One-dimensional Variational Problems whose Minimizers do not Satisfy the Euler-Lagrange Equation

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Dedicated to Walter Noll

§ 1. Introduction

In this paper we consider the problem of minimizing

$$I(u) = \int_a^b f(x, u(x), u'(x))\, dx \tag{1.1}$$

in thc set $\mathscr{A}$ of absolutely continuous functions $u: [a, b] \to \mathbb{R}$ satisfying the end conditions

$$u(a) = \alpha, \qquad u(b) = \beta, \tag{1.2}$$

where α and β are given constants. In (1.1), $[a, b]$ is a finite interval, $'$ denotes $\frac{d}{dx}$, and the integrand $f = f(x, u, p)$ is assumed to be smooth, nonnegative and to satisfy the *regularity condition*

$$f_{pp} > 0. \tag{1.3}$$

The significance of the regularity condition (1.3) is that, as is well known, it ensures the existence of at least one absolute minimizer for I in $\mathscr{A}$, provided f also satisfies an appropriate growth condition with respect to p. Further, it implies that any Lipschitz solution u of the integrated form

$$f_p = \int_a^x f_u\, dy + \text{const.} \qquad \text{a.e. } x \in [a, b] \tag{IEL}$$

of the Euler-Lagrange equation is in fact smooth in $[a, b]$. Notwithstanding these facts and the status of (IEL) as a classical necessary condition for a minimizer, we present a number of examples in which I attains a minimum at some $u \in \mathscr{A}$ but u is *not* smooth and does *not* satisfy (IEL).

To see where the classical argument leading to (IEL) may break down, recall that the argument relies on calculating the derivative

$$\frac{d}{dt} I(u + t\varphi)|_{t=0}$$

$$= \lim_{t \to 0} \int_a^b \frac{f(x, u(x) + t\varphi(x), u'(x) + t\varphi'(x)) - f(x, u(x), u'(x))}{t} dx \tag{1.4}$$

for φ a smooth function satisfying $\varphi(a) = \varphi(b) = 0$, and concluding that since $I(u + t\varphi)$ is minimized at $t = 0$ the derivative is zero; *viz.*

$$\int_a^b [f_u \varphi + f_p \varphi'] \, dx = 0. \tag{1.5}$$

If $u \in W^{1,\infty}(a, b)$ this argument is clearly valid, since by the mean value theorem the integrand on the right-hand side of (1.4) is uniformly bounded independently of small t and consequently one may pass to the limit $t \to 0$ using the bounded convergence theorem. However, if it is known only that the minimizer u belongs to $\mathscr{A}$, the only readily available piece of information which may aid passing to the limit in (1.4) is that $I(u) < \infty$. Consequently one is typically forced into making assumptions on the derivatives of f, these assumptions being unnecessary for the existence of a minimizer, so as to pass to the limit. More alarmingly, a difficulty may arise at an earlier stage in the argument to due the possibility that near some $u \in \mathscr{A}$ with $I(u) < \infty$ there may be functions $v \in \mathscr{A}$ with $I(v) = \infty$; in fact, in two of our examples we are able to show that for a large class of $\varphi \in C_0^\infty(a, b)$ the minimizers u are such that $I(u + t\varphi) = \infty$ for *all* $t \neq 0$.

The possibility that a minimizer u of I in $\mathscr{A}$ might be singular was envisaged by TONELLI, who proved a striking and little known partial regularity theorem to the effect that u is a smooth solution of the Euler-Lagrange equation on the complement of a closed subset E of $[a, b]$ of measure zero, and that $|u'(x)| = \infty$ for all $x \in E$. He then gave a number of criteria ensuring that "the set E does not exist" and thus that $u \in C^\infty([a, b])$. Remarks in TONELLI [32] suggest that he did not know of any examples in which E is nonempty, and we believe that our examples are the first of this type. A precise statement and proof of a version of the partial regularity theorem is given in § 2, where we also gather together a number of results concerning the existence of minimizers and first order necessary conditions. In this connection we mention that we are unaware of any integral form of a first order necessary condition that is satisfied by every minimizer u in the absence of additional hypotheses on f.

Our first example, given in § 3, is that of minimizing

$$I(u) = \int_0^1 [(x^2 - u^3)^2 (u')^{14} + \varepsilon(u')^2] \, dx \tag{1.6}$$

subject to

$$u(0) = 0, \quad u(1) = k, \tag{1.7}$$

where $\varepsilon > 0$, $k > 0$. (As we point out at the end of § 5, the power 14 is the lowest for which singular minimizers of (1.6) exist.) Note that if $0 < k \leqq 1$ and $\varepsilon = 0$ then the minimum of I is attained by $u(x) = \min(x^{\frac{2}{3}}, k)$; the results summarized below show that the singularity of u at $x = 0$ is not destroyed provided $\varepsilon > 0$ is sufficiently small. The integrand in (1.6) has a scale-invariance property which allows one to transform the Euler-Lagrange equation to an autonomous ordinary differential equation in the plane, and this makes it possible to give a very detailed and complete description of the absolute minimizers u of (1.6), (1.7) for all ε and k. Some of the main conclusions are the following (see especially Theorem 3.12). There exist numbers $\varepsilon_0 = .002474\ldots$, $\varepsilon^* = .00173\ldots$ such that (a) for $0 < \varepsilon < \varepsilon_0$ there exist two elementary solutions $\bar{k}_1(\varepsilon)\, x^{\frac{2}{3}}$, $\bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$ of the Euler-Lagrange equation on (0, 1]; (b) if $0 < \varepsilon < \varepsilon^*$ and k is sufficiently large I attains an absolute minimum at a unique function u which satisfies $u(x) \sim \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$ as $x \to 0+$, $u \in C^\infty((0, 1])$ and $f_u(\cdot, u(\cdot), u'(\cdot)) \notin L^1(0, 1)$, so that (IEL) does not hold: if $k = \bar{k}_2(\varepsilon)$ then $u(x) = \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$; (c) if $0 < \varepsilon < \varepsilon^*$ and k is sufficiently large (for example, $k \geqq 1$) there is no smooth solution of the Euler-Lagrange equation on [0, 1] satisfying the end conditions (1.7), and hence I does not attain a minimum among Lipschitz functions; (d) if $\varepsilon > \varepsilon^*$ then there is exactly one u that minimizes I and it is the unique smooth solution of the Euler-Lagrange equation on [0, 1] satisfying (1.7). The detailed structure of the phase portrait that leads to these conclusions would have been extremely difficult to determine without the aid of computer plots, though these do not form part of the proofs. Since the singular minimizers are smooth for $x > 0$ their "Tonelli set" E consists in the single endpoint $\{0\}$ and they *do* satisfy the Euler-Lagrange equation in the sense of distributions, *i.e.* in its "weak" form.

In § 4 we consider the case when $f = f(u, p)$ does not depend on x. We first construct an $f \in C^\infty(\mathbb{R}^2)$ satisfying (in addition to (1.3))

$$|p| \leqq f(u, p) \leqq \text{const.}\,(1 + p^2), \qquad (u, p) \in \mathbb{R}^2, \tag{1.8}$$

and

$$\frac{f(u, p)}{|p|} \to \infty \quad \text{as} \quad |p| \to \infty \quad \text{for each} \quad u \neq 0, \tag{1.9}$$

such that

$$I(u) = \int_{-1}^{1} f(u, u')\, dx \tag{1.10}$$

attains an absolute minimum subject to the end conditions

$$u(-1) = k_1, \qquad u(1) = k_2 \tag{1.11}$$

(for suitable k_1, k_2), at a unique function u_0 whose Tonelli set E is a single interior point $x_0 \in (-1, 1)$ and which satisfies

$$f_u(u_0, u_0') \notin L^1_{\text{loc}}(-1, 1); \tag{1.12}$$

hence (IEL) does not hold, with integration in the Lebesgue sense, and neither is the weak form of the Euler-Lagrange equation satisfied. Next we construct, for *any* preassigned closed set $E \subset [-1, 1]$ of measure zero, a similar function

$f = f^E$ satisfying (1.8) such that for suitable k_1, k_2, I attains an absolute minimum subject to (1.11) at a unique function u_0 whose Tonelli set is precisely E. Again (1.12) holds. These two examples demonstrate the optimality of Corollary 2.12 and the Tonelli partial regularity theorem (Theorem 2.7), respectively. Awareness of conditions necessary for the validity of chain rule calculations ([34], [30], [27], [28]) influenced our initial construction of those examples, thoug the proofs presented here avoid this issue.

In § 5 we consider the problem of minimizing

$$I(u) = \int_{-1}^{1} [(x^4 - u^6)^2 |u'|^s + \varepsilon (u')^2] \, dx \tag{1.13}$$

in the set $\mathscr{A}$ of absolutely continuous functions on $[-1, 1]$ (*i.e.* functions in $W^{1,1} = W^{1,1}(-1, 1)$) satisfying the end conditions

$$u(-1) = k_1, \quad u(1) = k_2, \tag{1.14}$$

where $s > 3$ and $\varepsilon > 0$. (We allow s to take nonintegral values, even though the integrand is smooth only if s is an even integer.) We show (Theorem 5.1) that if $s \geqq 27$ then, provided $-1 \leqq k_1 < 0 < k_2 \leqq 1$ and ε is sufficiently small, every minimizer u_0 of I in $\mathscr{A}$ is such that $u_0(x) \sim |x|^{\frac{2}{3}} \operatorname{sign} x$ as $x \to 0$, $u_0 \in C^\infty([-1, 0) \cup (0, 1])$ and $u_0 \in W^{1,p}$ for $1 \leqq p < 3$. It follows that $E = \{0\}$ and that u_0 does not satisfy the Euler-Lagrange equation either in its weak or its integrated form. Furthermore, if $3 \leqq q \leqq \infty$,

$$\inf_{v \in W^{1,q} \cap \mathscr{A}} I(v) > \inf_{v \in \mathscr{A}} I(v) = I(u_0). \tag{1.15}$$

This remarkable fact is known as the *Lavrentiev phenomenon* (*cf.* LAVRENTIEV [22], MANIÀ [25], CESARI [11]), and its occurrence in a *regular* problem has not previously been noted; in the cited references only the case $q = \infty$ is considered. If $s > 27$ then an equally surprising property holds (Theorem 5.5), namely that for any sequence $\{v_m\} \subset W^{1,q} \cap \mathscr{A}$ such that $v_m(x) \to u_0(x)$ for each x in some set containing arbitrarily small positive and negative numbers one has $I(v_m) \to \infty$ as $m \to \infty$. In particular, no minimizing sequence for I in $W^{1,q} \cap \mathscr{A}$ can converge to u_0. Since conventional finite-element methods for minimizing I yield such sequences, it follows that they cannot in general detect singular minimizers. Similarly, if v_η is a minimizer of, for example, an apparently innocuous penalized functional such as

$$I_\eta(u) = \int_{-1}^{1} [(x^4 - u^6)^2 |u'|^s + \varepsilon (u')^2 + \eta |u'|^{3+\gamma}] \, dx \tag{1.16}$$

in $\mathscr{A}$, where $\gamma > 0$, then v_η cannot converge to u_0 as $\eta \to 0+$. Motivated by numerical experiments of BALL & KNOWLES [6] we show also that if $s > 27$, $3 \leqq q \leqq \infty$ and $\varepsilon > 0$, k_1, k_2 are arbitrary then (Theorem 5.8) I attains a minimum in $W^{1,q} \cap \mathscr{A}$ and any such minimizer u_1 is a smooth solution of the Euler-Lagrange equation on $[-1, 1]$. (Note that such "pseudominimizers" do not in general exist for (1.6), (1.7).) The pseudominimizers can be regarded as being "admissible" minimizers of I with respect to various penalty methods such as (1.16). Finally, we show (Theorem 5.9) that for $s < 26$ all minimizers of I in $\mathcal{A}$ are smooth, and that, at least for the corresponding problem posed on (0, 1), singular minimizers not satisfying the Lavrentiev phenomenon may exist for $26 \leqq s < 27$.

In all the examples considered we analyze whether or not the minimizers satisfy the weak or integrated forms of the DuBois-Reymond equation

$$\frac{d}{dx}(f - u'f_p) = f_x. \tag{DBR}$$

The examples in this paper were motivated by attempts to prove that minimizers of the total energy

$$I(u) = \int_\Omega W(x, Du(x))\, dx \tag{1.17}$$

of an elastic body subject to appropriate boundary conditions are weak solutions of the corresponding Euler-Lagrange equations

$$\frac{\partial}{\partial x^\alpha} \frac{\partial W}{\partial A^i_\alpha} = 0, \quad i = 1, \dots, n. \tag{1.18}$$

Here we have assumed that the body occupies the bounded open subset $\Omega \subset \mathbb{R}^n$ in a reference configuration and that there are no external forces. The particle at $x \in \Omega$ in the reference configuration is displaced to $u(x) \in \mathbb{R}^n$, and $Du(x)$ denotes the gradient of u at x. One of the complications of the problem, which is still open, is that the stored-energy function $W(x, A)$ of the material is defined only for $\det A > 0$ and is typically assumed to satisfy $W(x, A) \to \infty$ as $\det A \to 0+$. The existence of minimizers in appropriate subsets of the Sobolev space $W^{1,1} = W^{1,1}(\Omega; R^n)$ is established in BALL [2] for a class of realistic functions W, and conditions guaranteeing that these minimizers satisfy other first order necessary conditions are announced in BALL [5]. It is known (BALL [3], BALL & MURAT [8]) that even when W satisfies favorable constitutive hypotheses such as strong ellipticity, I may not attain its minimum within the class of smooth functions, and in fact that if $n \leqq q \leqq \infty$ then

$$\inf_{\substack{v \text{ smooth} \\ v|_{\partial\Omega} = \bar{u}|_{\partial\Omega}}} I(v) = \inf_{\substack{v \in W^{1,q} \\ v|_{\partial\Omega} = \bar{u}|_{\partial\Omega}}} I(v) > \inf_{\substack{v \in W^{1,1} \\ v|_{\partial\Omega} = \bar{u}|_{\partial\Omega}}} I(v) \tag{1.19}$$

can occur for appropriate boundary displacements $\bar{u}$. Of course (1.19) is a higher-dimensional version of the Lavrentiev phenomenon. The deformations responsible here for LAVRENTIEV'S gap are those for which cavitation occurs, that is, holes form in the body. Cavitation cannot occur if W satisfies the growth condition

$$W(x, A) \geqq \text{const.}\, |A|^p \quad \text{for } \det A > 0, \tag{1.20}$$

for some $p > n$, by the Sobolev embedding theorem (nor, in fact, if $p = n$). An intriguing possibilty raised by our one-dimensional examples is that singular minimizers and the Lavrentiev phenomenon may occur for (1.17) even when (1.20) holds, and that the singularities of Du might be connected with the initiation of fracture. More work needs to be done to decide whether this can happen under realistic hypotheses on W. Similar considerations may be relevant for other nonlinear elliptic systems (see, for example, GIAQUINTA [17] and several articles in BALL [4]).

In view of the potential physical significance of singular minimizers and the Lavrentiev phenomenon in elasticity and perhaps other fields, our general view is that they should be studied rather than exorcised. However, it is of course also interesting to determine conditions under which this behavior cannot occur. We mention in particular the theorem of ANGELL [1] concerning a sufficient condition for nonoccurrence of the Lavrentiev phenomenon, which generalizes earlier results of TONELLI [32], CINQUINI [12] and MANIÀ [25]. ANGELL'S theorem is presented in CESARI [11], who gives a wealth of related results. We also refer the reader to the result of GIAQUINTA & GIUSTI [18] (see also GIAQUINTA [17, p. 267]) giving conditions on f for minimizers of (1.1) to be smooth in the case when f satisfies $\lambda p^2 \leqq f(x, u, p) \leqq \Lambda p^2$ for all x, u, p, where $\lambda > 0$.

Many of the results in this paper were announced in BALL & MIZEL [7] and BALL [5].

We conclude the introduction with a remark concerning an abuse of notation in which we indulge. If, for example, we write $u \in W^{1,q}(0, \delta) \cap W^{1,2}(0, 1)$, where $0 < \delta < 1$, we mean that $u \in W^{1,2}(0, 1)$ and that u restricted to $(0, \delta)$ belongs to $W^{1,q}(0, \delta)$.

§ 2. Review of positive results concerning minimizers and first order necessary conditions

We consider integrals of the form

$$I(u) = \int_a^b f(x, u(x), u'(x))\, dx,$$

where $-\infty < a < b < \infty$, and where the competing functions $u: [a, b] \to \mathbb{R}$. We discuss the problem of minimizing I in the set

$$\mathscr{A} = \{u \in W^{1,1}(a, b): u(a) = \alpha, u(b) = \beta\},$$

where α, β are given real constants. By an appropriate choice of representatives, $W^{1,1}(a, b)$ can be identified with the set of absolutely continuous functions $u: [a, b] \to \mathbb{R}$ and we shall henceforth assume this to have been done. To avoid getting enmeshed in technical hypotheses that are unnecessary for our purposes, we make the standing assumptions that $f = f(x, u, p)$ is C^3 in its arguments and bounded below; the reader interested in optimal regularity hypotheses or the case $u: [a, b] \to \mathbb{R}^n$ can consult the cited references. Our aim in this section is to summarize for later reference the available information concerning the existence of minimizers and first order necessary conditions satisfied by them.

Theorem 2.1. (TONELLI'S existence theorem). *Suppose $f_{pp} \geqq 0$ and $f(x, u, p) \geqq \varphi(|p|)$, $x \in [a, b]$, $(u, p) \in \mathbb{R}^2$, where φ is bounded below and satisfies $\frac{\varphi(t)}{t} \to \infty$ as $t \to \infty$. Then I attains an absolute minimum on $\mathscr{A}$.*

For the proof see, for example, CESARI [11, pp. 112, 372], HESTENES [20], or EKELAND & TÉMAM [16]. The original proof (for the case $\varphi(t) = t^p$, $p > 1$) can be found in TONELLI [31 II, p. 282], and in TONELLI [33] for the general case. TONELLI [31 II, pp. 287, 296] and [33] also proved that minimizers exist when f has superlinear growth in p except in the neighborhood of finitely many points or absolutely continuous curves; significant extensions of some of these results, together with a more complete bibliography, are described in MCSHANE [24], and CESARI [11, Chapter 12]. These results imply, for example, that the functionals I considered in § 4 attain a minimum, but are not needed in our development there since the minimizer is constructed explicitly.

Definitions 2.2. A function $u \in \mathscr{A}$ is a *weak relative minimizer* of I if $I(u) < \infty$ and there exists $\delta > 0$ such that $I(u) \leqq I(v)$ for all $v \in \mathscr{A}$ with $\operatorname{ess\,sup}_{x \in [a,b]} [|u(x) - v(x)| + |u'(x) - v'(x)|] \leqq \delta$. We say that $u \in \mathscr{A}$ is a *strong relative minimizer* of I if there exists $\delta > 0$ such that $I(u) \leqq I(v)$ for all $v \in \mathscr{A}$ with $\max_{x \in [a,b]} |u(x) - v(x)| \leqq \delta$.

We consider the following forms of classical first order necessary conditions for a minimum. The *Euler-Lagrange equation* is

$$\frac{d}{dx} f_p = f_u. \tag{EL}$$

A function $u \in \mathscr{A}$ satisfies the *weak form of the Euler-Lagrange equation* if $f_u, f_p \in L^1_{\mathrm{loc}}(a, b)$ and (EL) holds in the sense of distributions, *i.e.*

$$\int_a^b [f_p \varphi' + f_u \varphi] \, dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(a, b). \tag{WEL}$$

A function $u \in \mathscr{A}$ satisfies the *integrated form of the Euler-Lagrange equation* provided $f_u \in L^1(a, b)$ and

$$f_p(x, u(x), u'(x)) = \int_a^x f_u \, dy + \text{const.} \qquad \text{a.e. } x \in [a, b]. \tag{IEL}$$

The *DuBois-Reymond equation* is

$$\frac{d}{dx}(f - u' f_p) = f_x. \tag{DBR}$$

A function $u \in \mathscr{A}$ satisfies the *weak form of the DuBois-Reymond equation* if $f - u' f_p$, $f_x \in L^1_{\mathrm{loc}}(a, b)$ and (DBR) holds in the sense of distributions, *i.e.*

$$\int_a^b [(f - u' f_p) \varphi' + f_x \varphi] \, dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(a, b). \tag{WDBR}$$

A function $u \in \mathscr{A}$ satisfies the *integrated form of the DuBois-Reymond equation* provided $f_x \in L^1(a, b)$ and

$$f(x, u(x), u'(x)) - u'(x) f_p(x, u(x), u'(x)) = \int_a^x f_x \, dy + \text{const.} \qquad \text{a.e. } x \in [a, b]. \tag{IDBR}$$

Of course, if u satisfies (IEL) (respectively (IDBR)) then u satisfies (WEL) (respectively (WDBR)). We will see later that the converse is false in general; what *is* true is that, by the fundamental lemma of the calculus of variations, (WEL) is equivalent to

$$f_p(x, u(x), u'(x)) = \int_c^x f_u \, dy + \text{const.} \qquad \text{a.e. } x \in [a, b],$$

for any $c \in (a, b)$, a similar statement holding for (WDBR).

Theorem 2.3.

(i) *Let* $u \in \mathcal{A}$ *be a weak relative minimizer of* I *and suppose that* $f_u(\cdot, \bar{u}(\cdot), u'(\cdot)) \in L^1(a, b)$ *whenever* $\bar{u} \in L^\infty(a, b)$ *with* $\operatorname*{ess\,sup}_{x \in [a,b]} |u(x) - \bar{u}(x)|$ *sufficiently small. Then* u *satisfies* (IEL).

(ii) *Let* $u \in \mathcal{A}$ *be a strong relative minimizer of* I *and suppose that* $f_x(\bar{x}(\cdot), u(\cdot), u'(\cdot)) \in L^1(a, b)$ *whenever* $\bar{x} \in L^\infty(a, b)$ *with* $\operatorname*{ess\,sup}_{x \in [a,b]} |\bar{x}(x) - x|$ *sufficiently small. Then* u *satisfies* (IDBR).

Proof.

(i) For $\delta > 0$ sufficiently small and $G \subset \mathbb{R}$ closed define

$$\gamma_G(x) = \sup_{t \in [-\delta, \delta] \cap G} |f_u(x, u(x) + t, u'(x))|,$$

$$E(x) = \{t \in [-\delta, \delta] : |f_u(x, u(x) + t, u'(x))| = \gamma_{\mathbb{R}}(x)\}.$$

We consider the set-valued mapping $E: x \mapsto E(x)$. Clearly $E(x)$ is closed for a.e. $x \in [a, b]$. Furthermore, for any closed $G \subset \mathbb{R}$ the set

$$\{x \in [a, b] : E(x) \cap G \text{ nonempty}\} = \{x \in [a, b] : \gamma_G(x) - \gamma_{\mathbb{R}}(x) = 0\}$$

is measurable (since $\gamma_G - \gamma_{\mathbb{R}}$ is a measurable function). By a standard measurable selection theorem (*cf.* CESARI [11, p. 283ff]) there exists a measurable function $(x \mapsto t(x)$ with $t(x) \in E(x)$ a.e. $x \in [a, b]$. Hence $\gamma_{\mathbb{R}}(x) = |f_u(x, u(x) + t(x), u'(x))|$ a.e. $x \in [a, b]$, so that our hypothesis is equivalent to the existence of $\gamma \in L^1(a, b)$ such that

$$|f_u(x, \bar{u}(x), u'(x))| \leqq \gamma(x) \qquad \text{a.e. } x \in [a, b]$$

for all $\bar{u} \in L^\infty(a, b)$ with $\operatorname*{ess\,sup}_{x \in [a,b]} |u(x) - \bar{u}(x)|$ sufficiently small. The result now follows from TONELLI [31] (see also CESARI [11, p. 61ff], HESTENES [20, p. 196ff]).

(ii) This follows in a similar way from TONELLI [31] (see also CESARI [11, p. 61ff]). Alternatively, one can deduce (ii) from (i) by a reduction based on the idea that $\varphi \equiv 0$ is a weak relative minimum of

$$J(\varphi) = \int_a^b f(x, u_\varphi(x), u'_\varphi(x)) \, dx$$

subject to $\varphi(a) = \varphi(b) = 0$, where $u_\varphi(x) \overset{\text{def}}{=} u(z), \; z + \varphi(z) = x.$ □

Corollary 2.4. *Let* $f = f_1(x, u) + f_2(x, p)$. *If* $u \in \mathscr{A}$ *is a weak relative minimizer of* I *then* u *satisfies* (IEL).

Proof. If $\bar{u} \in L^\infty(a, b)$ then $f_u(x, \bar{u}(x), u'(x)) = (f_1)_u(x, \bar{u}(x))$ is uniformly bounded. □

Corollary 2.5. *Let* $f = f_1(x, u) + f_2(u, p)$. *If* $u \in \mathscr{A}$ *is a strong relative minimizer of* I *then* u *satisfies* (IDBR).

Proof. If $\bar{x} \in L^\infty(a, b)$ then $f_x(\bar{x}(x), u(x), u'(x)) = (f_1)_x(\bar{x}(x), u(x))$ is uniformly bounded. □

The above results are notable for the lack of any convexity assumptions on f. The growth assumptions are also considerably weaker than those of corresponding theorems known for multiple integrals. For example, in Theorem 2.3(i) there is no hypothesis on f_p; that the result is true without such a hypothesis is suggested by the fact that f_p is bounded for any solution of (IEL). We are not aware of any counterexamples to Theorem 2.3 if the integrability hypotheses are weakened to read in part (i) $f_u(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$, and in part (ii) $f_x(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$.

We now describe results in which f is assumed convex with respect to p.

Theorem 2.6. *Let* $u \in W^{1,\infty}(a, b)$ (= *Lipschitz continuous functions on* $[a, b]$) *be a weak relative minimizer of* I, *and suppose that* $f_{pp}(x, u(x), p) > 0$ *for all* $x \in [a, b], p \in \mathbb{R}$. *Then* $u \in C^3([a, b])$ *and satisfies* (EL).

Proof. This is standard and can be found in Cesari [11, p. 57ff]. □

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ denote the extended real line with its usual topology. We define $C^1([a, b]; \bar{\mathbb{R}})$ to be the set of continuous functions $u: [a, b] \to \mathbb{R}$ such that for all $x \in [a, b]$

$$u'(x) \overset{\text{def}}{=} \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \tag{2.1}$$

exists as an element of $\bar{\mathbb{R}}$ (with the appropriate one-sided limit being taken if $x = a$ or $x = b$), and such that $u': [a, b] \to \bar{\mathbb{R}}$ is continuous.

Theorem 2.7 (Tonelli's *partial regularity theorem*). *Let* $f_{pp} > 0$. *If* $u \in \mathscr{A}$ *is a strong relative minimizer of* I *then* $u \in C^1([a, b]; \bar{\mathbb{R}})$.

Before proving Theorem 2.7 we note some consequences. Clearly $u'(x)$ as defined in (2.1) coincides almost everywhere with the derivative of u in the sense of distributions. Therefore under the hypotheses of the theorem the *Tonelli set* E defined by

$$E = \{x \in [a, b] : |u'(x)| = \infty\}$$

is a closed set of measure zero. The complement $[a, b] \setminus E$ is a union of disjoint relatively open intervals D_j. By the optimality principle and Theorem 2.6, u

is a C^3 solution of (EL) on each D_j. By Theorem 2.7, $u'(x)$ tends to $+\infty$ or $-\infty$ as x tends to the end-points of every such interval (unless $a \in D_j$ or $b \in D_j$). These consequences of Theorem 2.7 constitute TONELLI'S statement of his theorem (TONELLI [31 II, p. 359]); our formulation includes the extra remark that u' is continuous. The proof we give, like TONELLI'S, uses the local solvability of (EL), but we avoid his construction of auxiliary integrands by applying the field theory of the calculus of variations. Recently, CLARKE & VINTER [13, 14] have presented certain extensions of TONELLI'S theorem to the cases when f is not smooth and $u: [a, b] \to \mathbb{R}^n$. They have also shown [15] that if f is a polynomial then the Tonelli set E is at most countable with finitely many points of accumulation.

Lemma 2.8. *Let $A \subset \mathbb{R}^2$ be bounded, and let $M > 0$, $\delta > 0$. There exists $\varepsilon > 0$ such that if $(x_0, u_0) \in A$, $|\alpha| \leqq M$, $|\beta| \leqq M$, the solution $u(x; \alpha, \beta)$ of* (EL) *satisfying the initial conditions*

$$u(x_0; \alpha, \beta) = u_0 + \alpha, \qquad u'(x_0; \alpha, \beta) = \beta, \tag{2.2}$$

exists for $|x - x_0| \leqq \varepsilon$, is unique, and is such that

(a) *u and u' are C^1 functions of x, α, β in the set*

$$S \stackrel{\text{def}}{=} \{(x, \alpha, \beta): |x - x_0| \leqq \varepsilon, |\alpha| \leqq M, |\beta| \leqq M\},$$

(b)
$$|u'(x; \alpha, \beta) - \beta| < \delta, \tag{2.3}$$

$$\frac{\partial u}{\partial \alpha}(x; \alpha, \beta) > 0, \qquad \operatorname{sign} \frac{\partial u}{\partial \beta}(x; \alpha, \beta) = \operatorname{sign}(x - x_0), \tag{2.4}$$

for all $(x, \alpha, \beta) \in S$, where sign t *takes the values $-1, 0, 1$ for $t < 0$, $t = 0$, $t > 0$, respectively.*

Proof. Because $f_{pp} > 0$, solving (EL) is equivalent to solving the equation

$$u'' = F(x, u, u'),$$

where $F(x, u, p) \stackrel{\text{def}}{=} (f_u - f_{px} - p f_{pu})/f_{pp}$. Our hypotheses imply that $F \in C^1(\mathbb{R}^3)$. The existence, uniqueness and smoothness assertions follow from standard results (see, for example, HARTMAN [19, Chapter 5]). Furthermore, the derivatives appearing in (2.3), (2.4) depend continuously on x_0, u_0. That $\varepsilon > 0$ can be chosen sufficiently small for (b) to hold follows by a simple compactness argument, using the relations

$$u'(x_0; \alpha, \beta) = \beta, \quad \frac{\partial u}{\partial \alpha}(x_0; \alpha, \beta) = 1, \quad \frac{\partial u}{\partial \beta}(x_0; \alpha, \beta) = 0,$$

$$\left(\frac{\partial u}{\partial \beta}\right)'(x_0; \alpha, \beta) = 1. \quad \square$$

Proposition 2.9. (TONELLI [31 II, p. 344ff]). *Let $m > 0$, $\varrho > 0$, $M_1 > 0$. Then there exists $\varepsilon > 0$ such that if $x_0, x_1 \in [a, b]$, $0 < x_1 - x_0 \leqq \varepsilon$, $|u_0| \leqq m$ and $\left|\frac{u_1 - u_0}{x_1 - x_0}\right| \leqq M_1$ there is a unique solution $\tilde{u} \in C^2([x_0, x_1])$ of* (EL) *satisfying*

$\tilde{u}(x_0) = u_0$, $\tilde{u}(x_1) = u_1$ *and* $\max_{x \in [x_0, x_1]} |\tilde{u}(x) - u_0| \leqq \varrho$, *and* $\tilde{u}$ *is the unique absolute minimizer of*

$$\tilde{I}(u) = \int_{x_0}^{x_1} f(x, u(x), u'(x))\, dx$$

over the set

$$\tilde{\mathscr{A}} = \{u \in W^{1,1}(x_0, x_1) : u(x_0) = u_0,\ u(x_1) = u_1,\ \max_{x \in [x_0, x_1]} |\tilde{u}(x) - u_0| \leqq \varrho\}.$$

Proof. Let $\sigma = m + \varrho$, $A = [a, b] \times [-\sigma, \sigma]$, $M > \max(M_1, 2\varrho)$ and let $0 < \delta < M - M_1$. Let $\varepsilon > 0$ be chosen as in Lemma 2.8, and suppose in addition that $3M\varepsilon < \varrho$. Let $x_0, x_1 \in [a, b]$, $0 < x_1 - x_0 \leqq \varepsilon$, $|u_0| \leqq m$ and $\left|\frac{u_1 - u_0}{x_1 - x_0}\right| \leqq M_1$. Note that by integrating (2.3) we have that

$$|u(x; \alpha, \beta) - u_0 - \alpha - \beta(x - x_0)| \leqq \delta(x - x_0), \qquad x \in [x_0, x_1]. \tag{2.5}$$

Therefore

$$u(x_1; 0, M) \geqq u_0 + M_1(x_1 - x_0) + (M - M_1 - \delta)(x_1 - x_0) > u_1,$$

$$u(x_1; 0, -M) \leqq u_0 - M_1(x_1 - x_0) - (M - M_1 - \delta)(x_1 - x_0) < u_1.$$

Since $\frac{\partial u}{\partial \beta}(x_1; 0, \beta) > 0$ for $\beta \in [-M, M]$ there is a unique $\beta_0 \in [-M, M]$ such that $u(x_1; 0, \beta_0) = u_1$. Define $\tilde{u}(x) = u(x; 0, \beta_0)$. Setting $x = x_1$ in (2.5) we obtain

$$|\beta_0| \leqq \delta + M_1. \tag{2.6}$$

Therefore, again by (2.5), for $x \in [x_0, x_1]$

$$|\tilde{u}(x) - u_0| \leqq (\delta + |\beta_0|)(x - x_0)$$
$$\leqq (2\delta + M_1)\varepsilon < \varrho.$$

Now suppose that $v \in C^2([x_0, x_1])$ is also a solution of (EL) satisfying $v(x_0) = u_0$, $v(x_1) = u_1$ and $\max_{x \in [x_0, x_1]} |v(x) - u_0| \leqq \varrho$. Then $v'(\bar{x}) = \frac{u_1 - u_0}{x_1 - x_0}$ for some $\bar{x} \in (x_0, x_1)$ and $(\bar{x}, v(\bar{x})) \in A$, and so applying (2.3) with $(\bar{x}, v(\bar{x}))$ replacing (x_0, u_0) we deduce that

$$\left|v'(x) - \frac{u_1 - u_0}{x_1 - x_0}\right| \leqq \delta \qquad \text{for } x \in [x_0, x_1].$$

In particular,

$$|v'(x_0)| \leqq M_1 + \delta < M.$$

By the uniqueness of β_0 we therefore have that $v'(x_0) = \beta_0$, and thus $v = \tilde{u}$.

To show that $\tilde{u}$ minimizes $\tilde{I}$ in $\tilde{\mathscr{A}}$, we consider the one-parameter family of solutions $\{u(\cdot; \alpha, \beta_0), |\alpha| \leqq M\}$. By (2.5), (2.6) we have

$$u(x; M, \beta_0) - u_0 \geqq M + (\beta_0 - \delta)(x - x_0) \geqq M - (2\delta + M_1)\varepsilon > \varrho$$

and

$$u(x; -M, \beta_0) - u_0 \leqq -M + (\beta_0 + \delta)(x - x_0) \leqq -M + (2\delta + M_1)\varepsilon < -\varrho,$$

for $x \in [x_0, x_1]$. Since $\frac{\partial u}{\partial \alpha}(x; \alpha, \beta_0) > 0$ it follows that $\tilde{u}$ is embedded in a field of extremals that simply covers the region $[x_0, x_1] \times [u_0 - \varrho, u_0 + \varrho]$. Since $f_{pp} > 0$ it follows from Weierstrass's formula (*e.g.* BOLZA [9, p. 91], CESARI [11, p. 72]) that

$$\tilde{I}(u) > \tilde{I}(\tilde{u})$$

for all $u \in \tilde{\mathscr{A}}$, with equality if and only if $u = \tilde{u}$, which concludes the proof. □

Proof of Theorem 2.7. Let $u \in \mathscr{A}$ be a strong relative minimizer of I; thus there exists $\delta_1 > 0$ such that $I(u) \leqq I(v)$ for all $v \in \mathscr{A}$ with $\max_{x \in [a,b]} |u(x) - v(x)| \leqq \delta_1$. Let $\bar{x} \in [a, b]$, and suppose that

$$M(\bar{x}) \overset{\text{def}}{=} \liminf_{\substack{x \to \bar{x} \\ x \neq \bar{x}, x \in [a,b]}} \left| \frac{u(x) - u(\bar{x})}{x - \bar{x}} \right| < \infty. \tag{2.7}$$

Suppose that $\bar{x} \neq b$ and take $\bar{x}_1 > \bar{x}$ with $\bar{x}_1 - \bar{x}$ sufficiently small that $\max_{x \in [\bar{x}, \bar{x}_1]} |u(x) - u(\bar{x})| \leqq \frac{\delta_1}{2}$. Choose $M_1 > M(\bar{x})$. By (2.7) we can apply Proposition 2.9 with $x_0 = \bar{x}$, $u_0 = u(\bar{x})$, $\varrho = \frac{\delta_1}{2}$, $u_1 = u(x_1)$, where $x_1 \in (\bar{x}, \bar{x}_1)$ satisfies

$$x_1 - \bar{x} < \varepsilon, \qquad \left| \frac{u(x_1) - u(\bar{x})}{x_1 - \bar{x}} \right| \leqq M_1.$$

Let $\tilde{u}$ be the corresponding solution of (EL). Let $\hat{u} \in \mathscr{A}$ be defined by $\hat{u}(x) = \tilde{u}(x)$ if $x \in [\bar{x}, x_1]$, $\hat{u}(x) = u(x)$ otherwise. Then $\max_{x \in [a,b]} |\hat{u}(x) - u(x)| \leqq \frac{\delta_1}{2} + \frac{\delta_2}{2} = \delta_1$ and so $I(\hat{u}) - I(u) = \tilde{I}(\tilde{u}) - \tilde{I}(u) \geqq 0$. Since $\tilde{u}$ is the unique minimizer of $\tilde{I}$ in $\tilde{\mathscr{A}}$ it follows that $\tilde{u} = u$ in $[\bar{x}, x_1]$ and hence that $u \in C^2([\bar{x}, x_1])$. Similarly, if $\bar{x} \neq a$ then $u \in C^2([x_0, \bar{x}])$ for some $x_0 < \bar{x}$. In particular u is Lipschitz in the neighborhood of any $\bar{x} \in [a, b]$ with $M(\bar{x}) < \infty$, and thus by Theorem 2.6 is C^3 in a neighborhood of any such $\bar{x}$. Since u is differentiable almost everywhere in $[a, b]$ it follows that $D \overset{\text{def}}{=} \{x \in [a, b] : M(x) < \infty\}$ is a relatively open subset of $[a, b]$ of full measure, and that $u \in C^3(D)$.

Let $E = [a, b] \setminus D$, and let $x_0 \in E$, so that $M(x_0) = \infty$. Suppose that $x_0 \in (a, b)$. By an appropriate reflection of the variables x and/or u we can suppose without loss of generality that there exist points $y_j \to x_0-$ with

$$\lim_{j \to \infty} \frac{u(x_0) - u(y_j)}{x_0 - y_j} = +\infty.$$

Let $M > 0$, $\delta > 0$ be arbitrary and apply Lemma 2.8 with $u_0 = u(x_0)$. The solutions $\{u(\cdot; \alpha, M) : |\alpha| \leqq M\}$ of (EL) form a field of extremals simply covering some neighborhood of (x_0, u_0) in $\mathbb{R}^2$. Thus, for $|x - x_0|$ sufficiently small there exists a unique $\alpha(x)$ with $|\alpha(x)| \leqq M$ such that $u(x) = u(x; \alpha(x), M)$, and by the implicit function theorem and (2.4) α depends continuously on x. Clearly $\alpha(x_0) = 0$. We claim that $\alpha(x)$ is nondecreasing near x_0. In fact suppose there exist sequences $a_j \to x_0, b_j \to x_0, c_j \to x_0$ with $a_j < b_j < c_j$ and $\alpha(a_j) = \alpha(c_j) \neq \alpha(b_j)$. Then for large enough j the solution $v_j(x) \stackrel{\text{def}}{=} u(x; \alpha(a_j), M)$, $a_j \leqq x \leqq c_j$, satisfies $v_j(a_j) = u(a_j)$, $v_j(b_j) \neq u(b_j)$, $v_j(c_j) = u(c_j)$ and $\max_{x \in [a_j, c_j]} |u(x) - v_j(x)| \leqq \delta_1$. Since v_j is embedded in a field of extremals, Weierstrass's formula gives

$$\int_{a_j}^{c_j} f(x, u(x), u'(x))\, dx > \int_{a_j}^{c_j} f(x, v_j(x), v_j'(x))\, dx,$$

contradicting our hypothesis that u is a strong relative minimizer. Thus α is either nondecreasing or nonincreasing near x_0; the latter possibility is excluded by noting that by integrating (2.3) (*cf.* (2.5)) we obtain

$$\frac{\alpha(y_j)}{x_0 - y_j} \leqq \delta + M - \frac{u(x_0) - u(y_j)}{x_0 - y_j},$$

so that $\alpha(y_j) < 0$ for j sufficiently large. This proves our claim. Now let $x_j \to x_0$, $z_j \to x_0$ with $x_j > z_j$. Then for large enough j,

$$\begin{aligned} \frac{u(x_j) - u(z_j)}{x_j - z_j} &= \frac{u(x_j; \alpha(x_j), M) - u(z_j; \alpha(z_j), M)}{x_j - z_j} \\ &\geqq \frac{u(x_j; \alpha(z_j), M) - u(z_j; \alpha(z_j), M)}{x_j - z_j} \\ &= u'(w_j; \alpha(z_j), M) \\ &\geqq M - \delta, \end{aligned}$$

where $x_j \geqq w_j \geqq z_j$ and we have used (2.3). Thus, since M, δ are arbitrary,

$$\lim_{\to \infty} \frac{u(x_j) - u(z_j)}{x_j - z_j} = +\infty. \tag{2.8}$$

In particular $u'(x_0)$ exists in the sense of (2.1) and equals $+\infty$. A similar argument applies if $x_0 = a$ or $x_0 = b$. We have thus shown that $u'(x)$ exists in the sense of (2.1) for all $x \in [a, b]$. The continuity of u' at x_0 is obvious if $x_0 \in D$, and follows simply from (2.8) otherwise. □

As an application of Theorem 2.7 we prove the following version of results of TONELLI [31, Vol. II, pp. 361, 366], which should be compared with Theorem 2.3.

Theorem 2.10. *Let* $f_{pp} > 0$ *and suppose that*

$$\lim_{|p| \to \infty} \frac{f(x, u, p)}{|p|} = \infty \quad \text{for each } x \in [a, b], u \in \mathbb{R}.$$

Let $u(\cdot) \in \mathscr{A}$ *be a strong relative minimizer of* I *and suppose either that* $f_u(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$ *or that* $f_x(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$. *Then* $u \in C^3([a, b])$ *and satisfies* (EL) *and* (DBR) *on* $[a, b]$.

Proof. Let D_1 be a maximal relatively open interval in $D = [a, b] \setminus E$. By Theorem 2.7, $u \in C^3(D_1)$ and satisfies (EL) and thus (DBR) on D_1. If $f_u(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$ then by (EL)

$$|f_p(x, u(x), u'(x))| \leqq \text{const.}, \qquad x \in D_1. \tag{2.9}$$

If $f_x(\cdot, u(\cdot), u'(\cdot)) \in L^1(a, b)$ then by (DBR)

$$|u'(x) f_p(x, u(x), u'(x)) - f(x, u(x), u'(x))| \leqq \text{const.}, \qquad x \in D_1. \tag{2.10}$$

By the following lemma, either (2.9) or (2.10) implies that u' is bounded in D_1, and thus that $D_1 = D = [a, b]$. □

Lemma 2.11. *Let* f *satisfy the hypotheses of Theorem 2.10. Then*

$$|f_p(x, u, p)| \to \infty, \ \ pf_p(x, u, p) - f(x, u, p) \to \infty$$

as $|p| \to \infty$, *uniformly for* $x \in [a, b]$ *and for* u *in compact sets of* $\mathbb{R}$.

Proof. By the convexity of $f(x, u, \cdot)$ we have that

$$f(x, u, 0) \geqq f(x, u, p) - pf_p(x, u, p),$$

and hence, for $p \neq 0$,

$$\frac{p}{|p|} f_p(x, u, p) \geqq \frac{f(x, u, p)}{|p|} - \frac{f(x, u, 0)}{|p|}.$$

Therefore, for fixed x, u,

$$\lim_{p \to \infty} f_p(x, u, p) = \infty, \qquad \lim_{p \to -\infty} f_p(x, u, p) = -\infty. \tag{2.11}$$

But $f_p(x, u, p)$ is increasing in p. Thus if $x_j \to x$, $u_j \to u$, $p_j \to \infty$ we have for $p_j \geqq M$,

$$f_p(x_j, u_j, p_j) \geqq f_p(x_j, u_j, M),$$

and so

$$\liminf_{j \to \infty} f_p(x_j, u_j, p_j) \geqq f_p(x, u, M).$$

Letting $M \to \infty$ we deduce that the first limit in (2.11) is uniform for x, u in compact sets; otherwise there would exist a convergent sequence (x_j, u_j) and a sequence $p_j \to \infty$ such that $\liminf f_p(x_j, u_j, p_j) < \infty$. The case $p \to -\infty$ is treated similarly.

To prove the second assertion of the lemma we note that

$$f(x, u, 1) \geqq f(x, u, p) - (p - 1) f_p(x, u, p),$$

and hence, provided $p > 1$,

$$pf_p(x, u, p) - f(x, u, p) \geqq \frac{f(x, u, p)}{p} \cdot \frac{p}{p-1} - f(x, u, 1) \cdot \frac{p}{p-1}.$$

Therefore, for fixed x, u,

$$\lim_{p \to \infty} [pf_p(x, u, p) - f(x, u, p)] = \infty. \tag{2.12}$$

That the limit in (2.12) is uniform for x, u in compact sets follows as above using the fact that $pf_p(x, u, p) - f(x, u, p)$ is increasing in p for $p > 0$. The case $p \to -\infty$ is handled similarly. □

Corollary 2.12. *Let* $f = f(u, p)$ *satisfy* $f_{pp} > 0$ *and*

$$\lim_{|p| \to \infty} \frac{f(u, p)}{|p|} = \infty \quad \text{for each } u \in \mathbb{R}. \tag{2.13}$$

If $u(\cdot) \in \mathscr{A}$ *is a strong relative minimizer of* I *then* $u(\cdot) \in C^3([a, b])$ *and satisfies* (EL) *and* (DBR) *on* $[a, b]$.

Finally, we remark that if $1 < q < \infty$ then Theorem 2.7 still holds (with the same proof) if we replace $\mathscr{A}$ by $\mathscr{A} \cap W^{1,q}(a, b)$ both in the statement of the theorem and in the definition of a strong relative minimizer. This is perhaps of interest since in § 5 we show that minimizers in $\mathscr{A}$ and $\mathscr{A} \cap W^{1,q}(a, b)$ may be different.

§ 3. An integral with a scale invariance property

In this section we consider the problem of minimizing

$$I(u) = \int_0^1 [(x^2 - u^3)^2 (u')^{14} + \varepsilon(u')^2] \, dx \tag{3.1}$$

subject to

$$u(0) = 0, \quad u(1) = k, \tag{3.2}$$

where $\varepsilon > 0$ and $k > 0$ are given.

Note that the integrand

$$f(x, u, p) = (x^2 - u^3)^2 p^{14} + \varepsilon p^2 \tag{3.3}$$

in (3.1) satisfies

$$f_{pp} \geqq 2\varepsilon > 0. \tag{3.4}$$

The Euler-Lagrange equation corresponding to (3.1) is

$$\frac{d}{dx}(7(x^2 - u^3)^2 (u')^{13} + \varepsilon u') = -3u^2(x^2 - u^3)(u')^{14}. \tag{3.5}$$

It is easily verified that (3.5) has an exact solution $u = \bar{k}x^{\frac{2}{3}}$ on (0, 1] provided

$$\varepsilon = \left(\frac{2\bar{k}}{3}\right)^{12} (1 - \bar{k}^3)(13\bar{k}^3 - 7). \tag{3.6}$$

Define

$$\theta(\tau) = (\tfrac{2}{3})^{12} \tau^4 (1 - \tau)(13\tau - 7).$$

Differentiating θ we see that θ attains its maximum in the interval $(\frac{7}{13}, 1)$ at the point $\tau^* = \dfrac{25 + \sqrt{79}}{39} = .868928\ldots$, and that $\theta'(\tau) > 0$ for $\frac{7}{13} < \tau < \tau^*$, $\theta'(\tau) < 0$ for $\tau^* < \tau < 1$. Define

$$\varepsilon_0 = \theta(\tau^*) = .002474\ldots$$

We have thus proved

Proposition 3.1. *If* $0 < \varepsilon < \varepsilon_0$ *the Euler-Lagrange equation* (3.5) *has exactly two solutions in* (0, 1] *of the form* $u = \bar{k}x^{\frac{2}{3}}$, $\bar{k} > 0$; *the corresponding values of* $\bar{k}$ *satisfy* $\frac{7}{13} < \bar{k}_1(\varepsilon)^3 < \tau^* < \bar{k}_2(\varepsilon)^3 < 1$. *If* $\varepsilon = \varepsilon_0$ *there is just one such solution, namely* $u = (\tau^*)^{\frac{1}{3}} x^{\frac{2}{3}}$; *if* $\varepsilon > \varepsilon_0$ *there are no such solutions.*

The integrand f in (3.3) satisfies the scale invariance property

$$f(\lambda x, \lambda^\gamma u, \lambda^{\gamma-1} p) = \lambda^\varrho f(x, u, p) \tag{3.7}$$

for all $\lambda > 0$ and all (x, u, p), where $\gamma = \frac{2}{3}$ and $\varrho = -\frac{2}{3}$. We exploit this by making the change of variables

$$v = u^{1/\gamma}, \quad z = \frac{v}{x}, \quad q = v', \quad x = e^t. \tag{3.8}$$

Setting $\lambda = 1/x$ in (3.7) we obtain

$$f(x, u, p) = x^\varrho F(z, q), \tag{3.9}$$

where

$$F(z, q) \stackrel{\text{def}}{=} f(1, z^\gamma, \gamma z^{\gamma-1} q). \tag{3.10}$$

It is easily verified that, for any smooth integrand satisfying (3.7), (EL) is transformed into the *autonomous* system

$$\begin{aligned} \frac{dz}{dt} &= q - z, \\ \frac{dF_q}{dt} &= F_z - \varrho F_q. \end{aligned} \tag{3.11}$$

More precisely, if $0 < a < b < \infty$ and u is a smooth solution of (EL) on (a, b) satisfying $u(x) > 0$ for all $x \in (a, b)$, then

$$q(t) = \gamma^{-1}[u(e^t)]^{(1-\gamma)/\gamma} u'(e^t), \quad z(t) = e^{-t}[u(e^t)]^{1/\gamma} \tag{3.12}$$

is a smooth solution of (3.11) for $\log a < t < \log b$. Conversely, if (q, z) is a smooth solution of (3.11) defined for $\alpha < t < \beta$ and satisfying $z(t) < 0$ for all $x \in (\alpha, \beta)$ then

$$u(x) = [x \cdot z(\log Ax)]^{\gamma} \tag{3.13}$$

is a smooth solution of (EL) for $e^{\alpha} < Ax < e^{\beta}$, where $A > 0$ is arbitrary. The arbitrary constant in (3.13) arises from the fact that, since (3.11) is autonomous, if $z(t)$ is a solution so is $z(t + \log A)$; equivalently, if $u(x)$ is a positive solution of (EL) so is $A^{-\gamma} u(Ax)$. Note that (3.11 b) is the Euler-Lagrange equation for the integral

$$\hat{I}(v) = \int_0^1 x^{\varrho} F\left(\frac{v(x)}{x}, v'(x)\right) dx,$$

obtained by making the change of variables (3.8) in (3.1). As has been pointed out to us by P. J. OLVER, the fact that the scale invariance property (3.7) implies the existence of a change of variables making (EL) autonomous is a consequence of the theory of Lie groups (*cf.* INCE [21, Chap. 4]). We remark that the above reduction to an autonomous system is used in BALL [3] as a tool for studying the radial equation of nonlinear elasticity in n space dimensions, the appropriate values of γ, ϱ being $\gamma = 1$, $\varrho = n - 1$.

From now on we assume that f is given by (3.3), although it will be apparent to the reader that much of what we have to say applies to a general class of integrands satisfying (3.7) for suitable γ, ϱ. For later use we note that since

$$F(z, q) = (\tfrac{2}{3})^{14} (1 - z^2)^2 z^{-14/3} q^{14} + (\tfrac{2}{3})^2 \varepsilon z^{-2/3} q^2, \tag{3.14}$$

(3.11) takes the form

$$\begin{aligned} \frac{dz}{dt} &= q - z, \\ \frac{dq}{dt} &= G(z, q), \end{aligned} \tag{3.15}$$

$$\begin{aligned} G(z, q) &\overset{\text{def}}{=} \frac{F_z + \frac{2}{3} F_q - (q - z) F_{qz}}{F_{qq}} \\ &= \frac{q^2}{3z} \left[\frac{(\frac{2}{3})^{12} (1 - z^2) [13q(7 - z^2) - 84z] q^{11} + \varepsilon z^4}{91 (\frac{2}{3})^{12} (1 - z^2)^2 q^{12} + \varepsilon z^4}\right]. \end{aligned} \tag{3.16}$$

We study (3.15) in the first quadrant of the (z, q) plane. Note that solutions of (3.15) in the first quadrant correspond to positive solutions u of (3.5) with $u'(x) \geqq 0$. It is clear that any minimizer of (3.1), (3.2) satisfies $u'(x) \geqq 0$ a.e. $x \in [0, 1]$, since otherwise the value of I could be reduced by making u constant on some interval.

Before proceeding with the details of our phase-plane analysis, the reader may wish to look at Figure 3.1 so as to see where we are heading.

We begin by examining the rest points of (3.15) in $z > 0$, $q > 0$. From (3.15), (3.16) these are easily seen to be given by $q = z = \bar{k}^{-\frac{3}{2}}$, where $\bar{k} > 0$ satisfies

(3.6), and correspond to the solutions $u = \bar{k}x^{\frac{2}{3}}$ discussed in Proposition 3.1. Thus, for $0 < \varepsilon < \varepsilon_0$, there are precisely two rest points, namely $q = z = \bar{k}_1(\varepsilon)^{\frac{3}{2}}$ and $q = z = \bar{k}_2(\varepsilon)^{\frac{3}{2}}$, with $\frac{7}{13} < \bar{k}_1(\varepsilon)^3 < \bar{k}_2(\varepsilon)^3 < 1$. We denote these points by P_1 and P_2 respectively. We study the nature of the rest points by linearization. Thus let P denote a rest point $q = z = \bar{k}^{\frac{3}{2}}$. Setting $z = \bar{k}^{\frac{3}{2}} + a$, $q = \bar{k}^{\frac{3}{2}} + b$ gives (3.15) the form

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix} + O(|a|^2 + |b|^2), \tag{3.17}$$

where

$$A \stackrel{\text{def}}{=} \begin{pmatrix} -1 & 1 \\ \sigma(\bar{k}) & \frac{2}{3} \end{pmatrix},$$

and

$$\sigma(\bar{k}) \stackrel{\text{def}}{=} \frac{2}{9}\,\frac{(31\bar{k}^3 - 28)}{(1 - \bar{k}^3)(14 - 13\bar{k}^3)}.$$

The eigenvalues of A are given by

$$\lambda_\pm = \tfrac{1}{6}\left(-1 \pm \sqrt{25 + 36\sigma(\bar{k})}\right).$$

Thus,

(i) if $\sigma(\bar{k}) < -\frac{25}{36}$, λ_+, λ_- are complex,

(ii) if $\sigma(\bar{k}) = -\frac{25}{36}$, $\lambda_+, \lambda_- = -\frac{1}{6}$ and A has a double elementary divisor,

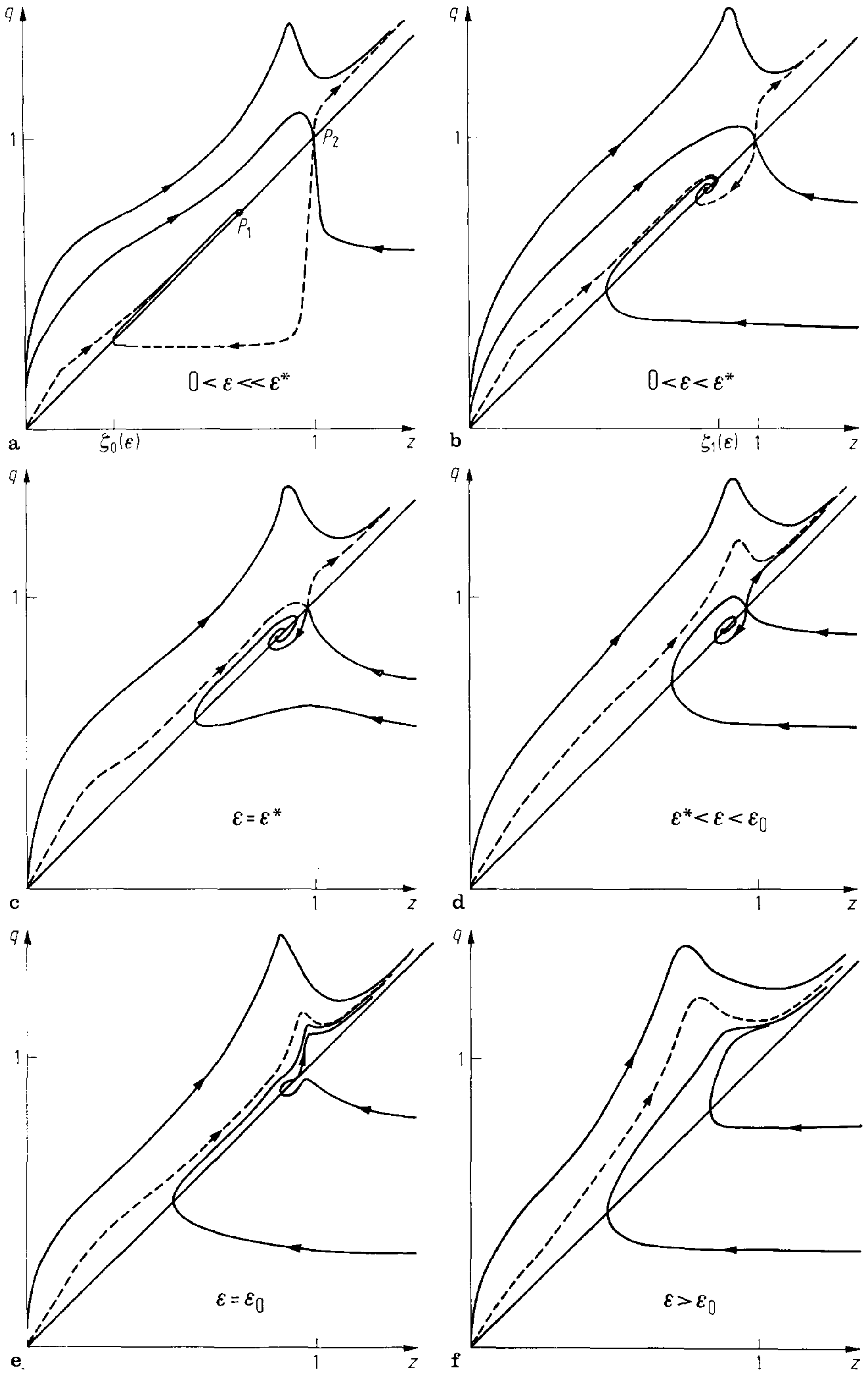
(iii) if $-\frac{25}{36} < \sigma(\bar{k}) < -\frac{2}{3}$, $\lambda_- < \lambda_+ < 0$,

(iv) if $\sigma(\bar{k}) = -\frac{2}{3}$, $\lambda_- = -\frac{1}{3}, \lambda_+ = 0$,

(v) if $-\frac{2}{3} < \sigma(\bar{k})$, $\lambda_- < 0 < \lambda_+$.

As is well known (*cf.* HARTMAN [19, p. 212, ff.]), cases (i)–(iii) correspond to P being a sink, and case (v) to a saddle-point. Case (iv) is a critical case where the stability is determined by the nonlinear terms in (3.17), and we discuss this presently. In case (i), P is a focus. In case (ii) P is an improper node, all solutions of (3.15) near P approaching P with slope $\frac{5}{6}$ as $t \to \infty$. In case (iii) P is an improper node with a single pair of solutions approaching P with slope $\lambda_- + 1 \in (\frac{2}{3}, \frac{5}{6})$ as $t \to \infty$, and all other nearby solutions approaching P with slope $\lambda_+ + 1 \in (\frac{5}{6}, 1)$ as $t \to \infty$. In case (v) the slope of the stable manifold of P at P is $\lambda_- + 1 < \frac{2}{3}$, that of the unstable manifold $\lambda_+ + 1 > 1$. We now note that $\sigma(\bar{k}) > -\frac{2}{3}$

Fig. 1. The phase-plane diagram for (3.15). Shown in particular are the smooth solution orbit, which leaves the origin with slope 3/2, and the stable and unstable manifolds of P_2. The absolute minimizers of I correspond to appropriate portions of the dashed curves (see Theorem 3.12).


q
1
P_2
P_1
$0<\varepsilon\ll\varepsilon^*$
a
$\xi_0(\varepsilon)$
1
z
q
1
$0<\varepsilon<\varepsilon^*$
b
$\xi_1(\varepsilon)$
1
z
q
1
$\varepsilon=\varepsilon^*$
c
1
z
q
1
$\varepsilon^*<\varepsilon<\varepsilon_0$
d
1
z
q
1
$\varepsilon=\varepsilon_0$
e
1
z
q
1
$\varepsilon>\varepsilon_0$
f
1
z

(respectively $\sigma(\bar{k}) < -\frac{2}{3}$) if and only if

$$39\tau^2 - 50\tau + 14 > 0 \quad (\text{respectively } <0)$$

where $\tau = \bar{k}^3$, and since $\dfrac{25-\sqrt{79}}{39} < \frac{7}{13}$ this holds if and only if $\tau > \tau^* = \dfrac{25+\sqrt{79}}{39}$ (respectively $\tau < \tau^*$). The case $\sigma(\bar{k}) = -\frac{2}{3}$ corresponds to $\tau = \tau^*$. Similarly, $\sigma(\bar{k}) > -\frac{25}{36}$ (respectively $\sigma(\bar{k}) < -\frac{25}{36}$) if and only if

$$325\tau^2 - 427\tau + 126 > 0 \quad (\text{respectively } <0),$$

which holds if and only if $\tau > \tau_1 = .86634\ldots$ (respectively $\tau < \tau_1$). We let $\varepsilon_1 = \theta(\tau_1) = .002473\ldots$. We have thus proved

Proposition 3.2. *Let* $0 < \varepsilon < \varepsilon_0$. *Then* P_1 *is a sink and* P_2 *is a saddle point.*

Since $\dfrac{dz}{dt} = q - z$ the flow in the region $0 \leqq q < z$ is to the left, that in the region $0 < z < q$ to the right. We also make frequent use of the direction of flow on the diagonal $q = z$, where $\dfrac{dz}{dt} = 0$, given in the following lemma.

Lemma 3.3.

(i) *Let* $0 < \varepsilon < \varepsilon_0$. *Then* $G(z, z) > 0$ *for* $0 < z < \bar{k}_1(\varepsilon)^{\frac{3}{2}}$ *and for* $\bar{k}_2(\varepsilon)^{\frac{3}{2}} < z < \infty$, *while* $G(z, z) < 0$ *for* $\bar{k}_1(\varepsilon)^{\frac{3}{2}} < z < \bar{k}_2(\varepsilon)^{\frac{3}{2}}$.

(ii) *Let* $\varepsilon = \varepsilon_0$. *Then* $G(z, z) > 0$ *for all* $z > 0$, $z \neq (\tau^*)^{\frac{1}{2}}$.

(iii) *Let* $\varepsilon > \varepsilon_0$. *Then* $G(z, z) > 0$ *for all* $z > 0$.

For the purpose of studying the existence of periodic orbits it is convenient to introduce the new variable $r = F_q(z, q)$. It is easily verified, using the fact that $F_{qq} > 0$, that $(z, q) \to (z, r)$ maps $z > 0$, $q > 0$ onto $z > 0$, $r > 0$ and has a smooth inverse. Thus (3.11) is equivalent to

$$\begin{aligned}\frac{dz}{dt} &= q(z, r) - z \overset{\text{def}}{=} Z(z, r),\\ \frac{dr}{dt} &= F_z(z, q(z, r)) + \tfrac{2}{3} r \overset{\text{def}}{=} R(z, r).\end{aligned} \tag{3.18}$$

An easy computation shows that

$$\frac{\partial Z}{\partial z} + \frac{\partial R}{\partial r} = -\tfrac{1}{3}. \tag{3.19}$$

Integration of (3.19) over the region enclosed by a nontrivial periodic or homoclinic orbit gives a contradiction. We have thus proved

Proposition 3.4. *The system* (3.15) *has no nontrivial periodic orbit and no homoclinic orbit in* $z > 0$, $q > 0$.

We next study the continuation and asymptotic properties of solutions.

Proposition 3.5. *Let* $z_0 > 0$, $q_0 > 0$, *and let* $(z(t), q(t))$ *denote the unique solution of* (3.15) *with* $z(0) = z_0$, $q(0) = q_0$. *Then* $(z(t), q(t))$ *exists and remains in* $z > 0$, $q > 0$ *on a maximal interval* $(t_{\min}, \infty)$, *where* $-\infty \leqq t_{\min} < 0$. *As* $t \to \infty$, *either* $z(t) \to \infty$ *and* $q(t) \to \infty$ *or* $(z(t), q(t)) \to (k^{-\frac{3}{2}}, k^{-\frac{3}{2}})$, *a rest point. As* $t \to t_{\min}+$ *either* $(z(t), q(t)) \to (0, 0)$ *or* $z(t) \to \infty$ *and* $q(t) \to c = c(z_0, q_0) \in [0, \infty)$† *or* $(z(t), q(t)) \to (k^{-\frac{3}{2}}, k^{-\frac{3}{2}})$, *a rest point.*

Proof. Let the maximal interval in which the solution $(z(t), q(t))$ exists and remains in $z > 0$, $q > 0$ be $(t_{\min}, t_{\max})$, where $-\infty \leqq t_{\min} < 0 < t_{\max} \leqq \infty$. Observe first that if $(z(t), q(t))$ remains in a compact subset of $z > 0$, $q \geqq 0$ for all $t \in [0, t_{\max})$ (respectively $t \in (t_{\min}, 0]$) then $t_{\max} = \infty$ (respectively $t_{\min} = -\infty$), and we can apply the Poincaré-Bendixson theory (*cf.* HARTMAN [19, p. 151 ff.]). By Proposition 3.4 the only possibilities are that $(z(t), q(t))$ tends to a rest point as $t \to \infty$ (respectively $t \to -\infty$), or that the ω-limit set (respectively α-limit set) of $(z(\cdot), q(\cdot))$ contains more than one rest point (and thus $0 < \varepsilon < \varepsilon_0$). The latter case cannot occur since P_1 is asymptotically stable.

Next we note that on any open t-interval where $q(t) \neq z(t)$ we have $\frac{dz}{dt} \neq 0$, and thus the orbit has the representation $q = q(z)$, where by (3.15)

$$\frac{dq}{dz} = \frac{q^2}{3z(q-z)} \left[\frac{(\frac{2}{3})^{12}(1-z^2)[13q(7-z^2) - 84z]q^{11} + \varepsilon z^4}{91(\frac{2}{3})^{12}(1-z^2)^2 q^{12} + \varepsilon z^4}\right] \stackrel{\text{def}}{=} H(z, q, \varepsilon). \tag{3.20}$$

We first eliminate the possibility that $q(z)$ becomes unbounded as $z \to \bar{z} \in (0, \infty)$ either from above or below. By general results on ordinary differential equations we would then have $q(z) \to +\infty$ as $z \to \bar{z}+$ or $q(z) \to +\infty$ as $z \to \bar{z}-$. If $\bar{z} \neq 1$, then for q large and for z near $\bar{z}$ we have

$$\left|\frac{dq}{dz}\right| = \left|\frac{q}{3z\left(1 - \frac{z}{q}\right)}\left[\frac{(\frac{2}{3})^{12}(1-z^2)\left[13(7-z^2) - 84\frac{z}{q}\right] + \varepsilon\frac{z^4}{q^{12}}}{91(\frac{2}{3})^{12}(1-z^2)^2 + \frac{z^4}{q^{12}}}\right]\right| \leqq Cq,$$

where here and below C denotes a generic constant. Thus q is bounded near $\bar{z}$, a contradiction. If $\bar{z} = 1$, we observe that $q(z)$ satisfies

$$\frac{d}{dz}((q-z)F_q - F) = -\tfrac{1}{3}F_q, \tag{3.21}$$

† It will be shown in Proposition 3.6 that in this case $t_{\min} = -\infty$ and $c(z_0, q_0) > 0$.

where F is given by (3.14). (This is essentially the DuBois-Reymond equation for $\hat{I}$.) Now

$$F_q = \tfrac{28}{3}\left(\frac{2q}{3}\right)^{13} z^{-\frac{14}{3}}(1-z^2)^2 + \frac{4\varepsilon}{3}\cdot\frac{2q}{3} z^{-\frac{2}{3}},$$

and

$$\begin{aligned}\psi(z,q) &\overset{\text{def}}{=} (q-z)F_q - F \\ &= \left(\frac{2q}{3}\right)^{14} z^{-\frac{14}{3}}(1-z^2)^2\left(13-\frac{14z}{q}\right) + \varepsilon\left(\frac{2q}{3}\right)^2 z^{-\frac{2}{3}}\left(1-\frac{2z}{q}\right).\end{aligned}$$

Thus, for z near 1 and q large,

$$\begin{aligned}|F_q| &\leqq C(q^{13}(1-z^2)^{\frac{13}{7}} + q) \\ &\leqq C(q^{14}(1-z^2)^2 + q^{\frac{14}{13}})^{\frac{13}{14}} \\ &\leqq C(q^{14}(1-z^2)^2 + q^2)^{\frac{13}{14}},\end{aligned}$$

and so by (3.21)

$$\left|\frac{d\psi}{dz}(z,q(z))\right| \leqq C\,|\psi(z,q(z))|^{\frac{13}{14}}.$$

Thus $\psi(z, q(z))$ is bounded near $z = 1$, which is a contradiction.

The case when (z_0, q_0) is a rest point being trivial, we now consider the remaining cases. First suppose that $q_0 < z_0$. Note that $q = 0$, $0 < z < \infty$ is an orbit of (3.15), and that $G(z, q) > 0$ if $z > 0$, $q > 0$ and $z + q$ is sufficiently small. Since $\frac{dz}{dt} < 0$ for $q < z$ it now follows that either $(z(t), q(t))$ remains below the line $q = z$ on $[0, t_{\max})$, and hence by the first part of the proof tends to a rest point, or that $z(t_0) = q(t_0)$ for some $t_0 > 0$. In the latter case it may happen that $z(t_1) = q(t_1)$ for some $t_1 > t_0$, with $q(t) > z(t)$ for $t_0 < t < t_1$. If so, then by Lemma 3.3, $0 < \varepsilon < \varepsilon_0$ and $z(t_0) < \bar{k}_1(\varepsilon)^{\frac{3}{2}} < z(t_1) < \bar{k}_2(\varepsilon)^{\frac{3}{2}}$, so that, unless $(z(t), q(t)) \to P_1$ as $t \to \infty$ without a further crossing of $q = z$, $z(t_2) = q(t_2)$ for some $t_2 > t_1$. If $z(t_2) < z(t_0)$ the orbit $(z(t), q(t))$ would remain in a compact subset of $z > 0$, $q \geqq 0$ for $t_{\min} < t \leqq 0$ and hence tend to P_1 as $t \to -\infty$; this is impossible as P_1 is a sink. Thus by Proposition 3.4, $z(t_2) > z(t_0)$, which implies that $(z(t), q(t))$ remains in a compact subset of $z > 0$, $q \geqq 0$ for $0 \leqq t < t_{\max}$, and thus tends to P_1 as $t \to \infty$.

The above considerations show that, as regards the behavior for $t \geqq 0$, it suffices to examine the case when $q(t) > z(t)$ for all $t \in [0, t_{\max})$ and the corresponding solution curve $q(z)$ is defined for all $z \geqq z_0$. To show that $t_{\max} = \infty$ we examine the slope of the vector field on the line $q = \mu z$, where $\mu > 1$. On this line, as $z \to \infty$,

$$\frac{dq}{dz} = \frac{\mu^2}{3(\mu-1)}\left[\frac{(\frac{2}{3})^{12}\left(\frac{1}{z^2}-1\right)\left(\frac{91}{z^2}-13-\frac{84}{z^2\mu}\right)+\frac{\varepsilon}{\mu^{12}z^{12}}}{91(\frac{2}{3})^{12}\left(\frac{1}{z^2}-1\right)^2+\frac{\varepsilon}{\mu^{12}z^{12}}}\right]$$

$$= \frac{\mu^2}{21(\mu-1)}\left[1+o\left(\frac{1}{z}\right)\right],$$

where the $o\left(\frac{1}{z}\right)$ term is independent of μ. Hence, provided $\mu_0 > \frac{21}{20}$, there exists $\hat{z} > 0$ such that if $z \geqq \hat{z}$ and $\mu \geqq \mu_0$ then $\frac{dq}{dz}(z) < \mu$ on $q = \mu z$. Choosing $\mu > \frac{q(\hat{z})}{\hat{z}}$ we deduce that

$$\dot{z}(t) \leqq (\mu - 1)\, z(t)$$

whenever $z(t) \geqq \hat{z}$, and hence that $t_{\max} = \infty$.

We consider now the behavior of $(z(t), q(t))$ for $t \in (t_{\min}, 0]$. Suppose first that $q_0 > z_0$. If $q(t) > z(t)$ for all $t \in (t_{\min}, 0]$, then either $\inf_{t \in (t_{\min}, 0]} z(t) > 0$ or $z(t) \to 0$ as $t \to t_{\min}+$. In the former case, since $q(t)$ cannot become unbounded as $t \to t_{\min}+$, the curve lies in a compact set of $z > 0$, $q \geqq 0$ and we must have that $t_{\min} = -\infty$ and $(z(t), q(t))$ tends to a rest point as $t \to -\infty$. If $z(t) \to 0$ as $t \to t_{\min}+$ then by (3.20) the corresponding curve $q(z)$ satisfies $\frac{dq}{dz} > 0$ for sufficiently small $z > 0$, so that $q(t_{\min}) \stackrel{\text{def}}{=} \lim_{t \to t_{\min}+} q(t)$ exists. If $q(t_{\min}) > 0$ then by (3.20) $\frac{dq}{dz} \geqq \frac{C}{z}$ for sufficiently small $z > 0$, where $C > 0$ is a constant, and integration of this inequality gives a contradiction. Thus $(z(t), q(t)) \to (0, 0)$ as $t \to t_{\min}+$. On the other hand, if $q(t) = z(t)$ for some $t \in (t_{\min}, 0]$ then $q(t_1) < z(t_1)$ for some earlier time.

It only remains, therefore, to consider the case when $q_0 < z_0$. First, if $q(t) < z(t)$ for all $t \in (t_{\min}, 0]$ then either $z(t)$ remains bounded as $t \to t_{\min}+$, in which case $t_{\min} = -\infty$ and $(z(t), q(t))$ tends to a rest point as $t \to -\infty$, or $\lim_{t \to t_{\min}+} z(t) = \infty$. In the latter case, by (3.20), $\frac{dq}{dz} < 0$, for $z^2 > 7$, $q < z$, and so as $t \to t_{\min}+$ $q(t)$ tends to a nonnegative limit, which we denote by $c(z_0, q_0)$. Next, if $q(t_0) = z(t_0)$ for some $t_0 \in (t_{\min}, 0]$ then $q(\bar{t}) > z(\bar{t})$ for some $\bar{t} \in (t_{\min}, t_0)$. We have already treated the case when $q(t) > z(t)$ for all $t \in (t_{\min}, \bar{t}]$ and thus it remains to eliminate the possibility that $q(t_j) = z(t_j)$ for an infinite sequence $t_j \to t_{\min}+$, and of course this can only occur for $0 < \varepsilon < \varepsilon_0$. The corresponding orbit would spiral either inwards or outwards as $t \to t_{\min}+$. If it spiralled inwards then clearly we would have $t_{\min} = -\infty$ and $(z(t), q(t)) \to P_1$ as $t \to -\infty$, which is impossible since P_1 is a sink. It must thus spiral outwards, and of course it cannot remain in a compact subset of $z > 0$, $q \geqq 0$, since otherwise it would have to tend to P_2 as $t \to t_{\min}+$, which is clearly impossible. Furthermore the orbit must remain under that part of the stable manifold of P_2 lying in $q > z$, and so $z(t_j) \to 0$ as $t_j \to t_{\min}+$. But the solution curve $(z_r(t), q_r(t))$ of (3.15) satisfying $z_r(0) = 1$, $q_r(0) = \frac{1}{r}$ approaches the z-axis as $r \to \infty$, crossing $q = z$ arbitrarily close to the origin, which implies that $z(t_j)$ is bounded away from zero. □

Note that Propositions 3.2, 3.5 together imply that when $\varepsilon = \varepsilon_0$ the unique fixed point $q = z = \bar{k}^{\frac{3}{2}}$ is unstable.

It is possible to specify more precisely the asymptotic behavior of those solutions of (3.15) satisfying $z(t) \to \infty$, $q(t) \to \infty$ as $t \to \infty$. For such a solution we have seen in the proof of Proposition 3.5 that $\frac{q(z)}{z(t)}$ is bounded for large t. Setting $\zeta(t) = \frac{1}{z(t)}$, we see that (3.15) becomes

$$\dot{\zeta} = \zeta(1 - \varphi),$$

$$\dot{\varphi} = (1 - \varphi) + \frac{\varphi^2}{3}\left[\frac{(\frac{4}{3})^{12}(\zeta^2 - 1)\,[13\varphi(7\zeta^2 - 1) - 84\zeta^2]\,\varphi^{11} + \varepsilon\zeta^{12}}{91(\frac{4}{3})^{12}\,(\zeta^2 - 1)^2\,\varphi^{12} + \varepsilon\zeta^{12}}\right], \tag{3.22}$$

where $\varphi(t) \stackrel{\text{def}}{=} \frac{q(t)}{z(t)}$, and hence as $t \to \infty$,

$$\dot{\varphi} = \varphi(1 - \tfrac{20}{21}\varphi) + o(1).$$

Hence $\varphi(t) \to \frac{21}{20}$ as $t \to \infty$. Linearizing about the rest point $\zeta = 0$, $\varphi = \frac{21}{20}$ of (3.22) we obtain

$$\zeta(t) \geqq C_1 e^{-\frac{1}{20}t},$$

$$|\varphi(t) - \tfrac{21}{20}| \leqq C_2 e^{-t},$$

for sufficiently large t, where C_1 and C_2 are positive constants. It follows that

$$|q(t) - \tfrac{21}{20} z(t)| \leqq \frac{C_2}{C_1} e^{-\frac{19}{20}t}$$

for sufficiently large t, so that the solution curve rapidly approaches the line $q = \frac{21}{20} z$. Since $\dot{z} = q - z$ we deduce that

$$z(t) = Ae^{\frac{t}{20}} + O\,(e^{-\frac{19}{20}t})$$

as $t \to \infty$, where $A = A(z_0, q_0)$ is a constant, and hence that the corresponding solution u of (3.5) satisfies

$$u(x) = Ax^{\frac{7}{10}} + O(x^{\frac{1}{30}}) \quad \text{as } x \to \infty.$$

We now study the behavior of solutions in a neighborhood of the q and z axes, and in particular near the origin.

Proposition 3.6. *Every smooth solution u of* (3.5) *with* $u(0) = 0$, $u'(0) > 0$ *corresponds to a single orbit of* (3.15) *in* $z > 0$, $q > 0$ *that leaves the origin* $z = q = 0$ *with slope* $\frac{3}{2}$. *The only other orbits of* (3.15) *leaving the origin correspond to solutions* u *of* (3.5) *with* $u(x_0) = 0$ *for some* $x_0 > 0$; *these orbits satisfy*

$$\lim_{t \to \log x_0 +} z(t) = \lim_{t \to \log x_0 +} q(t) = 0, \quad \lim_{t \to \log x_0 +} \frac{q(t)}{z(t)} = \infty. \tag{3.23}$$

Solutions $(z(\cdot), q(\cdot))$ *of* (3.15) *whose orbits have an unbounded intersection with* $0 < q < z$ *correspond precisely to solutions* u *of* (3.5) *with* $u(0) > 0$, $u'(0) > 0$, *and thus satisfy* $\lim_{t \to -\infty} z(t) = \infty$, $\lim_{t \to -\infty} q(t) = c > 0$, *where* $c = c(z(0), q(0))$ *is a constant.*

Proof. Let u be a smooth solution of (3.5) on some interval $[0, a]$, $a > 0$, satisfying $u(0) = 0$, $u'(0) = \alpha > 0$. Then $u(x) = \alpha x + o(x)$, $u'(x) = \alpha + o(1)$, as $x \to 0+$, and hence $z = \alpha^{\frac{3}{2}} x^{\frac{1}{2}} + o(x^{\frac{1}{2}})$, $q = \frac{3}{2}(\alpha^{\frac{3}{2}} x^{\frac{1}{2}} + o(x^{\frac{1}{2}}))$. Thus the corresponding solution $(z(t), q(t))$ satisfies

$$\lim_{t \to -\infty} z(t) = \lim_{t \to -\infty} q(t) = 0, \quad \lim_{t \to -\infty} \frac{q(t)}{z(t)} = \frac{3}{2}.$$

That this solution is the same for any $\alpha > 0$ (up to adding a constant to t) follows from the similarity transformation (3.13) and the uniqueness of solutions to the initial value problem for (3.5).

Let $u_\beta(x)$ denote the unique solution to (3.5) satisfying $u_\beta(1) = 0$, $u_\beta'(1) = \beta > 0$; this corresponds to a solution $(z_\beta(\cdot), q_\beta(\cdot))$ satisfying

$$z_\beta(t) = \frac{[\beta(e^t - 1) + o(e^t - 1)]^{\frac{3}{2}}}{e^t} = o(1),$$

and

$$q_\beta(t) = \tfrac{3}{2}[\beta(e^t - 1) + o(e^t - 1)]^{\frac{1}{2}} (\beta + o(1)) = o(1),$$

as $t \to 0+$. Also

$$\lim_{t \to 0+} \frac{q_\beta(t)}{z_\beta(t)} = \lim_{t \to 0+} \frac{3}{2(e^t - 1)} = \infty.$$

Let $\delta > 0$ be sufficiently small. It follows from Proposition 3.5 that $z_\beta(t_\beta) = \delta$ for some minimal $t_\beta > 0$. Also, since $q_\beta(t_\beta) > z_\beta(t_\beta)$, the corresponding intersection at $x = e^{t_\beta}$ of the graph of u_β with $\delta^{\frac{2}{3}} x^{\frac{1}{3}}$ is transversal, and thus by the implicit function theorem t_β depends continuously on β. Hence also $q_\beta(t_\beta)$ depends continuously on β. We examine the behavior of $q_\beta(t_\beta)$ as β varies from 0 to ∞. We first show that

$$\lim_{\beta \to \infty} q_\beta(t_\beta) = \infty. \tag{3.24}$$

Since $u_\beta'(x) > 0$ for all $x \geqq 0$, u_β is invertible; denote the inverse function by $x_\beta(u)$. By (3.5) $x_\beta(\cdot)$ satisfies the transformed equation

$$[19(x^2 - u^3)^2 + \varepsilon x_u^{12}]\, x_{uu} = x_u(x^2 - u^3)\,(28xx_u - 39u^2), \tag{3.25}$$

where the subscripts denote derivatives with respect to u. This equation has the solution $\bar{x}(u) \equiv 1$, $u \in [0, \frac{1}{2}]$, in the neighborhood of which (3.25) can be written in the form $x_{uu} = h(u, x, x_u)$ with h continuously differentiable. Since $\bar{x}(0) = x_\beta(0) = 1$, $\bar{x}_u(0) = 0$, $(x_\beta)_u\,(0) = \dfrac{1}{\beta}$, it follows that $x_\beta \to 1$ in $C^1([0, \frac{1}{2}])$ as $\beta \to \infty$. In particular, $t_\beta \to 0$ as $\beta \to \infty$. Since $q_\beta(t_\beta) = \frac{3}{2}\, u_\beta(e^{t_\beta})^{\frac{1}{2}}\, u_\beta'(e^{t_\beta}) = \frac{3}{2}\,\dfrac{\delta^{\frac{1}{3}}\, e^{\frac{t}{3}\beta}}{(x_\beta)_u\,(\delta^{\frac{2}{3}}\, e^{\frac{2}{3}t_\beta})}$ this gives (3.24).

Next, let $\tilde{u}_\beta(x) = \beta^2 u_\beta(\beta^{-3}\, x)$, which also solves (3.5) and satisfies $\tilde{u}_\beta(\beta^3) = 0$, $\tilde{u}_\beta'(\beta^3) = 1$. Clearly $\tilde{u}_\beta \to \tilde{u}$ in $C^1([0, 1])$ as $\beta \to 0+$, where $\tilde{u}$ is the unique solution of (3.5) satisfying $\tilde{u}(0) = 0$, $\tilde{u}'(0) = 1$. But $\beta^3 e^{t_\beta}$ is the least value of $x > \beta^3$ such that $\tilde{u}_\beta(x) = \delta^{\frac{2}{3}}\, x^{\frac{2}{3}}$, and thus tends to the least positive root $\tilde{x}$ of $\tilde{u}(x) = \delta^{\frac{2}{3}} x^{\frac{2}{3}}$ as $\beta \to 0+$. Thus

$$\lim_{\beta \to 0+} q_\beta(t_\beta) = \tfrac{3}{2}(\delta\tilde{x})^{\frac{1}{3}}\, \tilde{u}'(\tilde{x}),$$

which is the value of q at the intersection of $z = \delta$ with the smooth solution orbit leaving $q = z = 0$ with slope $\frac{3}{2}$. We have thus shown that the region above this orbit in the strip $0 < z \leqq \delta$ is completely filled by the orbits $(z_\beta(\cdot), q_\beta(\cdot))$. If $x_0 > 0$ is given then $(z(t), q(t)) = (z_\beta(t - \log x_0),\ q_\beta(t - \log x_0))$ corresponds by (3.13)ff to the solution u of (3.5) satisfying $u(x_0) = 0$, $u'(x_0) = \beta x_0^{-\frac{1}{3}}$, and thus (3.23) holds.

Let $u_{\gamma,\nu}$ be the unique solution of (3.5) satisfying $u(0) = \gamma > 0$, $u'(0) = \nu > 0$. Then the corresponding solution $(z_{\gamma,\nu}(\cdot), q_{\gamma,\nu}(\cdot))$ of (3.15) satisfies $\lim\limits_{t \to -\infty} z_{\gamma,\nu}(t) = \infty$, $\lim\limits_{t \to -\infty} q_{\gamma,\nu}(t) = \frac{3}{2}\gamma^{\frac{1}{2}}\nu$. As $\gamma \to 0+$, $u_{\gamma,1} \to \tilde{u}$ in $C^1([0, 1])$ and hence, for each *fixed* t, $z_{\gamma,1}(t) \to \tilde{z}(t) \overset{\text{def}}{=} \dfrac{\tilde{u}^{\frac{3}{2}}(e^t)}{e^t}$ and $q_{\gamma,1}(t) \to \tilde{q}(t) \overset{\text{def}}{=} \frac{3}{2}\, \tilde{u}^{\frac{1}{2}}(e^t)\, \tilde{u}'(e^t)$.

Conversely, suppose that $(z(\cdot), q(\cdot))$ is a solution of (3.15) whose orbit has an unbounded intersection with $0 < q < z$. By Proposition 3.5, $\lim\limits_{t \to t_{\min}+} q(t) = c \geqq 0$. Let $x_0 = e^{t_{\min}}$. Suppose $t_{\min} > -\infty$, so that $x_0 > 0$. Then the corresponding solution u of (3.5) would satisfy

$$\lim_{x \to x_0+} v(x) = \lim_{t \to t_0} z(t) = \infty, \qquad \lim_{x \to x_0+} v'(x) = c,$$

where $v = u^{\frac{3}{2}}$, which is impossible. Thus $t_{\min} = -\infty$, $x_0 = 0$, and since $\lim\limits_{x \to 0+} v'(x) = c$ we have $v(x) \to d$ as $x \to x_0+$, where $d \geqq 0$ is a constant. But if d were zero then we would have

$$\infty = \lim_{x \to 0+} \frac{v(x)}{x} = \lim_{x \to 0+} \frac{v'(x)}{1} = c,$$

a contradiction. Hence $u(0) > 0$, $u'(0) = \dfrac{2c}{3u(0)^{\frac{1}{2}}} \geqq 0$. Now if $c = 0$, $u(x) \equiv u(0)$ by uniqueness of solutions to (3.5), and hence $q(t) \equiv 0$. Hence $u'(0) > 0$.

It follows immediately from the above that for $\delta > 0$ sufficiently small the region in $0 < z < \delta$, $q > 0$ below the smooth solution orbit is completely filled with orbits corresponding to solutions of (3.5) with $u(0) > 0$, $u'(0) > 0$. In particular there are no other orbits leaving the origin. □

We next apply the results of Section 2.

Theorem 3.7. *I attains an absolute minimum on the set $\mathscr{A} = \{u \in W^{1,1}(0, 1): u(0) = 0, u(1) = k\}$. Let u be any minimizer. If $\varepsilon > \varepsilon_0$ then u is a C^∞ solution of* (3.5) *on* $[0, 1]$. *If $0 < \varepsilon \leqq \varepsilon_0$ then either u is a C^∞ solution of* (3.5) *on* $[0, 1]$ *or u is a C^∞ solution of* (3.5) *on* $(0, 1]$ *with $u(x) \sim \bar{k}x^{\frac{2}{3}}$, $u'(x) \sim \frac{2}{3}\bar{k}x^{-\frac{1}{3}}$ as $x \to 0+$, where $\bar{k}$ satisfies* (3.6). *In all cases u corresponds to a single semi-orbit $(z(t), q(t))$, $t \in (-\infty, 0]$, of* (3.15), *with $z(t) > 0$, $q(t) > 0$ for all $t \in (-\infty, 0]$.*

Proof. That I attains a minimum on $\mathscr{A}$ follows immediately from Theorem 2.1. Let u be any minimizer. By Theorem 2.7 and the subsequent discussion there is a closed set E of measure zero on the complement of which u is a C^3, and hence smooth, solution of (3.5). Let D_1 be a maximal relatively open interval in $[0, 1] \setminus E$, and denote by x_0, x_1 the left and right hand endpoints of D_1 respectively. We have already noted that $u'(x) \geqq 0$ a.e., and it thus follows from Theorem 2.7 that if $x_0 \neq 0$ (respectively $x_1 \neq 1$) then $\lim_{x \to x_0+} u'(x) = +\infty$ (respectively $\lim_{x \to x_1-} u'(x) = +\infty$). If $u'(x)$ were zero for some $x \in (x_0, x_1)$ we would have, by uniqueness of solutions to (3.5), that $u = \text{const.}$ in (x_0, x_1) and thus in $D_1 = [0, 1]$, contradicting $k > 0$. Thus $u'(x) > 0$ for all $x \in (x_0, x_1)$ and u generates a solution $(z(t), q(t))$, $t \in (\log x_0, \log x_1)$, to (3.15) with $z(t) > 0$, $q(t) > 0$ for all $t \in (\log x_0, \log x_1)$. But by Proposition 3.5 the solution $(z(t), q(t))$ exists for all $t > \log x_0$, and therefore

$$\lim_{x \to x_1-} u'(x) = \lim_{t \to \log x_1 -} \tfrac{2}{3}\, q(t)\, z(t)^{-\frac{1}{3}}\, x_1^{-\frac{1}{3}} < \infty.$$

Hence $x_1 = 1$. Suppose that $-\infty \leqq t_{\min} < \log x_0$. Then

$$\lim_{x \to x_0+} u'(x) = \lim_{t \to \log x_0 +} \tfrac{2}{3}\, q(t)\, z(t)^{-\frac{1}{3}}\, x_0^{-\frac{1}{3}} < \infty,$$

since $x_0 > 0$, yielding a contradiction. Therefore $t_{\min} = \log x_0$. By Proposition 3.5 there are three cases to consider. First, we may have $(z(t), q(t)) \to (0, 0)$ as $t \to \log x_0+$. If $x_0 > 0$ this is impossible since we would then have $u(x_0) = 0$ and hence $u(x) = 0$ for all $x \in [0, 1]$. If $x_0 = 0$ then by Proposition 3.6 u is C^∞ on $[0, 1]$. Second, we may have $z(t) \to \infty$ and $q(t) \to c \geqq 0$ as $t \to \log x_0+$. In this case, by Proposition 3.6 $x_0 = 0$ and $u(0) > 0$, which is impossible. Third, we may have $x_0 = 0$ and $\lim_{t \to -\infty} (z(t), q(t)) = (\bar{k}^{-\frac{3}{2}}, \bar{k}^{-\frac{3}{2}})$, a rest point. In this case u is C^∞ on $(0, 1]$ with $u(x) \sim \bar{k}x^{\frac{2}{3}}$, $u'(x) \sim \frac{2}{3}\bar{k}x^{-\frac{1}{3}}$ as $x \to 0+$. □

As a preliminary result showing that every minimizer must in certain cases be singular we prove

Lemma 3.8. *Let u minimize I on $\mathscr{A}$, and suppose that* $0 < \alpha < \beta < \min(1, k)$ *and*

$$\frac{4\varepsilon}{3}\beta^{\frac{1}{2}} < (\tfrac{9}{13})^{13}(1-\beta^3)^2(\beta-\alpha)^{14}. \tag{3.26}$$

Then $u(x) > \alpha x^{\frac{2}{3}}$ *for all* $x \in (0, 1]$.

Proof. We modify an argument of MANIÀ [25] (see also CESARI [11] and Section 4). If the conclusion of the lemma were false then there would exist a subinterval (x_1, x_2) of $[0, 1]$ such that

$$\alpha x^{\frac{2}{3}} \leqq u(x) \leqq \beta x^{\frac{2}{3}} \quad \text{for all } x_1 \leqq x \leqq x_2$$

and $u(x_1) = \alpha x_1^{\frac{2}{3}}$, $u(x_2) = \beta x_2^{\frac{2}{3}}$. Thus

$$\int_0^{x_2} f(x, u, u')\,dx \geqq \int_{x_1}^{x_2} x^4\left(1 - \frac{u^3}{x^2}\right)^2 (u')^{14}\,dx$$

$$\geqq (1-\beta^3)^2 \int_{x_1}^{x_2} x^4 (u')^{14}\,dx.$$

Let $x = y.^{\frac{13}{9}}$ Then

$$\int_{x_1}^{x_2} x^4 (u')^{14}\,dx = (\tfrac{9}{13})^{13} \int_{x_1^{\frac{9}{13}}}^{x_2^{\frac{9}{13}}} \left(\frac{du}{dy}\right)^{14} dy,$$

and by Jensen's inequality the minimizer of this integral subject to $u\big|_{y=x_1^{\frac{9}{13}}} = \alpha x_1^{\frac{2}{3}}$, $u\big|_{y=x_2^{\frac{9}{13}}} = \beta x_2^{\frac{2}{3}}$ is given by the linear function $u = \alpha x_1^{\frac{2}{3}} + \left(\dfrac{\beta x_2^{\frac{2}{3}} - \alpha x_1^{\frac{2}{3}}}{x_2^{\frac{9}{13}} - x_1^{\frac{9}{13}}}\right)(y - x_1^{\frac{9}{13}})$.
Therefore

$$\int_0^{x_2} f(x, u, u')\,dx \geqq (\tfrac{9}{13})^{13}(1-\beta^3)^2 \frac{\left(\beta x_2^{\frac{2}{3}} - \alpha x_1^{\frac{2}{3}}\right)^{14}}{\left(x_2^{\frac{9}{13}} - x_1^{\frac{9}{13}}\right)^{13}}$$

$$= (\tfrac{9}{13})^{13}(1-\beta^3)^2 x_2^{\frac{1}{3}} \frac{\left(\beta - \alpha\left(\dfrac{x_1}{x_2}\right)^{\frac{2}{3}}\right)^{14}}{\left(1 - \left(\dfrac{x_1}{x_2}\right)^{\frac{9}{13}}\right)^{13}} \tag{3.27}$$

$$\geqq (\tfrac{9}{13})^{13}(1-\beta^3)^2 x_2^{\frac{1}{3}}(\beta-\alpha)^{14}.$$

Define $v \in \mathscr{A}$ by

$$v(x) = \begin{cases} x^{\frac{2}{3}}, & 0 \leqq x \leqq \beta^{\frac{3}{2}} x_2 \\ \beta x_2^{\frac{2}{3}}, & \beta^{\frac{3}{2}} x_2 \leqq x \leqq x_2 \\ u(x), & x_2 \leqq x \leqq 1. \end{cases}$$

Then

$$\begin{aligned} I(v) &= \int_0^{\beta^{\frac{3}{2}} x_2} \varepsilon(\tfrac{2}{3} x^{-\frac{1}{3}})^2 \, dx + \int_{x_2}^{1} f(x, u, u') \, dx \\ &= \frac{4\varepsilon}{3} \beta^{\frac{1}{2}} x_2^{\frac{1}{3}} + I(u) - \int_0^{x_2} f(x, u, u') \, dx. \end{aligned}$$

Hence if (3.26), (3.27) hold then $I(v) < I(u)$, a contradiction. □

Remark. Although MANIÀ'S device, which he developed in connection with the Lavrentiev phenomenon, is used in the proof of Lemma 3.8, our minimization problem does not exhibit this phenomenon. In fact if u is a minimizer then by Theorem 3.7 we have $|u(x)| \leqq C x^{\frac{2}{3}}$ for x near zero. Thus if

$$u_\delta(x) = \begin{cases} \dfrac{u(\delta)\, x}{\delta}, & 0 \leqq x \leqq \delta \\ u(x), & \delta \leqq x \leqq 1, \end{cases}$$

then $\lim\limits_{\delta \to 0+} \int_0^{\delta} f(x, u_\delta, u_\delta') \, dx = 0$ and so

$$\inf_{v \in W^{1,\infty}(0,1) \cap \mathscr{A}} I(v) = I(u).$$

In order to identify the minimizer from among the various geometrically possible trajectories in the phase-plane we make use of the following lemma.

Lemma 3.9. *Let* $u \in \mathscr{A}$ *be a smooth solution of* (3.5) *on* (0, 1] *with* $u(x) > 0$, $u'(x) > 0$ *for all* $x \in (0, 1]$. *Let* $(z(\cdot), q(\cdot))$ *be the corresponding solution of* (3.15). *Then*

$$I(u) = -3\psi(k^{\frac{3}{2}}, q(0)),$$

where $\psi(z, q) = (q - z) F_q - F$.

Proof. From (3.11), (3.21) we have that

$$x^{-\frac{2}{3}} F = -3 \frac{d}{dx} [x^{\frac{1}{3}} \psi], \qquad x \in (0, 1].$$

By Proposition 3.6 (see the formula for ψ in the proof of Theorem 3.7) $\lim_{x \to 0+} \psi(x)$ exists and is finite. Therefore

$$I(u) = \int_0^1 x^{-\frac{2}{3}} F\,dx = -3\psi(z(0), q(0))$$

$$= -3\psi(k^{\frac{3}{2}}, q(0)). \quad \square$$

Since $\psi_q(z, q) = (q - z) F_{qq}$ and $F_{qq} > 0$, it follows from Lemma 3.9 that, of all trajectories $(z(\cdot), q(\cdot))$ of (3.15) satisfying $z(0) = k$ and $\lim_{t \to -\infty} (z(t), q(t)) = (0, 0)$ or $(\overline{k^{\frac{3}{2}}}, \overline{k^{\frac{3}{2}}})$ (a rest point), that corresponding to an absolute minimum of I has either the greatest value of $q(0) \geqq k^{\frac{3}{2}}$ or the least value of $q(0) \leqq k^{\frac{3}{2}}$. So as to decide between these two possibilities it is convenient to restate Lemma 3.9 in the following way. Define

$$\Gamma(z, q) = \psi(z, q) + \tfrac{1}{3} \int_0^z F_q(\zeta, \zeta)\,d\zeta. \tag{3.28}$$

Then if u_1, u_2 satisfy the hypotheses of Lemma 3.9 with corresponding solutions $(z_1(\cdot), q_1(\cdot)), (z_2(\cdot), q_2(\cdot))$ of (3.15),

$$I(u_1) - I(u_2) = -3[\Gamma(k^{\frac{3}{2}}, q_1(0)) - \Gamma(k^{\frac{3}{2}}, q_2(0))]. \tag{3.29}$$

Note that by (3.21) we have that along solutions of (3.20)

$$\frac{d}{dz}\Gamma(z, q) = -\tfrac{1}{3}(F_q(z, q) - F_q(z, z))$$

$$= -(q - z)\,M(z, q, \varepsilon), \tag{3.30}$$

where $M(z, q, \varepsilon) > 0$ for $z, q > 0$. As an application of this idea we prove the following proposition.

We denote by $(z_{sm}(\cdot), q_{sm}(\cdot))$ the smooth solution orbit, which by Proposition 3.6 leaves the origin with slope $\frac{3}{2}$; this orbit is unique modulo adding an arbitrary constant to t, and we choose for convenience the normalization corresponding to the smooth solution u of (3.5) satisfying $u(0) = 0$, $u'(0) = 1$.

Proposition 3.10. *If* $z_{sm}(t) \to \infty$, $q_{sm}(t) \to \infty$ *as* $t \to \infty$ *then for any* $k > 0$ *there exists precisely one solution* u *of* (3.5) *belonging to* $C^\infty([0, 1])$ *and satisfying the boundary conditions* (3.2), *and* u *is the unique minimizer of* I *in* $\mathscr{A}$.

Proof. If u is a smooth solution of (3.5) on [0, 1] satisfying (3.2) then $u(x) > 0$, $u'(x) > 0$ for all $x \in (0, 1]$. Otherwise there would exist some $x_0 \in (0, 1)$ with $u'(x_0) = 0$, and hence $u(x) \equiv u(x_0)$ by uniqueness, a contradiction. Thus any such solution is represented by an appropriate portion of the smooth solution orbit $(z_{sm}(\cdot), q_{sm}(\cdot))$, and since this orbit cuts the line $z = k^{\frac{3}{2}}$ exactly, once the existence and uniqueness of u is assured.

It remains to prove that $I(u) < I(v)$ for every other $v \in \mathscr{A}$. If this were false there would exist by Theorem 3.7 an absolute minimizer u_1 of I in $\mathscr{A}$ with $u_1 \neq u$, $I(u_1) \leqq I(u)$. Let $(z_1(\cdot), q_1(\cdot))$ be the corresponding solution of (3.15); thus $z_1(0) = k^{\frac{3}{2}}$. We know by Theorem 3.7 that we must have $0 < \varepsilon \leqq \varepsilon_0$ and $\lim_{t\to-\infty} (z_1(t), q_1(t)) = (\bar{k}^{\frac{3}{2}}, \bar{k}^{\frac{3}{2}})$, and since P_1 is a sink we also have $\bar{k} = \bar{k}_2$ if $0 < \varepsilon < \varepsilon_0$. Since the smooth solution orbit lies entirely above any such solution, by our preceding discussion we know that $q_1(0)$ has the least value of $q(0) \leqq k^{\frac{3}{2}}$ of all solutions $(z(\cdot), q(\cdot))$ of (3.15) with $z(0) = k^{\frac{3}{2}}$, $\lim_{t\to-\infty} (z(t), q(t)) = (\bar{k}^{\frac{3}{2}}, \bar{k}^{\frac{3}{2}})$. It follows that $q_1(t) \leqq z_1(t) \leqq \bar{k}^{\frac{3}{2}}$ for all $t \in (-\infty, 0]$. Let the smooth solution orbit have graph $q = q_{\text{sm}}(z)$, $z > 0$. Then by (3.28)–(3.30) and the fact that $\psi_q(z, q) > 0$ for $q > z$,

$$\Gamma(k^{\frac{3}{2}}, q_1(0)) \leqq \Gamma(\bar{k}^{\frac{3}{2}}, \bar{k}^{\frac{3}{2}}) < \Gamma(\bar{k}^{\frac{3}{2}}, q_{\text{sm}}(\bar{k}^{\frac{3}{2}})) \leqq \Gamma(k^{\frac{3}{2}}, q_{\text{sm}}(k^{\frac{3}{2}})),$$

and thus $I(u) < I(u_1)$, a contradiction. □

We give now an alternative proof of the assertion in Proposition 3.10 that the unique smooth solution u of (3.5) minimizes I, since it illustrates the various connections between the phase-plane diagram and the field theory of the calculus of variations. We note that $u_A(x) \overset{\text{def}}{=} A^{-\frac{2}{3}} u(Ax)$ is a smooth solution of (3.5) for any $A > 0$, and that $\dfrac{\partial u_A(x)}{\partial A} = A^{\frac{1}{3}} u'(Ax) > 0$. Also, for any $x > 0$ we have

$$\lim_{A\to0+} u_A(x) = \lim_{A\to0+} A^{\frac{1}{3}} x \frac{u(Ax)}{Ax} = 0 \qquad \text{and} \qquad \lim_{A\to\infty} u_A(x) = \lim_{A\to\infty} x^{\frac{2}{3}} \left[\frac{u^{\frac{3}{2}}(Ax)}{Ax}\right]^{\frac{2}{3}}$$

$= x^{\frac{2}{3}} \lim_{t\to\infty} z_{\text{sm}}(t)^{\frac{2}{3}} = \infty$. Define $u_0(x) \equiv 0$. Then $\{u_A\}_{0\leqq A<\infty}$ is a field of extremals that simply covers the region $x > 0$, $u \geqq 0$. Let $v \in \mathscr{A}$, $v \neq u$, with $v(x) > 0$ for all $x \in (0, 1]$ and $v(x) \leqq Cx^{\frac{2}{3}}$ as $x \to 0+$ (we have already seen that any minimizer of I has these properties). In order to handle the singularity of the field at the origin define for $\delta > 0$.

$$v_\delta(x) = \begin{cases} u(x), & 0 \leqq x \leqq \delta, \\ u(\delta) + \dfrac{x-\delta}{\delta}(v(2\delta) - u(\delta)), & \delta \leqq x \leqq 2\delta, \\ v(x), & 2\delta \leqq x \leqq 1. \end{cases}$$

Then

$$I(v_\delta) - I(u) = \int_0^1 [f(x, v_\delta(x), v_\delta'(x)) - f(x, v_\delta(x), p(x, v_\delta(x)))$$
$$- (v_\delta'(x) - p(x, v_\delta(x))) f_p(x, v_\delta(x), p(x, v_\delta(x)))]\, dx,$$

where $p(x, v)$ denotes the slope function of the field. Since the integrand on the right-hand side is positive by convexity, and since it can be verified that

$\lim_{\delta\to 0+} \int_\delta^{2\delta} f(x, v_\delta, v_\delta')\,dx = 0$, it follows that $I(v_\delta) \to I(v)$ as $\delta \to 0+$, and we obtain by Fatou's Lemma that

$$I(v) - I(u) \geqq \int_0^1 [f(x, v(x), v'(x)) - f(x, v(x), p(x, v(x)))$$
$$- (v'(x) - p(x, v(x)), f_p(x, v(x), p(x, v(x))))]\,dx > 0,$$

as required.

Theorem 3.11. *There exists a number* ε^* *satisfying* $0 < \varepsilon^* < \varepsilon_1 < \varepsilon_0$ *such that*

(i) *if* $0 < \varepsilon < \varepsilon^*$ *then* $(z_{sm}(t), q_{sm}(t)) \to (\bar{k}_1^{-\frac{3}{2}}, \bar{k}_1^{-\frac{3}{2}})$ as $t \to \infty$,

(ii) *if* $\varepsilon = \varepsilon^*$ *then* $(z_{sm}(t), q_{sm}(t)) \to (\bar{k}_2^{-\frac{3}{2}}, \bar{k}_2^{-\frac{3}{2}})$ *as* $t \to \infty$, *and*

(iii) *if* $\varepsilon > \varepsilon^*$ *then* $z_{sm}(t) \to \infty$, $q_{sm}(t) \to \infty$ *as* $t \to \infty$.

Proof. We first show that there exists a minimal number ε^* with $0 \leqq \varepsilon^* < \varepsilon_1$ such that (iii) holds. If $\varepsilon > \varepsilon_0$ then $z_{sm}(t) \to \infty$, $q_{sm}(t) \to \infty$ as $t \to \infty$ by Proposition 3.5. Thus suppose $0 < \varepsilon \leqq \varepsilon_0$, and let $\bar{k} = \bar{k}_2(\varepsilon)$ if $0 < \varepsilon < \varepsilon_0$, $\bar{k} = (\tau^*)^{\frac{1}{3}}$ if $\varepsilon = \varepsilon_0$ (for τ^* as in (3.6)), and set $\tau = \bar{k}^3$. For $\gamma > \frac{39}{40}$ define $v_\gamma(x) = \bar{k} x^{\frac{2\gamma}{3}}$. Then by direct calculation

$$\begin{aligned} J(\gamma) &\overset{\text{def}}{=} \tfrac{1}{3}(\tfrac{3}{2})^{14}\,\bar{k}^{-14}(I(v_\gamma) - I(v_1)) \\ &= \tau^2\left[\frac{\gamma^{14}}{40\gamma - 39} - \frac{13\gamma^2}{4\gamma - 3} + 12\right] + \tau\left[-\frac{2\gamma^{14}}{34\gamma - 33} + \frac{20\gamma^2}{4\gamma - 3} - 18\right] \\ &\quad + \frac{\gamma^{14}}{28\gamma - 27} - \frac{7\gamma^2}{4\gamma - 3} + 6. \end{aligned}$$

Therefore

$$J(1.1) = a\tau^2 + b\tau + c,$$

where $a = 1.52378\ldots$, $b = -2.44042\ldots$, $c = .94934\ldots$. It now follows that $J(1.1)$ is negative if $\tau_- < \tau < \tau_+$, where $\tau_- = .66576\ldots$, $\tau_+ = .93578\ldots$. Since $\tau^* > \tau_-$ it follows that $\bar{k}x^{\frac{2}{3}}$ does not minimize I if $\varepsilon > \theta(\tau_+) = .0019603\ldots$. Therefore if $\varepsilon > \theta(\tau_+)$, there is some solution $(z(\cdot), q(\cdot))$ of (3.15) with $z(0) = \bar{k}^{-\frac{3}{2}}$, $q(0) \neq \bar{k}^{-\frac{3}{2}}$ and $\lim_{t\to-\infty} (z(t), q(t)) = (0, 0)$ or a rest point, and this clearly implies that $z_{sm}(t) \to \infty$, $q_{sm}(t) \to \infty$ as $t \to \infty$. Define ε^* to be the least nonnegative number such that (iii) holds. Since $\varepsilon_1 = \theta(\tau_1) = .0024735\ldots$ it follows that $0 \leqq \varepsilon^* \leqq \varepsilon_1$, as claimed.

We next prove that $\varepsilon^* > 0$. If not we would have $z_{sm}(t) \to \infty$, $q_{sm}(t) \to \infty$ as $t \to \infty$ for every $\varepsilon > 0$. By Proposition 3.10 all minimizers of I in $\mathscr{A}$ would then be smooth for any $k > 0$. But by Lemma 3.8 this is false for $\varepsilon > 0$ sufficiently small.

For the remainder of the proof it is convenient to make the dependence on ε explicit by writing $z_{sm}(t) = z_{sm}(t, \varepsilon)$, $q_{sm}(t) = q_{sm}(t, \varepsilon)$, and where appropriate $q_{sm}(z) = q_{sm}(z, \varepsilon)$. Using the implicit function theorem it is easily shown that if $z_{sm}(\bar{t}, \bar{\varepsilon}) = \bar{z} > 0$, $q_{sm}(\bar{t}, \bar{\varepsilon}) \neq \bar{z}$ then there exists a smooth function $t(\varepsilon)$ defined for ε near $\bar{\varepsilon}$ such that $z_{sm}(t(\varepsilon), \varepsilon) = \bar{z}$. Thus if $z_{sm}(t, \varepsilon^*) \to \infty$ as $t \to \infty$ we also have $z_{sm}(t, \varepsilon) \to \infty$ as $t \to \infty$ for ε near ε^*, contradicting the minimality of ε^*. Likewise, if $(z_{sm}(t, \varepsilon^*), q_{sm}(t, \varepsilon^*)) \to (\bar{k}_1(\varepsilon^*)^{\frac{3}{2}}, \bar{k}_1(\varepsilon^*)^{\frac{3}{2}})$ as $t \to \infty$ then since $\varepsilon^* < \varepsilon_1$ we have $q_{sm}(t, \varepsilon^*) < z_{sm}(t, \varepsilon^*)$ for some t; thus $q_{sm}(t', \varepsilon) < z_{sm}(t', \varepsilon)$ for some $\varepsilon > \varepsilon^*$ and some t', a contradiction. There remains only one possibility, that $(z_{sm}(t, \varepsilon^*), q_{sm}(t, \varepsilon^*)) \to (\bar{k}_2(\varepsilon^*)^{\frac{3}{2}}, \bar{k}_2(\varepsilon^*)^{\frac{3}{2}})$ as $t \to \infty$, which proves (ii).

We next remark that for any $\varepsilon > 0$ the slope of the vector field on the curve $q = \frac{14}{13z}$ equals, by (3.20),

$$H\left(z, \frac{14}{13z}, \varepsilon\right) = \frac{14^2}{39z^2(14 - 13z^2)},$$

which is positive if $0 < z \leqq 1$. In particular $q_{sm}(z, \varepsilon^*) < \frac{14}{13z}$ for all $z \in (0, \bar{k}_2(\varepsilon^*)^{\frac{3}{2}})$. An easy computation also shows that $\frac{\partial H}{\partial \varepsilon}(z, q, \varepsilon) > 0$ for $0 < z < 1$, $z < q < \frac{14}{13z}$, $\varepsilon > 0$. Suppose that $0 < \varepsilon < \varepsilon^*$ but that $(z_{sm}(t, \varepsilon), q_{sm}(t, \varepsilon)) \not\to (\bar{k}_1(\varepsilon)^{\frac{3}{2}}, \bar{k}_1(\varepsilon)^{\frac{3}{2}})$ as $t \to \infty$. Since $\bar{k}_2(\varepsilon)$ is decreasing in ε we must then have $q_{sm}(\bar{k}_2(\varepsilon^*)^{\frac{3}{2}}, \varepsilon) > \bar{k}_2(\varepsilon^*)^{\frac{3}{2}}$. Choose any q_0 with $q_{sm}(\bar{k}_2(\varepsilon^*)^{\frac{3}{2}}, \varepsilon) > q_0 > \bar{k}_2(\varepsilon^*)^{\frac{3}{2}}$ and consider the solution $(z(t, \varepsilon), q(t, \varepsilon))$ of (3.15) satisfying $z(0, \varepsilon) = \bar{k}_2(\varepsilon^*)^{\frac{3}{2}}$, $q(0, \varepsilon) = q_0$. For $t < 0$ this solution curve cannot cross the $(z_{sm}(\cdot, \varepsilon), q_{sm}(\cdot, \varepsilon))$ orbit and hence by Proposition 3.6 it crosses $q = z$. Therefore there exists $\hat{z} \in (0, \bar{k}_2(\varepsilon^*)^{\frac{3}{2}})$ with $q(\hat{z}, \varepsilon) = q_{sm}(\hat{z}, \varepsilon^*)$, $\frac{dq}{dz}(\hat{z}, \varepsilon) \geqq \frac{dq_{sm}}{dz}(\hat{z}, \varepsilon^*)$, where $q = q(z, \varepsilon)$ denotes the graph of $(z(\cdot, \varepsilon), q(\cdot, \varepsilon))$. But this contradicts the monotonicity of $H(\hat{z}, q_{sm}(\hat{z}, \varepsilon), \cdot)$. Therefore (i) holds. □

Remarks. The numerical evidence is that $\varepsilon^* = .00173\ldots$. That $(z_{sm}(t), q_{sm}(t)) \to (\bar{k}_1^{\frac{3}{2}}(\varepsilon), \bar{k}_1^{\frac{3}{2}}(\varepsilon))$ as $t \to \infty$ for $\varepsilon > 0$ sufficiently small can also be proved by trapping the smooth solution orbit in an appropriate triangular invariant region, but the calculations are rather tedious.

For $0 < \varepsilon < \varepsilon^*$ we denote by $\zeta_1(\varepsilon)$ the maximum value of $z_{sm}(t)$, $t \in \mathbb{R}$, which is achieved when the smooth solution orbit cuts $q = z$, $z > 0$, for the first time. It follows immediately from Theorem 3.11 that if $0 < \varepsilon < \varepsilon^*$, $k^{\frac{3}{2}} > \zeta_1(\varepsilon)$ (or if $\varepsilon = \varepsilon^*$ and $k \geqq \bar{k}_2(\varepsilon)$) then *there is no smooth solution to the Dirichlet problem consisting in the Euler-Lagrange equation* (3.5) *and the boundary conditions* (3.2).

In the following theorem we identify the absolute minimizer of I in $\mathscr{A}$ for every $k > 0$, $\varepsilon > 0$. If $0 < \varepsilon < \varepsilon_1$ we denote by $\zeta_0(\varepsilon)$ the minimum value of z on the unstable manifold of P_2, which is achieved when that part of the unstable manifold in $q \leqq z$ cuts $q = z$ for the first time.

Theorem 3.12

(a) *Let* $0 < \varepsilon < \varepsilon^*$. *There exists a number* $\zeta(\varepsilon)$ *with* $\zeta_0(\varepsilon) < \zeta(\varepsilon) < \zeta_1(\varepsilon)$ *such that*

(i) *if* $0 < k < \zeta(\varepsilon)^{\frac{2}{3}}$ *there is exactly one u that minimizes I in* $\mathscr{A}$ *and u is the unique smooth solution of* (3.5) *on* $[0, 1]$ *satisfying* (3.2),

(ii) *if* $k = \zeta(\varepsilon)^{\frac{2}{3}}$ *there are exactly two functions* u_1, u_2 *that minimize I in* $\mathscr{A}$; u_1 *is the unique smooth solution of* (3.5) *on* $[0, 1]$ *satisfying* (3.2), *and* $u_2(x) \sim \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$ *as* $x \to 0+$ *and corresponds to that connected part of the unstable manifold of* P_2 *defined by* $q \leqq z$, $\zeta(\varepsilon) \leqq z \leqq \bar{k}_2^{\frac{3}{2}}(\varepsilon)$,

(iii) *if* $k > \zeta(\varepsilon)^{\frac{2}{3}}$ *there is exactly one u that minimizes I in* $\mathscr{A}$; $u(x) \sim \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$ *as* $x \to 0+$ *and corresponds to that part of the unstable manifold of* P_2 *defined by* $q \leqq z, k^{\frac{3}{2}} \leqq z \leqq \bar{k}_2^{\frac{3}{2}}(\varepsilon)$ *if* $k \leqq \bar{k}_2(\varepsilon)$ *and by* $q \geqq z, \bar{k}_2^{\frac{3}{2}}(\varepsilon) \leqq z \leqq k^{\frac{3}{2}}$ *if* $k \geqq \bar{k}_2(\varepsilon)$, *so that in particular if* $k = \bar{k}_2(\varepsilon)$ *then* $u(x) = \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$.

(b) *Let* $\varepsilon = \varepsilon^*$. *Then there is exactly one u that minimizes I in* $\mathscr{A}$. *If* $k < \bar{k}_2(\varepsilon^*)$ *then u is the unique smooth solution of* (3.5) *on* $[0, 1]$ *satisfying* (3.2), *and, if* $k \geqq \bar{k}_2(\varepsilon^*)$, $u(x) \sim \bar{k}_2(\varepsilon^*)\, x^{\frac{2}{3}}$ *as* $x \to 0+$ *and corresponds to that connected part of the unstable manifold of* P_2 *defined by* $q \geqq z$, $\bar{k}_2^{\frac{3}{2}}(\varepsilon) \leqq z \leqq k^{\frac{3}{2}}$. *In particular if* $k = \bar{k}_2(\varepsilon^*)$ *then* $u(x) = \bar{k}_2(\varepsilon^*)\, x^{\frac{2}{3}}$.

(c) *Let* $\varepsilon > \varepsilon^*$. *Then there is exactly one u that minimizes I in* $\mathscr{A}$ *and u is the unique smooth solution of* (3.5) *on* $[0, 1]$ *satisfying* (3.2).

Proof. Part (c) follows immediately from Theorem 3.11(iii) and Proposition 3.10. If $0 < \varepsilon < \varepsilon^*$ and $k \in (0, \zeta_0(\varepsilon)^{\frac{2}{3}}) \cup (\zeta_1(\varepsilon)^{\frac{2}{3}}, \infty)$ or if $\varepsilon = \varepsilon^*$ and $k \in (0, \zeta_0(\varepsilon^*)^{\frac{2}{3}}) \cup [\bar{k}_2(\varepsilon^*), \infty)$ then the solution specified in the theorem is the only geometrically possible one, and perforce by Theorem 3.7 is the unique minimizer. Suppose $0 < \varepsilon < \varepsilon^*$ and $k \in [\zeta_0(\varepsilon)^{\frac{2}{3}}, \zeta_1(\varepsilon)^{\frac{2}{3}}]$. By Lemma 3.9 and the subsequent discussion there are only two possibilities for a minimizer, a smooth solution $u = u_1(x, k)$ represented by part of the smooth solution orbit $q = q_{\text{sm}}(z)$, $0 \leqq z \leqq k^{\frac{3}{2}}$, $q \geqq z$ and a singular solution $u = u_2(x, k)$ represented by a part of the unstable manifold of P_2 which we denote by $q = q_{\text{un}}(z)$, $k^{\frac{3}{2}} \leqq z \leqq \bar{k}_2(\varepsilon)^{\frac{3}{2}}$,

$q \leqq z$. Define

$$R(k) = I(u_1(\cdot, k)) - I(u_2(\cdot, k)).$$

By Lemma 3.9 we have

$$R(\zeta_0(\varepsilon)^{\frac{2}{3}}) = -3(\psi(\zeta_0(\varepsilon), q_{sm}(\zeta_0(\varepsilon))) - \psi(\zeta_0(\varepsilon), \zeta_0(\varepsilon))) < 0,$$

and

$$R(\zeta_1(\varepsilon)^{\frac{2}{3}}) = -3(\psi(\zeta_1(\varepsilon), \zeta_1(\varepsilon)) - \psi(\zeta_1(\varepsilon), q_{un}(\zeta_1(\varepsilon)))) > 0.$$

Also, by (3.29), (3.30),

$$\frac{dR}{dk}(k) > 0 \quad \text{for } \zeta_0(\varepsilon)^{\frac{2}{3}} \leqq k \leqq \zeta_1(\varepsilon)^{\frac{2}{3}}.$$

Hence $R(\zeta(\varepsilon)^{\frac{2}{3}}) = 0$ for a unique $\zeta(\varepsilon) \in (\zeta_0(\varepsilon), \zeta_1(\varepsilon))$ and part (a) follows.

In the case $\varepsilon = \varepsilon^*$, $k \in [\zeta_0(\varepsilon^*)^{\frac{2}{3}}, \bar{k}_2(\varepsilon^*)]$ we define $R(k)$ as above and note that $R(\zeta_0(\varepsilon^*)^{\frac{2}{3}}) < 0$, $\lim\limits_{k \to \bar{k}_2(\varepsilon^*)-} R(k) = 0$, $\dfrac{dR}{dk}(k) > 0$ for $k \in [\zeta_0(\varepsilon^*)^{\frac{2}{3}}, \bar{k}_2(\varepsilon^*))$. Hence part (b) holds. □

The results of Theorems 3.11, 3.22 are summarized pictorially in Figure 3.1. Note that whenever the minimizer is singular at the origin neither (IEL) nor (IDBR) holds, since then both $\lim\limits_{x \to 0+} f_p(x, u(x), u'(x))$ and $\lim\limits_{x \to 0+} [u'(x) f_p(x, u(x), u'(x)) - f(x, u(x), u'(x))]$ are $+\infty$. However, in all cases (WEL) and (WDBR) are satisfied. Note also that if $0 < \varepsilon < \varepsilon_0$ then $u(x) = \bar{k}_1(\varepsilon)\, x^{\frac{2}{3}}$ is never a minimizer. It is interesting to observe from the figure how for fixed ε the number of solutions $u \in C^\infty((0, 1])$ of (3.5) satisfying $u(0) = 0$, $u(1) = k$ varies with k. For example, if $0 < \varepsilon < \varepsilon_1$ then as k approaches k_1 the number of such solutions tends to infinity. An alternative proof that for $k = \bar{k}_2(\varepsilon)$ and $\varepsilon > 0$ sufficiently small $u(x) = \bar{k}_2(\varepsilon)\, x^{\frac{2}{3}}$ minimizes I in $\mathscr{A}$ has been given by Clarke & Vinter [14].

We conclude our discussion of the phase portrait with a few remarks concerning the behavior of the branch of the unstable manifold of P_2 that near P_2 lies in $q < z$. Since, by Lemma 3.8, if $k > 0$ is arbitrary but fixed then any minimizer of I in $\mathscr{A}$ is singular provided $\varepsilon > 0$ is sufficiently small, it follows from Theorem 3.12 that $\zeta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence also $\zeta_0(\varepsilon) \to 0$ as $\varepsilon \to 0$; this is consistent with the fact that the slope of the unstable manifold at P_2 tends to infinity as $\varepsilon \to 0$. As $t \to \infty$ the above branch of the unstable manifold tends to $(\bar{k}_1(\varepsilon)^{\frac{3}{2}}, \bar{k}_1(\varepsilon)^{\frac{3}{2}})$; in fact it cannot tend to $(\bar{k}_2(\varepsilon)^{\frac{3}{2}}, \bar{k}_2(\varepsilon)^{\frac{3}{2}})$ by Proposition 3.4, and it cannot tend to infinity because the upper branch of the stable manifold at P_2 would then have nowhere to go as $t \to -\infty$. For $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0 - \varepsilon$ very small an application of center manifold theory (see, for example, Carr [10]) shows that the connecting orbit from P_2 to P_1 is almost parallel to the line $q = z$.

§ 4. The case with no x-dependence

In this section we consider the problem of minimizing

$$I(u) = \int_{-1}^{1} f(u(x), u'(x))\, dx \tag{4.1}$$

in

$$\mathscr{A} = \{u \in W^{1,1}(-1, 1) : u(-1) = k_1, u(1) = k_2\}, \tag{4.2}$$

where $k_1, k_2 \in \mathbb{R}$. Concerning the integrand $f = f(u, p)$ we will require that

$$\begin{gathered} f \in C^\infty(\mathbb{R}^2), \quad f_{pp} > 0, \\ |p| \leqq f(u, p) \leqq \text{const.}\,(1 + p^2), \quad (u, p) \in \mathbb{R}^2. \end{gathered} \tag{4.3}$$

We will show that an absolute minimizer u_0 of I over $\mathscr{A}$ need not satisfy (WEL) or (IEL), although by Corollary 2.5 u_0 must satisfy (IDBR). In our examples u_0 is constructed directly, though as we remarked in Section 2, for the functions f appearing below the existence of u_0 also follows from known extensions of Theorem 2.1.

We first give an example where the Tonelli set $E = \{x_0\}$ is a singleton.

Theorem 4.1. *There exist an f satisfying* (4.3) *and*

$$\frac{f(u, p)}{|p|} \to \infty \quad \text{as } |p| \to \infty \text{ for each } u \neq 0 \tag{4.4}$$

and a number $k_0 > 0$ such that whenever $-k_1, k_2 > k_0$ then (4.1), (4.2) *has a unique global minimizer u_0, but $E = \{x_0\}$ for some $x_0 = x_0(k_1, k_2) \in (-1, 1)$, and*

$$f_u(u_0, u_0') \notin L^1_{\text{loc}}(-1, 1),$$

so that neither (WEL) *nor* (IEL) *is satisfied.*

Remark. The theorem shows that if (2.13) fails for just *one* value of u then the conclusion of Corollary 2.12 need not hold.

Proof of Theorem 4.1. The proof splits naturally into two parts. Part I is devoted to the construction of a strictly monotone function $g \in C^1(\mathbb{R})$ satisfying

(g1) $g \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$,

(g2) $g'' \notin L^1(-\delta, \delta)$ for any $\delta > 0$,

and to the solution of the minimization problem on $\mathscr{A}$ for a certain functional J involving g. Part II then presents the construction of an integrand f satisfying (4.3), (4.4) such that the corresponding functional I has the same global minimizer as J over $\mathscr{A}$.

Part I. Select an even function $h \in C(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ such that

$$\left.\begin{array}{llll} h(0)=0, & 0 \leqq h \leqq 1, & h(s)>0 & \text{for } s \neq 0, \\ h(s)=1 & \text{for } |s| \geqq 1/2, & h' \notin L^1(-\delta,\delta) & \text{for any } \delta>0. \end{array}\right\} \tag{4.5}$$

For instance,

$$h(s) = s^2(2 + \sin(s^{-2}))\,\eta(s) + (1-\eta(s)), \quad s \in \mathbb{R},$$

with $\eta \in C^\infty$ an even function satisfying

$$0 \leqq \eta \leqq 1, \quad \eta(s) = 0 \quad \text{for} \quad |s| \geqq 1/2, \quad \eta(s) = 1 \quad \text{for} \quad |s| \leqq 1/4,$$

defines such a function.

Now specify $g \in C^1(\mathbb{R})$ by

$$g' = h, \qquad g(0) = 0, \tag{4.6}$$

and note that g is odd, strictly monotone, and satisfies (g1), (g2). Put

$$J(u) = \int_{-1}^{1} [g'(u(x))\,u'(x)]^2\,dx, \qquad u \in \mathscr{A}. \tag{4.7}$$

Since g is C^1 it readily follows that

$$g \circ u \in W^{1,1}(-1,1) \qquad \text{for all } u \in W^{1,1}(-1,1). \tag{4.8}$$

Hence given $l \in \mathbb{R}$ one can decompose J as follows:

$$\begin{aligned} J(u) &= \int_{-1}^{1} [(g'(u(x))\,u'(x) - l)^2 + 2l g'(u(x))\,u'(x) - l^2]\,dx \\ &= \int_{-1}^{1} (g'(u(x))\,u'(x) - l)^2\,dx + 2l(g(k_2) - g(k_1)) - 2l^2, \end{aligned} \tag{4.9}$$

for all $u \in \mathscr{A}$. Thus it is clear that if $u \in \mathscr{A}$ satisfies for some l

$$g'(u(x))\,u'(x) = l, \qquad \text{a.e. } x \in [-1,1], \tag{4.10}$$

then u is a global minimizer of J in $\mathscr{A}$. By (4.8) this last condition requires that

$$g(u(x)) = lx + m, \qquad x \in [-1,1], \tag{4.11}$$

and the end conditions on u imply that

$$l = \tfrac{1}{2}(g(k_2) - g(k_1)), \qquad m = \tfrac{1}{2}(g(k_1) + g(k_2)). \tag{4.12}$$

Now since g is strictly increasing and has range $\mathbb{R}$, (4.11), (4.12) determine a unique strictly increasing function $u_0 \in C([-1,1])$. Moreover, by the inverse function theorem $\left(\text{applied to } \frac{1}{l}(g - m)\right)$ it follows that

$$u_0 \in C([-1,1]) \cap C^\infty([-1,1] \setminus \{x_0\}), \tag{4.13}$$

where x_0 is the unique point such that $u_0(x_0) = 0$. Finally, by (4.12) it follows that

$$u_0(-1) = k_1, \quad u_0(1) = k_2.$$

Therefore if the function u_0 defined by (4.11), (4.12) is absolutely continuous, then u_0 belongs to $\mathscr{A}$ and provides a (unique) global minimizer for J. The absolute continuity of u_0 is now verified by making use of the monotonicity of u_0 and (4.13):

$$\int_{-1}^{1} |u_0'(x)|\, dx = \int_{-1}^{x_0} u_0'(x)\, dx + \int_{x_0}^{1} u_0'(x)\, dx = (u_0(x_0) - u_0(-1)) + (u_0(1) - u_0(x_0))$$
$$= k_2 - k_1 < \infty.$$

Part II. Write

$$f^0(u, p) = (g'(u)\, p)^2, \quad (u, p) \in \mathbb{R}^2,$$

so that

$$f^0_{pp}(u, p) > 0 \quad \text{if and only if} \quad u \neq 0.$$

Using (4.11)–(4.13) and (g1), we have

$$f^0_u(u_0, u_0') = 2(g'(u_0)\, u_0')\, g''(u_0)\, u_0'$$
$$= (g(k_2) - g(k_1))\, g''(u_0)\, u_0' \quad \in C^\infty([-1, 1] \setminus \{x_0\}).$$

Therefore, by (g2), if $-1 < a < x_0 < b < 1$ then

$$\int_a^b |f^0_u(u_0, u_0')|\, dx = \lim_{h \to 0+} \left[\int_a^{x_0 - h} |f^0_u(u_0, u_0')|\, dx + \int_{x_0 + h}^b |f^0_u(u_0, u_0')|\, dx \right]$$
$$= (g(k_2) - g(k_1)) \lim_{h \to 0+} \left[\int_{u_0(a)}^{u_0(x_0 - h)} |g''(u)|\, du + \int_{u_0(x_0 + h)}^{u_0(b)} |g''(u)|\, du \right]$$
$$= \infty,$$

so that

$$f^0_u(u_0, u_0') \notin L^1_{\text{loc}}(-1, 1). \tag{4.14}$$

A function $f \in C^\infty(\mathbb{R}^2)$ satisfying (4.3) as well as

$$\left.\begin{aligned} &f(u, p) \geqq f^0(u, p) + p, \quad (u, p) \in \mathbb{R}^2, \\ &f(u, p) = f^0(u, p) + p \quad \text{when} \quad g'(u)\, p \geqq l - \delta, \end{aligned}\right\} \tag{4.15}$$

where l is given by (4.12) and $\delta > 0$, is constructed below. Obviously for f satisfying (4.15) and for u_0 as in (4.11), (4.12),

$$I(u_0) \stackrel{\text{def}}{=} \int_{-1}^{1} f(u_0, u_0')\, dx = \int_{-1}^{1} [f^0(u_0, u_0') + u_0']\, dx = J(u_0) + k_2 - k_1, \tag{4.16}$$

so that u_0 is also the unique global minimizer for I over $\mathscr{A}$. Moreover, by (4.14), (4.15),

$$f_u(u_0, u_0') = f^0_u(u_0, u_0') \notin L^1_{\text{loc}}(-1, 1). \tag{4.17}$$

Also, since (4.10) implies that

$$u_0'(x) = \frac{l}{g'(u_0(x))} \to \infty \quad \text{as} \quad x \to x_0,$$

it follows by TONELLI'S partial regularity theorem (Theorem 2.7) that

$$u_0'(x_0) = \infty,$$

so that the Tonelli set of u_0 is the singleton $E = \{x_0\}$, completing the conclusions of the theorem.

To construct f satisfying (4.3) and (4.15) we first construct an appropriate function $e \in C^\infty([0, 1] \times \mathbb{R})$ such that the formula

$$f(u, p) \stackrel{\text{def}}{=} e((g'(u))^2, p) + p \tag{4.18}$$

yields a function f with the desired properties. Let $\varrho \in C^\infty(\mathbb{R})$ be a nonnegative even function with $\operatorname{supp} \varrho \subset (-1, 1)$, $\int_{-\infty}^{\infty} \varrho(p)\, dp = 1$, and put

$$\alpha = \int_{-\infty}^{\infty} p^2 \varrho(p)\, dp.$$

Thus $\varrho_\varepsilon(p) = \varepsilon^{-1} \varrho(p/\varepsilon)$ satisfies

$$\operatorname{supp} \varrho_\varepsilon \subset (-\varepsilon, \varepsilon), \quad \int_{-\infty}^{\infty} \varrho_\varepsilon(p)\, dp = 1, \quad \int_{-\infty}^{\infty} p^2 \varrho_\varepsilon(p)\, dp = \varepsilon^2 \alpha. \tag{4.19}$$

Now let $\theta \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ be given by

$$\theta(p) = \begin{cases} 2p^2 - p + 1, & p \leqq 0 \\ (p + 1)^{-1}, & p > 0. \end{cases} \tag{4.20}$$

Note that θ is strictly convex, with $\theta''(p) > 0$ for $p \neq 0$. We claim that for $\varepsilon > 0$ small and $b \in (0, 1]$ the graphs of θ and of $v(p) = b(p^2 - \alpha\varepsilon^2)$, $p \in \mathbb{R}$, intersect at a unique point $p_b \in [\frac{1}{2}, \infty)$. The existence and uniqueness of the intersection follows from the strict monotonicity, in opposing senses, of θ and v on $0 \leqq p < \infty$. The condition for intersection:

$$(p + 1)(p^2 - \alpha\varepsilon^2) = b^{-1}, \tag{4.21}$$

implies when $b = 1$ that

$$(p_1 + 1)\, p_1^2 > 1,$$

so that $p_1 > \frac{1}{2}$. Since the left-hand side of (4.21) is strictly increasing on $\frac{1}{2} \leqq p < \infty$ when $\alpha\varepsilon^2 < 1$, it follows that then $p_b \in [1/2, \infty)$ as required. Note also that (4.21) yields the asymptotic estimate

$$p_b \sim b^{-\frac{1}{3}} \quad \text{for} \quad b \sim 0. \tag{4.22}$$

By the inverse function theorem p_b is C^∞ on $(0, 1]$. Therefore on defining

$$e(b,p) = (\varrho_\varepsilon * \max\{\theta, v\})(p)$$
$$= \int_{-\infty}^{p_b} \varrho_\varepsilon(p-q)\,\theta(q)\,dq + \int_{p_b}^{\infty} \varrho_\varepsilon(p-q)\,v(q)\,dq, \tag{4.23}$$

one obtains e as the sum of two functions in $C^\infty((0,1]\times\mathbb{R})$, so that $e \in C^\infty((0,1]\times\mathbb{R})$. Moreover, by (4.19), if $p > p_b + \varepsilon$ then

$$e(b,p) = b\int_{-\infty}^{\infty} \varrho_\varepsilon(q)\,[(p-q)^2 - \alpha\varepsilon^2]\,dq$$
$$= b\int_{-\infty}^{\infty} \varrho_\varepsilon(q)\,[q^2 - 2pq + p^2 - \alpha\varepsilon^2]\,dq = bp^2, \tag{4.24}$$

while if $p < p_b - \varepsilon$ then

$$e(b,p) = \int_{-\infty}^{\infty} \varrho_\varepsilon(p-q)\,\theta(q)\,dq \overset{\text{def}}{=} \varphi(p). \tag{4.25}$$

It is easily verified that φ has the following properties:

$$\left.\begin{aligned} \varphi > 0; \quad \varphi(p) &= 2p^2 - p + 1 + 2\alpha\varepsilon^2 \quad \text{for } p \leqq -\varepsilon, \\ \varphi(p) &\geqq \theta(p+\varepsilon) = (p+\varepsilon+1)^{-1} \quad \text{for } p \geqq -\varepsilon. \end{aligned}\right\} \tag{4.26}$$

Thus, since by (4.23)

$$e(b,p) \geqq \max\{(\varrho_\varepsilon * \theta)(p), (\varrho_\varepsilon * v)(p)\} = \max\{\varphi(p), bp^2\}, \quad p \in \mathbb{R}, \tag{4.27}$$

it follows from (4.26) that for $\varepsilon > 0$ sufficiently small

$$e(b,p) + p \geqq \varphi(p) + p \geqq |p|, \quad (b,p) \in (0,1]\times\mathbb{R}. \tag{4.28}$$

Now set

$$e(0,p) = \varphi(p), \quad p \in \mathbb{R}, \tag{4.29}$$

and define f by (4.18). It is immediate that $f \in C^\infty((\mathbb{R}\setminus\{0\})\times\mathbb{R})$. Furthermore, since for any interval $-A \leqq p \leqq A$ there exists by (4.22) a number $\delta_A > 0$ such that

$$e(g'(u)^2, p) + p = \varphi(p) + p, \quad |g'(u)| < \delta_A, \quad p \in [-A, A],$$

it is seen that actually $f \in C^\infty$ as required. The proof that f satisfies (4.3) is straightforward: the property $f_{pp} > 0$ follows from the facts that $\theta''(p) > 0$ for all $p \neq 0$, $v''(p) > 0$ if $b \in (0,1]$, $p \in \mathbb{R}$, while the growth condition follows from (4.23), (4.26) and (4.28). The growth condition (4.4) is a consequence of (4.24). The inequality in (4.15) results directly from (4.27), (4.29). To establish the equation in (4.15) we choose $k_0 = 3/2$, so that by (4.5), (4.6) there exists $\delta > 0$ such that

$$l = \tfrac{1}{2}(g(k_2) - g(k_1)) > g(k_0) > 1 + 2\delta$$

whenever $-k_1, k_2 > k_0$. Next, we note that by (4.21)

$$p_b^3 \leqq p_b^3 + \left(p_b - \frac{\alpha\varepsilon^2}{2}\right)^2 = b^{-1} + \alpha\varepsilon^2 + \frac{\alpha^2\varepsilon^4}{4},$$

so that for sufficiently small $\varepsilon > 0$ we have

$$l - \delta > b^{\frac{1}{2}} p_b + \varepsilon b^{\frac{1}{2}} \quad \text{for all } b \in (0, 1]. \tag{4.30}$$

Now suppose that $b^{\frac{1}{2}} p \geqq l - \delta$ for some $b \in (0, 1]$, $p \in \mathbb{R}$. By (4.30) we have

$$p \geqq b^{-\frac{1}{2}}(l - \delta) > p_b + \varepsilon,$$

so that $e(b, p) = bp^2$ by (4.24). This completes the proof. □

Remarks. 1. If the construction in Theorem 4.1 is repeated with a function $h' \in L^2(-1, 1)$ then it is easily verified that the minimizer u_0 does satisfy (IEL), (WEL) even though $u'(x_0) = \infty$.

2. Let $\varphi \in C_0^\infty(-1, 1)$ be nonzero in a neighborhood of x_0. Then for any $t \neq 0$ there exist constants $c(t) > 0$, $\alpha(t) > 0$, such that

$$I(u_0 + t\varphi) \geqq c(t) \int_{x_0-\alpha(t)}^{x_0+\alpha(t)} (u_0'(x))^2 \, dx.$$

Since

$$\int_{x_0-\alpha(t)}^{x_0+\alpha(t)} (u_0'(x))^2 \, dx = \int_{u_0(x_0-\alpha(t))}^{u_0(x_0+\alpha(t))} l \frac{du}{h(u)},$$

it follows that if $\frac{1}{h} \notin L^1(-\delta, \delta)$, for any $\delta < 0$, which is clearly consistent with (4.5), then $I(u_0 + t\varphi) = +\infty$.

We now give an example where the Tonelli set E is any prescribed closed Lebesgue null set; this shows that the Tonelli partial regularity theorem (Theorem 2.7) is in a certain sense optimal.

Theorem 4.2. *Given any closed subset* $E \subset [-1, 1]$ *of measure zero, there exists a function* $f = f^E$ *satisfying* (4.3) *and*

$$\frac{f(u, p)}{|p|} \to \infty \quad \text{as } |p| \to \infty \quad \text{for all } u \notin F, \tag{4.4$'$}$$

with F *a Lebesgue null set, such that for certain scalars* $k_1, k_2 \in \mathbb{R}$, *there variational problem* (4.1), (4.2) *has a unique global minimizer* u_0, *and* u_0 *is strictly increasing with*

$$u_0'(x) = +\infty \quad \text{if and only if } x \in E.$$

Furthermore

$$f_u(u_0, u_0') \notin L^1_{\text{loc}}(-1, 1),$$

so that neither (WEL) *nor* (IEL) *is satisfied.*

Proof. Again the proof splits naturally into two parts, with Part II identical with the argument for Part II in Theorem 4.1. Hence only Part I is given here.

Part I. The construction *begins* with the global minimizer u_0 and then yields a function $g \in C^1(\mathbb{R})$ satisfying

(g1)′ $g \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus F)$, with $F \subset \mathbb{R}$ a compact Lebesgue null set,

(g2)′ $g'' \notin L^1(a, b)$ for any (a, b) such that $F \cap (a, b) \neq \emptyset$.

Let $k \in C(\mathbb{R}) \cap C^\infty(0, 1)$ satisfy

$$\begin{aligned} &k(t) = 0 \quad \text{for } t \in (-\infty, 0], \qquad k(t) = 2 \quad \text{for } t \in [1, \infty), \\ &k'(t) > 1 \quad \text{for } t \in (0, 1), \qquad \lim_{t \to 0+} k'(t) = \lim_{t \to 1-} k'(t) = +\infty. \end{aligned} \tag{4.31}$$

We take the harder case when E is an infinite set such that neither -1 nor 1 belongs to E. The modifications necessary when E is finite and/or one or both endpoints belong to E are easily made. Let $x_- = \min_{x \in E} x$, $x_+ = \max_{x \in E} x$, so that $-1 < x_- < x_+ < 1$. Pick $c < -1$, $d > 1$. Then

$$(c, d) \setminus E = \bigcup_{j=1}^{\infty} \mathcal{O}_j,$$

where the $\mathcal{O}_j = (a_j, b_j)$, $j \geqq 1$, are disjoint and open, with

$$(a_1, b_1) = (c, x_-), \quad (a_2, b_2) = (x_+, d).$$

Clearly

$$\sum_{j=1}^{\infty} |\mathcal{O}_j| = d - c, \tag{4.32}$$

where $|\mathcal{O}_j| = b_j - a_j$. It follows that

$$\alpha \overset{\text{def}}{=} \sum_{j=1}^{\infty} \varphi(|\mathcal{O}_j|) < \infty \tag{4.33}$$

for some increasing continuous function $\varphi : (0, \infty) \to (0, \infty)$ satisfying

$$\frac{\varphi(t)}{t} \geqq 1, \quad t > 0; \quad \lim_{t \to 0+} \frac{\varphi(t)}{t} = \infty. \tag{4.34}$$

Define $\bar{u}_0 : [c, d] \to \mathbb{R}$ by

$$\bar{u}_0(x) = \sum_{j=1}^{\infty} \varphi(|\mathcal{O}_j|)\, k\left(\frac{x - a_j}{b_j - a_j}\right). \tag{4.35}$$

By (4.31), (4.33) it follows that this series is uniformly convergent on $\mathbb{R}$. Moreover, for $x \in \mathcal{O}_j$,

$$\begin{aligned} \bar{u}_0(x) &= \sum_{i \neq j} \varphi(|\mathcal{O}_i|)\, k\left(\frac{x - a_i}{b_i - a_i}\right) + \varphi(|\mathcal{O}_j|)\, k\left(\frac{x - a_j}{b_j - a_j}\right) \\ &= \bar{u}_0(a_j) + \varphi(|\mathcal{O}_j|)\, k\left(\frac{x - a_j}{b_j - a_j}\right), \end{aligned} \tag{4.36}$$

so that

$$|\bar{u}_0(\mathcal{O}_j)| = \lim_{x\to b_j^-} \bar{u}_0(x) - \bar{u}_0(a_j) = 2\varphi(|\mathcal{O}_j|). \tag{4.37}$$

It follows from (4.36), (4.37) that $\bar{u}_0$ is strictly increasing on $[c, d]$, and $\bar{u}_0 \in C^\infty([c, d] \setminus E)$ with

$$\bar{u}_0(c) = 0, \quad \bar{u}_0(d) = 2\alpha. \tag{4.38}$$

Furthermore,

$$\int_c^d |\bar{u}_0'(x)|\, dx = \sum_{j=1}^\infty \int_{a_j}^{b_j} \frac{\varphi(|\mathcal{O}_j|)}{b_j - a_j} k' \left(\frac{x - a_j}{b_j - a_j}\right) dx$$

$$= \sum_{j=1}^\infty 2\varphi(|\mathcal{O}_j|) = 2\alpha = \bar{u}_0(d) - \bar{u}_0(c) < \infty,$$

so $\bar{u}_0 \in W^{1,1}((c, d) \setminus E)$, and since $\bar{u}_0 \in C([c, d])$ it follows ([29, p. 224]) that

$$\bar{u}_0 \in W^{1,1}(c, d). \tag{4.39}$$

Now define u_0 to be the restriction of $\bar{u}_0$ to $[-1, 1]$ and let

$$k_1 = u_0(-1), \quad k_2 = u_0(1),$$

so that $0 < k_1 < k_2 < 2\alpha$. Define $g: [k_1, k_2] \to [-1, 1]$ by $g = u_0^{-1}$. It follows from (4.36) that

$$g(u) = (b_j - a_j)\, k^{-1} \left(\frac{u - u_0(a_j)}{\varphi(|\mathcal{O}_j|)}\right) + a_j \quad \text{for} \quad u \in \bar{u}_0(\mathcal{O}_j) \cap [k_1, k_2], \tag{4.40}$$

where $\bar{u}_0(\mathcal{O}_j) = (\bar{u}_0(a_j), \bar{u}_0(b_j))$, $j \geqq 1$. Consequently $g \in C^\infty(\bar{u}_0(\mathcal{O}_j) \cap [k_1, k_2])$ and

$$g'(u) = \frac{b_j - a_j}{\varphi(b_j - a_j)} (k^{-1})' \left(\frac{u - \bar{u}_0(a_j)}{\varphi(b_j - a_j)}\right), \quad u \in \bar{u}_0(\mathcal{O}_j) \cap [k_1, k_2]. \tag{4.41}$$

By (4.31), (4.40)

$$\begin{aligned} &\lim_{u\to u_0(a_j)+} g'(u) = 0, \quad j \neq 1, \\ &\lim_{u\to u_0(b_j)-} g'(u) = 0, \quad j \neq 2, \\ &0 < g'(u) < \frac{b_j - a_j}{\varphi(b_j - a_j)} \quad \text{for} \quad u \in \bar{u}_0(\mathcal{O}_j) \cap [k_1, k_2], \quad j \geqq 1. \end{aligned} \tag{4.42}$$

By (4.34), (4.42),

$$0 < \sup_{u\in\bar{u}_0(\mathcal{O}_j)\cap[k_1,k_2]} g'(u) \to 0 \quad \text{as} \quad j \to \infty, \tag{4.43}$$

so that g' can be extended to a function $g^* \in C([k_1, k_2])$ by setting

$$g^*(u) \overset{\text{def}}{=} \begin{cases} 0, & u \in F, \\ g'(u), & u \in [k_1, k_2] \setminus F, \end{cases} \tag{4.44}$$

where $F \overset{\text{def}}{=} [k_1, k_2] - \bigcup_{j=1}^{\infty} \bar{u}_0(\mathcal{O}_j) = u_0(E)$. To show that $g \in C([k_1, k_2]) \cap C^\infty([k_1, k_2] \setminus F)$ is in $C^1([k_1, k_2])$ note that

$$\begin{aligned}\int_{k_1}^{k_2} |g'(u)|\, du &= \sum_{j=1}^{\infty} \int_{\mathcal{O}_j \cap [k_1,k_2]} g'(u)\, du \\ &= \sum_{j=3}^{\infty} [g(u_0(b_j)) - g(u_0(a_j))] + g(u_0(b_1)) - g(u_0(-1)) \\ &\qquad + g(u_0(1)) - g(u_0(a_2)) \\ &= \sum_{j=1}^{\infty} (b_j - a_j) - (d - c) + 2 \\ &= 2 = g(k_2) - g(k_1) < \infty,\end{aligned}$$

where we have used the fact that, by (4.36)–(4.38), F has measure zero. Hence $g \in W^{1,1}((k_1, k_2) \setminus F)$ and thus by the continuity of g on $[k_1, k_2]$ ([29, p. 224]) $g \in W^{1,1}(k_1, k_2)$. Therefore, for each $u \in [k_1, k_2]$,

$$g(u) - g(k_1) = \int_{k_1}^{u} g'(y)\, dy = \int_{k_1}^{u} g^*(y)\, dy.$$

Since $g^* \in C([k_1, k_2])$ we deduce that $g \in C^1([k_1, k_2])$. Moreover, by (4.34), (4.42) g can be extended to a function in $C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus F)$ satisfying

$$0 < g'(u) \leqq 1, \quad u \in \mathbb{R} \setminus [k_1, k_2].$$

Thus

$$0 \leqq g'(u) \leqq 1, \quad u \in \mathbb{R},$$

$$g'(u) = 0 \quad \text{if and only if } u \in F.$$

Now $g = u_0^{-1}$ on $[k_1, k_2]$ implies that

$$g'(u_0(x))\, u_0'(x) = 1 \quad \text{a.e. } x \in [-1, 1].$$

It follows as in (4.7)–(4.9) that u_0 is a (unique) global minimizer for J in $\mathcal{A}$.

It remains only to repeat the proof given in Part II of the argument of Theorem 4.1 in order to construct an f satisfying (4.3), (4.15) relative to $f^0(u, p) \overset{\text{def}}{=} (g'(u)\, p)^2$, $(u, p) \in \mathbb{R}^2$. □

§ 5. A case exhibiting the Lavrentiev phenomenon

In this section we consider the problem of minimizing

$$I(u) = \int_{-1}^{1} [(x^4 - u^6)^2\, |u'|^s + \varepsilon (u')^2]\, dx \tag{5.1}$$

over

$$\mathcal{A} = \{u \in W^{1,1}(-1, 1) : u(-1) = k_1, \quad u(1) = k_2\},$$

where $s > 3$, $\varepsilon > 0$ and $k_1, k_2 \in \mathbb{R}$. Note that the integrand

$$f(x, u, p) = (x^4 - u^6)^2 |p|^s + \varepsilon p^2 \tag{5.2}$$

is C^3, nonnegative, and satisfies $f_{pp} \geqq 2\varepsilon > 0$; furthermore, in the case when $s = 2m$ is an even integer f is a polynomial.

By Theorem 2.1 there exists at least one absolute minimizer u_0 of I in $\mathscr{A}$. Any minimizer u_0 of I in $\mathscr{A}$ is either nondecreasing or nonincreasing, since otherwise the value of I could be reduced by making u_0 constant on some interval. (In fact, if $k_1 \neq k_2$ then u_0 is *strictly* increasing or decreasing, since if u_0 were constant on some interval then, as constants satisfy (EL), by Theorem 2.7 we would have u_0 constant everywhere.)

Our first aim is to prove the following theorem. (In the statement of the theorem and below we abbreviate $W^{1,p}(-1, 1)$ by $W^{1,p}$ where convenient.)

Theorem 5.1. *Let* $s \geqq 27$. *Let* $-1 \leqq k_1 < 0 < k_2 \leqq 1$ *and* $0 < \alpha < 1$. *Then there is an* $\varepsilon_1 = \varepsilon_1(\alpha, k_1, k_2, s) > 0$ *such that when* $0 < \varepsilon < \varepsilon_1$ *each minimizer* u_0 *of* I *in* $\mathscr{A}$ *satisfies*

(i) *the Tonelli set for* u_0 *is* $E = \{0\}$,

(ii) $u_0 \in W^{1,p}$ *for all* p *such that* $1 \leqq p < 3$, *and*

$$u_0(x) \sim |x|^{\frac{2}{3}} \operatorname{sign} x \quad \text{as } x \to 0, \tag{5.3}$$

$$-|x|^{\frac{2}{3}} < u_0(x) < \alpha k_1 |x|^{\frac{2}{3}} \quad \text{for } -1 < x < 0, \tag{5.4}$$

and

$$\alpha k_2 x^{\frac{2}{3}} < u_0(x) < x^{\frac{2}{3}} \quad \text{for } 0 < x < 1, \tag{5.5}$$

(iii) u_0 *satisfies none of* (WEL), (WDBR), (IEL), (IDBR),

(iv) *for any* q, $3 \leqq q \leqq \infty$,

$$\inf_{v \in W^{1,q} \cap \mathscr{A}} I(v) > \inf_{v \in \mathscr{A}} I(v) = I(u_0) \quad \textit{(the Lavrentiev phenomenon).}$$

The proof of Theorem 5.1 depends on some lemmas.

Lemma 5.2. *Let* $0 < \alpha < \beta < 1$, $0 < k \leqq 1$, $\gamma \geqq \frac{2}{3}$ *and* $s > 9$. *Let* $v \in W^{1,1}(0, 1)$ *satisfy*

$$\alpha k x^\gamma \leqq v(x) \leqq \beta k x^\gamma \quad \text{for } x_1 \leqq x \leqq x_2,$$

$$v(x_1) = \alpha k x_1^\gamma, \quad v(x_2) = \beta k x_2^\gamma,$$

where $0 \leqq x_1 < x_2 \leqq 1$. *Then*

$$\int_{x_1}^{x_2} (x^4 - v^6)^2 |v'|^s \, dx \geqq (1 - (\beta k)^6)^2 \, \theta^{s-1} k^s (\beta - \alpha)^s \, x_2^{(\gamma-1)s+9}$$

where $\theta = \dfrac{s-9}{s-1}$.

Proof (*cf.* MANIÀ [25]). We have

$$\int_{x_1}^{x_2} (x^4 - v^6)^2 |v'|^s \, dx = \int_{x_1}^{x_2} \left(1 - \frac{v^6}{x^4}\right)^2 x^8 |v'|^s \, dx$$
$$\geqq (1 - (\beta k)^6)^2 \int_{x_1}^{x_2} x^8 |v'|^s \, dx.$$

Setting $y = x^\theta$, $\tilde{v}(x^\theta) = v(x)$, we obtain by Jensen's inequality

$$\int_{x_1}^{x_2} x^8 |v'|^s \, dx = \theta^{s-1} \int_{x_1^\theta}^{x_2^\theta} \left|\frac{d\tilde{v}}{dy}\right|^s dy$$
$$\geqq \theta^{s-1} \frac{[v(x_2) - v(x_1)]^s}{[x_2^\theta - x_1^\theta]^{s-1}}$$
$$= \theta^{s-1} k^s x_2^{(\gamma-1)s+9} \frac{\left[\beta - \alpha\left(\frac{x_1}{x_2}\right)^\gamma\right]^s}{\left[1 - \left(\frac{x_1}{x_2}\right)^\theta\right]^{s-1}}$$
$$\geqq \theta^{s-1} k^s (\beta - \alpha)^s x_2^{(\gamma-1)s+9},$$

and the result follows. □

Lemma 5.3. *Let k_1, k_2 be arbitrary, $s > 3$, and let u_0 minimize I in $\mathscr{A}$. Then either the Tonelli set E of u_0 is empty, or $E = \{0\}$ and $u_0(0) = 0$.*

Proof. Suppose first that $x_0 \in [-1, 1]$ with $u_0(x_0)^6 \neq x_0^4$. Then there is a non-trivial interval $[c, d] \subset [-1, 1]$ containing x_0 and such that

$$u_0(x)^6 \neq x^4, \quad x \in [c, d]. \tag{5.6}$$

Now u_0 minimizer the integral

$$J(v) = \int_c^d [(x^4 - v^6)^2 |v'|^s + \varepsilon(v')^2] \, dx$$

in

$$\mathscr{B} = \{v \in W^{1,1}(c, d) : v(c) = u_0(c), v(d) = u_0(d)\}.$$

But by (5.6),

$$|f_u(x, u_0(x), u_0'(x))| \leqq \text{const.} f(x, u_0(x), u_0'(x)), \quad x \in [c, d],$$

and therefore, since $J(u_0) < \infty$, $f_u(\cdot, u_0(\cdot), u_0'(\cdot)) \in L^1(c, d)$. By Theorem 2.10, u_0 is smooth in $[c, d]$, and in particular $x_0 \notin E$.

It remains to consider the possibility that $x_0 \in E$ with $u(x_0)^6 = x_0^4 \neq 0$. Suppose $k_2 \geqq k_1$; the case $k_2 \leqq k_1$ is treated similarly. Then u is nondecreasing and so we must have $u_0'(x_0) = +\infty$. Suppose that $x_0 > 0$ and $u(x_0) = x_0^{\frac{2}{3}}$; the other three cases $x_0 > 0$, $u(x_0) = -x_0^{\frac{2}{3}}$ and $x_0 < 0$, $u(x_0) = \pm |x_0|^{\frac{2}{3}}$

are treated similarly. Then there exists $x_1 \in (0, x_0)$ such that $0 < u(x) < x^{\frac{2}{3}}$ for all $x \in (x_1, x_0)$. By the preceding argument u_0 is a C^3 (in fact C^∞) solution of the Euler-Lagrange equation

$$\frac{d}{dx}[s(x^4 - u^6)^2 |u'|^{s-1} \operatorname{sign} u' + 2\varepsilon u'] = -12u^5(x^4 - u^6)|u'|^s \tag{5.7}$$

in (x_1, x_0). Since the right-hand side of (5.7) is negative in (x_1, x_0) this contradicts $u_0'(x_0) = +\infty$. □

Lemma 5.4. *Let k_1, k_2 be arbitrary, $s > 3$ and let u_0 minimize I in $\mathscr{A}$. Suppose that $0 \leqq x_1 < x_2 \leqq 1$ and that $u_0(x_1) = x_1^{\frac{2}{3}}$, $u_0(x_2) = x_2^{\frac{2}{3}}$. Then $u_0(x) \leqq x^{\frac{2}{3}}$ for all $x \in [x_1, x_2]$.*

Proof. Suppose first that $u_0(x) > x^{\frac{2}{3}}$ for all $x \in (x_1, x_2)$. Define

$$\bar{u}(x) = \begin{cases} u_0(x) & x \notin [x_1, x_2] \\ x^{\frac{2}{3}} & x \in [x_1, x_2]. \end{cases}$$

Then

$$I(u_0) - I(\bar{u}) = \int_{x_1}^{x_2} [(x^4 - u_0^6)^2 |u_0'|^s + \varepsilon(u_0')^2 - \varepsilon(\bar{u}')^2]\, dx$$

$$= \int_{x_1}^{x_2} (x^4 - u_0^6)^2 |u_0'|^s\, dx + \varepsilon \int_{x_1}^{x_2} [(\tfrac{2}{3}x^{-\frac{1}{3}} + v'(x))^2 - (\tfrac{2}{3}x^{-\frac{1}{3}})^2]\, dx,$$

where $v = u_0 - \bar{u}$. Note that $v(x_1) = v(x_2) = 0$, $v(x) > 0$ for all $x \in (x_1, x_2)$. The first integral on the right-hand side is positive, and for the second integral we obtain, using integration by parts,

$$\int_{x_1}^{x_2} \tfrac{4}{3}x^{-\frac{1}{3}} v'\, dx + \int_{x_1}^{x_2} (v')^2\, dx$$

$$= (\tfrac{4}{3}x^{-\frac{1}{3}} v)\Big|_{x_1+}^{x_2} + \int_{x_1}^{x_2} \tfrac{4}{9}x^{-\frac{4}{3}} v\, dx + \int_{x_1}^{x_2} (v')^2\, dx > 0.$$

This contradicts the minimum property for u_0. (When $x_1 = 0$ the validity of the integration by parts stems from the fact that finiteness of $I(u_0)$ ensures that $v' \in L^2(-1, 1)$, so that $v(x) = \int_0^x v'(y)\, dy$ is $o(x^{\frac{1}{2}})$ as $x \to 0+$.)

More generally, if $u_0(\bar{x}) > \bar{x}^{\frac{2}{3}}$ for any $\bar{x} \in (x_1, x_2)$ then there is an interval $(\bar{x}_1, \bar{x}_2) \subset (x_1, x_2)$ such that $u_0(\bar{x}_1) = \bar{x}_1^{\frac{2}{3}}$, $u_0(\bar{x}_2) = \bar{x}_2^{\frac{2}{3}}$ and $u_0(x) > x^{\frac{2}{3}}$ for all $x \in (\bar{x}_1, \bar{x}_2)$. Applying the preceding argument to $(\bar{x}_1, \bar{x}_2)$ gives a contradiction. □

Proof of Theorem 5.1. Fix α, β with $0 < \alpha < \beta < 1$. Let $v \in \mathscr{A}$, and suppose for the moment that

$$v(\bar{x}) \leqq \alpha k_2 \bar{x}^{\frac{2}{3}} \quad \text{for some } \bar{x} \in (0, 1]. \tag{5.8}$$

Then there exists an interval $[x_1, x_2] \subset (0, 1)$ such that

$$\left.\begin{aligned} \alpha k_2 x^{\frac{2}{3}} < v(x) < \beta k_2 x^{\frac{2}{3}}, \quad x \in (x_1, x_2) \\ v(x_1) = \alpha k_2 x_1^{\frac{2}{3}}, \quad v(x_2) = \beta k_2 x_2^{\frac{2}{3}}. \end{aligned}\right\} \tag{5.9}$$

By Lemma 5.2, with $k = k_2$ and $\gamma = 2/3$,

$$I(v) > (1 - (\beta k_2)^6)^2 \, \theta^{s-1} k_2^s (\beta - \alpha)^s \, x_2^{-\frac{1}{3}(s-27)}. \tag{5.10}$$

Since $s \geqq 27$, (5.10) implies that

$$I(v) > (1 - (\beta k_2)^6)^2 \, \theta^{s-1} k_2^s (\beta - \alpha)^s. \tag{5.11}$$

Similarly, if in place of (5.8) we assume that

$$v(\bar{x}) \geqq \alpha k_1 \, |\bar{x}|^{\frac{2}{3}} \quad \text{for some } \bar{x} \in [-1, 0), \tag{5.12}$$

then by applying Lemma 5.2 to the function $-v(-x)$ we obtain

$$I(v) > (1 - (\beta k_1)^6)^2 \, \theta^{s-1} (-k_1)^s (\beta - \alpha)^s. \tag{5.13}$$

We now note that one of (5.8), (5.12) holds if either $v(0) \neq 0$ or $v \in W^{1,q} \cap \mathscr{A}$ for some q with $3 \leqq q \leqq \infty$, since if $v(0) = 0$ and $v \in W^{1,q} \cap \mathscr{A}$, $q \in [3, \infty)$, then for all $x \in [-1, 1]$

$$|v(x)| = \left| \int_0^x v'(y) \, dy \right| \leqq \left| \int_0^x |v'|^q \, dy \right|^{1/q} \left| \int_0^x 1^{q'} \, dy \right|^{1/q'} = o(1) \, |x|^{1-1/q},$$

while if $q = \infty$,

$$|v(x)| \leqq \text{const.} \, |x|.$$

In either case we therefore have

$$I(v) > \min \{h_\beta(k_1), h_\beta(k_2)\} \, \theta^{s-1} (\beta - \alpha)^s, \tag{5.14}$$

where $h_\beta(k) \stackrel{\text{def}}{=} (1 - (\beta k)^6) \, |k|^s$, this estimate being independent of $\varepsilon > 0$ and of q.

Now consider the following function $\hat{u} \in \mathscr{A}$:

$$\hat{u}(x) = \begin{cases} k_1, & -1 \leqq x \leqq -|k_1|^{\frac{3}{2}} \\ -|x|^{\frac{2}{3}}, & -|k_1|^{\frac{3}{2}} \leqq x \leqq 0 \\ x^{\frac{2}{3}}, & 0 \leqq x \leqq k_2^{\frac{3}{2}} \\ k_2, & k_2^{\frac{3}{2}} \leqq x \leqq 1. \end{cases}$$

A direct computation yields

$$I(\hat{u}) = 0 + \int_{-|k_1|^{\frac{3}{2}}}^{k_2^{\frac{3}{2}}} \varepsilon (\hat{u}')^2 \, dx = \frac{4\varepsilon}{9} \int_{-|k_1|^{\frac{3}{2}}}^{k_2^{\frac{3}{2}}} |x|^{-\frac{2}{3}} \, dx = \frac{4\varepsilon}{3} (|k_1|^{\frac{1}{2}} + k_2^{\frac{1}{2}}). \tag{5.15}$$

Together (5.14), (5.15) ensure that for $\beta = \beta(\alpha, k_1, k_2, s)$ chosen to maximize the right side of (5.14) and for $\varepsilon > 0$ sufficiently small, *i.e.* $\varepsilon < \varepsilon_0(\alpha, k_1, k_2, s)$,

$$\inf_{v \in W^{1,q} \cap \mathscr{A}} I(v) > I(\hat{u}) \geqq \inf_{v \in \mathscr{A}} I(v)$$

for all q with $3 \leqq q \leqq \infty$; furthermore any minimizer u_0 of I in $\mathscr{A}$ satisfies $u_0(0) = 0$, and

$$\left.\begin{aligned} u_0(x) &< \alpha k_1 |x|^{\frac{2}{3}}, \quad -1 \leqq x < 0 \\ u_0(x) &> \alpha k_2 x^{\frac{2}{3}}, \quad 0 < x \leqq 1. \end{aligned}\right\} \tag{5.16}$$

It follows from (5.16) that $u_0'(0) = +\infty$, so that by Lemma 5.3 we have $E = \{0\}$. Also, since $|k_i| \leqq 1$, $i = 1, 2$, it follows from Lemma 5.4 (applied to $u_0(x)$ and $-u_0(-x)$) that

$$\left.\begin{aligned} u_0(x) &\geqq -|x|^{\frac{2}{3}}, \quad -1 \leqq x \leqq 0 \\ u_0(x) &\leqq x^{\frac{2}{3}}, \quad 0 \leqq x \leqq 1. \end{aligned}\right\} \tag{5.17}$$

Since by Theorem 2.7 $u_0': [-1, 1] \to \overline{\mathbb{R}}$ is continuous, we have

$$\lim_{x \to 0+} f_p(x, u_0(x), u_0'(x)) = \lim_{x \to 0+} [u_0'(x) f_p(x, u_0(x), u_0'(x)) - f(x, u_0(x), u_0'(x))]$$
$$= +\infty,$$

and it follows immediately that none of (WEL), (WDBR), (IEL), (IDBR) hold.

We next show that

$$|u_0'(x)| \leqq \text{const.}\, |x|^{-\frac{1}{3}}, \quad x \in [-1, 1], \tag{5.18}$$

which ensures that $u_0 \in W^{1,p}$ for all p such that $1 \leqq p < 3$. For $\zeta \in [0, 1]$ define $u_\zeta \in \mathscr{A}$ by

$$u_\zeta(x) = \begin{cases} u_0(x), & -1 \leqq x \leqq -\zeta \\ u_0(-\zeta), & -\zeta \leqq x \leqq -|u_0(-\zeta)|^{\frac{3}{2}} \\ |x|^{\frac{2}{3}} \operatorname{sign} x, & -|u_0(-\zeta)|^{\frac{3}{2}} \leqq x \leqq u_0(\zeta)^{\frac{3}{2}} \\ u_0(\zeta), & u_0(\zeta)^{\frac{3}{2}} \leqq x \leqq \zeta \\ u_0(x), & \zeta \leqq x \leqq 1. \end{cases}$$

Then

$$0 \geqq I(u_0) - I(u_\zeta) = \int_{-\zeta}^{\zeta} [(x^4 - u_0^6)^2 (u_0')^s + \varepsilon (u_0')^2]\, dx$$
$$- \int_{-|u_0(-\zeta)|^{\frac{3}{2}}}^{u_0(\zeta)^{\frac{3}{2}}} \varepsilon \left(\tfrac{2}{3} |x|^{-\frac{1}{3}}\right)^2 dx,$$

and hence by (5.17)

$$\int_{-\zeta}^{\zeta} [(x^4 - u_0^6)^2 (u_0')^s + \varepsilon(u_0')^2]\, dx \leqq \text{const.}\, \{u_0(\zeta)^{\frac{1}{2}} + |u_0(-\zeta)|^{\frac{1}{2}}\} \leqq C\zeta^{\frac{1}{3}}, \quad \zeta \in [0, 1], \tag{5.19}$$

for some constant $C > 0$. Now define

$$g(x) = u_0'(x) f_p(x, u_0(x), u_0'(x)) - f(x, u_0(x), u_0'(x)),$$

so that

$$g(x) = (s-1)(x^4 - u_0^6)^2 (u_0')^s + \varepsilon(u_0')^2.$$

Since $u_0(x)$ is smooth for $x \neq 0$, by (DBR),

$$g'(x) = -8x^3(x^4 - u_0^6)(u_0')^s, \quad x \in [-1, 0) \cup (0, 1],$$

and so by (5.17) g is increasing on $[-1, 0)$, decreasing on $(0, 1]$. Thus for $\zeta \in (0, 1]$,

$$g(\zeta)\,\zeta = \int_0^{\zeta} g(\zeta)\, dx \leqq \int_0^{\zeta} g(x)\, dx \leqq (s-1) \int_0^{\zeta} [(x^4 - u_0^6)^2 (u_0')^s + \varepsilon(u_0')^2]\, dx, \tag{5.20}$$

and it follows from (5.19), (5.20) that

$$g(\zeta) \leqq \text{const.}\, |\zeta|^{-\frac{2}{3}}. \tag{5.21}$$

The same argument applied on $[-1, 0)$ shows that (5.21) holds also for $\zeta \in [-1, 0)$, and (5.18) follows by the formula for g. Clearly (5.18) implies that $u_0 \in W^{1,p}$ for $1 \leqq p < 3$.

It now only remains to prove (5.3) and the strictness of the inequality in (5.17). For this we make the same substitutions as in Section 3, namely

$$z = \frac{u^{\frac{3}{2}}}{x}, \quad q = \tfrac{3}{2} u^{\frac{1}{2}} u', \quad x = e^t,$$

which for $u > 0$, $x > 0$ reduce the Euler-Lagrange equation (5.7) to the system

$$\begin{aligned} \frac{dz}{dt} &= q - z, \\ \frac{dq}{dt} &= \frac{q^2}{3z} \\ &\times \frac{\left(\frac{2q}{3}\right)^{s-3} (1 - z^4) \left[(s-1)\, q \left(\frac{s}{3}(1 - z^4) + 8z^4\right) - 8sz\right] + \varepsilon z^{\frac{s-2}{3}} e^{\left(\frac{s-26}{3}\right) t}}{\frac{s}{2}(s-1)(1 - z^4)^2 \left(\frac{2q}{3}\right)^{s-2} + \varepsilon z^{\frac{s-2}{3}} e^{\left(\frac{s-26}{3}\right) t}}, \end{aligned} \tag{5.22}$$

which, of course, is not autonomous since $s \neq 26$. We compute the sign of $\frac{dq}{dt}$ on the diagonal $q = z$; this is the same as the sign of

$$\left(\frac{2z}{3}\right)^{s-3}(1-z^4)\left[(s-1)\,z\left(\frac{s}{3}(1-z^4)+8z^4\right)-8sz\right]+\varepsilon z^{\frac{s-2}{3}}e^{\left(\frac{s-26}{3}\right)t}$$

$$=\left(\frac{2z}{3}\right)^{s-3}(1-z^4)\,z(s-24)\left(\frac{s-1}{3}\right)(\sigma_s-z^4)+\varepsilon z^{\frac{s-2}{3}}e^{\left(\frac{s-26}{3}\right)t},$$

where $\sigma_s = \frac{s(s-25)}{(s-1)(s-24)} < 1$. Fix $\varrho \in (\sigma_s^{\frac{1}{4}}, 1)$. Then if $\varrho < z_0 < 1$ we have that, for t sufficiently large and negative, $\frac{dq}{dt} < 0$ whenever $q = z$, $\varrho < z < z_0$.

Now let $(z(\cdot), q(\cdot))$ be the solution of (5.22) corresponding to u_0 on $(0, 1]$; the behavior of u_0 on $[-1, 0)$ is handled similarly. By our results so far (taking $\alpha_\varrho = \varrho^{\frac{2}{3}}$) it follows that if $\varepsilon > 0$ is sufficiently small, *i.e.* $\varepsilon < \varepsilon_0(\alpha_\varrho, k_1, k_2, s)$, then $\varrho < z(t) \leqq 1$, $0 < q(t) \leqq$ const. for all $t \in (-\infty, 0]$. Note that $\frac{dz}{dt} < 0$ for $q < z$, $\frac{dz}{dt} > 0$ for $q > z$. In view of the bound on z, this implies that we cannot have $z(t) = 1 \neq q(t)$ for any $t \in (-\infty, 0)$. If $z(t) = q(t) = 1$ for some $t \in (-\infty, 0)$ then by (5.22), $\frac{dq}{dt}(t) = \frac{1}{3}$, $\frac{dz}{dt}(t) = 0$, $\frac{d^2z}{dt^2}(t) = \frac{1}{3}$, which by the same reasoning is impossible. Thus strict inequality holds in (5.17) for $x \neq 0$, ± 1. If $z(t) \leqq q(t)$ for all sufficiently large and negative t, or if $z(t) \geqq q(t)$ for all sufficiently large and negative t, then for some $z^* \in [0, 1]$, $z(t) \to z^*$ as $t \to -\infty$; in these cases we must have $z^* = 1$, since for $z^* < 1$ the relations $u_0(x) \sim (z^*x)^{\frac{2}{3}}$ as $x \to 0+$ and $I(u_0) < \infty$ imply that

$$\int_0^1 x^8 (u_0')^s\,dx < \infty,$$

and hence by Hölder's inequality

$$u_0(x) = \int_0^x u_0'(y)\,dy \leqq \text{const.}\left(\int_0^x y^8 u_0'(y)^s\,dy\right)x^{1-\frac{9}{s}} = o(1)\,x^{1-\frac{9}{s}},$$

which contradicts $z(t) > \varrho$ since $s \geqq 27$. We therefore need only consider the case when there exists a sequence $t_j \to -\infty$ with $z(t_j) = q(t_j) < 1$, $\frac{dq}{dt}(t_j) \geqq 0$. By our analysis of the sign of $\frac{dq}{dt}$ on the diagonal, it follows that for any such sequence $z(t_j) \to 1$ as $j \to \infty$, and the sign of $\frac{dz}{dt}$ then implies that $z(t) \to 1$ as $t \to -\infty$. This proves (5.3). □

We remark that the relations (5.3)–(5.5) imply that the inverse function $x_0(u)$ of u_0 is not C^2, and in particular that the graph of u_0 is not a smooth curve in the plane. We mention this fact because of its relevance to attempts to elucidate the phenomenon of singular minimizers by consideration of some parametric problem of the calculus of variations.

We remark also that if $\varphi \in C_0^\infty(-1, 1)$ is given with $\varphi(0) \neq 0$, and if $t \neq 0$, then $I(u_0 + t\varphi) = \infty$. In fact, since $u_0'(x) \to \infty$ as $x \to 0$ by Theorem 2.7, there exist constants $c(t) > 0$, $\alpha(t) > 0$ such that

$$I(u_0 + t\varphi) \geqq c(t) \int_{-\alpha(t)}^{\alpha(t)} |u_0'(x)|^s \, dx.$$

Since by (i) and (iv) $u_0 \notin W^{1,s}(-\alpha(t), \alpha(t))$ the assertion follows.

The reader can easily verify that an appropriate version of Theorem 5.1 holds when the signs of k_1, k_2 are reversed. Note also that the comparison argument used in the proof requires that the value of ε approach zero as $k_1, k_2 \to 0$. In fact if $\varepsilon > 0$ is fixed then for sufficiently small $|k_1|, |k_2|$ the minimizer of I in $\mathscr{A}$ is unique and *smooth*. This can be proved by noting that constants satisfy (EL), and thus, by an argument similar to that used in the proof of Lemma 2.8, for $|k_1|, |k_2|$ sufficiently small there is a unique smooth solution u_1 of (EL) in $\mathscr{A}$ and u_1 can be embedded in a field of extremals simply covering the region $S = \{(x, v): |x| \leqq 1, k_1 \leqq v \leqq k_2\}$. But any minimizer $\bar{u}_0$ of I in $\mathscr{A}$ is monotone and thus has graph lying in S. By the field theory of the calculus of variations $I(u_1) \leqq I(\bar{u}_0)$ with equality if and only if $\bar{u}_0 = u_1$. Hence $\bar{u}_0 = u_1$, as claimed.

It is important to note the significance of the Lavrentiev phenomenon for numerical schemes designed to approximate minimizers of variational problems such as (5.1). Such schemes, for instance those using finite elements, are often associated with the use of approximating functions that are Lipschitz. Hence the existence of Lavrentiev's gap ensures that *no such scheme can yield a minimizing sequence* for I. On the other hand, one might suppose that a sequence $\{v_m\} \subset W^{1,\infty} \cap \mathscr{A}$ could be found satisfying the pseudo-minimizing condition

$$I(v_m) \to \inf_{v \in W^{1,\infty} \cap \mathscr{A}} I(v),$$

and such that v_m converges to the actual minimizer u_0 in some mild sense. Our next result shows that even this cannot happen.

Theorem 5.5. *Let* $s > 27$, $-1 \leqq k_1 < 0 < k_2 \leqq 1$, $0 < \alpha < 1$, $0 < \varepsilon < \varepsilon_1(\alpha, k_1, k_2, s)$ *and* $3 \leqq q \leqq \infty$. *Let* u_0 *be an absolute minimizer of* I *in* $\mathscr{A}$. *For any sequence* $\{v_m\} \subset W^{1,q} \cap \mathscr{A}$ *such that* $v_m(x) \to u_0(x)$ *for each* x *in some set containing arbitrarily small positive and negative numbers one necessarily has*

$$I(v_m) \to \infty \quad \text{as} \quad m \to \infty.$$

Proof. Let $l_j \in [-1, 0)$, $r_j \in (0, 1]$ satisfy $l_j \to 0$, $r_j \to 0$ as $j \to \infty$, and suppose that $v_m(l_j) \to u_0(l_j)$, $v_m(r_j) \to u_0(r_j)$ as $m \to \infty$ for each $j = 1, 2, \ldots$. Fix $\bar{\alpha}$ with $0 < \bar{\alpha} < \alpha$. Given j we have by (5.5) that for all sufficiently large m

$$v_m(l_j) < \alpha k_1 |l_j|^{\frac{2}{3}}, \quad v_m(r_j) > \alpha k_2 r_j^{\frac{2}{3}}.$$

Since v_m changes sign in $[l_j, r_j]$, $v_m(x_0) = 0$ for some $x_0 \in (l_j, r_j)$. If $x_0 \geqq 0$, say, then by the argument preceding (5.14) there is an interval $[y_1, y_2] \subset (0, r_j)$ such that

$$\bar{\alpha} k_2 x^{\frac{2}{3}} < v_m(x) < \alpha k_2 x^{\frac{2}{3}}, \quad x \in (y_1, y_2),$$

$$v_m(y_1) = \bar{\alpha} k_2 y_1^{\frac{2}{3}}, \quad v_m(y_2) = \alpha k_2 y_2^{\frac{2}{3}}.$$

By Lemma 5.2,

$$I(v_m) > (1 - (\alpha k_2)^6)^2\, \theta^{s-1} k_2^s (\alpha - \bar{\alpha})^s\, y_2^{-\frac{1}{3}(s-27)}$$
$$> C r_j^{-\frac{1}{3}(s-27)},$$

where $C > 0$ is a constant. If $x_0 \leqq 0$ we obtain similarly

$$I(v_m) > C\, |l_j|^{-\frac{1}{3}(s-27)}.$$

Letting $j \to \infty$ we obtain that $I(v_m) \to \infty$ as $m \to \infty$, as required. $\square$

A more quantitative version of Theorem 5.5 may be proved. If $s, k_1, k_2, \alpha, \varepsilon, q$ and u_0 are as in the theorem and if $1 \leqq \sigma \leqq \infty$ then

$$\inf_{v \in \mathscr{A} \cap W^{1,q}} (I(v) - I(u_0))\, \|v - u_0\|_{L^\sigma(-1,1)}^{\frac{s-27}{2+3/\sigma}} > 0.$$

This has some of the features of an uncertainty principle. By Theorem 5.1(iv) it suffices for the proof to show that if $\delta = \|v - u_0\|_{L^\sigma(-1,1)}$ then

$$I(v)\, \delta^{\frac{s-27}{2+3/\sigma}} \geqq \text{const.} > 0$$

for $\delta > 0$ sufficiently small. But it is easily shown using (5.3) that for $\delta > 0$ sufficiently small there exist points $-l, r \in (0, \text{const.}\ \delta^{\frac{3}{2+3/\sigma}})$ such that

$$v(l) < \alpha k_1\, |l|^{\frac{2}{3}}, \quad v(r) > \alpha k_2 r^{\frac{2}{3}},$$

and the result follows using the same proof as for Theorem 5.5.

Theorem 5.5 contrasts strongly with a claim of LEWY [23]. There it was asserted that the sequence $\{u_M\}$ constructed through the following constrained minimization procedure:

$$I(u_M) = \inf_{v \in \mathscr{A}_M} I(v),$$

where $\mathscr{A}_M = \{v \in W^{1,\infty} \cap \mathscr{A} : \|v\|_{W^{1,\infty}} \leqq M\}$, would yield a sequence $\{u_M\}$ converging to the global minimizer u_0 as $M \to \infty$. Note that the existence of a constrained minimizer u_M for M sufficiently large follows from the precompactness of $\mathscr{A}_M$ in $C([-1, 1])$. Theorem 5.5 reveals that in our example no subsequence of $\{u_M\}$ can converge to u_0 pointwise, even on a two-sided sequence of points $x_j \to 0$. A similar comment applies to any "penalty method" which involves adding to the integrand a term such as $\eta\, |u'|^{3+\gamma}$ or $\eta\, |u''|^{1+\gamma}$, $\gamma > 0$, and examining the limiting behavior of the corresponding minimizers as $\eta \to 0+$.

The above predictions for numerical methods have been confirmed experimentally in BALL & KNOWLES [6], where numerical methods are described and developed that are capable of detecting the absolute minimizer u_0. Their numerical experiments also indicate that in an example due to MANIÀ [25] (which is not regular, but to which the ideas below also apply) minimizing sequences in $W^{1,\infty}$ converge to a "pseudominimizer" $u_1 \neq u_0$. We now examine the existence of pseudominimizers corresponding to the regular integrand (5.2).

We first discuss the problem of minimizing

$$J(u) = \int_0^1 [(x^4 - u^6)^2 |u'|^s + \varepsilon(u')^2]\, dx \tag{5.23}$$

in various subsets of

$$\mathscr{A}_0 = \{u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = k\}.$$

We begin by stating an analogue of Theorem 5.1 for this problem. Note that there is no condition on the size of k.

Theorem 5.6. *Let $s \geqq 27$, $k > 0$, and $0 < \alpha < 1$. Then there is an $\varepsilon_1 = \varepsilon_1(\alpha, k, s) > 0$ such that if $0 < \varepsilon < \varepsilon_1$ each minimizer u_0 of J in $\mathscr{A}_0$ (at least one such existing by Theorem 2.1 for any $\varepsilon > 0$) satisfies*

(i) *the Tonelli set for u_0 is $E = \{0\}$,*

(ii) *$u_0 \in W^{1,p}(0, 1)$ for $1 \leqq p < 3$ and satisfies*

$$u_0(x) \sim x^{\frac{2}{3}} \quad \text{as } x \to 0+; \tag{5.24}$$

if $k \leqq 1$ then

$$\alpha k x^{\frac{2}{3}} < u_0(x) < x^{\frac{2}{3}} \quad \text{for } 0 < x < 1, \tag{5.25}$$

while if $k > 1$ then there exists exactly one $\bar{x} \in (0, 1)$ with $u_0(\bar{x}) = \bar{x}^{\frac{2}{3}}$ and

$$\alpha x^{\frac{2}{3}} < u_0(x) < x^{\frac{2}{3}} \quad \text{for } 0 < x < \bar{x}, \tag{5.26}$$

$$x^{\frac{2}{3}} < u_0(x) < k \quad \text{for } \bar{x} < x < 1. \tag{5.27}$$

(iii) *u_0 does not satisfy* (IEL) *or* (IDBR),

(iv) *for any q, $3 \leqq q \leqq \infty$,*

$$\inf_{v \in W^{1,q}(0,1) \cap \mathscr{A}_0} J(v) > \inf_{v \in \mathscr{A}_0} J(v) = J(u_0).$$

Proof. If $k \leqq 1$ then the proof follows the same lines as that of Theorem 5.1. We therefore suppose that $k > 1$. Choose β with $\alpha < \beta < 1$. Let $v \in \mathscr{A}_0$ with $v(x_0) \leqq \alpha x_0^{\frac{2}{3}}$ for some $x_0 \in (0, 1)$; we have seen that such an x_0 exists if $v \in W^{1,q}(0, 1)$ with $3 \leqq q \leqq \infty$. As in the proof of Theorem 5.1 there exists an interval $[x_1, x_2] \subset (0, 1)$ such that

$$\alpha x^{\frac{2}{3}} < v(x) < \beta x^{\frac{2}{3}}, \quad x \in (x_1, x_2),$$

$$v(x_1) = \alpha x_1^{\frac{2}{3}}, \quad v(x_2) = \beta x_2^{\frac{2}{3}}$$

and thus by Lemma 5.2

$$\int_0^{x_2} [(x^4 - v^6)^2 |v'|^s + \varepsilon(v')^2]\, dx > (1 - \beta^6)^2\, \theta^{s-1}(\beta - \alpha)^s\, x_2^{-\frac{1}{3}(s-27)}. \tag{5.28}$$

Now define $\hat{v} \in \mathscr{A}_0$ by

$$\hat{v}(x) = \begin{cases} x^{\frac{2}{3}}, & 0 \leqq x \leqq v(x_2)^{\frac{3}{2}}, \\ v(x_2), & v(x_2)^{\frac{3}{2}} \leqq x \leqq x_2, \\ v(x), & x_2 \leqq x \leqq 1. \end{cases}$$

Then

$$J(v) - J(\hat{v}) = \int_0^{x_2} [(x^4 - v^6)^2 |v'|^s + \varepsilon(v')^2]\, dx - \int_0^{\beta^{\frac{3}{2}} x_2} \varepsilon(\tfrac{2}{3} x^{-\frac{1}{3}})^2\, dx,$$

and so by (5.28), with $\beta = \beta(\alpha, s)$ chosen to maximize $(1 - \beta^6)^2 (\beta - \alpha)^s$, one has

$$J(v) - J(\hat{v}) > \tfrac{1}{2}(1 - \beta^6)^2\, \theta^{s-1}(\beta - \alpha)^s$$

for ε sufficiently small (independently of v). In particular, (iv) holds. Also, any minimizer u_0 of J in $\mathscr{A}_0$ satisfies

$$u_0(x) > \alpha x^{\frac{2}{3}} \quad \text{for } 0 < x \leqq 1,$$

and so $E \supset \{0\}$ with $u_0'(0) = +\infty$. By inspection of (5.7) it is seen to be impossible that $u_0(x) \geqq x^{\frac{2}{3}}$ for all sufficiently small $x \in [0, 1]$; hence there exists some $\bar{x} \in (0, 1)$ with $u_0(\bar{x}) = \bar{x}^{\frac{2}{3}}$ and we may assume that $\bar{x}$ is maximal. By Lemma 5.4, $u_0(x) > x^{\frac{2}{3}}$ for $\bar{x} < x \leqq 1$ and $u_0(x) \leqq x^{\frac{2}{3}}$ for $0 \leqq x \leqq \bar{x}$. It follows as in the proof of Theorem 5.5 that $u_0(x) < x^{\frac{2}{3}}$ for $0 < x < \bar{x}$, so that $\bar{x}$ is unique. The remaining assertions in the theorem follow as before. □

We now prove the existence of a pseudominimizer for (5.23).

Theorem 5.7. *Let* $s > 27$, $k > 0$ *and* $3 \leqq q \leqq \infty$. *Then* $J(u)$ *attains an absolute minimum on* $W^{1,q}(0, 1) \cap \mathscr{A}_0$, *and any such minimizer* u_1 *belongs to* $C^\infty([0, 1])$ *and satisfies* (EL) *on* $[0, 1]$.

Proof. We first note that it suffices to prove the theorem for $q = 3$, since any minimizer for this q value is by the theorem smooth and thus a minimizer for all $q > 3$.

Let $\{v_j\}$ be a minimizing sequence for J in $W^{1,3}(0, 1) \cap \mathscr{A}_0$. Since $v_j \in W^{1,3}(0, 1)$ we have by Hölder's inequality that, as $x \to 0$,

$$v_j(x) = o(1)\, x^{\frac{2}{3}}, \text{ all } j \geqq 1.$$

We claim that there exists a number $\delta > 0$ such that

$$v_j(x) \leqq \tfrac{1}{2} x^{\frac{2}{3}} \quad \text{for all } x \in [0, \delta], \text{ all } j \geqq 1. \tag{5.29}$$

If not there would exist a subsequence $\{v_\mu\}$ of $\{v_j\}$ and a sequence $x_\mu \to 0+$ with $v_\mu(x_\mu) > \frac{1}{2}x_\mu^{\frac{2}{3}}$. Therefore there would exist numbers $x_{1\mu}, x_{2\mu} \in (0, x_\mu)$ such that

$$\tfrac{1}{4}x^{\frac{2}{3}} \leqq v_\mu(x) \leqq \tfrac{1}{2}x^{\frac{2}{3}}, \quad x_{1\mu} \leqq x \leqq x_{2\mu}$$
$$v_\mu(x_{1\mu}) = \tfrac{1}{4}x_{1\mu}^{\frac{2}{3}}, \quad v_\mu(x_{2\mu}) = \tfrac{1}{2}x_{2\mu}^{\frac{2}{3}}.$$

Applying Lemma 5.2 we deduce that

$$J(v_\mu) > (1 - 2^{-6})^2\,\theta^{s-1}\,4^{-s}x_{2\mu}^{\frac{1}{3}(27-s)}.$$

Since $s > 27$ and $x_{2\mu} \to 0+$, it follows that $J(v_\mu) \to \infty$ as $\mu \to \infty$. This contradiction establishes (5.29). By (5.29),

$$J(v_j) \geqq \int_0^\delta x^8\left(1 - \frac{v_j^6}{x^4}\right)^2 |v_j'|^s\,dx \geqq (1 - 2^{-6})^2 \int_0^\delta x^8\,|v_j'|^s\,dx. \tag{5.30}$$

But

$$\int_0^\delta |v_j'|^3\,dx \leqq \left(\int_0^\delta x^{-\frac{24}{s-3}}\,dx\right)^{1-\frac{3}{s}} \left(\int_0^\delta x^8\,|v_j'|^s\,dx\right)^{\frac{3}{s}}$$
$$\leqq \text{const.}\left(\int_0^\delta x^8\,|v_j'|^s\,dx\right)^{\frac{3}{s}},$$

and therefore by (5.30)

$$\int_0^\delta |v_j'|^3\,dx \leqq M < \infty, \quad j \geqq 1.$$

Since $v_j(0) = 0$ it follows that $\{v_j\}$ is bounded in $W^{1,3}(0, \delta)$. Moreover, it is obvious from the form of J that $\{v_j\}$ is also bounded in $W^{1,2}(0, 1)$. Therefore there exist a subsequence $\{v_\varrho\}$ of $\{v_j\}$ and a function $u \in W^{1,3}(0, \delta) \cap W^{1,2}(0, 1) \cap \mathscr{A}_0$ such that in the sense of weak convergence,

$$v_\varrho \rightharpoonup u \quad \text{in } W^{1,3}(0, \delta) \text{ and in } W^{1,2}(0, 1);$$

in particular, $v_\varrho(x) \to u(x)$, for all x, $0 \leqq x \leqq 1$. Since J is sequentially weakly lower semicontinuous in $W^{1,1}(0, 1)$ (see, for example, CESARI [11, p. 104]) it follows that

$$J(u) \leqq \liminf_{\varrho\to\infty} J(v_\varrho) = \inf_{v\in W^{1,3}(0,1)\cap\mathscr{A}_0} J(v). \tag{5.31}$$

For given $\bar{x} \in (0, 1)$, however small, the integral

$$J_{\bar{x}}(v) = \int_{\bar{x}}^1 [(x^4 - v^6)^2\,|v'|^s + \varepsilon(v')^2]\,dx$$

attains a minimum on the set

$$\mathscr{A}_{\bar{x}} = \{v \in W^{1,1}(\bar{x}, 1) : v(\bar{x}) = u(\bar{x}),\ v(1) = k\}$$

and, by the proof of Lemma 5.3 (reformulated for the interval $(\bar{x}, 1)$), any minimizer $\bar{u}$ belongs to $C^\infty([\bar{x}, 1])$. Given any such minimizer $\bar{u}$, define

$$\bar{v}_\varrho(x) = \begin{cases} v_\varrho(x), & 0 \leqq x \leqq \bar{x} \\ v_\varrho(\bar{x}) + \eta_\varrho^{-1}[\bar{u}(\bar{x} + \eta_\varrho) - v_\varrho(\bar{x})]\,(x - \bar{x}), & \bar{x} \leqq x \leqq \bar{x} + \eta_\varrho \\ \bar{u}(x), & \bar{x} + \eta_\varrho \leqq x \leqq 1, \end{cases}$$

where $\eta_\varrho = |v_\varrho(\bar{x}) - \bar{u}(\bar{x})|$. For sufficiently large ϱ and small $\bar{x}$, $\bar{v}_\varrho$ is well defined and belongs to $W^{1,3}(0, 1) \cap \mathscr{A}_0$. Notice that $|\bar{v}_\varrho'(x)|$ is uniformly bounded in $[\bar{x}, \bar{x} + \eta_\varrho]$, independently of ϱ. Therefore, since $\eta_\varrho \to 0$,

$$\lim_{\varrho \to \infty} J_{\bar{x}}(\bar{v}_\varrho) = J_{\bar{x}}(\bar{u}).$$

By lower semicontinuity,

$$\liminf_{\varrho \to \infty} J_{\bar{x}}(v_\varrho) \geqq J_{\bar{x}}(u),$$

and hence

$$\begin{aligned} 0 \leqq \limsup_{\varrho \to \infty} [J(\bar{v}_\varrho) - J(v_\varrho)] &= \limsup_{\varrho \to \infty} [J_{\bar{x}}(\bar{v}_\varrho) - J_{\bar{x}}(v_\varrho)] \\ &\leqq J_{\bar{x}}(\bar{u}) - J_{\bar{x}}(u). \end{aligned}$$

Therefore $J_{\bar{x}}(u) = J_{\bar{x}}(\bar{u})$ and thus u minimizes $J_{\bar{x}}$ in $\mathscr{A}_{\bar{x}}$. In particular $u \in C^\infty([\bar{x}, 1])$ and satisfies (EL) in $[\bar{x}, 1]$. Since $\bar{x}$ was arbitrary it follows that $u \in C^\infty((0, 1])$ and satisfies (EL) in $(0, 1]$. Since $u \in W^{1,3}(0, \delta)$ we also have $u \in W^{1,3}(0, 1)$ and therefore by (5.31) u minimizes J in $W^{1,3}(0, 1) \cap \mathscr{A}_0$. Clearly $u'(x) > 0$ for $x \in (0, 1]$.

Our final task is to show that $u \in C^\infty([0, 1])$, and by (EL) it suffices for this to show that $u'(0)$ is finite. Passing to the limit $\varrho \to \infty$ in (5.29) we obtain

$$u(x) \leqq \tfrac{1}{2} x^{\frac{2}{3}} \quad \text{for all } x \in [0, \delta], \tag{5.32}$$

and since $J(u) < \infty$ it follows that

$$\int_0^1 x^8 \, |u'|^s \, dx < \infty. \tag{5.33}$$

Since u is a smooth solution of (DBR) on $(0, 1]$ we have

$$\begin{aligned} \frac{d}{dx}(u' f_p - f) &= \frac{d}{dx}((s - 1)(x^4 - u^6)^2 (u')^s + (u')^2) \\ &= -8x^3(x^4 - u^6)(u')^s \end{aligned}$$

for $0 < x \leqq 1$, and therefore by (5.32), (5.33)

$$x \frac{d}{dx}(u' f_p - f) \in L^1(0, 1).$$

But $J(u) < \infty$ implies that

$$u' f_p - f = (s - 1)(x^4 - u^6)^2 (u')^s + \varepsilon (u')^2 \in L^1(0, 1)$$

and thus

$$\frac{d}{dx}(x(u'f_p - f)) = x\frac{d}{dx}(u'f_p - f) + (u'f_p - f)$$

belongs to $L^1(0, 1)$. Hence

$$x(u'f_p - f) = (s-1)\left(1 - \frac{u^6}{x^4}\right)^2 x^9(u')^s + \varepsilon x(u')^2$$

is uniformly bounded, and by (5.32) this implies that $x^9(u')^s$ is bounded. Hence

$$u'(x) \leqq \text{const. } x^{-\frac{9}{s}}, \quad u(x) \leqq \text{const. } x^{1-\frac{9}{s}}, \quad x \in (0, 1]. \tag{5.34}$$

Note that since $s > 27$, $1 - 9/s > \frac{2}{3}$. Pick $\sigma_0 \in (\frac{2}{3}, 1 - 9/s)$ such that if

$$\sigma_n \stackrel{\text{def}}{=} (\sigma_0 - \tfrac{2}{3})\left(\frac{s+5}{s-1}\right)^n + \tfrac{2}{3} \tag{5.35}$$

then $\sigma_n \neq \frac{s-5}{s+5}$ for any $n = 0, 1, 2, \ldots$. This is clearly possible. We prove by induction that for any $n = 0, 1, 2, \ldots$ there is a constant $c_n > 0$ such that

$$u'(x) \leqq c_n(1 + x^{\sigma_n - 1}), \quad u(x) \leqq c_n(x + x^{\sigma_n}), \quad x \in (0, 1]. \tag{5.36}$$

This is true for $n = 0$ by (5.34). Suppose the assertion is true for n. We prove that it holds for $n + 1$. This is obvious if $\sigma_n \geqq 1$, so we consider the case $\sigma_n < 1$. Now by (EL)

$$\frac{d}{dx}f_p = -12u^5(x^4 - u^6)(u')^s, \quad x \in (0, 1],$$

and so by (5.32), (5.36)

$$\left|\frac{d}{dx}f_p\right| \leqq \text{const. } x^{5\sigma_n + 4 + s(\sigma_n - 1)}, \quad x \in (0, 1].$$

Since $\sigma_n \neq \frac{s-5}{s+5}$ it follows that

$$f_p = s(x^4 - u^6)^2(u')^{s-1} + 2\varepsilon u' \leqq \text{const. } (1 + x^{5-s+(s+5)\sigma_n}), \quad x \in (0, 1]. \tag{5.37}$$

If $\sigma_n > \frac{s-5}{s+5}$ then (5.37) implies that u' is bounded on $(0, 1]$ and thus that (5.36) holds for $n + 1$. Otherwise, $\sigma_n < \frac{s-5}{s+5}$ and we deduce from (5.37) that

$$x^8(u')^{s-1} \leqq \text{const. } x^{5-s+(s+5)\sigma_n}, \quad x \in (0, 1].$$

Therefore

$$u'(x) \leqq \text{const. } x^{\left(\frac{s+5}{s-1}\right)\sigma_n - \left(\frac{s+3}{s-1}\right)} = \text{const. } x^{\sigma_{n+1} - 1},$$

so that (5.36) holds for $n + 1$. This proves our assertion.

Since $\sigma_0 > \frac{2}{3}$, $\sigma_n \geqq 1$ for large enough n, and hence by (5.36) $u'(x)$ is bounded in $(0, 1]$. Therefore $u'(0)$ is finite and $u \in C^\infty([0, 1])$. Finally, if $\tilde{u}$ is *any* minimizer of J in $W^{1,q}(0, 1) \cap \mathscr{A}_0$, $q \geqq 3$, then the above arguments applied to the minimizing sequence in $W^{1,3}(0, 1) \cap \mathscr{A}_0$ given by $v_j \equiv \tilde{u}$ show that $\tilde{u} \in C^\infty([0, 1])$ and satisfies (EL) on $[0, 1]$. $\square$

Note that the proof of Theorem 5.7 shows that any minimizing sequence for J in $W^{1,q}(0, 1) \cap \mathscr{A}_0$, $q \geqq 3$, has a subsequence converging weakly in $W^{1,q}(0, \delta)$ and $W^{1,2}(0, 1)$ to a minimizer.

Remark. Theorems 5.6 and 5.7 apply equally to the problem of minimizing

$$J_-(u) = \int_{-1}^{0} [(x^4 - u^6)^2 |u'|^s + \varepsilon(u')^2]\, dx$$

over various subsets of

$$\mathscr{A}_- = \{u \in W^{1,1}(-1, 0) : u(-1) = k,\ u(0) = 0\}, \quad \text{with } k < 0,$$

as can be seen by noting that $v(\cdot) \in \mathscr{A}_-$ if and only if $\hat{v} = -v(-\cdot) \in \mathscr{A}_0$, and $J_-(v) = J(\hat{v})$.

We next prove the existence of pseudominimizers for our original functional $I(u)$ given by (5.1).

Theorem 5.8. *Let $s > 27$, $3 \leqq q \leqq \infty$, and let k_1, k_2 be arbitrary. Then $I(u)$ attains an absolute minimum in $\mathscr{A} \cap W^{1,q}$, and each such minimizer u_1 belongs to $C^\infty([-1, 1])$ and satisfies* (EL) *on* $[-1, 1]$.

Proof. If $k_1 = k_2$ then the unique minimizer of I in $\mathscr{A} \cap W^{1,q}$ is $u_1 \equiv k_1$ and there is nothing to prove. If k_1, k_2 are not equal and have the same sign then any minimizer u_0 of I in $\mathscr{A}$ is strictly monotone and by Lemma 5.3 is a smooth solution of (EL) in $[-1, 1]$, and again we have finished. We therefore suppose that $k_1 < 0 < k_2$; the case $k_1 > 0 > k_2$ is treated similarly. Let $\{v_j\}$ be a minimizing sequence for I in $\mathscr{A} \cap W^{1,q}$. By extracting an appropriate subsequence, again denoted by $\{v_j\}$, we may suppose that $v_j \rightharpoonup u_1$, say, in $W^{1,2}(-1, 1)$ and that either (a) $v_j(0) = 0$ for all j, or (b) $v_j(0) < 0$, for all j, or (c) $v_j(0) > 0$ for all j. If (a) holds then clearly $\{v_j\}$ (restricted to $[0, 1]$) is a minimizing sequence for J (given by (5.23)) in $\mathscr{A}_0 \cap W^{1,q}(0, 1)$, where $\mathscr{A}_0 = \{u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = k_2\}$, and therefore by the proof of Theorem 5.7 u_1 minimizes J in $W^{1,q}(0, 1) \cap \mathscr{A}_0$. A similar argument holds on $[-1, 0]$, and so by Theorem 5.7 and lower semicontinuity u_1 is smooth on $[-1, 0]$ and $[0, 1]$ and minimizes I in $\mathscr{A} \cap W^{1,q}$. Standard arguments then show that u_1 satisfies (EL) and is smooth in $[-1, 1]$.

Suppose (b) holds; case (c) is treated similarly. Suppose first that $\lim_{j\to\infty} v_j(0) = u_1(0) < 0$. Let u_2 be any minimizer of I in $\bar{\mathscr{A}} \stackrel{\text{def}}{=} \{u \in \mathscr{A} : u(0) = u_1(0)\}$. Then by Lemma 5.3 u_2 is smooth in $[-1, 0]$ and $[0, 1]$, and so

$$\inf_{\mathscr{A} \cap W^{1,q}} I \leqq I(u_2) \leqq I(u_1).$$

But by lower semicontinuity $I(u_1) \leqq \inf_{\mathscr{A} \cap W^{1,q}} I$, and it follows that u_1 minimizes

I in $\mathscr{A}$ also. Hence u_1 is smooth in $[-1, 0]$ and $[0, 1]$, minimizes I in $\mathscr{A} \cap W^{1,q}$, and by standard arguments is a smooth solution of (EL) in $[-1, 1]$.

It remains to consider the case when (b) holds and $\lim_{j\to\infty} v_j(0) = u_1(0) = 0$. Let u_3 be any minimizer of

$$J_-(u) = \int_{-1}^{0} [(x^4 - u^6)^2 \, |u'|^s + \varepsilon(u')^2] \, dx$$

in $\mathscr{A}_-$, where

$$\mathscr{A}_- = \{u \in W^{1,1}(-1, 0) : u(-1) = k_1, u(0) = 0\}.$$

On the other hand let u_4 be any minimizer of $J(u)$ in $\mathscr{A}_0 \cap W^{1,q}(0, 1)$; the existence and smoothness of u_4 is guaranteed by Theorem 5.7 and the remark following it. Define $\bar{u} \in \mathscr{A}$ by

$$\bar{u}(x) = \begin{cases} u_3(x), & -1 \leqq x \leqq 0 \\ u_4(x), & 0 \leqq x \leqq 1. \end{cases}$$

We first show that

$$I(\bar{u}) \geqq \inf_{\mathscr{A} \cap W^{1,q}} I. \tag{5.38}$$

To this end consider the sequence

$$w_j(x) = \begin{cases} u_3(x), & -1 \leqq x \leqq -\dfrac{1}{j}, \\ u_3\left(-\dfrac{1}{j}\right), & -\dfrac{1}{j} \leqq x \leqq 0, \\ u_3\left(-\dfrac{1}{j}\right) + Mx, & 0 \leqq x \leqq \beta_j, \\ u_4(x), & \beta_j \leqq x \leqq 1. \end{cases} \tag{5.39}$$

In (5.39), M is chosen greater than $\max\{u_4'(0), k_2 + |k_1|\}$ so $\beta_j \to 0+$ satisfies $u_4(\beta_j) = u_3\left(-\dfrac{1}{j}\right) + M\beta_j$. The existence of β_j follows from the intermediate value theorem. Note that by a version of Lemma 5.3 which applies to J_-, $u_3 \in C^\infty([-1, 0))$ and so $w_j \in \mathscr{A} \cap W^{1,q}$. Since, as is easily checked,

$$\lim_{j\to\infty} I(w_j) = I(\bar{u}),$$

(5.38) follows. Next, let $\delta_j > 0$ be the largest root of $v_j(x) = 0$ in $(0, 1)$, and define

$$\bar{v}_j(x) = \begin{cases} 0, & 0 \leqq x \leqq \delta_j, \\ v_j(x), & \delta_j \leqq x \leqq 1. \end{cases}$$

Then $\bar{v}_j \in \mathscr{A}_0 \cap W^{1,q}(0, 1)$ and so $J(v_j) \geqq J(\bar{v}_j) \geqq J(u_4)$. Also, by the sequential lower semicontinuity of J_-,

$$J_-(u_3) \leqq J_-(u_1) \leqq \liminf_{j\to\infty} J_-(v_j).$$

Therefore

$$\inf_{\mathscr{A}\cap W^{1,q}} I = \lim_{j\to\infty} I(v_j) = \lim_{j\to\infty} (J_-(v_j) + J(v_j))$$
$$\geqq \liminf_{j\to\infty} J_-(v_j) + \liminf_{j\to\infty} J(v_j) \tag{5.40}$$
$$\geqq J_-(u_3) + J(u_4) = I(\bar{u}).$$

Combining (5.38), (5.40) we obtain

$$I(\bar{u}) = \inf_{\mathscr{A}\cap W^{1,q}} I. \tag{5.41}$$

Suppose first that $u_3'(0)$ is finite so that u_3 is smooth on $[-1, 0]$. Then $\bar{u}$ minimizes I in $\mathscr{A} \cap W^{1,q}$, is smooth on $[-1, 0]$ and $[0, 1]$, and so is a smooth solution of (EL) on $[-1, 1]$. On the other hand by Theorem 5.6 and the Remark, we know that for $\varepsilon > 0$ sufficiently small $u_3'(0) = +\infty$. For such ε, $\lim_{j\to\infty} v_j(0) = u_1(0) = 0$ cannot occur. Indeed, solving (EL) with initial data

$$u(0) = \delta, \quad u'(0) = M, \tag{5.42}$$

for $|\delta|$ small generates by Lemma 2.8 a field of extremals covering a neighborhood of the origin. For δ sufficiently small and negative the solution u_δ of (EL) satisfying (5.42) intersects the graphs of both u_3 and u_4 at points $r_\delta < 0$ and $s_\delta > 0$ respectively, where $r_\delta, s_\delta \to 0$ as $\delta \to 0-$. It then follows from the field theory that

$$v_\delta(x) = \begin{cases} u_3(x), & -1 \leqq x \leqq r_\delta, \\ u_\delta(x), & r_\delta \leqq x \leqq s_\delta, \\ u_4(x), & s_\delta \leqq x \leqq 1, \end{cases}$$

satisfies $I(v_\delta) < I(\bar{u})$. But $v_\delta \in \mathscr{A} \cap W^{1,q}$, contradicting (5.41).

Summarizing, we have shown that in all cases I attains a minimum on $\mathscr{A} \cap W^{1,q}$ at some smooth solution u_1 of (EL). If u_1 is *any* minimizer in $\mathscr{A} \cap W^{1,q}$ then applying the proof to $v_j \equiv u_1$ shows that u_1 is smooth (the case when (b) holds and $\lim_{j\to\infty} v_j(0) = 0$ does not occur). □

We now examine what happens if $s < 27$. If $s = 26$ the integrand f given by (5.2) satisfies the scale invariance property (3.7) with $\gamma = \frac{2}{3}$ and $\varrho = -\frac{2}{3}$ and the phase-plane techniques of Section 3 are applicable to the one-sided problem of minimizing J in $\mathscr{A}_0$. We confine attention here to the observation that the same argument as in Lemma 3.8 shows that if $s \geqq 26$, $0 < \alpha < \min(1, k)$ and $\varepsilon > 0$ is sufficiently small then any minimizer u_0 of J in $\mathscr{A}_0$ satisfies $u_0(x) > \alpha x^{\frac{2}{3}}$ for all $x \in (0, 1]$ and is thus singular. If, further, $s < 27$ then the Lavrentiev phenomenon does not occur; this follows by noting that, by the proof of Theorem 5.6, $u_0(x) \leqq x^{\frac{2}{3}}$ for x sufficiently small, and by using the argument in the remark following Lemma 3.8. It remains, therefore, to consider the case $s < 26$.

Theorem 5.9. *Let* $3 < s < 26$, $\varepsilon > 0$.

(i) *Let* $k_1, k_2 \in \mathbb{R}$ *and let* u_0 *minimize* I *in* $\mathscr{A}$. *Then* $u_0 \in C^\infty([-1, 1])$ *and satisfies* (EL) *on* $[-1, 1]$.

(ii) *Let* $k \in \mathbb{R}$ *and let* u_0 *minimize* J *in* $\mathcal{A}_0$. *Then* $u_0 \in C^\infty([0, 1])$ *and satisfies* (EL) *on* $[0, 1]$.

Proof. It suffices to prove (ii), since in case (i) if $u_0(0) \neq 0$ then u_0 is smooth by Lemma 5.3.

To prove (ii) we may as before assume that $k > 0$. We note that by the same arguments as in the proofs of Theorems 5.1, 5.6 any minimizer u_0 satisfies $0 \leqq u_0(x) \leqq x^{\frac{2}{3}}$ for x sufficiently small and $|u_0'(x)| \leqq \text{const.}\, x^{-\frac{1}{3}}$, $x \in (0, 1]$. It follows from (5.7) that

$$\left|\frac{d}{dx}(s(x^4 - u_0^6)^2 (u_0')^{s-1} + 2\varepsilon u_0')\right| \leqq \text{const.}\, x^{\frac{22-s}{3}}, \quad x \in (0, 1]. \tag{5.43}$$

Hence, by integration, $u_0'(x)$ is bounded for $x \in (0, 1]$ if $s < 25$. If $25 \leqq s < 26$ we deduce by integrating (5.43) that

$$u_0'(x) \leqq \text{const.}\, x^{\tau_0 - 1}, \quad u_0(x) \leqq \text{const.}\, x^{\tau_0}, \quad x \in (0, 1], \tag{5.44}$$

for some $\tau_0 \in (\frac{2}{3}, 1)$, and we may clearly choose τ_0 such that if

$$\tau_n \stackrel{\text{def}}{=} \left(\tau_0 - \frac{s-6}{s+4}\right)(s+5)^n + \frac{s-6}{s+4}, \tag{5.45}$$

then $\tau_n \neq \dfrac{s-5}{s+5}$ for any $n = 0, 1, 2, \ldots$. We prove by induction that for any $n = 0, 1, 2, \ldots$ there is a constant $d_n > 0$ such that

$$u_0'(x) \leqq d_n(1 + x^{\tau_n - 1}), \quad u_0(x) \leqq d_n(x + x^{\tau_n}), \quad x \in (0, 1]. \tag{5.46}$$

This is true for $n = 0$ by (5.44). Suppose it is true for n. We prove that it holds for $n + 1$. This is obvious if $\tau_n \geqq 1$, so we consider the case $\tau_n < 1$. By (5.7)

$$2\varepsilon u_0'(x) \leqq \text{const.}\, (1 + x^{5-s+(s+5)\tau_n}), \quad x \in (0, 1]. \tag{5.47}$$

But $5 - s + (s+5)\tau_n = \tau_{n+1} - 1$, so that (5.46) holds for $n + 1$.

Since $s < 26$ it follows that $\tau_0 > \dfrac{s-6}{s+4}$, and thus $\tau_n \geqq 1$ for large enough n. Hence in all cases u_0' is bounded in $(0, 1]$ and thus u_0 is a smooth solution of (EL) in $[0, 1]$. □

We end by remarking that the methods of this section apply also to the problem of minimizing

$$I(u) = \int_0^1 [(x^2 - u^3)^2 |u'|^s + \varepsilon(u')^2]\, dx$$

in

$$\mathcal{A} = \{u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = k\},$$

the special case $s = 14$ having been exhaustively discussed in Section 3. For this problem any absolute minimizer is smooth for $3 < s < 14$, singular minimizers can exist without the Lavrentiev phenomenon for $14 \leqq s < 15$, singular minimizers and the Lavrentiev phenomenon can exist for $s \geqq 15$, and smooth pseudominimizers exist if $s > 15$.

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