mathbb{R}^m \times \mathbb{R} \times \Omega \to \mathbb{R}$ be continuous ($n = 1, 2, \ldots$) and let $F^n \to F$ uniformly on compact subsets of $\mathbb{R}^m \times \mathbb{R} \times \Omega$. Let $u^n, u \in C^1(\Omega)$ with $u^n \to u$ uniformly on Ω, where u^n satisfies*

$$F^n(Du^n(x), u^n(x), x) = 0 \qquad \textit{for all } x \in \Omega,\ n = 1, 2, \ldots;$$

then

$$F(Du(x), u(x), x) = 0 \qquad \textit{for all } x \in \Omega.$$

Proof. This is the same as that of Theorem 2. As before we find a point $x_\varepsilon \in B(x_0, r_\varepsilon)$ such that

$$2\varepsilon^{1/2}(x_\varepsilon - x_0) + Du(x_\varepsilon) - Du^n(x_\varepsilon) = 0.$$

Thus

$$F^n(Du(x_\varepsilon) + 2\varepsilon^{1/2}(x - x_0), u^n(x_\varepsilon), x_\varepsilon) = 0,$$

so that, letting $\varepsilon \searrow 0$, $n \to \infty$, $n \geqslant N(\varepsilon)$, we obtain

$$F(Du(x_0), u(x_0), x_0) = 0$$

for any $x_0 \in \Omega$.

COROLLARY. *Let $u^n \in C^1(\Omega)$ satisfy*

$$F(Du^n(x)) = f^n(x), \qquad \text{for all } x \in \Omega,\ n = 1, 2, \ldots,$$

where $F : \mathbb{R}^m \to \mathbb{R}$ is continuous, f^n is continuous, and $f^n \to f$ uniformly on Ω. If $u^n \to u$ uniformly on Ω then

$$F(Du(x)) = f(x) \qquad \text{for all } x \in \Omega.$$

Remark. Theorem 4 and the corollary are false if the u^n take values in $\mathbb{R}^s$, $s > 1$. For example, let $m = 1$, $s = 2$, let $F : \mathbb{R}^2 \to \mathbb{R}$ be smooth with $F(0) = 0$, $F = 1$ on the unit circle, and let $u^n(x) = \frac{1}{n}(\sin nx, \cos nx)$. Then $F(u^n_x(x)) = 1$ for all x, $u^n \to 0$ uniformly, but $F(0) = 0 \neq 1$.

The next result shows in particular that Theorem 4 and the corollary are false if the u^n are Lipschitz rather than C^1, or if the hypotheses $F^n \to F$, $f^n \to f$ uniformly are weakened only slightly.

THEOREM 5. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be continuous, such that either*

(i) *if $u^n \xrightarrow{*} u$ weak $*$ in $W^{1,\infty}(\Omega)$ and, for some $k \in \mathbb{R}$, $F(Du^n(x)) = k$ for all $x \in \Omega$, $n = 1, 2, \ldots$, then $F(Du(x)) = k$ for all $x \in \Omega$, or*

(ii) *if $u^n, u \in C^\infty(\Omega)$, $u^n \xrightarrow{*} u$ weak $*$ in $W^{1,\infty}(\Omega)$ and $F(Du^n(x)) = f^n(x) \to f(x)$ a.e. $x \in \Omega$, then $F(Du(x)) = f(x)$ a.e. $x \in \Omega$.*

Then there exists $a \in \mathbb{R}^m$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$, which is neither nonincreasing or nondecreasing, such that

$$F(y) = g(a \cdot y) \qquad \text{for all } y \in \mathbb{R}^m.$$

Proof. Let F satisfy (i). Let b, $c \in \mathbb{R}^m$ with $F(b) = F(c)$, and consider the sequence

$$u^n(x) \equiv c \cdot x + \frac{1}{n} h(n(b-c) \cdot x), \qquad n = 1, 2, \ldots,$$

where $h : \mathbb{R} \to \mathbb{R}$ is continuous, h' is periodic with period 1, and

$$h'(t) = \begin{cases} 0, & 0 < t < \theta, \\ 1, & \theta < t < 1, \end{cases}$$

where $\theta \in (0, 1)$ is fixed. Then $Du^n(x)$ takes the values b or c a.e., and $u^n \xrightarrow{*} u$ in $W^{1,\infty}(\Omega)$ (that is, $u^n \xrightarrow{*} u$, $Du^n \xrightarrow{*} u$ in $L^\infty(\Omega)$), where $u(x) = (\theta c + (1-\theta)b) \cdot x$. By assumption

$$F(\theta c + (1-\theta)b) = F(b) = F(c), \tag{2.16}$$

and hence $F^{-1}(k)$ *is convex for every* $k \in \mathbb{R}$.

If F satisfies (ii), we 'round off the corners' of h to obtain a sequence $v^n \in C^\infty(\Omega)$ such that $v^n \xrightarrow{*} u$ in $W^{1,\infty}(\Omega)$ and

$$F(Dv^r(x)) \equiv f^r(x) \to F(b) \qquad \text{a.e. } x \in \Omega .$$

Therefore (2.16) again holds. The result now follows from the following proposition.

PROPOSITION 2. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be continuous and such that $F^{-1}(k)$ is convex for every $k \in \mathbb{R}$. Then there exist $a \in \mathbb{R}^m$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ which is either nonincreasing or nondecreasing and such that*

$$F(y) = g(a \cdot y) \qquad \textit{for all } y \in \mathbb{R}^m .$$

Proof. The proof is quite similar to that of a result of Ball and Tartar (Tartar [**21**; p. 196]) and Jensen [**10**] and proceeds by considering successively the cases in which $m = 1$, $m = 2$ and $m > 2$. If $m = 1$ the result is obvious; in this case F itself is nondecreasing or nonincreasing. Let $m = 2$. By the first case F is nondecreasing or nonincreasing along every line. Given $x_0 \in \mathbb{R}^m$, take any line through x_0 and rotate it continuously through an angle π; since the end result of such a rotation is to reverse the orientation of the line it follows by the continuity of F that there exists some line l_{x_0} through x_0 on which F is constant. If F is not identically constant then l_{x_0} is parallel to l_{x_1} for any x_0, x_1, since otherwise there would be two such lines which intersected and on which F had different constant values. Hence $F^{-1}(k)$ is a strip $-\infty \leqslant c(k) \leqslant x \cdot a \leqslant d(k) \leqslant \infty$ for each k, and the result follows since F is nondecreasing or nonincreasing on the line $\{ta : t \in \mathbb{R}\}$.

Let $m > 2$. We show that for any $k \in \mathbb{R}$, $F^{-1}(k)$ contains a hyperplane (that is, an affine subspace of dimension $m-1$). Since any two nonparallel hyperplanes intersect the result then follows as for the case when $m = 2$. We may suppose that F is not identically constant, so that the closed convex set $F^{-1}(k)$ contains a boundary point x_0. Let $\pi = \{x \in \mathbb{R}^m : (x-x_0) \cdot a = 0\}$ be a supporting hyperplane to $F^{-1}(k)$ at x_0, and suppose without loss of generality that $F^{-1}(k) \subset \pi_- \equiv \{x \in \mathbb{R}^m : (x-x_0) \cdot a \leqslant 0\}$ and that $F(x) > k$ for $x \in \pi_+ \equiv \{x \in \mathbb{R}^m : (x-x_0) \cdot a > 0\}$. Clearly π_+ cannot contain both points where $F > k$ and $F < k$. Let l be any line in π passing through x_0 and let $\bar{l}$ be the line through x_0 parallel to a. By the case in which $m = 2$, $F^{-1}(k) \cap \operatorname{span}\{l, \bar{l}\}$ is a strip. Since $F(x_0) = k$, $F^{-1}(k) \subset \pi_-$, it follows that l is one of the bounding lines of the strip, and hence $l \subset F^{-1}(k)$. Since l is arbitrary, $\pi \subset F^{-1}(k)$ as required.

Our final theorem shows that the necessary conditions on F established in Theorem 5 are sufficient.

THEOREM 6. *Let $g, g^n : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ $(n = 1, 2, \ldots)$ satisfy these conditions:*

(a) *g, g^n are Carathéodory functions, that is, for each $(p, z) \in \mathbb{R} \times \mathbb{R}$ the functions $g(p, z, \cdot)$, $g^n(p, z, \cdot)$ are measurable, and for almost all $x \in \Omega$ the functions $g(\cdot, \cdot, x), g^n(\cdot, \cdot, x)$ are continuous;*

(b) *for almost all $x \in \Omega$, as $n \to \infty$*

$$g^n(p, z, x) \to g(p, z, x)$$

uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}$;

(c) *for almost all* $x \in \Omega$ *the functions* $g^n(\cdot, z, x)$, $z \in \mathbb{R}$, $n = 1, 2, \ldots$ *are all nondecreasing or all nonincreasing.*

Let $u^n \to u$ *weakly in* $W^{1,1}(\Omega)$, *where* u^n *satisfies*

$$g^n(a^n \cdot Du^n(x), u^n(x), x) = 0 \qquad a.e.\ x \in \Omega,\ n = 1, 2, \ldots,$$

and $|a^n| = 1$, $a^n \to a \in \mathbb{R}^m$. *Then*

$$g(a \cdot Du(x), u(x), x) = 0 \qquad a.e.\ x \in \Omega.$$

Proof. Let $S = \{x \in \Omega : g^n(\cdot, z, x)$ is nondecreasing for all $n, z\}$; S is measurable. By redefining g^n for $x \in \Omega \setminus S$ to be $g^n(-p, z, x)$ we may and shall without loss of generality suppose that $g^n(\cdot, z, x)$ is nondecreasing for all n, z and almost all $x \in \Omega$.

Given $\delta > 0$, there exists a measurable subset $G \subset \Omega$ with

$$\text{meas}(\Omega \setminus G) \leqslant \delta, \tag{2.17}$$

such that

(i) $Du^n(x), u^n(x), Du(x), u(x)$ are defined and

$$g^n(a^n \cdot Du^n(x), u^n(x), x) = 0 \qquad (n = 1, 2, \ldots)$$

for all $x \in G$,

(ii) g^n, g are continuous on $S_\delta \equiv \mathbb{R} \times \mathbb{R} \times G$ and $g^n \to g$ uniformly on compact subsets of S_δ (Lemma 2),

(iii) Du, u are continuous on G (Lusin's theorem),

(iv) there is a subsequence $n_k \to \infty$ for which $u^{n_k} \to u$ uniformly on G, (Sobolev embedding theorem, Egorov's theorem).

Let K be any closed measurable subset of G. We employ the well-known device of Minty, and for notational simplicity reindex so that (iv) holds for the full sequence $n = 1, 2, \ldots$. For any $v \in W^{1,1}(\Omega)$ such that Dv is bounded on G we have that

$$\int_K [g^n(a^n \cdot Du^n(x), u^n(x), x) - g^n(a^n \cdot Dv(x), u^n(x), x)]\, a \cdot (Du^n(x) - Dv(x))\,dx \geqslant 0.$$

Since $g^n(a^n \cdot Dv(x), u^n(x), x) \to g(a \cdot Dv(x), u(x), x)$ uniformly on G we obtain

$$\int_K g(a \cdot Dv(x), u(x), x)\, a \cdot (Du(x) - Dv(x))\,dx \geqslant 0.$$

Setting $v(x) \equiv u(x) + ta \cdot x$, dividing by t and letting $t \to 0$, we deduce that

$$\int_K g(a \cdot Du(x), u(x), x)\,dx = 0.$$

Since K was arbitrary it follows that

$$g(a \cdot Du(x), u(x), x) = 0 \qquad \text{a.e. } x \in X ,$$

as required.

Remark. It would be interesting to characterize the F such that $u^n \to u$ weakly in $W^{1,1}(\Omega)$, $f^n \to f$ a.e. in Ω and $F(Du^n(x)) = f^n(x)$ a.e. $x \in \Omega$, $n = 1, 2, \ldots$, imply that $F(Du(x)) = f(x)$ a.e. $x \in \Omega$, in the case when the u^n take values in $\mathbb{R}^s$, $s > 1$. For information on the related problem when the f^n are required only to converge *weakly* to f (in some suitable L^p space) see Ball [**1, 2**], Reshetnyak [**18, 19**]; for generalizations see Murat [**16**], Tartar [**21**].

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