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A Numerical Method for Detecting Singular Minimizers

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Summary. A numerical method for computing minimizers in one-dimensional problems of the calculus of variations is described. Such minimizers may have unbounded derivatives, even when the integrand is smooth and regular. In such cases, because of the Lavrentiev phenomenon, standard finite element methods may fail to converge to a minimizer. The scheme proposed is shown to converge to an absolute minimizer and is tested on an example. The effect of quadrature is analyzed. The implications for higher-dimensional problems, and in particular for fracture in nonlinear elasticity, are discussed.

Subject Classifications: AMS(MOS): Primary 65K10, 49A10, 49D99; Secondary 73G05; CR: G1.6.

1. Introduction

In this paper we describe a numerical method for computing minimizers of integrals in the calculus of variations. Our work has its origin in recent studies by Ball and Mizel (1984, 1985) of some regular one-dimensional integrals whose minimizers, in appropriate classes of absolutely continuous functions, have unbounded derivatives at certain points. These singularities may prevent the minimizers satisfying classical necessary conditions of the calculus of variations, such as the usual weak form of the Euler-Lagrange equation. To compute such singular minimizers, the most obvious initial approach is to directly approximate the integral by a finite-element or finite-difference scheme, leading at each stage to a finite-dimensional minimization problem which can be solved by nonlinear programming techniques. However, for certain integrals *such approximation schemes typically fail both to converge to a minimizer and to determine the minimum value of the integral.* To illustrate this consider the problem of minimizing

$$I(u)=\int_0^1 (u^3-x)^2 (u')^6\,dx \tag{1.1}$$

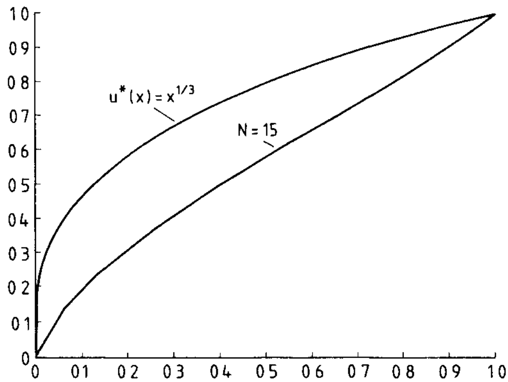

Fig. 1. Numerical minimization of $\int_0^1 (u^3-x)^2(u')^6\,dx$ subject to $u(0)=0$, $u(1)=1$; the result of direct minimization using piecewise linear finite elements and the mid-point rule. The method converges to the pseudo-minimizer $\bar{u}$

in the set of admissible functions

$$\mathscr{A}=\{u\in W^{1,1}(0,1)\colon \mathrm{u}(0)=0,\ u(1)=1\}. \tag{1.2}$$

It is easily seen that the unique minimizer of I in $\mathscr{A}$ is $u^*(x)=x^{1/3}$, and that $I(u^*)=0$. It was shown by Manià (1934) that

$$\inf_{u\in\mathscr{A}\cap W^{1,\infty}(0,1)} I(u)>\inf_{u\in\mathscr{A}} I(u)=0. \tag{1.3}$$

This remarkable property is known as the *Lavrentiev phenomenon* (Lavrentiev 1926; Cesari 1983). The technique in Ball and Mizel (1985, Theorem 5.5) shows further that if $p\geqq 3/2$

$$\lim_{j\to\infty} I(u_j)=\infty \tag{1.4}$$

for any sequence of functions $u_j\in\mathscr{A}\cap W^{1,p}(0,1)$ converging almost everywhere to u^*! In view of (1.3), (1.4) it is clear that any numerical scheme based on sufficiently accurate computation of $I(u_j)$ for Lipschitz functions u_j will fail both to find u^* and the correct minimum value of I. This is borne out by numerical experiments. For instance, the simplest finite-element method is to approximate u by $u^h\in S^h$, the space of piecewise linear splines in $\mathscr{A}$ on a uniform mesh subdividing $[0,1]$ with mesh spacing $h=1/N$, and to minimize $I(u^h)$ in S^h. If the internal nodal values of u^h are $\{a_1,\dots,a_{N-1}\}$, then

$$I(u^h)=I_N(a_1,\dots,a_{N-1}), \tag{1.5}$$

and we are left with the programming problem

$$\operatorname*{minimize}_{a\in\mathbb{R}^{N-1}} I_N(a). \tag{1.6}$$

This has been done for a sequence of values of $h\searrow 0$ and a plot of the numerical minimizer u^h of (1.5) is shown in Fig. 1. As predicted by (1.3), (1.4) the scheme does not converge to $u^*(x)=x^{1/3}$, but rather to a completely different function $\bar{u}$. (The effect of quadrature, which was neglected in the above description, is discussed later.) In fact the 'pseudominimizer' $\bar{u}$ can be identified as a solution of the Euler-Lagrange equation for (1.1) that is smooth in (0, 1) and minimizes I not in $\mathscr{A}$ but in $\mathscr{A} \cap W^{1,p}(0,1)$ for any $p \in [3/2, 2)$ (see § 4). There is nothing in the numerical results to alert the unwary to the fact that the minimum in $\mathscr{A}$ has not been found. The integrand $f(x,u,v)=(u^3-x)^2 v^6$ in (1.1) is not regular, i.e., f_{vv} is not strictly positive for all x, u, v. However the examples in Ball and Mizel (1985) show that singular minimizers, the Lavrentiev phenomenon and (1.4) can occur for regular integrands also [see also the interesting examples of Davie (1987)]. The results in Ball and Mizel (1985) concerning the existence of pseudominimizers were motivated by our numerical experiments.

An example of more practical interest occurs in nonlinear elasticity in connection with the experimentally observed phenomenon of cavitation [Gent and Lindley (1958)]. Consider deformations of an elastic body occupying in a reference configuration the bounded domain $\Omega \in \mathbb{R}^3$. Suppose that the material is homogeneous with stored-energy function $W\colon M_+^{3\times 3} \to \mathbb{R}$, where $M_+^{3\times 3}$ denotes the set of real 3×3 matrices with positive determinant. We seek a deformation $x\colon \Omega \to \mathbb{R}^3$ minimising

$$I(x)=\int_\Omega W(Dx(X))\,dX \tag{1.7}$$

in the set of admissible functions

$$\mathscr{A}=\{x \in W^{1,1}(\Omega;\mathbb{R}^3)\colon\ x|_{\partial\Omega}=AX\}, \tag{1.8}$$

where $A \in M_+^{3\times 3}$ is given. It is shown in Ball (1982), Sivaloganathan (1986a, b), Stuart (1985) that there are functions W such that for any A the absolute minimum of I in $\mathscr{A} \cap W^{1,\infty}(\Omega;\mathbb{R}^3)$ is attained by the linear deformation $x(X)=AX$ [that is, W is quasiconvex in the sense of Morrey (1966)], but that for some A the Lavrentiev phenomenon

$$\inf_{\mathscr{A} \cap W^{1,\infty}(\Omega;\mathbb{R}^3)} I = W(A)\cdot \operatorname{vol}\Omega > \inf_{\mathscr{A}} I \tag{1.9}$$

holds. If Ω is a ball and $A=\lambda 1$ then for sufficiently large λ the minimum of I among radial deformations $x(X)=\dfrac{r(R)}{R}X$, $R=|X|$, is attained by a function $r(R)$ with $r(0)>0$, so that a cavity forms at the origin. For further information concerning cavitation, radial equilibria, and the possible implications of singular minimizers for other modes of fracture see Ball and Murat (1984), Knops and Stuart (1984) and Ball and Mizel (1985). For numerical methods for computing regular deformations in elasticity see Glowinski and Le Tallec (1982).

The numerical procedure developed here avoids the Lavrentiev phenomenon and can detect singular minimizers. The basic idea is to decouple the unknown function u from its gradient in a manner reminiscent of control theory (or mixed finite element methods). Of course, for problems such as (1.1) it would be possible to make a change of variables so that a standard finite-element method would successfully minimize I, or, what is virtually the same idea, to use basis functions with singularities. Our procedure does not suffer from an important disadvantage of such methods, namely that they prejudge both the spatial location and order of the singularities. A stochastic approach to minimization of an integral similar to (1.1) has been shown by Heinricher and Mizel (1986) to inherit the Lavrentier phenomenon. In this paper we concentrate on one-dimensional variational problems. Preliminary numerical results indicate that the methods here do locate cavitating solutions in multi-dimensional elasticity problems, and this will be expanded in a later paper. We note that the convergence proofs given are based on the direct method of the calculus of variations and are in some respects independent of the space dimension.

The plan of the paper is as follows. In §2 we describe our numerical method and establish its convergence, neglecting the effects of quadrature, under various hypotheses on f. In §3 we analyse the effects of quadrature. Finally, in §4 we describe our numerical results for the Manià example.

2. The Numerical Method

We consider the problem of minimizing

$$I(u)=\int_a^b f(x,u(x),u'(x))\,dx \tag{2.1}$$

in the set of admissible functions

$$\mathscr{A}=\{u\in W^{1,1}(a,b)\colon\ u(a)=\alpha,\ \ u(b)=\beta\} \tag{2.2}$$

where $-\infty<a<b<\infty$ and α, β are constants.

Let $\phi\colon \mathbb{R}\to\mathbb{R}$ be an even continuous function satisfying

$$|v|^s\leqq\phi(v) \qquad \text{for all } v\in\mathbb{R}, \tag{2.3}$$

$$\phi(v_1+v_2)\leqq C(\phi(v_1)+\phi(v_2)) \qquad \text{for all } v_1,v_2\in\mathbb{R}, \tag{2.4}$$

where $1\leqq s<\infty$ and $C>0$ (for example, $\phi(v)=|v|$).

Our approach is to minimize numerically the decoupled integral

$$I(u,v)=\int_a^b f(x,u(x),v(x))\,dx \tag{2.5}$$

among pairs of functions $(u,v)\in\mathscr{A}\times L^1(a,b)$ satisfying the constraint

$$\int_a^b \phi(u'(x)-v(x))\,dx\leqq\varepsilon \tag{2.6}$$

and to let $\varepsilon \to 0+$. Note that this problem is similar to that obtained from the usual method for transforming a problem in the calculus of variations into control form, except that the *state constraint*

$$u'(x) = v(x), \qquad \text{a.e. } x \in [a, b] \tag{2.7}$$

is replaced by the inequality constraint (2.6). When $\varepsilon = 0$ the two problems are equivalent.

We will make use of the following hypotheses on the integrand f:

(H1) $f\colon [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous,

(H2) there exist constants $k_0 > 0$, k_1 such that

$$f(x, u, v) \geqq k_0 \phi(v) + k_1 \qquad \text{for all } x \in [a, b],\ u, v \in \mathbb{R}, \tag{2.8}$$

(H3) $$\lim_{|v| \to \infty} \frac{f(x, u, v)}{|v|} = \infty \qquad \text{for all } (x, u) \notin S, \tag{2.9}$$

where S is a closed *slender* subset of $[a, b] \times \mathbb{R}$, that is a closed subset satisfying $\operatorname{meas}\{u \in \mathbb{R}\colon (x, u) \in S \text{ for some } x \in E\} = 0$ whenever $E \subset [a, b]$, $\operatorname{meas} E = 0$,

(H4) $f(x, u, \cdot)$ is convex for all $x \in [a, b]$, $u \in \mathbb{R}$.

Note that if

$$f(x, u, v) \geqq \psi(v) \qquad \text{for all } x \in [a, b],\ u, v \in \mathbb{R}, \tag{2.10}$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{|v| \to \infty} \frac{\psi(v)}{|v|} = \infty$ then (H3) holds with S empty. The Manià integrand $f = (u^3 - x)^2 v^6$ satisfies (H1), (H3) and (H4) with $S = \{(x, u)\colon x \in [a, b],\ u^3 = x\}$, but not (H2); we return to this point later. Theorem 2.1 below asserts in particular that under hypotheses (H1)–(H4) I attains its absolute minimum on $\mathscr{A}$. However these hypotheses do not imply in general that $I(u, v)$ attains a minimum subject to (2.6), as is shown by the following example.

Example. Let $f(x, u, v) = u^2 + v^2$, $\phi(v) = |v|$, $\varepsilon = 1$, $\mathscr{A} = \{u \in W^{1,1}(0, 1)\colon u(0) = 0,\ u(1) = 1\}$. Let $u_\delta(x) = \max\{0, 1 + (x - 1)/\delta\}$, $v_\delta(x) = 0$. Then

$$I(u_\delta, v_\delta) = \int_0^1 (u_\delta^2 + v_\delta^2)\, dx = \frac{\delta}{3},$$

$$\int_0^1 |u_\delta' - v_\delta|\, dx = 1,$$

so that the infimum of $I(u, v)$ subject to (2.6) is zero, which is clearly not attained.

If ϕ is convex and satisfies (2.10) then (H1)–(H4) imply that $I(u, v)$ does attain an absolute minimum on $\mathscr{A} \times L^1(a, b)$ subject to (2.6); this can be proved using similar arguments to Theorem 2.1 below.

Let $h>0$ be the approximation parameter. We suppose that for each h there are finite-dimensional affine subspaces $\mathcal{A}^h$, V^h of $\mathcal{A}$, $L^\infty(a,b)$ respectively such that

(i) given any $u\in\mathcal{A}$ with $\phi(u')\in L^1(a,b)$ there exist functions $u^h\in\mathcal{A}^h$ with $\lim_{h\to 0}\int_a^b \phi((u^h)'-u')\,dx=0$, and

(ii) given any $v\in L^\infty(a,b)$ there exist functions $v^h\in V^h$ with $|v^h(x)|\leqq K$ for a.e. $x\in[a,b]$, all h and some constant K, and $v^h(x)\to v(x)$ as $h\to 0$, a.e. $x\in[a,b]$; if, further, $v\geqq 0$ a.e. in $[a,b]$ (resp. $v\leqq 0$ a.e.) then we assume that the v^h may be chosen such that $v^h\geqq 0$ a.e. (resp. $v^h\leqq 0$ a.e.) (this last property is used only in Theorem 2.4).

Typical examples include for $\mathcal{A}^h$ the space of piecewise affine splines in $\mathcal{A}$ on a grid covering $[a,b]$ with mesh size h, and for V^h the space of piecewise constants with respect to the same mesh.

The natural discretization of (2.5), (2.6) is to minimize

$$I(u,v)=\int_a^b f(x,u(x),v(x))\,dx \tag{2.5}$$

among pairs $(u,v)\in\mathcal{A}^h\times V^h$ satisfying

$$\int_a^b \phi(u'(x)-v(x))\,dx\leqq\varepsilon. \tag{2.6}$$

(In practice the integral $I(u,v)$ will need to be further approximated by quadrature - this is considered in § 3.) This is a finite-dimensional optimization problem with a closed constraint set that is nonempty for h sufficiently small. It follows easily from (2.3), (H2) that for pairs $(u,v)\in\mathcal{A}^h\times V^h$ satisfying (2.6) we have $I(u,v)\to\infty$ as $\|(u,v)\|_{\mathcal{A}^h\times V^h}\to\infty$. Since f is continuous the minimum value I_ε^h is therefore attained by a pair $(u_\varepsilon^h, v_\varepsilon^h)\in\mathcal{A}^h\times V^h$.

Theorem 2.1. *Assume* (H1)-(H4). *Then there is a nondecreasing function* γ: $(0,\infty)\to(0,\infty)$ *such that*

$$\lim_{\substack{h,\varepsilon\to 0\\ 0<h<\gamma(\varepsilon)}} I_\varepsilon^h=\inf_{u\in\mathcal{A}} I(u). \tag{2.11}$$

Let $h_j\to 0$, $\varepsilon_j\to 0$ *be sequences with* $0<h_j<\gamma(\varepsilon_j)$, *and let* $(u_{\varepsilon_j}^{h_j}, v_{\varepsilon_j}^{h_j})$ *be a minimizing pair for* $I(u,v)$ *in* $\mathcal{A}^{h_j}\times V^{h_j}$ *subject to the constraint*

$$\int_a^b \phi(u'-v)\,dx\leqq\varepsilon_j. \tag{2.12}$$

Then there exist a subsequence (h_μ,ε_μ) *of* (h_j,ε_j) *and a minimizer* u^* *of* $I(u)$ *in* $\mathcal{A}$ *such that as* $\mu\to\infty$

$$u_{\varepsilon_\mu}^{h_\mu}\to u^* \qquad \textit{uniformly in } [a,b], \tag{2.13a}$$

$$v_{\varepsilon_\mu}^{h_\mu} \overset{*}{\rightharpoonup} (u^*)' \quad \textit{weak } * \textit{ in the sense of measures} \tag{2.13b}$$

$\left(\text{i.e., } \int_a^b \theta\, v_{\varepsilon_\mu}^{h_\mu}\,dx \to \int_a^b \theta(u^*)'\,dx \textit{ for all } \theta\in C([a,b])\right).$

Proof. Let $\varepsilon>0$. There exists $u\in\mathscr{A}$ with

$$I(u)\leqq \inf_{\mathscr{A}} I+\varepsilon<\infty. \tag{2.14}$$

Since u is continuous, it follows from (2.3) and (H2) that there exists $M>0$ such that if $|v|>M$

$$f(x,u(x),v)\geqq f(x,u(x),0) \qquad \text{for all } x\in[a,b]. \tag{2.15}$$

For any $\delta>M$ define

$$v_\delta(x)=\begin{cases}0 & \text{if } |u'(x)|>\delta\\ u'(x) & \text{otherwise.}\end{cases} \tag{2.16}$$

By (2.15)

$$I(u,v_\delta)\leqq I(u). \tag{2.17}$$

Also

$$\int_a^b \phi(u'-v_\delta)\,dx=\int_{E_\delta}\phi(u')\,dx, \tag{2.18}$$

where $E_\delta=\{x\in[a,b];\ |u'(x)|>\delta\}$.

Choose $\delta>M$ sufficiently large so that

$$C\int_a^b \phi(u'-v_\delta)\,dx\leqq\frac{\varepsilon}{2}, \tag{2.19}$$

where C is the constant in (2.4); this is possible by (2.18), since $\phi(u')\in L^1(a,b)$ by (2.8). There exist functions $u^h\in\mathscr{A}^h$, $v^h\in V^h$ with

$$\lim_{h\to 0}\int_a^b \phi((u^h)'-u')\,dx=0, \tag{2.20}$$

$$v^h(x)\to v_\delta(x) \quad \text{as } h\to 0, \qquad |v^h(x)|\leqq K, \quad \text{a.e. } x\in[a,b]. \tag{2.21}$$

Now, by (2.4),

$$\int_a^b \phi((u^h)'-v^h)\,dx\leqq C\int_a^b \phi(u'-v_\delta)\,dx+C^2\int_a^b \phi((u^h)'-u')\,dx+C^2\int_a^b \phi(v^h-v_\delta)\,dx. \tag{2.22}$$

The last two integrals in (2.22) tend to zero as $h\to 0$ by (2.20), (2.21). Further, since (2.3), (2.20) imply that $u^h\to u$ uniformly in $[a,b]$ as $h\to 0$, it follows from the bounded convergence theorem that

$$\lim_{h\to 0} I(u^h,v^h)=I(u,v_\delta). \tag{2.23}$$

From (2.19)–(2.23) we see that there exists $\bar{\gamma}(\varepsilon)\in(0,1)$ such that for $h<\bar{\gamma}(\varepsilon)$

$$I(u^h, v^h) \leqq I(u, v_\delta) + \varepsilon \tag{2.24}$$

and

$$\int_a^b \phi((u^h)' - v^h)\, dx \leqq \varepsilon. \tag{2.25}$$

From (2.14), (2.17), (2.24) we then have that

$$I(u^h, v^h) \leqq \inf_{\mathscr{A}} I + 2\varepsilon, \tag{2.26}$$

and thus

$$I_\varepsilon^h \leqq \inf_{\mathscr{A}} I + 2\varepsilon. \tag{2.27}$$

Now let $\gamma(\varepsilon) = \frac{1}{2}\sup\{\bar{\gamma}(\varepsilon)\colon 0 < \bar{\varepsilon} \leqq \varepsilon\}$. Then $0 < \gamma(\varepsilon) < \infty$ and γ is nondecreasing. If $h < \gamma(\varepsilon)$ then $h < \gamma(\bar{\varepsilon})$ for some $\bar{\varepsilon} \in (0, \varepsilon]$, and

$$I_\varepsilon^h \leqq I_{\bar{\varepsilon}}^h \leqq \inf_{\mathscr{A}} I + 2\bar{\varepsilon} \leqq \inf_{\mathscr{A}} I + 2\varepsilon,$$

so that (2.27) still holds.

We complete the proof of the theorem first under the assumption that ϕ satisfies the superlinear growth condition (2.10), since the proof in this case is much simpler and since we will refer to it later. Let $h_j \to 0$, $\varepsilon_j \to 0$ with $0 < h_j < \gamma(\varepsilon_j)$, and set $u_j = u_{\varepsilon_j}^{h_j}$, $v_j = v_{\varepsilon_j}^{h_j}$. By (2.27), (2.10)

$$\sup_j \int_a^b \psi(v_j)\, dx < \infty, \tag{2.28}$$

so that by de la Vallée Poussin's criterion (Natanson 1964, p. 158; Cesari 1983, p. 329) there exists a subsequence v_μ of v_j with

$$v_\mu \rightharpoonup v^* \quad \text{in } L^1(a, b) \tag{2.29}$$

for some v^*. Since

$$\int_a^b |u_\mu' - v_\mu|^s\, dx \leqq \int_a^b \phi(u_\mu' - v_\mu)\, dx \leqq \varepsilon_\mu,$$

it follows that $u_\mu' \rightharpoonup v^*$ in $L^1(a, b)$. Hence

$$u_\mu \rightharpoonup u^* \quad \text{in } W^{1,1}(a, b), \tag{2.30}$$

where $u^*(x) \stackrel{\text{def}}{=} \alpha + \int_a^x v^*(y)\, dy$. In particular, $u_\mu \to u^*$ uniformly in $[a, b]$, and thus $u^* \in \mathscr{A}$. Since f is convex in v, we deduce from standard lower semicontinuity results (Cesari 1983, p. 352; Ekeland and Témam 1974, p. 226; Eisen 1979) that

$$\inf_{\mathscr{A}} I \leqq I(u^*) \leqq \liminf_{\mu \to \infty} I(u_\mu, v_\mu). \tag{2.31}$$

Since $I(u_\mu, v_\mu) = I_{\varepsilon_\mu}^{h_\mu}$ it follows from (2.27), (2.31) that u^* is a minimizer and $\lim_{\mu \to \infty} I_{\varepsilon_\mu}^{h_\mu} = \inf_{\mathscr{A}} I$. Since (2.13b) follows from (2.29), it remains to prove (2.11). But this follows from the above argument applied to sequences h_j, ε_j assumed for contradiction to satisfy $\lim_{j \to \infty} I_{\varepsilon_j}^{h_j} < \inf_{\mathscr{A}} I$.

To complete the proof in the general case we again suppose that $h_j \to 0$, $\varepsilon_j \to 0$ with $h_j < \gamma(\varepsilon_j)$ and set $u_j = u_{\varepsilon_j}^{h_j}$, $v_j = v_{\varepsilon_j}^{h_j}$. We first show that the u_j are uniformly bounded and equicontinuous. The uniform boundedness is a consequence of (2.27), (2.3), (2.8) and (2.13a). For the equicontinuity we use the following lemma.

Lemma 2.2. *Suppose that the rectangle $Q = [c, d] \times [u_1, u_2]$, $a \leqq c < d \leqq b$, $u_1 < u_2$, contains no points of S. Then*

$$f(x, u, v) \geqq \phi_Q(v) \quad \textit{for all } (x, u) \in Q,\ v \in \mathbb{R} \tag{2.32}$$

for some function $\phi_Q\colon \mathbb{R} \to \mathbb{R}$ with $\lim_{|v| \to \infty} \frac{\phi_Q(v)}{|v|} = \infty$. Furthermore, there exist a constant $\delta > 0$ and a positive integer j_0 such that

$$|x_{2j} - x_{1j}| \geqq \delta \tag{2.33}$$

for any $j \geqq j_0$ such that $u_j(x_{1j}) = u_1$, $u_j(x_{2j}) = u_2$, $x_{1j}, x_{2j} \in [c, d]$.

Proof. Define

$$\phi_Q(v) = \min_{(x, u) \in Q} f(x, u, v) \tag{2.34}$$

and suppose for contradiction that there exist $(x^{(j)}, u^{(j)}) \in Q$ and $v^{(j)}$ with $|v^{(j)}| \to \infty$ as $j \to \infty$ and

$$\phi_Q(v^{(j)}) = f(x^{(j)}, u^{(j)}, v^{(j)}) \leqq K_0 |v^{(j)}| \tag{2.35}$$

for all j and some constant K_0. We can assume that $(x^{(j)}, u^{(j)}) \to (x, u) \in Q$ as $j \to \infty$. Since $f(x^{(j)}, u^{(j)}, \cdot)$ is convex, for each j the function

$$v \mapsto \frac{f(x^{(j)}, u^{(j)}, v) - f(x^{(j)}, u^{(j)}, 0)}{|v|}$$

is nondecreasing for $v > 0$ and nonincreasing for $v < 0$. Therefore, for any $v \neq 0$,

$$\begin{aligned}\frac{f(x, u, v) - f(x, u, 0)}{|v|} &= \lim_{j \to \infty} \frac{f(x^{(j)}, u^{(j)}, v) - f(x^{(j)}, u^{(j)}, 0)}{|v|} \\ &\leqq \lim_{j \to \infty} \frac{f(x^{(j)}, u^{(j)}, v^{(j)}) - f(x^{(j)}, u^{(j)}, 0)}{|v^{(j)}|} \leqq K_0,\end{aligned}$$

contradicting (H3).

To establish (2.33) we can suppose without loss of generality that $x_{1j} < x_{2j}$ and that $u_j(x) \in [u_0, u_1]$ for all $x \in [x_{1j}, x_{2j}]$. By (2.27), (2.32), there exists a constant C_0 such that

$$\int_{x_{1j}}^{x_{2j}} \phi_Q(v_j)\, dx \leqq C_0 \quad \text{for all } j. \tag{2.36}$$

Choose j_0 such that $\varepsilon_j < \frac{1}{2}(u_1 - u_0)$ for all $j \geqq j_0$ and $\sigma > 0$ such that $\sigma C_0 < \frac{1}{2}(u_1 - u_0)$. Then choose M_0 such that $\frac{\phi_Q(v)}{|v|} \geqq \frac{1}{\sigma}$ whenever $|v| \geqq M_0$. Set $E_j = \{x \in [x_{1j}, x_{2j}] \colon |v_j(x)| \geqq M_0\}$. Thus

$$u_1-u_0\leqq\int_{x_{1j}}^{x_{2j}}|u_j'|\,dx\leqq\int_{x_{1j}}^{x_{2j}}|v_j|\,dx+\varepsilon_j, \tag{2.37}$$

and

$$\begin{aligned}\int_{x_{1j}}^{x_{2j}}|v_j|\,dx&\leqq\int_{E_j}|v_j|\,dx+M_0(x_{2j}-x_{1j})\\&\leqq\sigma\,C_0+M_0(x_{2j}-x_{1j}).\end{aligned} \tag{2.38}$$

Combining (2.37), (2.38) we obtain (2.33) with $\delta=M_0^{-1}[\frac{1}{2}(u_1-u_0)-\sigma\,C_0]$. (This argument is a modification of that in Natanson 1964, p. 159.) □

Continuation of Proof of Theorem 2.1

Suppose for contradiction that the u_j are not equicontinuous. Then there exist a subsequence u_{j_k} of u_j and points $y_{1k}, y_{2k}\in[a,b]$ such that $y_{1k}\to x_0$, $y_{2k}\to x_0$, $u_{j_k}(y_{1k})\to w_1$, $u_{j_k}(y_{2k})\to w_2$ as $k\to\infty$ with $w_1<w_2$. The line segment $L=\{x_0\}\times[w_1,w_2]$ is not entirely contained in S since S is slender, and since S is closed this implies that there is a rectangle $Q=[c,d]\times[u_1,u_2]$ with $a\leqq c<d\leqq b$, $c\leqq x_0\leqq d$, $w_1\leqq u_1<u_2\leqq w_2$, $Q\cap S$ empty, and points $x_{1k}, x_{2k}\in[c,d]$ such that $x_{1k}, x_{2k}\to x_0$ and $u_{j_k}(x_{1k})=u_1$, $u_{j_k}(x_{2k})=u_2$. This contradicts Lemma 2.2.

Since the u_j are uniformly bounded and equicontinuous, by the Arzela-Ascoli theorem there exists a subsequence u_μ of u_j converging uniformly to a continuous function u^* on $[a,b]$. Since by (2.37), (2.3), (H2) and the constraint we have $\sup_j\int_a^b|u_j'|\,dx<\infty$, it follows that u^* has bounded variation on $[a,b]$. We claim that u^* maps sets of measure zero to sets of measure zero. If not there would exist a subset E of $[a,b]$ with $\operatorname{meas}E=0$ and $\operatorname{meas}u^*(E)>0$. Define $A=\{x\in[a,b]\colon\ (x,u^*(x))\in S\}$. Then either $\operatorname{meas}u^*(E\cap A)>0$ or $\operatorname{meas}u^*(E\setminus A)>0$. The first case is impossible since S is slender. The second case is impossible since the uniform convergence, the estimate (2.32), and the de la Vallée Poussin criterion show that u^* is absolutely continuous on any closed interval J disjoint from A, and thus $\operatorname{meas}u^*(J)=0$. Since u^* maps sets of measure zero to sets of measure zero and is continuous of bounded variation, the Banach-Zarecki theorem (Natanson 1964, p. 250) implies that u^* is absolutely continuous on $[a,b]$. Hence $u^*\in\mathscr{A}$.

Since $u^*\in\mathscr{A}$, $u_\mu'\to(u^*)'$ in the sense of distributions on (a,b), and hence, using the uniform boundedness of $\int_a^b|u_\mu'|\,dx$ and a standard approximation argument, we have also that $u_\mu'\overset{*}{\rightharpoonup}(u^*)'$ in the sense of measures. Since $\|u_\mu'-v_\mu\|_{L^s(a,b)}\to0$ as $\mu\to\infty$ this proves (2.13b). We now use the following result of Reshetnyak (1967, Theorem 1.1). (We remark that in the English translation of Reshetnyak's proof the phrase 'a point close to' should read throughout 'the point closest to'.)

Lemma 2.3. *Let $\Omega\in\mathbb{R}^m$ be open and $F_\mu\colon\Omega\times\mathbb{R}^l\to\mathbb{R}$ be a sequence of nonnegative continuous functions such that $F_\mu\to F$ locally uniformly in $\Omega\times\mathbb{R}^l$ as $\mu\to\infty$. Then, if the functions $F_\mu(x,\cdot)$ are convex, the inequality*

$$\int_{\Omega} F(x, v(x))\,dx \leqq \liminf_{\mu\to\infty} \int_{\Omega} F_\mu(x, v_\mu(x))\,dx \tag{2.39}$$

holds for every sequence $v_\mu \in L^1(\Omega; \mathbb{R}^l)$ *converging weak* $*$ *in the sense of measures to a function* $v \in L^1(\Omega; \mathbb{R}^l)$.

Applying Lemma 2.3 *to the functions*

$$F_\mu(x, v) \stackrel{\text{def}}{=} f(x, u_\mu(x), v) - k_1,$$

we deduce that (2.31) *holds. The remainder of the proof is as before.* □

Remark. The assertion in Theorem 2.1 that I attains a minimum on $\mathscr{A}$ is essentially a special case of a result of Cesari et al. (1971) (see Cesari 1983, p. 412) and is similar to theorems in McShane (1938). The parts of the proof concerned with slender sets use many of the ideas in Cesari (1983, Chap. 12).

We now give a variant of Theorem 2.1 applying in particular to the Manià example (1.1).

Theorem 2.4. *Assume* (H1), (H3), (H4) *and that*

$$0 = f(x, u, 0) = \min_{v\in\mathbb{R}} f(x, u, v) \tag{2.40}$$

for all $x \in [a, b]$, $u \in \mathbb{R}$. *Suppose that* ϕ: $\mathbb{R} \to \mathbb{R}$ *is continuous, with*

$$|v| \leqq \phi(v) \leqq C_1 |v| \quad \textit{for all } v \in \mathbb{R}, \tag{2.41}$$

where C_1 *is a constant.*

Let $\alpha \leqq \beta$ (resp. $\alpha \geqq \beta$). *Then there exists a nondecreasing function* γ: $(0, \infty) \to (0, \infty)$ *such that if* $0 < h < \gamma(\varepsilon)$ *then* $I(u, v)$ *attains an absolute minimum among pairs* $(u, v) \in \mathscr{A}^h \times V^h$ *satisfying* $v(x) \geqq 0$ (resp. $v(x) \leqq 0$) *for* a.e. $x \in [a, b]$ *and the constraint* (2.6), *and such that the minimum value* I_ε^h *satisfies* (2.11). *Let* $h_j \to 0$, $\varepsilon_j \to 0$ *be sequences with* $0 < h_j < \gamma(\varepsilon_j)$ *and let* $(u_{\varepsilon_j}^{h_j}, v_{\varepsilon_j}^{h_j})$ *be a corresponding minimizing pair. Then there exist a subsequence* (h_μ, ε_μ) *of* (h_j, ε_j) *and a minimizer* u^* *of* $I(u)$ *in* $\mathscr{A}$ *such that, as* $\mu \to \infty$, (2.13) *holds.*

Remark. If $\alpha = \beta$ then of course $u(x) \equiv \alpha$ is an absolute minimizer of $I(u)$ in $\mathscr{A}$, so that the minimization problem is trivial.

Proof of Theorem 2.4

Let $\alpha \leqq \beta$; the case $\alpha \geqq \beta$ is treated similarly. First note that for any $(u, v) \in \mathscr{A}^h \times V^h$ satisfying $v \geqq 0$ a.e. and (2.6),

$$\int_a^b v\,dx = \int_a^b (v - u' + u')\,dx \leqq \varepsilon + \beta - \alpha \tag{2.42}$$

and hence

$$\int_a^b |u'|\,dx \leqq 2\varepsilon + \beta - \alpha. \tag{2.43}$$

The finite-dimensional optimization problem therefore has a compact constraint set that is nonempty for h sufficiently small (depending on ε), and so the minimum is attained.

Let $\varepsilon>0$. There exists $\tilde{u}\in\mathscr{A}$ with

$$I(\tilde{u})\leqq\inf_{\mathscr{A}} I+\varepsilon<\infty. \tag{2.44}$$

Define u: $[a,b]\to[\alpha,\beta]$ by

$$u(x)=\min\{\beta, \max_{a\leqq y\leqq x} \tilde{u}(y)\}, \qquad x\in[a,b]. \tag{2.45}$$

Then u is continuous and for some $\bar{b}\in(a,b]$,

$$\begin{aligned} \alpha\leqq u(x)<\beta \quad & \text{for } x\in[a,\bar{b}), \\ u(x)=\beta \quad & \text{for } x\in[\bar{b},b]. \end{aligned} \tag{2.46}$$

Since u is nondecreasing it is differentiable a.e.. Let $x\in(a,\bar{b})$ be a point at which both u and $\tilde{u}$ are differentiable. If $u(x)>\tilde{u}(x)$ then clearly $u'(x)=0$, while if $u(x)=\tilde{u}(x)$ then $u-\tilde{u}$ has a minimum at x and thus $u'(x)=\tilde{u}'(x)$. The set $E\overset{\text{def}}{=}\{x\in(a,\bar{b})\colon u(x)>\tilde{u}(x))\}$ is open and is therefore a countable union of disjoint open intervals E_k. Clearly

$$\int_E u'(x)\,dx=\sum_k \int_{E_k} \tilde{u}'(x)\,dx=0,$$

and so

$$\int_a^{\bar{b}} u'(x)\,dx=\int_{(a,\bar{b})\setminus E} u'(x)\,dx=\int_{(a,\bar{b})\setminus E} \tilde{u}'(x)\,dx=\int_a^{\bar{b}} \tilde{u}'(x)\,dx=\beta-\alpha.$$

Since $u'\geqq 0$ a.e. it follows (Saks 1937, p. 224) that u is absolutely continuous, and hence $u\in\mathscr{A}$. By (2.40), (2.44)

$$I(u)\leqq I(\tilde{u})\leqq\inf_{\mathscr{A}} I+\varepsilon. \tag{2.47}$$

For $\delta>0$ we define v_δ by (2.16). Then (2.17) holds by (2.40). The proof now follows that of Theorem 2.1. Note that $\phi(u')\in L^1(a,b)$ by (2.41), (2.43) and that by assumption the v^h can be chosen to be nonnegative. $\square$

A variant of Theorem 2.1 can also be proved in which the convexity hypothesis (H4) is dropped. The function u^* in (2.13) is in this case a minimizer of

$$J(u)\overset{\text{def}}{=}\int_a^b f^{**}(x,u(x),u'(x))\,dx \tag{2.48}$$

in $\mathscr{A}$, where $f^{**}(x,u,v)$ denotes the lower convex envelope of $f(x,u,v)$ with respect to v. In order to carry over the proof use is made of the relation

$$\inf_{u\in\mathscr{A}} I(u)=\inf_{u\in\mathscr{A}} J(u). \tag{2.49}$$

Unfortunately, (2.49) seems only to be known under rather strong growth hypotheses on f such as

$$c_1+d_1|v|^\alpha \leqq f(x,u,v) \leqq c_2+\sigma|u|^\alpha+d_2|v|^\alpha, \tag{2.50}$$

where $1<\alpha<\infty$, $\sigma \geqq 0$, $d_2 \geqq d_1>0$ and c_1, c_2 are constants (cf. Ekeland and Témam 1974, p. 314; Marcellini and Sbordone 1980). With such growth hypotheses the Lavrentiev phenomenon cannot occur, and a proof similar to that of Theorem 2.1 shows that direct minimization of $I(u)$ in $\mathscr{A}^h$ as $h \to 0$ suffices to determine $\inf_{\mathscr{A}} I$ and a minimizer u^* of J. We therefore omit the details.

Our numerical scheme can be adapted to find the minimum value and minimizers for I in $\mathscr{A} \cap W^{1,\alpha}(a,b)$ for any $\alpha \in [1,\infty]$. The idea is to add the constraint

$$\|v\|_{L^\alpha(a,b)} \leqq M \tag{2.51}$$

to the minimization problem (2.5), (2.6) and let M take larger and larger values. For α such that the minimum of I in $\mathscr{A} \cap W^{1,\alpha}(a,b)$ is attained (cf. Ball and Mizel 1985) the computed u^* and value of I will be in general independent of all sufficiently large M.

3. The Effect of Quadrature

In Theorem 2.1 and its variants it is assumed that the integrals in (2.5), (2.6) are computed exactly. In this section we study the effect of various methods of computing these integrals on the outcome of the numerical scheme. We consider throughout the special case used in our numerical computations of $\mathscr{A}^h = S^h \subset \mathscr{A}$ piecewise linear and V^h piecewise constants on the grid with mesh points $\{a+ih\}_{0 \leqq i \leqq N}$, where $h=N^{-1}(b-a)$. Note that in this case the integral in (2.6) can be computed exactly (ignoring round-off error) since the integrand is piecewise constant; we therefore consider only the approximation of the integral (2.5).

In order to compute the integral (2.5) to arbitrary accuracy it is necessary to introduce further mesh points in each subinterval $\Delta_i=[a+ih, a+(i+1)h]$, and the number of such new mesh points in each subinterval will in general need to increase without limit as $h, \varepsilon \to 0+$ in order to compensate for the increasingly singular behaviour of the finite-dimensional minimizer $(u_\varepsilon^h, v_\varepsilon^h)$. Such a procedure, while feasible, has obvious disadvantages in programming and computation time, and it is more convenient to calculate the integral by means of a quadrature rule involving at most a fixed finite number of new mesh points in each subinterval. Let $a_i=a+ih$, so that $\Delta_i=[a_i, a_{i+1}]$, and suppose that $I(u,v)$ is calculated for $(u,v) \in \mathscr{A}^h \times V^h$ through the quadrature rule

$$I^h(u,v)=h \sum_{i=0}^{N-1} \sum_{k=1}^{M} \lambda_k f(a_i+t_k h, u(a_i+t_k h), v_{(i)}), \tag{3.1}$$

where $v_{(i)}$ is the (constant) value of v in (a_i, a_{i+1}) and where $M \geqq 1$, $\lambda_k>0$, $\sum_{k=1}^{M} \lambda_k = 1$ and $0 \leqq t_1 < \ldots < t_M \leqq 1$ are given. Special cases of (3.1) are the *mid-point rule* ($M=2$, $\lambda_1=\lambda_2=1/2$, $t_1=0$, $t_2=1$), and *Simpson's rule* ($M=3$, $\lambda_1=\lambda_3=1/6$, $\lambda_2=2/3$, $t_1=0$, $t_2=1/2$, $t_1=1$). The following example shows that for any such quadrature rule there are integrands f satisfying the hypotheses of Theorem 2.1 such that the numerical scheme

$$\text{minimize } I_h(u,v) \text{ for } (u,v)\in\mathscr{A}^h\times V^h \text{ satisfying (2.6)} \tag{3.2}$$

can lead as $h, \varepsilon\to 0$ to *too low* a value for $\inf_{\mathscr{A}} I$.

Example. Let

$$g(u)=\prod_{k=1}^{M}(u-t_k), \quad f(x,u,v)=(g(u)v)^2+|v|, \quad \phi(v)=|v|,$$

$a=\alpha=0$, $b=\beta=1$. Then (H1)-(H4) are satisfied with $S=[0,1]\times\{t_1,\dots,t_M\}$. Given $h=N^{-1}$ define

$$u^h(x)=\begin{cases} Nx, & 0\leqq x\leqq h\\ 1, & h\leqq x\leqq 1\end{cases} \tag{3.3}$$

and $v^h(x)=(u^h)'(x)$. Then the constraint (2.6) is satisfied for all $\varepsilon\geqq 0$ and

$$I_h(u^h,v^h)=h\cdot h^{-1}=1. \tag{3.4}$$

But by Theorem 2.1 I attains an absolute minimum on $\mathscr{A}$ and hence

$$\inf_{u\in\mathscr{A}} I(u)>\inf_{u\in\mathscr{A}}\int_0^1|u'|\,dx=1. \tag{3.5}$$

We now show that the scheme (3.2) works if f satisfies the superlinear growth condition (2.10).

Theorem 3.1. *Assume* (H1)-(H4) *and that* f *satisfies* (2.10). *Then there is a nondecreasing function* $\tilde{\gamma}$: $(0,\infty)\to(0,\infty)$ *such that if* $0<h<\tilde{\gamma}(\varepsilon)$ *then* $I_h(u,v)$ *attains an absolute minimum among pairs* $(u,v)\in\mathscr{A}^h\times V^h$ *satisfying the constraint* (2.6), *and such that the minimum value* I^h_ε *satisfies* (2.11). *Let* $h_j\to 0$, $\varepsilon_j\to 0$ *be sequences with* $0<h_j<\gamma(\varepsilon_j)$ *and let* $(u^{h_j}_{\varepsilon_j}, v^{h_j}_{\varepsilon_j})$ *be a corresponding minimizing pair. Then there exist a subsequence* (h_μ,ε_μ) *of* (h_j,ε_j) *and a minimizer* u^* *of* $I(u)$ *in* $\mathscr{A}$ *such that, as* $\mu\to\infty$,

$$\begin{aligned} u^{h_\mu}_{\varepsilon_\mu} &\rightharpoonup u^* && \text{weakly in } W^{1,1}(a,b),\\ v^{h_\mu}_{\varepsilon_\mu} &\rightharpoonup (u^*)' && \text{weakly in } L^1(a,b). \end{aligned} \tag{3.6}$$

Proof. Let $h=N^{-1}(b-a)$. Define

$$y_h(x)=a_i+t_k h \quad \text{for } x\in\left(a_i+h\sum_{r=1}^{k-1}\lambda_r,\ a^i+h\sum_{r=1}^{k}\lambda_r\right), \tag{3.7}$$

where $0\leqq i\leqq N-1$, $1\leqq r\leqq M$, and where $\sum_{r=1}^{0}$ is interpreted as zero. Then $y_h\in L^\infty(a,b)$ and $y_h(x)\to x$ a.e. as $h\to 0$. If $(u,v)\in\mathscr{A}^h\times V^h$ then

$$I_h(u,v)=\int_a^b f(y_h(x),u(y_h(x)),v(x))\,dx. \tag{3.8}$$

The proof now follows that of Theorem 2.1 for the superlinear growth case. From (3.8) and the bounded convergence theorem we see that for the functions

u^h, v^h in (2.20), (2.21), we have

$$\lim_{h\to 0} [I_h(u^h, v^h) - I(u^h, v^h)] = 0 \tag{3.9}$$

so that, by the argument following (2.23), (2.27) holds for $0<h<\tilde{\gamma}(\varepsilon)$, where $\tilde{\gamma}$: $(0,\infty)\to(0,\infty)$ is nondecreasing. Let $h_j\to 0$, $\varepsilon_j\to 0$ with $0<h_j<\tilde{\gamma}(\varepsilon_j)$ and set $u_j = u_{\varepsilon_j}^{h_j}$, $v_j = v_{\varepsilon_j}^{h_j}$. As before we extract the subsequence (h_μ, ε_μ) such that (2.29), (2.30) hold. Since the vector $(y_{h_\mu}(x), u_\mu(y_{h_\mu}(x)))$ converges a.e. to $(x, u^*(x))$ and f is continuous we deduce from (3.8) and the cited lower semicontinuity results that

$$I(u^*) \leqq \liminf_{\mu\to\infty} I_{h_\mu}(u_\mu, v_\mu), \tag{3.10}$$

and the theorem follows. □

The example preceding Theorem 3.1 suggests that a direct minimization of $I_h(u) \overset{\text{def}}{=} I_h(u, u')$ over $\mathscr{A}^h$ might in some problems bypass the Lavrentiev phenomenon and converge to the correct minimum. Indeed this occurs for the Manià problem (1.1), (1.2) if we use the trapezium rule; if $u^h \in \mathscr{A}^h$ satisfies $u^h(jh) = (jh)^{1/3}$, $0 \leqq j \leqq N$, then $I_h(u^h) = 0$. In this example the Lavrentiev phenomenon is not inherited by the numerical scheme since the trapezium rule does not see the points where $u^h \neq x^{1/3}$. This does not occur for the integral

$$I(u) = \int_0^1 [(u^3 - x)^2 (u')^6 + (x u' - \tfrac{1}{3} u)^2 (u')^{12}]\, dx, \tag{3.11}$$

with the same boundary conditions $u(0)=0$, $u(1)=1$. The absolute minimum of I in $\mathscr{A}$ is again attained by $u(x) = x^{1/3}$. However direct minimization of $I_h(u)$ over $\mathscr{A}^h$ using the trapezium rule leads numerically to a minimizer u_h satisfying

$$\lim_{h\to 0} I_h(u^h) > 0, \tag{3.12}$$

showing that in this example the numerical scheme does inherit the Lavrentiev phenomenon (this could probably be proved analytically). We expect that convergence to the correct minimum of direct minimization of I_h using the trapezium rule is rare, even when f is convex in u'.

4. Numerical Results

To test our method we have applied both it and direct minimization to the Manià example (1.1), (1.2). The result of direct minimization of $I_h(u)$ over $\mathscr{A}^h$ (piecewise affine functions) using the mid-point rule are shown in Fig. 1. The scheme converges to a function $\bar{u}$ that can be identified as the unique minimizer of I in $\mathscr{A} \cap W^{1,p}(0,1)$ for every $p \in [\frac{3}{2}, 2)$; this function is a smooth solution of the Euler-Lagrange equation for (1.1), (1.2) on $(0,1)$ with infinite slope at $x=0,1$. We do not give the complete details of this identification as they are lengthy and follow roughly the lines of calculations in Ball and Mizel (1985), and as for our purposes the essential point is that $\bar{u} \neq x^{1/3}$. The main ingredients are (i) the reduction of the Euler-Lagrange equation to an auto-

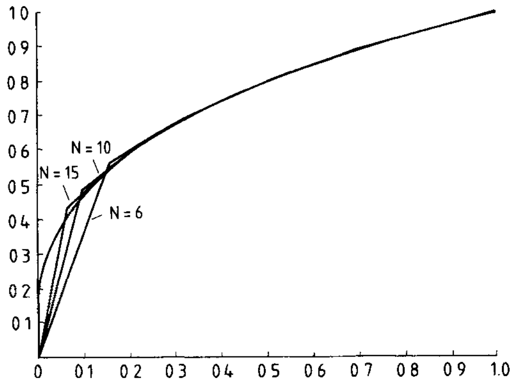

Fig. 2. Numerical minimization of $\int_0^1 (u^3 - x)^2 (u')^6 dx$ subject to $u(0)=0$, $u(1)=1$; the result of the numerical scheme (3.2) with $\mathscr{A}^h$ piecewise linear, V^h piecewise constant, $\phi(v)=|v|$ and using the mid-point rule. The scheme converges to the true minimizer $u^*(x)=x^{1/3}$

nomous ordinary differential equation in the (q, z) plane by means of the change of variables $w=u^3$, $z=w/x$, $q=dw/dx$ and $x=e^t$, (ii) the proof that this equation has a unique orbit $q=\bar{q}(z)$ leaving the origin $z=q=0$ with slope 9/5 and which satisfies $\bar{q}(z)\to\infty$ as $z\to 1-$, (iii) use of arguments such as in Ball and Mizel (1985, Theorems 5.8, Lemma 3.9) to show that the minimum is attained, meets $u=x^{1/3}$ only at $x=0, 1$, and corresponds to $\bar{q}(\cdot)$, (iv) correlation of the computed values in Fig. 1 with numerical computation of $\bar{q}(z)$.

We implemented the numerical scheme (3.2) in the following way. First, as in § 3, $\mathscr{A}^h$ and V^h were chosen to be piecewise linear and constant respectively. Second, $I_h(u, v)$ was calculated using the mid-point rule; as the example following (3.2) shows, we were perhaps fortunate that this did not in practice affect the convergence adversely. Third, the function ϕ in the constraint was chosen to be $\phi(v)=|v|$. Finally the finite-dimensional optimization was carried out using a modified penalty method to handle the constraint and a quasi-Newton unconstrained optimization routine; to avoid non-differentiability difficulties for the optimization routine the absolute value function was suitably 'rounded' near zero. The starting values for the minimization algorithm were taken on the straight line $u(x)=x$. The results are presented in Fig. 2, where the minimizing u_ε^h is plotted for various values of h for small values of ε. It can be seen that u_ε^h does approximate the minimizer $x^{1/3}$. An effect of using the mid-point rule can be seen – the graph of u_ε^h intersects $x^{1/3}$ close to the mid-point of each subinterval.

Strictly speaking the implemented numerical scheme is not covered by any of the convergence theorems proved; for example, we have not included any analysis of the finite-dimensional optimization routine in this paper. If we neglect the effects of quadrature and errors in the finite-dimensional optimization then Theorem 2.4 establishes convergence under the addition constraint $v\geqq 0$ a.e.; this constraint is seen to be verified a posteriori by the numerical results. Theorem 3.1 does not apply to the Manià example, which does not satisfy (2.10).

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