Quasiconvexity at the Boundary, Positivity of the Second Variation and Elastic Stability

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Dedicated to Jerry Ericksen

1. Introduction

For one dimensional problems of the calculus of variations, there is a well developed theory relating the positivity of the second variation at a critical point (*i.e.* 'linearized stability') to strong relative minima. These results, largely due to WEIERSTRASS, are usually proved by the method of 'fields of extremals' as is described in BOLZA [1904] and MORREY [1966] or by an argument involving a careful use of Taylor series, as in HESTENES [1966, Chapter 3, § 14]. We recall that if the integral is denoted

$$I(y) = \int_a^b f(x, y(x), y'(x))\, dx,$$

where y is a scalar, $y'(x) = \dfrac{d}{dx} y(x)$, and the admissible functions satisfy the boundary conditions $y(a) = \alpha$, $y(b) = \beta$, say, then the Weierstrass theory guarantees that a solution $u \in C^2([a, b])$ of the Euler-Lagrange equations is a strong relative minimum of $I(\cdot)$ (*i.e.* a local minimum in $C^0([a, b])$), provided f is smooth, $f_{y'y'} > 0$ and the second variation $\delta^2 I(u)$ is positive. (The theory gives more than this; see, for example, Theorem 3.1 below.) Without a convexity condition in y', easy examples, such as $\int_0^1 (y'^2 - y'^4)\, dx$, $y(0) = y(1) = 0$, show that positivity of the second variation does not give a strong relative minimum, although it does give a *weak* relative minimum (*i.e.* a local minimum in $C^1([a, b])$).

A principal result of this paper (Theorem 3.5) shows that for nonlinear elasticity in $n > 1$ space dimensions, positivity of the second variation does not imply a strong local minimum even under 'favorable' convexity hypotheses on the stored-energy function that reduce in one dimension to convexity in y'. This is done by developing in § 2 a new necessary condition for a minimum in mixed problems of the calculus of variations that we call *quasiconvexity at the boundary*; this condition applies at an appropriate boundary point of the spatial domain, and is a version of MORREY's quasiconvexity condition which is known to be necessary at an interior point.

The rough idea for the construction is the following. Realistic stored-energy functions may be strongly elliptic and polyconvex (BALL [1977a, b]) but cannot be convex. Certain such functions, of a type quite similar to those used to model natural rubbers, possess local minimizers (natural states) that are not global minimizers. The second variation at a homogeneous deformation corresponding to such a local minimizer is positive, but we can arrange a special deformation near a traction-free portion of the boundary in which the principal stretches are altered sufficiently to lower the overall energy, while keeping the displacement C^0 small.

The above negative results indicate the serious difficulties that arise in attempting to extend the Weierstrass theory to higher dimensions. See RUND [1963], [1974] and the discussion in MORREY [1966, p. 15]. In one dimension the usual field of extremals argument involves the addition to I of the integral of a suitable divergence that permits pointwise comparison of integrands. It might be thought that this technique could be used in higher dimensions, where the appropriate divergences are null Lagrangians (*cf.* BALL, CURRIE & OLVER [1981]); our failure, despite some efforts, to implement this idea under realistic hypotheses on the stored-energy function led to the examples in § 3.

On the positive side, we show in § 4 that for nonlinear elasticity any proper local minimum of the energy in $W^{1,1}$ (or certain other energy spaces) lies in a potential well, and thus, according to a well known theorem (*cf.* Proposition 4.3), is Lyapunov stable for any dynamical theory along whose trajectories the total energy (kinetic + potential) is nonincreasing. The applicability of this result, a preliminary version of which was announced in MARSDEN & HUGHES [1983], is limited by the present lack of an appropriate global existence theory for nonlinear elastodynamics or viscoelasticity.

We conclude the introduction with some comments on the literature. The question of whether or not positivity of the second variation is sufficient for stability was posed by KOITER [1945] and studied by several authors, such as SHIELD & GREEN [1963], KNOPS & WILKES [1973], KOITER [1976], BALL, KNOPS & MARSDEN [1978] and KNOPS & PAYNE [1978]. Under growth conditions on the stored-energy function allowing discontinuous equilibrium solutions and cavitation, BALL [1982, p. 609] showed that for incompressible materials the second variation criterion for stability is invalid, and presented plausible evidence that it fails also for compressible materials. The example of § 3 showing that positivity of the second variation is not sufficient for a strong relative minimum suggests, but does not prove, that the criterion fails too for displacements that are continuous and small in C^0 norm, under growth conditions incompatible with cavitation.

If viscoelastic dissipation is added, then for pure displacement boundary conditions POTIER-FERRY [1982] has shown that positivity of the second variation at an equilibrium implies dynamic stability (in fact, exponential asymptotic stability) in $W^{2,p}$, $p > n$. (See also BROWNE [1978].) Because of the different spaces used, this result is not strictly comparable with our Theorem 4.9. In this context we remark that a missing ingredient in elastic stability theory seems to be an understanding of in which metrics the perturbations due to the external world can reasonably be expected to be small; a case could be made for preferring metrics, such as those considered in § 4, that are closely related to the elastic energy.

In this paper we have ignored thermal effects, though a proper justification of energy methods is often made via thermodynamics. In the context of thermoelasticity, the relation between thermodynamics and energy methods has been clarified by ERICKSEN [1966a, b] and GURTIN [1975]. To extend our results to thermoelasticity is probably not difficult, as essentially it involves only addition of the temperature or entropy as an extra dependent variable in the energy.

The role of strong and weak relative minima in nonlinear elasticity has been studied by ERICKSEN [1975, 1980] and JAMES [1979, 1980, 1981] in their work on solid phase transitions and on twinning. In particular, these authors discuss metastable states, which correspond to weak but not strong relative minima of the elastic energy. The stored-energy functions considered by ERICKSEN and JAMES are not strongly elliptic, but the example of § 3 indicates that metastability is also to be expected in the strongly elliptic case. Perhaps metastable states can be understood mathematically as being dynamically unstable but having the compensating dynamical property that most orbits starting in a neighborhood of them with respect to an appropriate metric stay close for a long time.

Finally, we note that it is relatively easy to justify the second variation criterion for stability in finite dimensions or for semilinear equations, such as those occuring in beam and plate theories: *cf.* MARSDEN & HUGHES [1983], and SATTINGER [1969]. It would be interesting to understand the relation between three dimensional nonlinear elasticity, where the second variation criterion appears to fail, and its finite dimensional Galerkin, and infinite dimensional semilinear, models where the criterion is valid. For ideal fluids and plasmas, a convexity technique of ARNOLD [1969] has proved successful for particular models and equilibrium solutions; see HOLM *et al.* [1983] and WAN *et al.* [1983].

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2. Quasiconvexity at the Boundary

We now derive a new necessary condition for a minimum in mixed problems of the calculus of variations. This condition is an extension of MORREY's quasiconvexity condition (MORREY [1952, 1966, § 4.4]) to include boundary points. In Section 3 we use the condition to show that certain equilibrium solutions in nonlinear elasticity are not minima.

Let $\Omega \subset \mathbb{R}^m$ be a bounded strongly Lipschitz[†] domain with boundary $\partial\Omega$ We suppose that $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$ with $\partial\Omega_1, \partial\Omega_2$ disjoint and measurable (with respect to the standard $m-1$ dimensional surface measure denoted dA. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ with the usual topology and let

$$f\colon \overline{\Omega}\times\mathbb{R}^n\times M^{n\times m} \to \overline{\mathbb{R}}, \quad g: \partial\Omega_2\times\mathbb{R}^n \to \overline{\mathbb{R}}$$

be continuous functions, where $M^{n\times m}$ denotes the space of real $n\times m$ metrices with the norm induced by $\mathbb{R}^{nm}$. Let $W^{s,p}(\Omega;\mathbb{R}^n)$ denote the usual Sobolev space of $\mathbb{R}^n$ valued functions u on Ω. For $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ let

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,dx + \int_{\partial\Omega_2} g(x, u(x))\,dA, \tag{2.1}$$

provided this is well defined (*i.e.* provided both integrals exist as elements of $\overline{\mathbb{R}}$ and they are not both infinite with opposite signs). In (2.1) u restricted to $\partial\Omega_2$ is understood in the sense of trace.

Definitions 2.1. (a) *By a standard boundary region with normal $\nu \in \mathbb{R}^m$ we shall mean a bounded strongly Lipschitz domain $D \subset \mathbb{R}^m$ satisfying*

(i) *D is contained in the half-space $K_a^\nu = \{x \in \mathbb{R}^m \mid x\cdot\nu < a\}$ for some $a \in \mathbb{R}^m$, and* (ii) *the $m-1$ dimensional interior ∂D_2 of $\partial D \cap K_a^\nu$ is nonempty. We let $\partial D_1 = \partial D \setminus \partial D_2$.*

(b) *Let $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ be such that $I(u)$ exists and is finite, and let $x_0 \in \overline{\Omega}$. We say that u is a local minimum of I at x_0 in $W^{r,p} \cap C^0$ if there are numbers $\varrho > 0$, $\delta > 0$ such that $I(v)$ exists and $I(v) \geqq I(u)$ whenever $u - v \in C^\infty(\overline{\Omega};\mathbb{R}^n)$, $v(x) = u(x)$ for $|x - x_0| > \varrho$, $x \in \overline{\Omega}$, and $\|v - u\|_{W^{r,p}(\Omega;\mathbb{R}^n)} + \|v - u\|_{C^0} < \delta$.*

Remark. If $r > m/p$ then by the Sobolev inequality, the term $\|v - u\|_{C^0} = \max_{x\in\Omega} |v(x) - u(x)|$ is bounded above by const. $\times \|v - u\|_{W^{r,p}(\Omega;\mathbb{R}^n)}$ and thus can be omitted.

Theorem 2.2. *Let $1 \leqq p < \infty$ and let r be a positive integer satisfying $r < 1 + m/p$. Suppose $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ is a local minimum of I at $x_0 \in \overline{\Omega}$ in $W^{r,p} \cap C^0$ and that u is C^1 in a neighborhood of x_0 in $\overline{\Omega}$.*

(i) *If $x_0 \in \Omega$, then*

$$\int_D f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(y))\,dy \geqq \int_D f(x_0, u(x_0), \nabla u(x_0))\,dy$$
$$= (\text{meas } D) f(x_0, u(x_0), \nabla u(x_0)) \tag{2.2}$$

for any bounded open set $D \subset \mathbb{R}^m$ and all $\phi \in C_0^1(D;\mathbb{R}^n)$ (C^1 functions with compact support in D) such that $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(\cdot))$ is uniformly bounded in D.

(ii) *Let $x_0 \in \partial\Omega_2$, $\partial\Omega_2$ be C^∞ in a neighborhood of x_0, $g(x_0, u(x_0))$ be finite and suppose $g_u(\cdot,\cdot)$ exists and is continuous in a neighborhood of $(x_0, u(x_0))$. Let $\nu = \nu(x_0)$ be the outward normal to $\partial\Omega$ at x_0, and let D be a standard boundary*

[†] *Cf.* MORREY [1966, § 3.4].

region with normal ν. Then

$$\int_D f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(y))\,dy + \int_{\partial D_2} g_u(x_0, u(x_0))\cdot\phi(y)\,dA \geqq \int_D f(x_0, u(x_0), \nabla u(x_0))\,dy \tag{2.3}$$

for all $\phi \in C^1(\bar{D}; \mathbb{R}^n)$ vanishing in a neighborhood of ∂D_1 in $\bar{D}$ and such that $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(\cdot))$ is uniformly bounded in D.

Condition (2.2) is MORREY's quasiconvexity condition, and part (i) is closely related to his result (MORREY [1952 Theorem 2.1, 1966 Theorem 4.4.2]) giving (2.2) as a necessary condition for lower semicontinuity. That (2.2) is a necessary condition for a local minimum was proved in MEYERS [1965 pp. 128–131] (see also BUSEMANN & SHEPHARD [1965]), and our version of this result using $W^{r,p}$ spaces uses a similar method of proof. The main point of Theorem 2.2 is condition (2.3), which is a quasiconvexity condition at the boundary. The most important feature of both parts (i) and (ii) of the theorem is that the hypothesis of a *local* minimum implies that a corresponding auxiliary problem has a trivial *global* minimum. In the case of part (ii), for example, the auxiliary problem is to minimize

$$J(v) = \int_D f(x_0, u(x_0), \nabla v(y))\,dy + \int_{\partial D_2} g_u(x_0, u(x_0))\cdot v(y)\,dA \tag{2.4}$$

subject to the boundary conditions

$$v(x) = \nabla u(x_0)\,x \text{ if } x \in \partial D_1, \tag{2.5}$$

and (2.3) says that in an appropriate sense the minimum is given by $v(x) \equiv \nabla u(x_0)\,x$.

We recall the notions of *weak* and *strong relative minima*, which are classical in the calculus of variations. Let $u \in W^{1,1}(\Omega; \mathbb{R}^n)$ be such that $I(u)$ exists and is finite; we say that u is a weak (respectively strong) relative minimum of I if there exists $\delta > 0$ such that $I(v)$ exists and $I(v) \geqq I(u)$ whenever $v \in W^{1,1}(\Omega; \mathbb{R}^n)$, $v|_{\partial\Omega_1} = u|_{\partial\Omega_1}$ (in the sense of trace), and $\|v - u\|_{W^{1,\infty}(\Omega;\mathbb{R}^n)} < \delta$ (respectively, $\|v - u\|_{L^\infty(\Omega;\mathbb{R}^n)} < \delta$). Note that a strong relative minimum is also a local minimum at x_0 in $W^{r,p} \cap C^0$ for any $x_0 \in \bar{\Omega} \setminus \overline{\partial\Omega_1}$. If $r > 1 + m/p$ then a weak relative minimum of I is a local minimum at x_0 in $W^{r,p} \cap C^0$ for any $x_0 \in \bar{\Omega} \setminus \overline{\partial\Omega_1}$, since $\|v - u\|_{W^{r,p}(\Omega;\mathbb{R}^n)} \geqq \text{const.}\, \|v - u\|_{C^1(\bar{\Omega};\mathbb{R}^n)}$ by the Sobolev inequality. If f, g are C^2, and if $u \in C^1(\bar{\Omega}; \mathbb{R}^n)$ is a solution of the Euler-Lagrange equations for I, then a sufficient condition for u to be a (proper) weak relative minimum of I is that the second variation

$$\delta^2 I(u)\,(w, w) := \int_\Omega \frac{d^2}{d\varepsilon^2} f(x, u(x) + \varepsilon w(x), \nabla u(x) + \varepsilon\,\nabla w(x))\,|_{\varepsilon=0}\,dx + \int_{\partial\Omega_2} \frac{d^2}{d\varepsilon^2} g(x, u(x) + \varepsilon w(x))\,|_{\varepsilon=0}\,dA \tag{2.6}$$

be $W^{1,2}$-positive definite; *i.e.* there is a constant $c_0 > 0$ such that

$$\delta^2 I(u)(w, w) \geqq c_0 \int_\Omega [|w(x)|^2 + |\nabla w(x)|^2]\, dx \tag{2.7}$$

for all $w \in W^{1,2}(\Omega; \mathbb{R}^n)$ with $w|_{\partial\Omega_1} = 0$. This is easily proved by expanding f and g in Taylor series (*cf.* VAN HOVE [1949]). Since (2.7) depends only on the derivatives of f and g at the extremal u, and since the quasiconvexity conditions (2.2), (2.3) depend on the complete function $f(x_0, u(x_0), \cdot)$, it follows that (2.2), (2.3) are not necessary conditions for a weak relative minimum. In particular, the conclusion of Theorem 2.2 cannot hold if $r > 1 + m/p$.

The quasiconvexity conditions (2.2), (2.3) can be viewed as generalizations of the classical necessary condition for a strong relative minimum due to GRAVES [1939], which in turn is a generalization of the Legendre-Hadamard condition (HADAMARD [1902]) and of the Weierstrass condition of the one-dimensional calculus of variations. To see this, suppose for simplicity that f and g are finite and continuous. Then if $x_0 \in \Omega$, an approximation argument (see Remark 1 after the proof) and the method of MORREY [1952, p. 45] shows that if (2.2) holds then $f_{x_0,u}(A) := f(x_0, u(x_0), A)$ satisfies the inequality

$$f_{x_0,u}(\nabla u(x_0)) \leqq \lambda f_{x_0,u}\left(\nabla u(x_0) + \frac{1}{\lambda} c \otimes d\right) + (1-\lambda) f_{x_0,u}\left(\nabla u(x_0) - \frac{1}{1-\lambda} c \otimes d\right) \tag{2.8}$$

for all $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $0 < \lambda < 1$, which is the condition of GRAVES. Next suppose that $x_0 \in \partial\Omega_2$ and that $f_{x_0,u}(A)$ is continuously differentiable with respect to A in a neighborhood of $\nabla u(x_0)$. Then by differentiating (2.3), we get

$$\int_D Df_{x_0,u}(\nabla u(x_0)) \cdot \nabla\phi(y)\, dy + \int_{\partial D_2} g_u(x_0, u(x_0)) \cdot \phi(y)\, dy = 0$$

for all $\phi \in C^1(\bar{D}; \mathbb{R}^n)$ vanishing in a neighborhood of ∂D_1, and therefore

$$Df_{x_0,u}(\nabla u(x_0)) \cdot \nu(x_0) = g_u(x_0, u(x_0)). \tag{2.9}$$

(It is perhaps worth remarking that the natural boundary condition (2.9) is thus valid without any differentiability assumptions on f with respect to x, u.) Approximating x_0 by interior points $x_{0_j} \in \Omega$ and using the continuity of f we see that (2.8) still holds, and thus (replacing c by $\lambda(1-\lambda)c$, dividing by λ and letting $\lambda \to 0$) that for any $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$

$$f_{x_0,u}(\nabla u(x_0) + c \otimes d) \geqq f_{x_0,u}(\nabla u(x_0)) + Df_{x_0,u}(\nabla u(x_0)) \cdot c \otimes d.$$

Setting $d = \nu(x_0)$ we deduce from (2.9) that if

$$\theta(c) := f_{x_0,u}(\nabla u(x_0) + c \otimes \nu(x_0)) - g_u(x_0, u(x_0))\, c,$$

then

$$\theta(c) \geqq \theta(0) \text{ for all } c \in \mathbb{R}^n. \tag{2.10}$$

One of the consequences of the results of § 3 is that (2.3) is strictly stronger than (2.9) and (2.10) together.

It would, of course, be extremely interesting (*e.g.* for verifying our assumptions in § 4) if the necessary conditions (2.2), (2.3) could be shown to be part of a useful set of sufficient conditions for a strong minimum, but we know no result of this type.

Proof of Theorem 2.2. (i) Let $D \subset \mathbb{R}^m$ be a bounded open set, let $x_0 \in \Omega$ and suppose initially that $\phi \in C_0^\infty(D; \mathbb{R}^n)$. Choose $\varepsilon > 0$ such that $x_0 + \varepsilon D \subset \Omega$ and define

$$u_\varepsilon(x) = \begin{cases} u(x) + \varepsilon\phi\left(\dfrac{x - x_0}{\varepsilon}\right) & \text{if } x \in x_0 + \varepsilon D \\ u(x) & \text{otherwise.} \end{cases}$$

Then $u_\varepsilon - u \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ and $u_\varepsilon(x) = u(x)$ for $|x - x_0| > \text{const.}\,\varepsilon$. Clearly $\|u_\varepsilon - u\|_{C^0} \leqq \varepsilon \|\phi\|_{C^0}$ and

$$\begin{aligned} \|u_\varepsilon - u\|_{W^{r,p}(\Omega;\mathbb{R}^n)} &= \left(\sum_{|\beta| \leqq r} \int_\Omega \left|D^\beta \varepsilon\phi\left(\frac{x - x_0}{\varepsilon}\right)\right|^p dx\right)^{1/p} \\ &= \left(\sum_{|\beta| \leqq r} \varepsilon^{(1-|\beta|)p+m} \int_D |D^\beta\phi(y)|^p\, dy\right)^{1/p} \\ &\leqq \text{const.}\, \varepsilon^{1+(m/p)-r} \|\phi\|_{W^{r,p}(\Omega;\mathbb{R}^n)}. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, from the hypothesis that u is a local minimum we conclude that

$$\begin{aligned} 0 \leqq I(u_\varepsilon) - I(u) &= \int_{x_0+\varepsilon D} [f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) - f(x, u(x), \nabla u(x))]\, dx \\ &= \varepsilon^m \left[\int_D f(x_0 + \varepsilon y, u(x_0 + \varepsilon y) + \varepsilon\phi(y), \nabla u(x_0 + \varepsilon y) + \nabla\phi(y))\, dy \right. \\ &\qquad \left. - \int_D f(x_0 + \varepsilon y, u(x_0 + \varepsilon y), \nabla u(x_0 + \varepsilon y))\, dy\right]. \end{aligned}$$

Since u is C^1 near x_0, f is continuous, and $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(\cdot))$ is uniformly bounded, both integrands are bounded as $\varepsilon \to 0$. Dividing by ε^m and letting $\varepsilon \to 0$ we obtain (2.2) by the bounded convergence theorem.

Finally, if $\phi \in C_0^1(D; \mathbb{R}^n)$ there exists a sequence $\phi_k \in C_0^\infty(D; \mathbb{R}^n)$ such that $\phi_k \to \phi$ in C^1 as $k \to \infty$. Then $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi_k(\cdot))$ is uniformly bounded for k sufficiently large and so (2.2) holds for ϕ_k. Passing to the limit $k \to \infty$ we obtain (2.2) for ϕ.

(ii) We can assume that $x_0 = 0$, $\nu(x_0) = e_m = (0, 0, \dots, 0, 1)$ and that the standard boundary region D has the corresponding half-space $K = \{x = (x^1, \dots, x^m) \in \mathbb{R}^m \mid x^m < 0\}$. Let N be a neighborhood of zero in $\mathbb{R}^m$ such that

$$\partial\Omega_2 \cap N = \{x \in N \mid x^m = h(x')\}$$

and

$$\Omega \cap N = \{x \in N \mid x^m < h(x')\},$$

where $x' = (x^1, \dots, x^{m-1})$, $h \in C^\infty(\mathbb{R}^{m-1}; \mathbb{R})$ and $h(0) = 0$, $\nabla h(0) = 0$.

First, let $\phi \in C^\infty(\bar{D}, \mathbb{R}^n)$ vanish in a neighborhood of ∂D_1 and be such that $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi(\cdot))$ is uniformly bounded in D. In the calculations that follow we suppose that $\varepsilon > 0$ is chosen sufficiently small. For $x \in \bar{\Omega}$, define

$$u_\varepsilon(x) = \begin{cases} u(x) + \varepsilon\phi\left(\dfrac{x - h(x')\,e_m}{\varepsilon}\right) & \text{if } x \in N,\quad x - h(x')\,e_m \in \varepsilon\,\bar{D} \\ u(x) & \text{otherwise.} \end{cases}$$

Extend ϕ to $\tilde{\phi} \in C^\infty(\bar{K}; \mathbb{R}^n)$ by setting $\tilde{\phi}$ to zero in $\bar{K} \setminus \bar{D}$. Let $y_\varepsilon : \mathbb{R}^m \to \mathbb{R}^m$ be given by $y_\varepsilon(x) = \dfrac{1}{\varepsilon}(x - h(x')\,e_m)$. Clearly y_ε is C^∞ with the C^∞ inverse $x_\varepsilon(y) = \varepsilon y + h(\varepsilon y')\,e_m$, and $\nabla y_\varepsilon(x) = \dfrac{1}{\varepsilon}(1 - e_m \otimes \nabla h(x'))$, $\nabla x_\varepsilon(y) = \varepsilon(1 + e_m \otimes \nabla h(\varepsilon y'))$. Since $u_\varepsilon - u = \varepsilon\tilde{\phi} \circ y_\varepsilon$ in $\bar{\Omega} \cap N$ we have $u_\varepsilon - u \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$. Also, given $\varrho > 0$ and $\delta > 0$ we have $u_\varepsilon(x) = u(x)$ for $|x| > \varrho$, $x \in \bar{\Omega}$ and

$$\|u_\varepsilon - u\|_{C^0} \leqq \varepsilon\,\|\phi\|_{C^0} < \tfrac{1}{2}\,\delta.$$

Note that $x_\varepsilon(\bar{D}) \subset \bar{\Omega} \cap N$ is the portion of $\bar{\Omega} \cap N$ on which ϕ modifies $u(x)$ in the formula for $u_\varepsilon(x)$. Thus,

$$\begin{aligned}
\|u_\varepsilon - u\|_{W^{r,p}(\Omega;\mathbb{R}^n)} &= \left(\sum_{|\beta| \leqq r} \int_{x_\varepsilon(D)} \varepsilon^p\,|D^\beta(\tilde{\phi} \circ y_\varepsilon)(x)|^p\,dx\right)^{1/p} \\
&\leqq \text{const.}\left(\sum_{|\beta| \leqq r} \int_{x_\varepsilon(D)} \varepsilon^{p(1-|\beta|)}|(D^\beta\tilde{\phi})(y_\varepsilon(x))|^p\,dx\right)^{1/p} \\
&= \text{const.}\left(\sum_{|\beta| \leqq r} \int_D \varepsilon^{p(1-|\beta|)+m}|D^\beta\phi(y)|^p \det(1 + e_m \otimes \nabla h(\varepsilon y'))\,dy\right)^{1/p} \\
&\leq \text{const.}\,\varepsilon^{1+\frac{m}{p}-r} < \tfrac{1}{2}\delta.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 \leqq \varepsilon^{-m}(I(u_\varepsilon) - I(u)) &= \varepsilon^{-m}\int_{x_\varepsilon(D)} [f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) - f(x, u(x), \nabla u(x))]\,dx \\
&\quad + \varepsilon^{-m}\int_{x_\varepsilon(\partial D_2)} [g(x, u_\varepsilon(x)) - g(x, u(x))]\,dA \\
&= \int_D [f(x_\varepsilon(y), u(x_\varepsilon(y)) + \varepsilon\phi(y), \nabla_u(x_\varepsilon(y)) + \nabla\phi(y)\,(1 - e_m \otimes \nabla h(\varepsilon y'))) \\
&\quad - f(x_\varepsilon(y), u(x_\varepsilon(y)), \nabla u(x_\varepsilon(y)))]\det(1 + e_m \otimes \nabla h(\varepsilon y'))\,dy \\
&\quad + \int_{\partial D_2} \frac{g(x_\varepsilon(y), u(x_\varepsilon(y)) + \varepsilon\phi(y)) - g(x_\varepsilon(y), u(x_\varepsilon(y)))}{\varepsilon}(1 + |\nabla h(\varepsilon y')|^2)^{1/2}\,dA.
\end{aligned}$$

By our hypotheses of finiteness, both integrands are bounded independently of ε. Letting $\varepsilon \to 0$ and applying the bounded convergence theorem, we therefore obtain

$$0 \leqq \int_D f(0, u(0), \nabla u(0) + \nabla\phi(y))\,dy - \int_D f(0, u(0), \nabla u(0))\,dy$$

$$+ \int_{\partial D_2} g_u(0, u(0)) \cdot \phi(y)\,dA,$$

which is inequality (2.3).

If ϕ is only C^1, then we approximate ϕ in $C^1(\bar{D}; \mathbb{R}^n)$ by functions $\phi_k \in C^\infty(\bar{D}; \mathbb{R}^n)$ and pass to the limit as $k \to \infty$ using the bounded convergence theorem. □

Remarks 1. We first examine conditions under which (2.2) (resp. (2.3)) is valid when the functions ϕ are required only to be Lipschitz with $\phi|_{\partial D} = 0$ (respectively $\phi|_{\partial D_1} = 0$). This can be proved (a) if f is continuous and finite (by passing to the limit in (2.2), (2.3) using a suitable approximation of ϕ by C^1 functions) and (b) if $r = 1$, if in Definition 2.1(b) we change $v - u \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ to read $v - u \in W^{1,\infty}(\bar{\Omega}; \mathbb{R}^n)$, and if we add the hypothesis that $f(x_0, u(x_0), \nabla u(x_0))$ be finite. (The proof of (b) is the same as that of the theorem, and requires only that $\partial\Omega_2$ be C^1 near x_0.) The theorem is formulated to allow f to take on infinite values because such integrands occur in elasticity and in other subjects giving rise to constrained variational problems. Care then has to be taken when approximating a Lipschitz function ϕ by smooth ones ϕ_k lest values of y occur at which $f(x_0, u(x_0), \nabla u(x_0) + \nabla\phi_k(y))$ is infinite; an instructive example is given in BALL [1981, Remark 4, p. 324]. (The remark after Definition 3.1 in BALL [1977a] is incorrect without further hypotheses through lack of appreciation of this point.) Other variants of the theorem may be proved involving modifications to definition 2.1(b); for example, if $v - u \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ is changed to read $v - u \in W^{r,p}(\bar{\Omega}; \mathbb{R}^n) \cap C^0(\bar{\Omega}; \mathbb{R}^n)$, then part (ii) of the theorem is valid if $\partial\Omega_2$ is C^r near x_0. The theorem may also be proved for fractional r using interpolation inequalities.

2. Suppose that $x_1 \in \partial\Omega_2$ and that $\partial\Omega_2$ is not smooth at x_1 but has a conical singularity (*e.g.*, x_1 could be the vertex of a cube). Then we obtain a modified version of part (ii) of the theorem in which the standard boundary region D is replaced by a domain D' having the cone as $\partial D_2'$. Suppose, for example, that $g \equiv 0$, $f = f(\nabla u)$ with $f(AQ) = f(A)$ for all $Q \in SO(m)$; this is the case in homogeneous isotropic elasticity. Then the condition (2.3) does not depend on ν, and we can regard (2.3) as restricting the possible values of ∇u that can arise at a boundary point in a sufficiently regular minimizer u. If $\partial D_2'$ is smooth at some points then we can apply the original theorem to the auxiliary problem (2.4), (2.5) and deduce that (2.3) for D' implies (2.3) for D. Thus, the condition of quasiconvexity at the boundary restricts the values of ∇u that can arise at a singular point x_1 more than those at a regular boundary point.

3. A version of the theorem holds for the case when $f(x, \cdot, \cdot)$ and $\nabla u(x)$ may jump near $x_0 \in \Omega$ across a smooth surface S passing through x_0 and having normal $\nu = \nu(x_0)$ there. Denote the limits of f and ∇u as $x \to x_0$ from above

and below S by $f^{\pm}(x_0, \cdot, \cdot), \nabla u^{\pm}(x_0)$. The corresponding quasiconvexity condition then has the form

$$\int_{D_+} f^+(x_0, u(x_0), \nabla u^+(x_0) + \nabla\phi(y))\, dy$$
$$+ \int_{D_-} f^-(x_0, u(x_0), \nabla u^-(x_0) + \nabla\phi(y))\, dy$$
$$\geqq \int_{D_+} f^+(x_0, u(x_0), \nabla u^+(x_0))\, dy + \int_{D_-} f^-(x_0, u(x_0), \nabla u^-(x_0))\, dy \tag{2.11}$$

for all $\phi \in C_0^1(D(\mathbb{R}^n)$, where $D \subset \mathbb{R}^m$ is a bounded open set with $D_+ = \{y \in D \mid y \cdot v > a\}$, $D_- = \{y \in D \mid y \cdot v < a\}$. For the case when f is continuous (so that $f^+ = f^-$) but ∇u jumps, see GURTIN [1983].

3. Second Variations and Minimizers in Nonlinear Elasticity

In this section we study the following question: in which spaces is it true that for nonlinear elasticity, positivity of the second variation at an equilibrium solution u implies that u is a local minimum of the energy? We consider an elastic body occupying in a reference configuration the bounded strongly Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$. We suppose that $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$ with $\partial\Omega_1, \partial\Omega_2$ disjoint and measurable with respect to $n-1$ dimensional surface measure, and that $\partial\Omega_1$ has positive $n-1$ dimensional measure. The configurations of the body are described by mappings $u: \overline{\Omega} \to \mathbb{R}^n$. We suppose that the body has a stored-energy function

$$W: \overline{\Omega} \times M^{n\times n} \to \overline{\mathbb{R}}.$$

For definiteness we suppose that W is bounded below, continuous, satisfies $W(x, A) = +\infty$ whenever $x \in \overline{\Omega}$ and $\det A \leqq 0$, and that $W \in C^2(\overline{\Omega} \times M_+^{n\times n})$, where $M_+^{n\times n} = \{A \in M^{n\times n} \mid \det A > 0\}$. We consider the mixed displacement dead-load traction boundary conditions

$$u(x) = u_0(x) \qquad \text{a.e. } x \in \partial\Omega_1, \tag{3.1a}$$

$$\frac{\partial W}{\partial A}(x, \nabla u(x)) \cdot v(x) = t_R(x) \qquad \text{a.e. } x \in \partial\Omega_2, \tag{3.1b}$$

where u_0 and t_R are given smooth functions and $v(x)$ denotes the outward normal to $\partial\Omega$ at x. The corresponding total energy is given by

$$I(u) = \int_\Omega [W(x, \nabla u(x)) + \Psi(x, u(x))]\, dx - \int_{\partial\Omega_2} u(x) \cdot t_R(x)\, dA, \tag{3.2}$$

where $\Psi \in C^2(\overline{\Omega} \times \mathbb{R}^n)$ is the body-force potential. Formally, any weak relative minimum u of I subject to (3.1a) satisfies the equilibrium equations

$$\frac{\partial}{\partial x^\alpha} \frac{\partial W}{\partial A_\alpha^i}(x, \nabla u(x)) = \frac{\partial \Psi(x, u(x))}{\partial u^i}, \qquad x \in \Omega, \quad i = 1, \ldots, n \tag{3.3}$$

and the natural boundary condition (3.1 b). (In (3.3) and below repeated suffices are summed from 1 to n.) Note that (3.2) has the form (2.1) with $m = n$ and appropriate f, g. We can now phrase our question more precisely:

Question. *Let* $u \in C^2(\overline{\Omega}; \mathbb{R}^n)$ *satisfy* (3.1) *and* (3.3) *with* $\det \nabla u(x) > 0$ *for all* $x \in \overline{\Omega}$, *and suppose that the second variation*

$$\delta^2 I(u)(w, w) := \int_\Omega \left[\frac{\partial^2 W}{\partial A^i_\alpha \, \partial A^j_\beta}(x, \nabla u(x))\, w^i_{,\alpha} w^j_{,\beta} + \frac{\partial^2 \Psi}{\partial u^i \, \partial u^j}(x, u(x))\, w^i w^j \right] dx$$

s $W^{1,2}$*-positive definite*: i.e., *there exists a constant* $c_0 > 0$ *such that*

$$\delta^2 I(u)(w, w) \geqq c_0 \int_\Omega [|w(x)|^2 + |\nabla w(x)|^2]\, dx \tag{3.4}$$

for all $w \in W^{1,2}(\overline{\Omega}; \mathbb{R}^n)$ *with* $w|_{\partial\Omega_1} = 0$. *Then in what spaces is u a local minimum of I?*

As we have remarked in § 2, such an equilibrium solution u is a proper weak relative minimum of I, *i.e.* for some $\varepsilon > 0$ we have $I(v) > I(u)$ whenever $0 < \|v - u\|_{W^{1,\infty}(\Omega; \mathbb{R}^n)} \leqq \varepsilon$ and $v|_{\partial\Omega_1} = u|_{\partial\Omega_1}$. In one dimension, however, useful additional hypotheses are known guaranteeing that u is a *strong* relative minimum. In the following theorem we consider the case when $\Omega = (0, 1)$, $\partial\Omega_1 = \{0\}$, $\partial\Omega_2 = \{1\}$. Then $W: [0, 1] \times \mathbb{R} \to \overline{\mathbb{R}}$, $W(x, p) = \infty$ for all $x \in \overline{\Omega}$, $p \leqq 0$,

$$I(u) = \int_0^1 [W(x, u'(x)) + \Psi(x, u(x))]\, dx - t_R u(1), \tag{3.5}$$

where t_R is a constant, and (3.4) takes the form

$$\int_0^1 [W_{pp}(x, u'(x))\, w'(x)^2 + \Psi_{uu}(x, u(x))\, w(x)^2]\, dx \geqq c_0 \int_0^1 [w(x)^2 + w'(x)^2]\, dx \tag{3.6}$$

for all $w \in W_0^{1,2}(0, 1)$ with $w(0) = 0$.

Theorem 3.1. *Let* $u \in C^2([0, 1])$ *satisfy* $u'(x) > 0$ *for all* $x \in [0, 1]$, *be a solution of the equilibrium equation*

$$\frac{d}{dx} W_p(x, u'(x)) = \Psi_u(x, u(x)), \quad x \in [0, 1], \tag{3.7}$$

and satisfy the boundary conditions

$$u(0) = u_0, \quad W_p(1, u'(1)) = t_R, \tag{3.8}$$

where u_0, t_R *are constants. Suppose that* (3.6) *holds, that*

$$W_{pp}(x, u'(x)) > 0 \quad \text{for all } x \in [0, 1], \tag{3.9}$$

(*the strengthened Legendre condition*), *and that there exists* $\delta > 0$ *such that*

$$W(x, p) - W(x, q) - (p - q)\, W_p(x, q) \geqq 0 \tag{3.10}$$

for all $p > 0$, $x \in [0, 1]$ *and* q *with* $|q - u'(x)| \leqq \delta$ *(the strengthened Weierstrass condition). Then* u *is a proper strong relative minimum of* I, *i.e., for some* $\varepsilon > 0$, $I(v) > I(u)$ *whenever* $v \in W^{1,1}(0, 1)$, $v(0) = u_0$ *and* $0 < \|v - u\|_{C^0([0,1])} \leqq \varepsilon$.

Remark. The theorem is also valid for the pure displacement problem, *i.e.*, if we replace the boundary condition at $x = 1$ by $u(1) = u_1$ for some $u_1 > u_0$ and make the obvious modifications to (3.5), (3.6) *etc.* Indeed the theorem then reduces to one proved in numerous books *e.g.* BOLZA [1904, pp. 94–102] and HESTENES [1966, Chapter 3, § 6]. The case of mixed boundary conditions is harder to find explicitly in the literature, but the method of proof, which we give for the reader's convenience, is standard.

Proof of Theorem 3.1. We show that u can be embedded in an appropriate field of extremals satisfying the second of the boundary conditions (3.8). Thus for $|\alpha|$ sufficiently small, consider the initial-value problem set by the equation

$$\frac{d}{dx} W_p(x, z'(x)) = \Psi_u(x, z(x)) \tag{3.11}$$

and initial conditions

$$z(1) = u(1) + \alpha, \qquad z'(1) = u'(1). \tag{3.12}$$

On account of (3.9) and standard results on ordinary differential equations, (see HARTMAN [1964]) this problem has, for $|\alpha|$ sufficiently small, a unique solution $z = z(\cdot, \alpha) \in C^2([0, 1])$ with $z'(x, \alpha) > 0$ for all $x \in [0, 1]$ and the derivatives $\frac{\partial z}{\partial \alpha}, \frac{\partial z'}{\partial \alpha}$ and $\frac{\partial z''}{\partial \alpha}$ exist and are continuous in (x, α). Of course $z(x, 0) = u(x)$ for all $x \in [0, 1]$. To show that the extremals $z(\cdot, \alpha)$ simply cover a neighborhood of the graph of u it suffices to prove that the function

$$\theta(x) = \left.\frac{\partial z(x, \alpha)}{\partial \alpha}\right|_{\alpha=0}$$

is strictly positive in $[0, 1]$. Suppose otherwise; since $\theta(1) = 1$ by (3.12), it follows that $\theta(x_0) = 0$ for some $x_0 \in [0, 1)$. Differentiating (3.11) we see that θ satisfies the Jacobi equation

$$\frac{d}{dx}(a(x)\, \theta'(x)) = b(x)\, \theta(x), \qquad x \in [0, 1] \tag{3.13}$$

where $a(x) = W_{pp}(x, u'(x))$ and $b(x) = \Psi_{uu}(x, u(x))$. Now let

$$w(x) = \begin{cases} 0 & 0 \leqq x \leqq x_0 \\ \theta(x) & x_0 \leqq x \leqq 1. \end{cases}$$

Using (3.13) we obtain

$$\delta^2 I(u)(w, w) = \int_{x_0}^{1} \frac{d}{dx}(a(x)\,\theta'(x)\,\theta(x))\,dx = 0,$$

contradicting (3.6). We have thus shown that there exists $\varepsilon > 0$ such that if $x \in [0, 1]$ and $|y - u(x)| \leqq \varepsilon$ then there is a unique $\alpha = \alpha(x, y)$ with $y = z(x, \alpha)$. Let $p(x, y) = z'(x, \alpha)$ be the associated slope function. Clearly we may assume that ε is small enough so that also $|p(x, y) - u'(x)| \leqq \delta$ whenever $x \in [0, 1]$ and $|y - u(x)| \leqq \varepsilon$. Let

$$L(x, y, p) = \Psi(x, y) + W(x, p(x, y)) + (p - p(x, y))\,W_p(x, p(x, y)),$$

and

$$I^*(y) = \int_0^1 L(x, y(x), y'(x))\,dx.$$

As is well known and easily checked, the Euler-Lagrange equation for I^*, namely $\frac{d}{dx} L_p = L_y$, is satisfied identically for all $y \in W^{1,1}(0, 1)$ with $\|y - u\|_{C^0([0,1])} \leqq \varepsilon$. Let $v \in W^{1,1}(0, 1)$, $v(0) = u_0$, $0 < \|v - u\|_{C^0([0,1])} \leqq \varepsilon$ and let $0 \leqq t \leqq 1$. Then

$$\begin{aligned}
\frac{d}{dt} I^*(u + t(v - u)) &= \int_0^1 [L_y(v - u) + L_p(v' - u')]\,dx \\
&= \int_0^1 \frac{d}{dx}[L_p(v - u)]\,dx \\
&= [W_p(x, p(x, u + t(v - u)))\,(v - u)]_0^1 \\
&= t_R(v(1) - u(1)),
\end{aligned}$$

where we have used (3.8), (3.12). Therefore

$$I^*(v) = I^*(u) + t_R(v(1) - u(1)) = I(u) + t_R v(1),$$

and hence

$$\begin{aligned}
I(v) - I(u) = \int_0^1 [&W(x, v'(x)) - W(x, p(x, v(x))) \\
&- (v'(x) - p(x, v(x)))\,W_p(x, p(x, v(x)))]\,dx.
\end{aligned}$$

The result follows by observing that (3.9), (3.10) imply that (3.10) holds with strict inequality when $p \neq q$ (*cf.* HESTENES [1966, Chapter 3, Lemma 6.1]). □

An important case when the hypotheses (3.9) and (3.10) of Theorem 3.1 are satisfied is when $W_{pp}(x, p) > 0$ for all $x \in [0, 1]$ and $p > 0$, so that in particular $W(x, \cdot)$ is strictly convex. If $n > 1$, the stored-energy function $W(x, \cdot)$ should not be assumed convex (see *e.g.* BALL [1977a, b]); rather, the natural analogues of the condition $W_{pp}(x, p) > 0$ are the conditions of strong ellipticity, polyconvexity and quasiconvexity. We now show that under these latter conditions, (3.4) does *not* imply that u is a strong relative minimum of I, or even a local minimum in $W^{r,p} \cap C^0$ for $r < \frac{n}{p} + 1$.

Let $n > 1$, $\Psi \equiv 0$, $t_R \equiv 0$, and let $W(x, A) = W(A)$ be given by

$$W(A) = \Phi(v_1, \dots, v_n) = v_1^\alpha + \dots + v_n^\alpha + h(v_1 \cdots v_n) \tag{3.14}$$

(an isotropic "Hadamard material"), where $1 < \alpha < n$, where $v_1, \dots, v_n$ are the principal stretches (eigenvalues of $(A^TA)^{\frac{1}{2}}$), and where $h: \mathbb{R} \to \overline{\mathbb{R}}$ is convex, continuous, bounded below, $h(\delta) = \infty$ for $\delta \leqq 0$, and h is C^∞ for $\delta > 0$. This stored-energy function is strongly elliptic, strictly polyconvex and quasiconvex (BALL [1977a, b, 1983]), but can possess arbitrarily many natural states (BALL [1982, p. 592]).

Lemma 3.2. *If* $h'(1) = -\alpha$, *then* $u(x) = x$ *is a natural state and in particular is a solution of the Euler-Lagrange equations* (3.3) *and the natural boundary condition* (3.1 b).

Proof. The first Piola-Kirchhoff stress tensor at $A = \operatorname{diag}(v_1, \dots, v_n)$, $v_i > 0$, is given by

$$\frac{\partial W}{\partial A} = \operatorname{diag}(\Phi_1, \dots, \Phi_n),$$

where $\Phi_i := \frac{\partial}{\partial v_i} \Phi(v_1, \dots, v_n)$. But $h'(1) = -\alpha$ is exactly the condition that this vanishes at $v_1 = 1, \dots, v_n = 1$. $\square$

Lemma 3.3. *Let* $h'(1) = -\alpha$. *The second variation at* $u(x) = x$ *of*

$$I(u) = \int_\Omega W(\nabla u(x))\, dx$$

is

$$\delta^2 I(id)(w, w) = \int_\Omega D^2 W(1)(\nabla w, \nabla w)\, dx, \tag{3.15}$$

where

$$D^2 W(1)(G, G) = \sum_{i,j=1}^n \Phi_{ij}(1, \dots, 1)\, G_{ii} G_{jj}, \qquad \Phi_{ij} = \frac{\partial^2 \Phi}{\partial v_i\, \partial v_j},$$

and

$$\Phi_{ij}(1, \dots, 1) = \begin{cases} \alpha(\alpha - 1) + h''(1) & \text{if } i = j \\ -\alpha + h''(1) & \text{if } i \neq j \end{cases}$$

Proof. The formula for $D^2 W(1)$ follows by a straightforward argument using the chain rule and the fact that $\Phi_i(1, \dots, 1) = 0$; *cf.* BALL [1984]. $\square$

Lemma 3.4. *If* $h'(1) = -\alpha$, $h''(1) > \alpha$ *then there are positive constants* c_0 *and* d_0 *such that*

$$\delta^2 I(id)(w, w) \geqq d_0 \int_\Omega |\nabla w|^2 \, dx \geqq c_0 \int_\Omega [|w(x)|^2 + |\nabla w(x)|^2] \, dx,$$

for all $w \in W^{1,2}(\Omega; \mathbb{R}^n)$ *with* $w|_{\partial\Omega_1} = 0$.

Proof. This follows directly from Lemma 3.3 and a version of the Poincaré inequality (MORREY [1966, Theorem 3.6.4]). $\square$

From now on we assume that $h'(1) = -\alpha$, $h''(1) > \alpha$ and that $h'(\delta) \leqq -\beta < 0$ for $2 \leqq \delta \leqq k$, where $k > 2$ and $0 < \beta < \alpha$. It is easy to see that given any such k, β there exist convex functions k satisfying these and our previous hypotheses. By Lemma 3.4 the second variation at $u(x) = x$ is positive.

Theorem 3.5. *Let* $\beta k^{1-\alpha/n}$ *be sufficiently large. Then if* $x_0 \in \partial\Omega_2$ *and* $\partial\Omega_2$ *is* C^∞ *near* x_0, $u(x) = x$ *is not a local minimum of* I *at* x_0 *in* $W^{r,p} \cap C^0$ *for* $r < 1 + \dfrac{n}{p}$.

Proof. We construct a standard boundary region D and a mapping ϕ such that the necessary condition (2.3) of quasiconvexity at the boundary is violated. Let $x_0 \in \partial\Omega_2$ and suppose $\partial\Omega_2$ is C^∞ near x_0. Since W is isotropic we may orient the coordinate system so that $\nu = \nu(x_0) = (0, 0, \ldots, 1) = e_n$. Let D be the interior of the right circular cone with base on the plane $x^n = 1$ and lying in the half-space $\{x^n \leqq 1\}$. The base ∂D_2 is the disc in $\{x^n = 1\}$ satisfying

$$|x'| \leqq \eta, \quad x' = (x^1, \ldots, x^{n-1}),$$

where $\eta > 0$ is fixed arbitrarily, and the vertex is the origin. The sides of the cone comprise ∂D_1. For $x \in \bar{D}$ let $R = |x|$ and $\varrho = |x'|/x^n$, $x \neq 0$. Note that $D = \{x \in \mathbb{R}^n \mid 0 < x^n < 1 \text{ and } \varrho/R < \eta\}$.

To use (2.3) we need to construct a ϕ that vanishes in a neighborhood of ∂D_1; our ϕ will be C^∞. To this end, construct $\psi \in C^\infty(\mathbb{R}; \mathbb{R})$ such that $\psi' \geqq 0$, $0 \leqq \psi(t) \leqq 1$ and

$$\psi(t) = \begin{cases} 0 & \text{for } t \leqq \frac{1}{3} \\ 1 & \text{for } t \geqq \frac{2}{3} \end{cases}$$

Let $\zeta(\varrho, R) = 1 + (\lambda - 1)\, \psi\left(1 - \dfrac{\varrho}{\eta}\right) \psi(R)$, where $\lambda > 1$ is specified below, let $\tilde{u}(x) = \zeta(\varrho, R)\, x$, and set

$$\phi(x) = \tilde{u}(x) - x,$$

so that $\tilde{u}(x) = u(x) + \phi(x)$. We will show that (2.3) is violated for appropriately chosen values of β, k and λ.

Clearly $\phi \in C^\infty(\bar{D}; \mathbb{R}^n)$ and ϕ is zero in a neighborhood of ∂D_1. We compute

$$\nabla \tilde{u} = \zeta 1 + x \otimes (\zeta_\varrho \nabla \varrho + \zeta_R \nabla R),$$

where

$$\nabla \varrho = \frac{1}{x^n}\left(\frac{x'}{|x'|}, -\frac{|x'|}{x^n}\right) \quad \text{and} \quad \nabla R = \frac{x}{R}.$$

Since $\nabla \tilde{u}$ is $\zeta 1$ plus a rank one perturbation, we get

$$\begin{aligned} \det \nabla \tilde{u} &= \zeta^n + \zeta^{n-1} x \cdot (\zeta_\varrho \nabla \varrho + \zeta_R \nabla R) \\ &= \zeta^n + R\zeta^{n-1}\zeta_R. \end{aligned} \tag{3.16}$$

It follows from $\zeta_R \geqq 0$ that $\det \nabla \tilde{u} \geqq 1$ in $\bar{D}$, and in particular that $W(1 + \nabla\phi(\cdot))$ is uniformly bounded in $\bar{D}$. Let

$$A = \left\{ x \in D \,\middle|\, \frac{\varrho}{\eta} \leqq \tfrac{1}{3}, R \geqq \tfrac{2}{3} \right\},$$

so that on A, $\zeta(\varrho, R) = \lambda$ (corresponding to a homogeneous radial deformation of magnitude λ). Now

$$\begin{aligned} \Delta J := &\int_D W(\nabla \tilde{u}(y))\, dy - \int_D W(\nabla u(x_0))\, dy = (\text{meas } A)\, [n\lambda^\alpha + h(\lambda^n) - n - h(1)] \\ &+ \int_{D \setminus A} [W(\nabla \tilde{u}) - n - h(1)]\, dy \\ &\leqq (\text{meas } A)\, [n\lambda^\alpha + h(\lambda^n) - h(1)] \\ &+ (\text{meas } D \setminus A)\, [c + d(\lambda - 1)^\alpha + h(\det \nabla \tilde{u}) - h(1)], \end{aligned}$$

where c and d are constants. From (3.16),

$$1 \leqq \det \nabla \tilde{u} \leqq a + b(\lambda - 1)^n,$$

where a, b are constants. Choosing $\lambda = \varepsilon k^{1/n}$, where $\varepsilon > 0$ is sufficiently small and independent of k and β, we get $2 \leqq \lambda^n \leqq k$ and $1 \leqq \det \nabla \tilde{u} \leqq k$, provided k is sufficiently large. Therefore $h(\lambda^n) - h(1) \leqq -\beta(\lambda^n - 1)$ and $h(\det \nabla \tilde{u}) - h(1) \leqq 0$, and so

$$\begin{aligned} \Delta J &\leqq c_1(1 + \lambda^\alpha) - c_2\beta(\lambda^n - 1) \\ &= k^{\alpha/n}[c_1(k^{-\alpha/n} + \varepsilon^\alpha) - c_2\beta(\varepsilon^n k^{1-\alpha/n} - k^{-\alpha/n})], \end{aligned}$$

where $c_1 > 0$, $c_2 > 0$ are constants. This is negative if $\beta k^{1-\alpha/n}$ is sufficiently large, which violates (2.3). $\square$

Remarks 1. The actual deformation that lowers the energy can be viewed as a small protruding dimple on the surface of Ω which is C^0 small but contains large gradients.

2. By adding to the stored-energy function a suitable nonnegative $\omega(A)$ that vanishes on the compact subset $1 + \nabla\phi(\overline{D})$ of $M_+^{n\times n}$, where ϕ is as in the proof, we can arrange that W satisfy a variety of growth conditions as $\det A \to 0$ and $|A| \to \infty$; *e.g.* we can arrange that $W(A) > \text{const.} + |A|^p + (\det A)^{-s}$, $A \in M_+^{n\times n}$, for any $p > 1$, $s > 0$.

3. By considering a composite of two materials with stored energy functions W and εW, $\varepsilon > 0$ sufficiently small, joined together along a smooth surface in Ω (*cf.* BALL [1977b, p. 198]), one can arrange a similar violation of (2.11). This provides an example for the pure displacement problem.

4. The example works just as well for the pure traction problem where, of course, positive definiteness of the second variation has to be taken modulo infinitesimal rotations and rigid displacements.

5. The phenomenon in our construction is consistent with a lack of differentiability of the nonlinear Euler-Lagrange operator and the failure of I to be C^2 on $W^{r,p}$, $r < 1 + n/p$, even on deformations with $\det A$ bounded away from zero. In strong function spaces ($r > 1 + n/p$), Taylor's theorem shows that positivity of the second variation and I being C^2 implies a solution of the Euler-Lagrange equations is a (weak) local minimum. (See BALL, KNOPS & MARSDEN [1978], MARTINI [1979] and VALENT [1981].) In the latter case it is furthermore known that a Morse lemma holds, which provides a local representation in which I is quadratic (see GOLUBITSKY & MARSDEN [1983] and BUCHNER, MARSDEN & SCHECTER [1983] and references therein).

6. The above remarks suggest it might be hopeless to obtain any stability results in $W^{r,p}$, $r < 1 + n/p$. Indeed, using the methods of differential calculus, this may be so. Nevertheless, we will show in § 4 that if we *assume* we are at a minimum, then one can prove, under reasonable assumptions, that there *is* a potential barrier (compare KOITER [1981, p. 20]).

4. Potential wells and elastic stability

In this section we consider the mixed boundary value problem of nonlinear elasticity treated in § 3. As is well known, in order to prove that a given equilibrium state u is dynamically stable with respect to an appropriate topology by means of Lyapunov type arguments, it is not sufficient merely to know that u is a local minimum of the energy; rather, it is necessary that u lie in a *potential well*. A result of BALL, KNOPS & MARSDEN [1978] shows that in topologies stronger than $W^{1,\infty}$ this cannot happen. However we give below conditions guaranteeing that a local minimum in a naturally defined energy space *does* lie in a potential well, and is thus dynamically stable. Theorem 3.5 shows that if $n > 1$, positivity of the second variation is not sufficient for u to be a local minimum in this energy space even under favorable constitutive hypotheses, and so our hypothesis of a local minimum must be verified by some other means, *e.g.* by an as yet undiscovered set of sufficient conditions, or by an existence theorem of the calculus of variations.

We begin by recalling the notion of a potential well and its relevance to stability. Our development here is fairly standard; see, for example, KNOPS &

PAYNE [1978] and MARSDEN & HUGHES [1983, § 6.6]. Let (X, d) be a metric space and let Z be a set. Let $I: X \to \mathbb{R}$ and $K: Z \to [0, \infty)$ be given functions. A *trajectory* ϕ is a map $t \mapsto (y(t), v(t))$ from an interval $[0, t_{\max})$ to $X \times Z$, where $t_{\max} = t_{\max}(\phi) \in (0, \infty]$, such that $d(y(t), a)$ is a continuous function of t on $[0, t_{\max})$ for any $a \in X$, and such that $I(y(t)) + K(v(t))$ is a nonincreasing function of t on $[0, t_{\max})$. Let $\mathscr{S}$ be a set of trajectories.

Remark. In our intended interpretation X is the space of configurations, Z the velocity space, I the (potential) energy, K the kinetic energy, $\mathscr{S}$ the set of maximally defined solutions to the dynamical equations.

Definition 4.1. (a) *$u \in X$ is a proper local minimum of I if there exists $\varepsilon > 0$ such that $I(\tilde{u}) > I(u)$ whenever $0 < d(\tilde{u}, u) \leqq \varepsilon$.*

(b) *$u \in X$ lies in a potential well if for all $\varepsilon > 0$ sufficiently small there exists $\gamma(\varepsilon) > 0$ such that*

$$I(\tilde{u}) - I(u) \geqq \gamma(\varepsilon) \text{ whenever } d(\tilde{u}, u) = \varepsilon.$$

(A less precise way of saying this is

$$\inf_{d(\tilde{u},u)=\varepsilon} I(\tilde{u}) > I(u),$$

the point being that the set $\{\tilde{u} \in X \mid d(\tilde{u}, u) = \varepsilon\}$ may be empty.)

(c) *$u \in X$ lies in a uniform potential well if for all $\varepsilon > 0$ sufficiently small there exists $\gamma(\varepsilon) > 0$, with $\gamma(\varepsilon)$ a nondecreasing function of ε, such that*

$$I(\tilde{u}) - I(u) \geqq \gamma(\varepsilon) \text{ whenever } d(\tilde{u}, u) = \varepsilon.$$

Proposition 4.3. (*Lyapunov stability*). *Let $u \in X$ lie in a potential well. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi = (y, v) \in \mathscr{S}$ with*

$$d(y(0), u) < \delta \quad \text{and} \quad I(y(0)) + K(v(0)) < I(u) + \delta,$$

then $d(y(t), u) < \varepsilon$ for all $t \in [0, t_{\max})$.

Proof. Given $\varepsilon > 0$, let $\delta_1 > 0$ be sufficiently small with, in particular, $\delta_1 < \varepsilon$ and set $\delta = \min(\delta_1, \gamma(\delta_1))$. Let $d(y(0), u) < \delta$ and $I(y(0)) + K(v(0)) < I(u) + \delta$. If the conclusion of the proposition were false, by the continuity of $d(y(t), u)$ there would exist $t_1 \in (0, t_{\max})$ with $d(y(t_1), u) = \delta_1$. But then

$$\begin{aligned}
\gamma(\delta_1) + I(u) &\leqq I(y(t_1)) \text{ (by the definition of a potential well)} \\
&\leqq I(y(t_1)) + K(v(t_1)) \text{ (since } K(\cdot) \geqq 0) \\
&\leqq I(y(0)) + K(v(0)) \text{ (since } I + K \text{ is nonincreasing)} \\
&< I(u) + \gamma(\delta_1),
\end{aligned}$$

which is a contradiction. □

Now let ϱ be another metric on X satisfying the following two properties:

(i) ϱ is stronger than d, *i.e.* if $\varrho(a_j, a) \to 0$ then $d(a_j, a) \to 0$,

(ii) if $d(a_j, a) \to 0$ and $I(a_j) \to I(a)$ then $\varrho(a_j, a) \to 0$; equivalently, ϱ is weaker than $\bar{d}$, where $\bar{d}(a, b) := d(a, b) + |I(a) - I(b)|$.

Proposition 4.4. *Suppose I is lower semicontinuous with respect to d,* i.e. $d(a_j, a) \to 0$ *implies* $I(a) \leqq \liminf_{j \to \infty} I(a_j)$. *Then*

(i) *$u \in X$ is a proper local minimum with respect to d if and only if u is a proper local minimum with respect to ϱ, and*

(ii) *$u \in X$ lies in a uniform potential well with respect to d if and only if u lies in a uniform potential well with respect to ϱ.*

Proof. (i) If u is a proper local minimum with respect to d, then clearly it is a proper local minimum with respect to ϱ. Conversely, suppose u is a proper local minimum with respect to ϱ, but not with respect to d. Then there exists a sequence $u_j \neq u$ with $d(u_j, u) \to 0$ but $I(u_j) \leqq I(u)$. Hence

$$I(u) \leqq \liminf_{j \to \infty} I(u_j) \leqq I(u),$$

and so $I(u_j) \to I(u)$. Therefore $\varrho(u_j, u) \to 0$, which contradicts the fact that u is a proper local minimum with respect to ϱ.

(ii) Let u lie in a uniform potential well with respect to d. Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leqq d(\tilde{u}, u) \leqq \varepsilon_0$, then $I(\tilde{u}) \geqq I(u) + \gamma(\varepsilon)$. Since ϱ is stronger than d there exists $\varepsilon_1 > 0$ such that $d(\tilde{u}, u) \leqq \varepsilon_0$ whenever $\varrho(\tilde{u}, u) \leqq \varepsilon_1$. If u does not lie in a uniform potential well with respect to ϱ then by part (i), there exist $\delta > 0$ and a sequence u_j with $\delta \leqq \varrho(u_j, u) \leqq \varepsilon_1$ but $I(u_j) \to I(u)$. Since ϱ is weaker than $\bar{d}$, there exists $\varepsilon > 0$ such that $\varepsilon \leqq d(u_j, u) \leqq \varepsilon_0$, and thus $I(u_j) \geqq I(u) + \gamma(\varepsilon)$, a contradiction.

The converse is proved similarly. $\square$

We will describe our results concerning potential wells in elasticity for the case $n = 3$, though the method works for all dimensions $n \geqq 1$ and indeed for more general problems of the calculus of variations of the type considered in § 2.

Let $\Omega \subset \mathbb{R}^3$, $\partial\Omega_1$ and $\partial\Omega_2$ be as described at the beginning of § 3. We consider the same mixed boundary value problem as before (see (3.1)–(3.3)) but make different hypotheses on the functions W, Ψ, u_0 and t_R. These hypotheses are as follows:

(H1) $W: \bar{\Omega} \times M^{3\times 3} \to \bar{\mathbb{R}}$ is a Carathéodory function, *i.e.* $W(x, \cdot)$ is continuous for almost all $x \in \bar{\Omega}$, and $W(\cdot, A)$ is measurable for all $A \in M^{3\times 3}$.

(H2) $W(x, A) = +\infty$ if $x \in \bar{\Omega}$ and $\det A \leqq 0$.

(H3) W is *strictly polyconvex, i.e.* there exists a function $g : \bar{\Omega} \times E \to \mathbb{R}$, $E = M^{3\times 3} \times M^{3\times 3} \times (0, \infty)$, such that for almost all $x \in \bar{\Omega}$ we have

(i) $g(x, \cdot)$ is strictly convex, and

(ii) $W(x, A) = g(x, A, \operatorname{adj} A, \det A)$ for all $A \in M^{3\times 3}_{+}$.

(H4) $W(x, A) \geqq \sigma(x) + C_0(|A|^p + |\operatorname{adj} A|^q + \Gamma(\det A))$ for almost all $x \in \bar{\Omega}$ and all $A \in M^{3\times 3}_{+}$, where $\sigma \in L^1(\Omega)$, $C_0 > 0$ is constant, $p \geqq 2$, $q \geqq \dfrac{p}{p-1}$, and where $\Gamma : (0, \infty) \to \mathbb{R}^+$ is a convex function satisfying $\dfrac{\Gamma(\delta)}{\delta} \to \infty$ as $\delta \to \infty$.

(H5) $\Psi : \bar{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function satisfying $a(x) - k|y|^\alpha \leqq \Psi(x, y) \leqq b(x) + \theta(y)$ for all $y \in \mathbb{R}^3$ and a.e. $x \in \bar{\Omega}$, where $a, b \in L^1(\Omega)$, $k > 0$, $0 < \alpha < p$, and $\theta : \mathbb{R}^3 \to \mathbb{R}$ is continuous. If $2 \leqq p \leqq 3$ we assume further that $\theta(y) \leqq \text{const.}\, |y|^r$, where $1 \leqq r < \dfrac{3p}{3-p}$ if $\lambda \leqq p < 3$ and $r \geqq 1$ is arbitrary if $p = 3$.

(H6) $u_0 : \partial\Omega_1 \to \mathbb{R}^3$ is measurable.

(H7) $t_R \in L^s(\partial\Omega_2; \mathbb{R}^3)$, with $s \geqq 1$ if $p > 3$, and $s > \dfrac{2p}{3(p-1)}$ if $2 \leqq p \leqq 3$.

Remarks. 1. Examples of stored-energy functions satisfying (H1)–(H4) are given in BALL [1977a, b] and CIARLET & GEYMONAT [1982].

2. An example of a body-force potential satisfying (H5) is

$$\Psi(x, y) = \varrho_R(x)\, g_0 y^3, \qquad y = (y^1, y^2, y^3),$$

where $\varrho_R(\cdot)$ is the density in the reference configuration, and g_0 is the acceleration due to gravity near the earth's surface. Note that this Ψ is not bounded below.

3. If W is independent of x, and probably in general, the term $\Gamma(\det A)$ in (H4) can be omitted without affecting the results below, following the method of BALL & MURAT [1984].

We define the set X of admissible functions by

$$X = \{y \in W^{1,1}(\Omega; \mathbb{R}^3) \mid I[(y) \text{ exists and is finite, } y|_{\partial\Omega_1} = u_0 \text{ a.e.}\}$$

where $y|_{\partial\Omega_1}$ is understood in the sense of trace, and where $I(\cdot)$ is defined in (3.2). We suppose that X is not empty. Note that, by (H2), $\det \nabla y(x) > 0$ a.e. $x \in \bar{\Omega}$ for every $y \in X$. We give X the metric d induced by $W^{1,1}(\Omega; \mathbb{R}^3)$, namely

$$d(y_1, y_2) = \|y_1 - y_2\|_{W^{1,1}(\Omega; \mathbb{R}^3)}. \tag{4.1}$$

Lemma 4.5. *There exist constants* $d_0 > 0$, d_1 *such that*

$$I(y) \geqq d_0 \int_\Omega [|\nabla y|^p + |\operatorname{adj} \nabla y|^q + \Gamma(\det \nabla y) + |y|^p]\, dx + d_1$$

for all $y \in X$.

Proof. By a version of the Poincaré inequality (MORREY [1966, Theorem 3.6.4]) there exists a constant $c_1 > 0$ such that

$$\int_\Omega |\nabla y|^p \, dx + \left(\int_{\partial\Omega_1} |y| \, dA \right)^p \geqq c_1 \int_\Omega |y|^p \, dx \tag{4.2}$$

for all $y \in X$. By (H4), (H5),

$$\int_\Omega [W(x, \nabla y(x)) + \Psi(x, y(x))] \, dx \geqq \tfrac{1}{2} C_0 \int_\Omega [|\nabla y|^p + |\operatorname{adj} \nabla y|^q + \Gamma(\det \nabla y)] \, dx$$
$$+ \tfrac{1}{2} C_0 \int_\Omega |\nabla y|^p \, dx - k \int_\Omega |y|^\alpha \, dx + \text{const.}$$

for all $y \subset X$. Since $y|_{\partial\Omega_1} = u_0$, and since $0 < \alpha < p$ we deduce easily from (4.2) that

$$\int_\Omega [W(x, \nabla y(x)) + \Psi(x, y(x))] \, dx \geqq \tfrac{1}{2} C_0 \int_\Omega [|\nabla y|^p + |\operatorname{adj} \nabla y|^q + \Gamma(\det \nabla y)] \, dx$$
$$+ \tfrac{1}{4} C_0 c_1 \int_\Omega |y|^p \, dx + \text{const.} \tag{4.3}$$

for all $y \in X$. In particular $X \subset W^{1,p}(\Omega; \mathbb{R}^3)$. By trace theory, there exists a constant $c_2 > 0$ such that $\|y\|_{L^{s'}(\partial\Omega_2;\mathbb{R}^3)} \leqq c_2 \|y\|_{W^{1,p}(\Omega;\mathbb{R}^3)}$ for all $y \in X$, where $\dfrac{1}{s} + \dfrac{1}{s'} = 1$. Thus for all $y \in X$ we have for any $\varepsilon > 0$

$$\begin{aligned} - \int_{\partial\Omega_2} t_R(x) \cdot y(x) \, dA &\geqq \|t_R\|_{L^s(\partial\Omega_2;\mathbb{R}^3)} \|y\|_{L^{s'}(\partial\Omega_2;\mathbb{R}^3)} \\ &\geqq -c_2 \|t_R\|_{L^s(\partial\Omega_2;\mathbb{R}^3)} \|y\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \\ &\geqq -\varepsilon \|y\|^p_{W^{1,p}(\Omega;\mathbb{R}^3)} + c_\varepsilon, \end{aligned}$$

where c_ε is a constant. Combining this inequality with (4.3) we get the result. □

Lemma 4.6. *If $y_j \in X$ with $y_j \rightharpoonup y$ in $W^{1,1}(\Omega; \mathbb{R}^3)$ then $y \in X$ and*

$$I(y) \leqq \liminf_{j \to \infty} I(y_j).$$

Proof. Let $y_j \in X$ with $y_j \rightharpoonup y$ in $W^{1,1}(\Omega; \mathbb{R}^3)$. We assume without loss of generality that $I(y_j)$ is bounded. Hence, by Lemma 4.5, $\|y_j\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leqq$ const. and therefore $y_j \rightharpoonup y$ in $W^{1,p}(\Omega; \mathbb{R}^3)$. By standard compact embedding theorems, this implies that $y_j \to y$ in $L^\infty(\Omega; \mathbb{R}^3)$ if $p > 3$ and in $L^{\max(\alpha,r)}(\Omega; \mathbb{R}^3)$ if $2 \leqq p \leqq 3$, and that $y_j \to y$ in $L^{s'}(\partial\Omega_2; \mathbb{R}^3)$. It follows from (H5) and a suitable form of the dominated convergence theorem (such as Lemma 4.8 below) that

$$\int_\Omega \Psi(x, y_j(x)) \, dx \to \int_\Omega \Psi(x, y(x)) \, dx,$$

and that

$$\int_{\partial\Omega_2} t_R(x) \cdot y_j(x) \, dA \to \int_{\partial\Omega_2} t_R(x) \cdot y(x) \, dA,$$

both limits being finite. By Lemma 4.5, the de la Vallée Poussin criterion and the sequential weak continuity of Jacobians (*cf.* BALL [1977a, b]),

$$(\nabla y_j, \operatorname{adj} \nabla y_j, \det \nabla y_j) \rightharpoonup (\nabla y, \operatorname{adj} \nabla y, \det \nabla y)$$

in $L^1(\Omega; M^{3\times 3}\times M^{3\times 3}\times \mathbb{R})$. We claim that $\det \nabla y(x) > 0$ a.e. If not, there would exist a subset $\Omega_1 \subset \Omega$ with $\operatorname{meas} \Omega_1 > 0$ and $\det \nabla y(x) = 0$ for all $x \in \Omega_1$. But since $\det \nabla y_j \geqq 0$ and $\det \nabla y_j \rightharpoonup \det \nabla y$ in $L^1(\Omega_1)$, it follows that for a subsequence y_μ we have $\det \nabla y_\mu \to 0$ a.e. in Ω_1. From (H1), (H2) and (H4) it follows easily that $W(x, \nabla y_\mu(x)) - \sigma(x) \to \infty$ a.e. in Ω_1, and hence $\int_\Omega W(x, \nabla y_\mu(x))\, dx \to \infty$. This contradicts the boundedness of $I(y_j)$ and proves our claim. (This argument is taken from BALL, CURRIE & OLVER [1981, Theorem 6.2].)

Since $g(x, \cdot)$ is convex,

$$\begin{aligned}\int_\Omega W(x, \nabla y(x))\, dx &= \int_\Omega g(x, \nabla y(x), \operatorname{adj} \nabla y(x), \det \nabla y(x))\, dx \\ &\leqq \liminf_{j\to\infty} \int_\Omega g(x, \nabla y_j(x), \operatorname{adj} \nabla y_j(x), \det \nabla y_j(x))\, dx \\ &= \liminf_{j\to\infty} \int_\Omega W(x, \nabla y_j(x))\, dx,\end{aligned}$$

and the result follows. □

Lemma 4.7. *Let* $K \subset \mathbb{R}^M$ *be open and convex, and let* $f: K \to \mathbb{R}$ *be strictly convex. Let* $0 < \theta < 1$, *and suppose that* $a_j, a \in K$ *with*

$$\theta f(a_j) + (1-\theta) f(a) - f(\theta a_j + (1-\theta)\, a) \to 0 \text{ as } j \to \infty.$$

Then $a_j \to a$.

Proof. We first note that $h(b) := \theta f(b) + (1-\theta) f(a) - f(\theta b + (1-\theta)\, a)$ is strictly increasing along any ray starting from a. In fact, if the ray is given by $t \mapsto a + te$, $e \in \mathbb{R}^M$, then this is equivalent to saying that

$$\theta f(a+te) - f(a+\theta te) > \theta f(a+se) - f(a+\theta se) \text{ if } t > s > 0,$$

which follows from the strict convexity of f by writing $a + se$, $a + \theta te$ as convex combinations of $a + te$ and $a + \theta se$.

Since K is open and f convex, f is continuous. Therefore, given $\varepsilon > 0$ sufficiently small for the ball $\{|b-a| \leqq \varepsilon\}$ to be a subset of K,

$$\inf_{|b-a|=\varepsilon} h(b) > 0.$$

Since h is increasing along rays, this implies that

$$\inf_{\substack{|b-a|\geqq\varepsilon \\ b\in K}} h(b) > 0,$$

which gives the result. □

Lemma 4.8. *Let* f_j, f, h_j, h, H_j, $H \in L^1(\Omega)$ *with* $h_j \geqq f_j \geqq H_j$ *for all* j. *Suppose that* $f_j \to f$, $h_j \to h$ *and* $H_j \to H$ *a.e., and that* $\int_\Omega h_j(x)\,dx \to \int_\Omega h(x)\,dx$, $\int_\Omega H_j(x)\,dx \to \int_\Omega H(x)\,dx$ *as* $j \to \infty$. *Then*

$$\int_\Omega f_j(x)\,dx \to \int_\Omega f(x)\,dx.$$

Proof. This is an immediate consequence of the Vitali convergence theorem. Alternatively, the result follows by applying Fatou's lemma to the sequences $h_j - f_j$ and $f_j - H_j$. $\square$

We can now prove our main stability result.

Theorem 4.9. *Let* (H1)–(H7) *hold, and let* $u \in X$ *be a proper local minimum of* I *with respect to the metric* d *(see* (4.1)*). Then* u *lies in a uniform potential well with respect to* d.

Proof. Suppose for contradiction that there exist ε_0, ε with $0 < \varepsilon < \varepsilon_0$ such that

$$I(\tilde{u}) > I(u) \quad \text{for all} \quad \tilde{u} \in X \quad \text{with} \quad 0 < d(\tilde{u}, u) \leqq \varepsilon_0,$$

but

$$\inf_{\substack{\tilde{u} \in X \\ \varepsilon \leqq d(\tilde{u},u) \leqq \varepsilon_0}} I(\tilde{u}) = I(u).$$

Then there exists a sequence $\{u_j\} \subset X$ such that $\varepsilon \leqq d(u_j, u) \leqq \varepsilon_0$ and $I(u_j) \to I(u)$. By Lemma 4.5, u_j is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$ and thus there exists a subsequence, again denoted u_j, such that $u_j \rightharpoonup v$ in $W^{1,p}(\Omega;\mathbb{R}^3)$. By Lemma 4.6, $v \in X$ and $I(v) \leqq I(u)$. But by the sequential weak lower semicontinuity of $\|\cdot\|_{W^{1,1}(\Omega;\mathbb{R}^3)}$ it follows that $d(v, u) \leqq \varepsilon_0$, and hence $v = u$. Further, by the argument in Lemma 4.6,

$$\int_\Omega \Psi(x, u_j(x))\,dx \to \int_\Omega \Psi(x, u(x))\,dx,$$

and therefore, since $I(u_j) \to I(u)$,

$$\int_\Omega W(x, \nabla u_j(x))\,dx \to \int_\Omega W(x, \nabla u(x))\,dx.$$

Fix some $\theta \in (0, 1)$, let $a_j(x) = (\nabla u_j(x), \operatorname{adj} \nabla u_j(x), \det \nabla u_j(x))$, $a(x) = (\nabla u(x), \operatorname{adj} \nabla u(x), \det \nabla u(x))$, and define

$$h_j(x) = \theta g(x, a_j(x)) + (1-\theta)\, g(x, a(x)) - g(x, \theta a_j(x) + (1-\theta)\, a(x)).$$

As in Lemma 4.6, $a_j \rightharpoonup a$ in $L^1(\Omega; M^{3\times 3} \times M^{3\times 3} \times \mathbb{R})$. Since g is convex,

$$\int_\Omega g(x, a(x))\,dx \leqq \liminf_{j\to\infty} \int_\Omega g(x, \theta a_j(x) + (1-\theta)\, a(x))\,dx.$$

Hence

$$\begin{aligned}
0 &\leqq \limsup_{j\to\infty} \int_\Omega h_j(x)\,dx \\
&= \limsup_{j\to\infty} \left[\theta \int_\Omega W(x, \nabla u_j(x))\,dx + (1-\theta)\int_\Omega W(x, \nabla u(x))\,dx \right.\\
&\quad \left. - \int_\Omega g(x, \theta a_j(x) + (1-\theta)\,a(x))\,dx\right] \\
&= \int_\Omega W(x, \nabla u(x))\,dx - \liminf_{j\to\infty} \int_\Omega g(x, \theta a_j(x) + (1-\theta)\,a(x))\,dx \\
&\leqq 0.
\end{aligned}$$

Thus, for a further subsequence, again denoted u_j, we have $h_j \to 0$ a.e. in Ω. By Lemma 4.7, $\nabla u_j(x) \to \nabla u(x)$ a.e. in Ω. By (H4),

$$|\nabla u_j(x) - \nabla u(x)| \leqq CW(x, \nabla u_j(x)) + \varrho(x),$$

where $\varrho \in L^1(\Omega)$. Applying Lemma 4.8, we deduce that

$$\int_\Omega |\nabla u_j(x) - \nabla u(x)|\,dx \to 0.$$

Therefore $u_j \to u$ strongly in $W^{1,1}(\Omega; \mathbb{R}^3)$, contradicting $d(u_j, u) \geqq \varepsilon$. $\square$

An important special case to which Theorem 4.9 applies is when u is a global minimizer of I in X. That I attains a global minimum on X follows from Lemmas 4.5, 4.6 (*cf.* BALL, CURRIE & OLVER [1981, Theorem 6.2]). Note that the proof of Theorem 4.9 shows that minimizing sequences for I converge strongly. Other versions of Theorem 4.9 can be obtained by combining Theorem 4.9 and Proposition 4.4. For example, suppose $\Gamma(\delta) \geqq \delta^s$, $s \geqq 1$, and define

$$\begin{aligned}
\varrho(y_1, y_2) = \|y_1 - y_2\|_{W^{1,p}(\Omega;\mathbb{R}^3)} &+ \|\operatorname{adj} \nabla y_1 - \operatorname{adj} \nabla y_2\|_{L^q(\Omega; M^{3\times 3})} \\
&+ \|\det \nabla y_1 - \det \nabla y_2\|_{L^s(\Omega)}.
\end{aligned}$$

Then Theorem 4.9 holds with d replaced by ϱ. This follows from the following proposition.

Proposition 4.10. *The metric ϱ on X is strong than d and weaker than $\bar{d}$, where $\bar{d}(y_1, y_2) = d(y_1, y_2) + |I(y_1) - I(y_2)|$.*

Proof. Clearly ϱ is stronger than d. Conversely, suppose that $y_j \to y$ in $W^{1,1}(\Omega; \mathbb{R}^3)$ and $I(y_j) \to I(y)$, where $y_j, y \in X$, but that $\varrho(y_j, y) \geqq \varepsilon > 0$ for all j. By Lemma 4.5 and the argument in Lemma 4.6, $y_j \to y$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, hence $y_j \to y$ in $L^p(\Omega; \mathbb{R}^3)$, and

$$\int_\Omega W(x, \nabla y_j(x))\,dx \to \int_\Omega W(x, \nabla y(x))\,dx.$$

Let

$$f_j(x) = |\nabla y_j(x) - \nabla y(x)|^p + |\operatorname{adj} \nabla y_j(x) - \operatorname{adj} \nabla y(x)|^q + |\det \nabla y_j(x) - \det \nabla y(x)|^s.$$

Extracting a subsequence, again denoted by y_j, we can suppose than $\nabla y_j(x) \to \nabla y(x)$ a.e. in Ω, and hence $f_j(x) \to 0$ a.e. in Ω. But by (H4),

$$0 \leqq f_j(x) \leqq \text{const.}\ (W(x, \nabla y_j(x)) + W(x, \nabla y(x)) - 2\sigma(x)).$$

Applying Lemma 4.8, we deduce that $\int_\Omega f_j(x)\,dx \to 0$. Thus $\varrho(y_j, y) \to 0$, which is a contradiction. □

In order to apply Theorem 4.9 and Proposition 4.3 to deduce stability of a proper local minimizer u it is necessary that solutions exist to the appropriate dynamical equations. For the equations of elastodynamics in $n > 1$ space dimensions there is no global existence theory presently available; however, for pure displacement boundary conditions there is short-time existence in $W^{2,s}(\Omega; \mathbb{R}^n)$, $s > \frac{1}{2}n + 1$, due to HUGHES, KATO & MARSDEN [1977]. For $n = 1$, a global existence theorem has been established by DIPERNA [1983]. For nonlinear viscoelasticity of rate type in $n > 1$ space dimensions, with pure displacement boundary conditions, some information concerning existence is contained in the stability result of POTIER-FERRY [1982].

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