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$W^{1,p}$-Quasiconvexity and Variational Problems for Multiple Integrals

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Variational problems for the multiple integral $I_\Omega(u) = \int_\Omega g(\nabla u(x))\,dx$, where $\Omega \subset \mathbb{R}^m$ and $u: \Omega \to \mathbb{R}^n$ are studied. A new condition on g, called $W^{1,p}$-quasiconvexity is introduced which generalizes in a natural way the quasiconvexity condition of C. B. Morrey, it being shown in particular to be necessary for sequential weak lower semicontinuity of I_Ω in $W^{1,p}(\Omega; \mathbb{R}^n)$ and for the existence of minimizers for certain related integrals. Counterexamples are given concerning the weak continuity properties of Jacobians in $W^{1,p}(\Omega; \mathbb{R}^n)$, $p \leqslant n = m$. An existence theorem for nonlinear elastostatics is proved under optimal growth hypotheses.

1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^m$, let $M^{n\times m}$ denote the set of real $n \times m$ matrices, let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ denote the extended real line with the usual topology, and let $g: M^{n\times m} \to \overline{\mathbb{R}}$ be Borel measurable and bounded below. In this paper we consider variational problems for the multiple integral

$$I_\Omega(u) = \int_\Omega g(\nabla u(x))\,dx. \tag{1.1}$$

For $1 \leqslant p \leqslant \infty$, we introduce and study a new condition on g, called $W^{1,p}$-*quasiconvexity*, which generalizes in a natural way the quasiconvexity condition of Morrey [13] by allowing the competing functions to belong to the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^n)$ rather than to the smaller space

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$W^{1,\infty}(\Omega; \mathbb{R}^n)$. For g finite and continuous, Morrey showed that $W^{1,\infty}$-quasiconvexity of g is necessary and sufficient for I_Ω to be sequentially weak* lower semicontinuous (sw*lsc) on $W^{1,\infty}(\Omega; \mathbb{R}^n)$. He also proved that if g is $W^{1,\infty}$-quasiconvex and satisfies certain quite restrictive growth conditions related to p then g is sequentially weakly lower semicontinuous (swlsc) on $W^{1,p}(\Omega; \mathbb{R}^n)$; this result has recently been refined in an interesting paper of Acerbi and Fusco [3], who prove in particular that if g is continuous and satisfies

$$0 \leqslant g(A) \leqslant K(|A|^p + 1), \qquad A \in M^{n\times m}, \tag{1.2}$$

then I_Ω is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ if and only if g is $W^{1,\infty}$-quasiconvex.

Following the ideas of Morrey we show (Corollary 3.2) that $W^{1,p}$-quasiconvexity of g is a necessary condition for I_Ω to be swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc if $p=\infty$). Indeed we isolate a necessary condition for lower semicontinuity (Theorem 3.1) that for general g is strictly stronger than $W^{1,p}$-quasiconvexity (see Example 3.5). Furthermore (Corollary 5.2) we show that if meas $\partial\Omega = 0$ then $W^{1,p}$-quasiconvexity is a necessary condition for

$$J_\Omega(u) = \int_\Omega [\, g(\nabla u(x)) + \Psi(x, u(x))]\, dx \tag{1.3}$$

to attain a minimum on $X_A \stackrel{\text{def}}{=} \{u : u - Ax \in W_0^{1,p}(\Omega; \mathbb{R}^n)\}$ for *every* $A \in M^{n\times m}$ and *every* smooth nonnegative Ψ. It seems likely (see Conjecture 3.7) that $W^{1,p}$-quasiconvexity, or some slight variant of it, is also *sufficient* for I_Ω to be swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc if $p=\infty$) though we have not been able to prove this.

For upper semicontinuous functions g satisfying (1.2) $W^{1,p}$-quasiconvexity and $W^{1,\infty}$-quasiconvexity are equivalent (Proposition 2.4(i)), but in general this is not so. For example, let $m = n$, let $h: \mathbb{R} \to \overline{\mathbb{R}}$ be lower semicontinuous and bounded below, and let

$$g(A) = h(\det A), \qquad A \in M^{n\times n}. \tag{1.4}$$

Then (Theorem 4.1) if $n \leqslant p \leqslant \infty$, g is $W^{1,p}$-quasiconvex if and only if h is convex, but if $1 \leqslant p < n$ then g is $W^{1,p}$-quasiconvex if and only if h is *constant*. Another instructive example, drawn from a study of cavitation in nonlinear elasticity (Ball [6]) that partly motivated this work, is given by

$$g(A) = |A|^\alpha + h(\det A), \qquad A \in M^{n\times n}, \tag{1.5}$$

where $h: \mathbb{R} \to \overline{\mathbb{R}}$ is convex, lower semicontinuous, bounded below and satisfies $\lim_{|t|\to\infty} h(t)/|t| = \infty$. In this case (Theorem 4.5), g is $W^{1,p}$-quasiconvex if either $\alpha \geqslant 1$ and $n \leqslant p \leqslant \infty$ or $\alpha \geqslant n$ and $1 \leqslant p \leqslant \infty$, but is not $W^{1,p}$-quasiconvex for any $p<n$ if $1 \leqslant \alpha < n$ because (4.9) is not satisfied. In particular let $\alpha = 2$, $n > 2$ and suppose in addition that h is smooth. Then a calculation shows that g satisfies the *uniform strong ellipticity condition*

$$\sum_{i,j,\alpha,\beta=1}^{n} \frac{\partial^2 g(A)}{\partial A^i_\alpha \, \partial A^j_\beta} a^i b_\alpha a^j b_\beta \geqslant 2\,|a|^2\,|b|^2, \qquad a, b \in \mathbb{R}^n.$$

Thus strong ellipticity and the coercivity condition $\lim_{|A| \to \infty} g(A)/|A| = \infty$ do not together imply that I_Ω is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ for $1 \leqslant p < n$, nor that J_Ω attains a minimum on X_A for $1 \leqslant p < n$, arbitrary A and all smooth nonnegative Ψ.

There is a close relation between the examples described in the preceding paragraph and the weak continuity properties of the mapping $u \mapsto \det \nabla u(x)$. It is proved in Reshetnyak [18] that if $u_j \rightharpoonup u$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ then $\det \nabla u_j \rightharpoonup {}^*\det \nabla u$ in the sense of measures. We combine this and related results with a lower semicontinuity theorem (Proposition A.3) motivated by Reshetnyak [17] to show (Theorem 4.1) that if $h: \mathbb{R} \to \overline{\mathbb{R}}$ is lower semicontinuous and bounded below then $\int_\Omega h(\det \nabla u(x))\,dx$ is swlsc on $W^{1,n}(\Omega; \mathbb{R}^n)$ if and only if h is convex, and to prove a new existence theorem (Theorem 6.1) for nonlinear elastostatics with an optimal growth condition (see (6.3)). In Section 7 we give examples showing that $u \mapsto \det \nabla u$ is not sequentially weakly continuous from $W^{1,n}(\Omega; \mathbb{R}^n) \to L^1(\Omega)$, and has even worse properties with respect to weak convergence in $W^{1,p}(\Omega; \mathbb{R}^n)$ for $1 \leqslant p < n$.

To reduce technicalities and emphasize the essential difficulties we have restricted attention to integrals of the form (1.1), (1.3). Analogous results for the case of a general integral $\int_\Omega g(x, u(x), \nabla u(x))\,dx$ and other generalizations requiring technical feats are left for the courageous reader.

2. Definition and Elementary Properties of $W^{1,p}$-Quasiconvexity

Notation. If $E \subset \mathbb{R}^m$ is open we denote by $W^{1,p}(E; \mathbb{R}^n)$ the Sobolev space consisting of those measurable mappings $u: E \to \mathbb{R}^n$ with finite norm $\|u\|_{W^{1,p}(E;\mathbb{R}^n)} \overset{\text{def}}{=} \|u\|_{L^p(E;\mathbb{R}^n)} + \|\nabla u\|_{L^p(E;M^{n\times m})}$, and by $W^{1,p}_{\text{loc}}(E; \mathbb{R}^m)$ the space consisting of those measurable mappings $u: E \to \mathbb{R}^n$ with the property that every $x \in E$ possesses an open neighborhood $N_x \subset E$ such that $u \in W^{1,p}(N_x; \mathbb{R}^n)$. If $1 \leqslant p < \infty$, the closure of $C_0^\infty(E; \mathbb{R}^n)$ in $W^{1,p}(E; \mathbb{R}^n)$ is denoted by $W_0^{1,p}(E; \mathbb{R}^n)$. We define $W_0^{1,\infty}(E; \mathbb{R}^n)$ to be the closure of $C_0^\infty(E; \mathbb{R}^n)$ in the weak* topology of $W^{1,\infty}(E; \mathbb{R}^n)$, i.e., the subspace topology induced by regarding $W^{1,\infty}(E; \mathbb{R}^n)$ as a closed subspace of a finite product of $L^\infty(E)$ spaces each endowed with the weak* topology. Throughout we denote by C a generic constant whose value may vary from line to line.

Let $g: M^{n\times m} \to \bar{\mathbb{R}}$ be Borel measurable and bounded below, and let $1 \leqslant p \leqslant \infty$.

DEFINITION 2.1. The function g is *$W^{1,p}$-quasiconvex at* $A \in M^{n\times m}$ if

$$\int_E g(A + \nabla\phi(x))\, dx \geqslant (\text{meas}\, E)\, g(A) \tag{2.1}$$

for every bounded open set $E \subset \mathbb{R}^m$ with meas $\partial E = 0$ and all $\phi \in W_0^{1,p}(E; \mathbb{R}^n)$. We say that g is *$W^{1,p}$-quasiconvex* if it is $W^{1,p}$-quasiconvex at every $A \in M^{n\times m}$.

Remarks 2.2. (1) $W^{1,\infty}$-quasiconvexity was introduced by Morrey [13] for finite continuous functions g and called *quasiconvexity*. (Later, in [14], he changed his terminology.) We allow g to take the value $+\infty$ so as to include applications to nonlinear elasticity and optimal design (cf. Kohn and Strang [11]); note that in the definition, Eq. (2.1) is required to hold even if $g(A) = +\infty$.

(2) If g is $W^{1,p}$-quasiconvex then g is $W^{1,q}$-quasiconvex for all q with $p \leqslant q \leqslant \infty$. Thus $W^{1,1}$-quasiconvexity is the strongest condition, $W^{1,\infty}$-quasiconvexity the weakest.

Suppose that (2.1) holds for one nonempty bounded open subset $E \subset \mathbb{R}^m$, for some $A \in M^{n\times m}$, and for all $\phi \in W_0^{1,p}(E; \mathbb{R}^n)$. Suppose further that $g(A) < \infty$. Then following Meyers [12, p. 128], for any other bounded open subset $E_1 \subset \mathbb{R}^m$ there exist $a \in \mathbb{R}^m$ and $\varepsilon > 0$ such that $a + \varepsilon E_1 \subset E$. Thus we have, using (2.1), that for any $\phi \in W_0^{1,p}(E_1; \mathbb{R}^n)$

$$\int_{a+\varepsilon E_1} g\left(A + \nabla\phi\left(\frac{x-a}{\varepsilon}\right)\right) dx + \text{meas}(E \setminus \{a + \varepsilon E_1\})\, g(A) \geqslant \int_E g(A)\, dx,$$

from which it follows that

$$\int_{E_1} g(A + \nabla\phi(y))\, dy \geqslant (\text{meas}\, E_1)\, g(A). \tag{2.2}$$

This argument fails when $g(A) = \infty$. Nevertheless we have

PROPOSITION 2.3. *If* (2.1) *holds for one nonempty bounded open subset $E \subset \mathbb{R}^m$, some $A \in M^{n\times m}$ and all $\phi \in W_0^{1,p}(E; \mathbb{R}^n)$, then g is $W^{1,p}$-quasiconvex at A.*

Proof. Let $E_1 \subset \mathbb{R}^m$ be bounded open with meas $\partial E_1 = 0$. Consider the family of closed sets of the form $a + \varepsilon \bar{E}_1$ contained in E, where $a \in \mathbb{R}^m$, $\varepsilon > 0$. This family clearly covers E in the sense of Vitali, and hence by the

Vitali covering theorem [19, p. 109] there exists a finite or countable disjoint sequence $a_i + \varepsilon_i \bar{E}_1$ of subsets of E such that $\text{meas}(E \setminus \bigcup_i (a_i + \varepsilon_i \bar{E}_1)) = 0$. Let $\phi \in W_0^{1,p}(E_1; \mathbb{R}^n)$ and define

$$\tilde{\phi}(x) = \varepsilon_i \phi\left(\frac{x - a_i}{\varepsilon_i}\right) \qquad \text{for} \quad x \in a_i + \varepsilon_i E_1,$$
$$= 0 \qquad \text{otherwise.}$$

It is easily verified that $\tilde{\phi} \in W_0^{1,p}(E; \mathbb{R}^n)$. Hence, by (2.1),

$$\begin{aligned}(\text{meas } E)\, g(A) \leqslant \int_E g(A + \nabla\tilde{\phi}(x))\, dx &= \sum_i \int_{a_i + \varepsilon_i E_1} g\left(A + \nabla\phi\left(\frac{x - a_i}{\varepsilon_i}\right)\right) dx \\ &= \left(\sum_i \varepsilon_i^m\right) \int_{E_1} g(A + \nabla\phi(y))\, dy \\ &= \frac{\text{meas } E}{\text{meas } E_1} \int_{E_1} g(A + \nabla\phi(y))\, dy\end{aligned}$$

(note that in the first equality we use meas $\partial E_1 = 0$) and thus

$$\int_{E_1} g(A + \nabla\phi(y))\, dy \geqslant (\text{meas } E_1)\, g(A)$$

as required. ∎

PROPOSITION 2.4. (i) *Let g be upper semicontinuous, bounded below, and satisfy the estimate*

$$g(A) \leqslant K(|A|^p + 1) \qquad \textit{for all} \quad A \in M^{n \times m}, \tag{2.3}$$

where K is a constant and where $1 \leqslant p < \infty$. Then g is $W^{1,p}$-quasiconvex if and only if g is $W^{1,\infty}$-quasiconvex.

(ii) *Let g be Borel measurable and satisfy the estimate*

$$k|A|^p + k_1 \leqslant g(A) \qquad \textit{for all} \quad A \in M^{n \times m}, \tag{2.4}$$

where $k > 0$ and k_1 are constants and where $1 \leqslant p < \infty$. Then g is $W^{1,1}$-quasiconvex if and only if g is $W^{1,p}$-quasiconvex.

Proof. (i) Let $E \subset \mathbb{R}^m$ be bounded open, $\phi \in W_0^{1,p}(E; \mathbb{R}^n)$ and $A \in M^{n \times m}$. Suppose g is $W^{1,\infty}$-quasiconvex. There exists a sequence ϕ_j of

$C_0^\infty(E; \mathbb{R}^n)$ functions converging to ϕ in $W^{1,p}(E; \mathbb{R}^n)$. By (2.3), Fatou's lemma and the upper semicontinuity of g,

$$\liminf_{j\to\infty} \int_E [K\,|A+\nabla\phi_j|^p - g(A+\nabla\phi_j)]\,dx$$

$$\geqslant \int_E [K\,|A+\nabla\phi|^p - g(A+\nabla\phi)]\,dx.$$

Since g is $W^{1,\infty}$-quasiconvex it follows that

$$\int_E g(A+\nabla\phi)\,dx \geqslant \limsup_{j\to\infty} \int_E g(A+\nabla\phi_j)\,dx \geqslant (\text{meas}\,E)\,g(A),$$

so that g is $W^{1,p}$-quasiconvex.

(ii) Let $Q \subset \mathbb{R}^m$ be an open m-cube. Since ∂Q is Lipschitz $W_0^{1,1}(Q; \mathbb{R}^n) \cap W^{1,p}(Q; \mathbb{R}^n) = W_0^{1,p}(Q; \mathbb{R}^n)$. If $\phi \in W_0^{1,1}(Q; \mathbb{R}^n)$ but $\phi \notin W^{1,p}(Q; \mathbb{R}^n)$, by (2.4) we have $\int_Q g(A+\nabla\phi)\,dx = \infty$. Hence if g is $W^{1,p}$-quasiconvex

$$\int_Q g(A+\nabla\phi)\,dx \geqslant (\text{meas}\,Q)\,g(A)$$

for all $\phi \in W_0^{1,1}(Q; \mathbb{R}^n)$. Thus g is $W^{1,1}$-quasiconvex by Proposition 2.3. ∎

We remark that (i) is also a consequence of Acerbi and Fusco [3, Theorem II.4] and Corollary 3.2 below.

3. $W^{1,p}$-Quasiconvexity as a Necessary Condition for Lower Semicontinuity

Recall that $g: M^{n\times m} \to \overline{\mathbb{R}}$ is Borel measurable and bounded below.

Theorem 3.1. *Let $\Omega \in \mathbb{R}^m$ be a nonempty bounded open set. Define*

$$I_\Omega(u) = \int_\Omega g(\nabla u(x))\,dx.$$

Suppose that I_Ω is swlsc *(sequentially weakly lower semicontinuous) on $W^{1,p}(\Omega; \mathbb{R}^n)$; i.e., $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ implies $I_\Omega(u) \leqslant \liminf_{j\to\infty} I_\Omega(u_j)$. Then g is lower semicontinuous, and*

$$\frac{1}{\text{meas}\,Q}\int_Q g(\nabla v(x))\,dx \geqslant g\left(\frac{1}{\text{meas}\,Q}\int_Q \nabla v(x)\,dx\right) \tag{3.1}$$

for every m-cube Q and for all $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^n)$ *such that* ∇v *is Q-periodic.* (*If* $p = \infty$ *replace* $\rightharpoonup$ *by* $\rightharpoonup^*$ *and w by w**.)

Proof. Taking $\nabla u_j = A_j =$ constant shows immediately that g is lower semicontinuous. To prove (3.1) let $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^n)$ with ∇v Q-periodic. By Corollary A.2 of the appendix $v_j(x) \stackrel{\mathrm{def}}{=} j^{-1}v(jx)$ satisfies $v_j \rightharpoonup ((1/\mathrm{meas}\, Q)\int_Q \nabla v(y)\, dy)x$ in $W^{1,p}(\Omega; \mathbb{R}^n)$. Either $\int_Q g(\nabla v(x))\, dx = \infty$ or $g(\nabla v) \in L^1(Q)$ and so by Lemma A.1 $g(\nabla v_j) \rightharpoonup (1/\mathrm{meas}\, Q)\int_Q g(\nabla v(x))\, dx$ in $L^1(\Omega)$. Since I is swlsc we obtain (3.1). ∎

COROLLARY 3.2. *If* I_Ω *is* swlsc *on* $W^{1,p}(\Omega; \mathbb{R}^n)$ *then g is* $W^{1,p}$ *quasiconvex.* (*If* $p = \infty$, *replace w by w**.)

Proof. Let Q be an m-cube, let $A \in M^{n\times m}$ and let $\phi \in W^{1,p}_0(Q; \mathbb{R}^n)$. Let $\tilde{\phi}$ be the Q-periodic extension of ϕ to $\mathbb{R}^m$ and define $v(x) = Ax + \tilde{\phi}(x)$. By (3.1) we have that

$$\frac{1}{\mathrm{meas}\, Q}\int_Q g(A + \nabla\phi(x))\, dx \geqslant g(A).$$

It follows from Proposition 2.3 that g is $W^{1,p}$-quasiconvex. ∎

Remark 3.3. Suppose g takes only finite values and is continuous. Let $\partial\Omega$ and $u_0\colon \partial\Omega \to \mathbb{R}^n$ be sufficiently regular, and consider the *Dirichlet class*

$$\mathscr{C} = \{u \in W^{1,p}(\Omega; \mathbb{R}^n)\colon u\,|_{\partial\Omega} = u_0 \text{ in the sense of trace}\}.$$

Then (cf. Meyers [12, p. 129]) if I_Ω is swlsc on $\mathscr{C}$ g is $W^{1,p}$-quasiconvex. To prove this we assume without loss of generality that $\bar{Q} = [0, 1]^m \subset \Omega$, let w be a Lipschitz mapping satisfying $w\,|_{\partial\Omega} = u_0$, $w\,|_{\bar{Q}} = Ax$, and repeat the proof of Theorem 3.1 and Corollary 3.2 using the functions

$$\begin{aligned} v_j(x) &= Ax + j^{-1}\tilde{\phi}(jx) && \text{in } \bar{Q},\\ &= w(x) && \text{in } \Omega\backslash\bar{Q}, \end{aligned}$$

where $\phi \in W^{1,p}_0(Q; \mathbb{R}^n)$.

COROLLARY 3.4 (cf. Tartar [20, p. 164]). *If* I_Ω *is* sw*lsc *on* $W^{1,\infty}(\Omega; \mathbb{R}^n)$ *then g is* rank 1 convex, *i.e., g is convex on all straight line segments in* $M^{n\times m}$ *whose endpoints differ by a matrix of rank* 1.

Proof. Define $v(x) = Ax + f(\langle x, b\rangle)a$, where $A \in M^{n\times m}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $0 < \theta < 1$, $f(t) = \int_0^t \chi_\theta(s)\, ds$ and χ_θ is the real-valued periodic function given by

$$\begin{aligned} \chi_\theta(s) &= 1 && \text{on } [i, i+\theta),\\ &= 0 && \text{on } [i+\theta, i+1), \quad i \in \mathbb{Z}. \end{aligned}$$

Then $\nabla v(x) = A + \chi_\theta(\langle x, b\rangle) a \otimes b$ is periodic with respect to any unit m-cube with one pair of faces normal to b; applying (3.1) we obtain

$$g(A + \theta a \otimes b) \leqslant \theta g(A + a \otimes b) + (1 - \theta) g(A) \tag{3.2}$$

as required. ∎

For *continuous* $g: M^{n \times m} \to \bar{\mathbb{R}}$ rank 1 convexity follows from $W^{1,\infty}$-quasiconvexity; in fact the proof of Morrey (see Morrey [13, p. 45] and Ball [4, p. 353]) shows that (3.2) holds provided the left-hand side is finite, and by examining the behaviour of $g(A + \theta a \otimes b)$ for $\theta \in \mathbb{R}$ it is easily seen that (3.2) holds also when the left-hand side is infinite. However, for arbitrary g even $W^{1,1}$-quasiconvexity does not imply rank 1 convexity in general.

EXAMPLE 3.5. Define $g: M^{n \times m} \to \bar{\mathbb{R}}$ by

$$g(0) = g(a \otimes b) = 0, \qquad g(A) = \infty \text{ otherwise},$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are given nonzero vectors. Note that g is lower semicontinuous. Clearly g is not rank 1 convex, and therefore by Corollary 3.4 I_Ω is not sw*lsc on $W^{1,\infty}(\Omega; \mathbb{R}^n)$. However, if $m > 1$ then g is $W^{1,1}$-quasiconvex; in fact, the inequality (2.1) can be violated only if $A + \nabla\phi(x) = 0$ or $a \otimes b$ a.e. $x \in E$. Integrating over E we see that then $A = \lambda a \otimes b$ for some λ. Therefore ϕ is constant along lines perpendicular to b and is thus zero, so that (2.1) holds, giving a contradiction. Note, however, that g does not satisfy (3.1), as is clearly seen directly using v defined as in the proof of Corollary 3.4 with $A = 0$.

Remark 3.6. Example 3.5 shows that (3.1) is a stronger condition than $W^{1,p}$-quasiconvexity, and there is a case for making it the basic definition, following the lead of Ball, Currie, and Olver [7, p. 140]. Other results, however, such as those in Section 5, are perhaps more naturally expressed in terms of (2.1).

Example 3.5 shows that for a general Borel measurable (even lower semicontinuous) g, $W^{1,p}$-quasiconvexity does not imply that $I_\Omega(u) = \int_\Omega g(\nabla u)\,dx$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$. However, it is natural to make the

CONJECTURE 3.7. *If* $g: M^{n \times m} \to \bar{\mathbb{R}}$ *is continuous and bounded below then* I_Ω *is* swlsc *on* $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc *if* $p = \infty$) *if and only if* g *is* $W^{1,p}$-*quasiconvex.*

This conjecture is valid in the following cases:

(1) if g takes finite values and $p = \infty$ (Morrey [13]),

(2) if g satisfies (2.3) with $1 \leqslant p < \infty$ (Acerbi and Fusco [3, Statement II.5]),

(3) if $g(A) = h(\det A)$ (Theorem 4.1 below).

Another reasonable conjecture is the following. Let $g: M^{n \times m} \to \overline{\mathbb{R}}$ be lower semicontinuous and bounded below; then I_Ω is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc if $p = \infty$) if and only if (3.1) holds.

4. Examples of $W^{1,p}$-Quasiconvex Functions

Let $g: M^{n \times m} \to \overline{\mathbb{R}}$ be Borel measurable and bounded below. If g is convex and lower semicontinuous then by Jensen's inequality

$$\int_E g(A + \nabla\phi(x))\, dx \geqslant (\text{meas}\, E)\, g\left(\frac{1}{\text{meas}\, E} \int_E (A + \nabla\phi(x))\, dx\right)$$

$$= (\text{meas}\, E)\, g(A)$$

for all $\phi \in W_0^{1,1}(E; \mathbb{R}^n)$ and all bounded open sets $E \subset \mathbb{R}^m$, and so g is $W^{1,1}$-quasiconvex. (A proof of Jensen's inequality under these hypotheses follows immediately from the representation of g as a supremum of affine functions.) Examples in the case $m > 1$, $n > 1$ of $W^{1,\infty}$-quasiconvex functions g that are not convex are discussed at length in Ball [4], Ball, Currie, and Olver [7], Dacorogna [9], and Acerbi and Fusco [3]).

To see that $W^{1,p}$-quasiconvexity depends dramatically on p in general we consider the case $m = n > 1$ and the example

$$g(A) = h(\det A), \qquad A \in M^{n \times n}, \tag{4.1}$$

where $h: \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable and bounded below.

If $\Omega \subset \mathbb{R}^n$ is bounded and open we let

$$I_\Omega(u) = \int_\Omega h(\det \nabla u(x))\, dx.$$

Theorem 4.1. (i) *If g is $W^{1,\infty}$-quasiconvex then h is convex.*

(ii) *Let $n \leqslant p \leqslant \infty$. If h is lower semicontinuous then the following conditions are equivalent:*

(a) *I_Ω is* swlsc *on $W^{1,p}(\Omega; \mathbb{R}^n)$* (sw*lsc *if $p = \infty$),*

(b) *g is $W^{1,p}$-quasiconvex,*

(c) *h is convex.*

(iii) *Let $1 \leqslant p < n$. Then the following conditions are equivalent:*

(a) *I_Ω is* swlsc *on $W^{1,p}(\Omega; \mathbb{R}^n)$,*

(b) *g is $W^{1,p}$-quasiconvex,*

(c) *h is constant.*

Proof. (i) Let g be $W^{1,\infty}$-quasiconvex. Let $0 < t < 1$, $\lambda > \mu$, $t\lambda + (1-t)\mu > 0$, and consider the radial mapping $u(x) = (r(R)/R)x$, where $R = |x|$ and

$$\begin{aligned} r^n &= \lambda R^n - (\lambda - \mu)(1-t) \qquad &\text{for} \quad 1 \leqslant R < (1+t)^{1/n}, \\ &= \mu R^n + 2(\lambda - \mu)t \qquad &\text{for} \quad (1+t)^{1/n} \leqslant R \leqslant 2^{1/n}. \end{aligned}$$

It is easily verified that the right-hand side of (4.2) is positive, so that r can be chosen positive, and that r is Lipschitz. Let $E = \{x \in \mathbb{R}^n : 1 < |x| < 2^{1/n}\}$. Then $u(x) = [t\lambda + (1-t)\mu]^{1/n} x$ when $x \in \partial E$, and so by the $W^{1,\infty}$-quasiconvexity

$$\int_E g(\nabla u(x))\, dx \geqslant (\text{meas } E)\; g([t\lambda + (1-t)\mu]^{1/n}\, \mathbf{1}).$$

Since $\det \nabla u(x) = (r/R)^{n-1}\, r'$ it follows that

$$h(t\lambda + (1-t)\mu) \leqslant th(\lambda) + (1-t)\, h(\mu). \tag{4.3}$$

If $\bar{\lambda} < \bar{\mu}$, $t\bar{\lambda} + (1-t)\bar{\mu} \leqslant 0$ we set $\lambda = -\bar{\lambda}$, $\mu = -\bar{\mu}$ and define

$$v(x) = (-u^1(x), u^2(x), ..., u^n(x)),$$

where $u^i(x)$ denotes the ith component of $u(x)$. Then

$$v(x) = [t\lambda + (1-t)\mu]^{1/n} \operatorname{diag}(-1, 1, ..., 1)x \qquad \text{when } x \in \partial E,$$

and so

$$\int_E g(\nabla v(x))\, dx \geqslant (\text{meas } E)\, g([t\lambda + (1-t)\mu]^{1/n} \operatorname{diag}(-1, 1, ..., 1)).$$

We thus obtain (4.3) with $\bar{\lambda}, \bar{\mu}$ replacing λ, μ. Thus h is convex.

(ii) Let h be lower semicontinuous. By Corollary 3.2 we have (a) implies (b), and by Remark 2.2 and the first part (b) implies (c). So let h be convex, and suppose that $u_j \rightharpoonup u$ in $W^{1,n}(\Omega; \mathbb{R}^n)$. By Reshetnyak [16, 18], $\det \nabla u_j \overset{*}{\rightharpoonup} \det \nabla u$ in the sense of measures; this is also an immediate consequence of the facts that $\det \nabla u_j \to \det \nabla u$ in $\mathscr{D}'(\Omega)$ (see, e.g., Ball, Currie, and Olver [7, Theorem 3.4, p. 143] or (7.7) and Property 3 of distributional determinants below) and that $\int_\Omega |\det \nabla u_j(x)|\, dx$ is uniformly bounded. It follows from Proposition A.3 that I_Ω is swlsc on $W^{1,n}(\Omega; \mathbb{R}^n)$. Thus (c) implies (a).

(iii) Let $1 \leqslant p < n$. It suffices to show that (b) implies (c). Let $\lambda > 0$, $\lambda > \mu$, and consider the radial mapping $u(x) = (r(R)/R)x$ given by

$$r(R) = (\mu R^n + \lambda - \mu)^{1/n}, \qquad 0 < R \leqslant 1. \tag{4.4}$$

By Ball [6, Lemma 4.1, p. 566] $u(x) \in W^{1,p}(B; \mathbb{R}^n)$, where $B = \{|x| < 1\}$. Note that $\det \nabla u(x) = \mu$ a.e. Since $u(x) = \lambda^{1/n}x$ when $x \in \partial B$, the $W^{1,p}$-quasiconvexity implies that $h(\mu) \geqslant h(\lambda)$. By considering the mapping $v(x) = \operatorname{diag}(-1, 1, \ldots, 1)\, u(x)$ it follows similarly that $h(-\mu) \geqslant h(-\lambda)$. Hence $h(\delta) = c_0$ for all $\delta \neq 0$, $h(0) \geqslant c_0$ where $c_0 \in (-\infty, \infty]$ is a constant. But by (i) h is convex. Thus $h(0) = c_0$ and so h is constant. ∎

Remark 4.2. The following related result is proved in Fusco [10, Theorem 6, p. 405] (see also Acerbi, Buttazzo, and Fusco [2, Theorem 3.7: 1]. If $h: \mathbb{R} \to [0, \infty)$ is continuous, then $I_\Omega(u)$ is swlsc on $W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n) \cap C(\Omega; \mathbb{R}^n)$ with the topology of $L^\infty_{\text{loc}}(\Omega; \mathbb{R}^n)$ if and only if h is convex and $h(0) = \min_\lambda h(\lambda)$.

As a second example we consider the case $m = n > 1$ with

$$g(A) = f(A) + h(\det A), \tag{4.5}$$

where $f: M^{n\times n} \to \overline{\mathbb{R}}$ is Borel measurable and bounded below and where $h: \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable, bounded below and not identically $+\infty$. Let $\alpha \geqslant 1$. In what follows we will suppose that f satisfies various of the growth conditions

$$f \in C^2(M^{n\times n}) \quad \text{and}$$

$$|D^2 f(A)| \leqslant C_1 |A|^{\alpha-2} \quad \text{if } |A| \text{ sufficiently large}, \tag{4.6}$$

$$C_2 |A|^\alpha \leqslant f(A) \quad \text{if } |A| \text{ sufficiently large}, \tag{4.7}$$

$$f(A) \leqslant C_3(|A|^\alpha + 1) \quad \text{for all } A \in M^{n\times n}, \tag{4.8}$$

where $C_1, C_2 > 0$, and C_3 are constants. Note that (4.6) implies (4.8).

LEMMA 4.3. *Let $\alpha < 2n$ and let g given by* (4.5) *be rank* 1 *convex. Let f satisfy* (4.6). *Then h is convex.*

Remark 4.4. The hypothesis $\alpha < 2n$ in the lemma cannot be dropped. In fact, suppose h is smooth with $|h'| + |h''| \leqslant C$ and that h is zero in a neighborhood of the origin. Let $f(A) = [\operatorname{tr}(A^T A)]^{\alpha/2}$ with $\alpha \geqslant 2n$. Then it is easily verified that $g_\varepsilon(A) = f(A) + \varepsilon h(\det A)$ is convex for sufficiently small $\varepsilon > 0$, so that even $\int_\Omega g_\varepsilon(\nabla u)\, dx$ swlsc on $W^{1,1}(\Omega; \mathbb{R}^n)$ fails to imply h convex.

Proof of Lemma 4.3. Let $\lambda \neq 0$, $\mu \neq \lambda$ and $0 < t < 1$ be given. Let $e_1 = (1, 0, \ldots, 0)$. Let $\varepsilon \neq 0$ have the same sign as $\mu - \lambda$, let $v(\varepsilon) = \lambda\varepsilon/(\mu - \lambda)$ and define

$$A(\varepsilon) = \operatorname{diag}\left(v(\varepsilon), \left(\frac{\lambda}{v(\varepsilon)}\right)^{1/(n-1)}, \ldots, \left(\frac{\lambda}{v(\varepsilon)}\right)^{1/(n-1)}\right).$$

Then we have

$$\det A(\varepsilon) = \lambda, \qquad \det(A(\varepsilon) + \varepsilon e_1 \otimes e_1) = \mu,$$
$$\det(A(\varepsilon) + (1-t)\,\varepsilon e_1 \otimes e_1) = t\lambda + (1-t)\mu.$$

By rank 1 convexity,

$$[f(A(\varepsilon) + (1-t)\,\varepsilon e_1 \otimes e_1) - tf(A(\varepsilon)) - (1-t)\,f(A(\varepsilon) + \varepsilon e_1 \otimes e_1)]$$
$$+ [h(t\lambda + (1-t)\mu) - th(\lambda) - (1-t)\,h(\mu)] \leqslant 0.$$

Denoting the sum of the terms in the first square brackets by $\theta_\varepsilon(t)$, we note that $\theta_\varepsilon(0) = \theta_\varepsilon(1) = 0$ and hence $\theta'_\varepsilon(t_0) = 0$ for some $0 < t_0 < 1$. Therefore by (4.6)

$$|\theta_\varepsilon(t)| = \left|\int_0^t \int_{t_0}^s \theta''_\varepsilon(\tau)\,d\tau\,ds\right| \leqslant \int_0^1 |\theta''_\varepsilon(\tau)|\,d\tau$$
$$\leqslant C\,|\varepsilon|^2\,(|A(\varepsilon)|^{\alpha-2} + |\varepsilon|^{\alpha-2})$$

for $|\varepsilon|$ sufficiently small. But

$$|A(\varepsilon)| \leqslant C(|\varepsilon| + |\varepsilon|^{-1/(n-1)}),$$

and since $\alpha < 2n$ it follows that $\theta_\varepsilon(t) \to 0$ as $|\varepsilon| \to 0$. Therefore

$$h(t\lambda + (1-t)\mu) \leqslant th(\lambda) + (1-t)\,h(\mu),$$

as required. ∎

Define for $\Omega \subset \mathbb{R}^n$ bounded and open

$$I_\Omega(u) = \int_\Omega g(\nabla u(x))\,dx = \int_\Omega [f(\nabla u(x)) + h(\det \nabla u(x))]\,dx.$$

THEOREM 4.5. (i) *Let f be convex, lower semicontinuous and satisfy* (4.7), *and let h be convex and lower semicontinuous. Suppose* either *that $\alpha \geqslant 1$ and $n \leqslant p \leqslant \infty$* or *that $\alpha \geqslant n$ and $1 \leqslant p \leqslant \infty$. Then I_Ω is swlsc on* $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc *if $p = \infty$*) *and hence g is $W^{1,p}$-quasiconvex.*

(ii) *Let f satisfy* (4.8) *and let* $1 \leqslant \alpha < n$. *There exist constants ρ and K such that if* $|\lambda| \geqslant \rho$ *and g is* $W^{1,p}$*-quasiconvex at* $|\lambda|^{1/n}\operatorname{diag}(\operatorname{sign}\lambda, 1, 1, ..., 1)$ *for some* $p < n$ *then*

$$\frac{h(\lambda)}{|\lambda|^{\alpha/n}} \leqslant K. \tag{4.9}$$

In particular, if f satisfies (4.6), (4.7) *with* $1 \leqslant \alpha < n$ *and if* $I_\Omega(u)$ *is swlsc on* $W^{1,p}(\Omega; \mathbb{R}^n)$ *for some* $p < n$ *then h is constant.*

Proof. (i) Let $\alpha \geqslant 1$ and let $n \leqslant p \leqslant \infty$. The convex functional $\int_\Omega f(\nabla u)\,dx$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ (sw*lsc if $p = \infty$) by standard results (e.g., Proposition A.3). By Theorem 4.1(i) $\int_\Omega h(\det \nabla u)\,dx$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ also, and therefore so is the sum $I_\Omega(u)$.

Let $\alpha \geqslant n$, $1 \leqslant p \leqslant \infty$, and suppose that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ ($\overset{*}{\rightharpoonup}$ if $p = \infty$). We want to prove that

$$I_\Omega(u) \leqslant \liminf_{j \to \infty} I_\Omega(u_j), \tag{4.10}$$

and we can suppose that the right-hand side of (4.10) is finite. Then by (4.7) we have that $\int_\Omega |\nabla u_j|^n\,dx \leqslant C$. It follows by the Poincaré inequality that $u_j \rightharpoonup u$ in $W^{1,n}(\Omega'; \mathbb{R}^n)$ for any smooth subdomain Ω' of Ω. As in the proof of Theorem 4.1(i) we deduce that $\det \nabla u_j \overset{*}{\rightharpoonup} \det \nabla u$ in the sense of measures, and hence that

$$\int_\Omega h(\det \nabla u(x))\,dx \leqslant \liminf_{j \to \infty} \int_\Omega h(\det \nabla u_j(x))\,dx.$$

Since f is lower semicontinuous and convex we obtain (4.10).

(ii) Let $1 \leqslant \alpha < n$. By hypothesis there exists μ such that $h(\mu) < \infty$. Let $\lambda > |\mu|$ and consider the radial deformation (4.4) in the ball $B = \{|x| < 1\}$. For $0 < R \leqslant 1$ we have

$$|r(R)|^n \leqslant \max(\lambda - \mu, \lambda), \qquad |r'(R)| \leqslant |\mu|\,\lambda^{(1-n)/n},$$

and therefore

$$\int_B |\nabla u|^\alpha\,dx \leqslant C \int_0^1 R^{n-1} \left(|r'(R)|^\alpha + \left|\frac{r(R)}{R}\right|^\alpha\right) dR \leqslant C\lambda^{\alpha/n} \tag{4.11}$$

provided λ is sufficiently large. If g is $W^{1,p}$-quasiconvex at $\lambda^{1/n}\mathbf{1}$ for some $p < n$ then

$$\int_B |f(\lambda^{1/n}\mathbf{1}) + h(\lambda)|\,dx \leqslant \int_B |f(\nabla u) + h(\mu)|\,dx. \tag{4.12}$$

Combining (4.8), (4.11), and (4.12), we obtain $h(\lambda)/\lambda^{\alpha/n} \leqslant C$ for λ sufficiently large.

If $\lambda < -|\mu|$ we let $u(x) = (r(R)/R)x$ with $r(R) = (-\mu R^n - \lambda + \mu)^{1/n}$ and define $v(x) = (-u^1(x), u^2(x), ..., u^n(x))$. Proceeding as above we obtain $h(\lambda)/(-\lambda)^{\alpha/n} \leqslant C$ for $-\lambda$ sufficiently large. This proves (4.9).

Finally, if f satisfies (4.6), (4.7) and $I_\Omega(u)$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p < n$ then by Corollary 3.4 and Lemma 4.3 h is convex. But the only convex functions h satisfying (4.9) are constant. ∎

For the reader's convenience we rephrase some of the preceding results. Under the hypotheses (4.6) and (4.7) (thus $f(A) \sim |A|^\alpha$), if $I_\Omega(u) = \int_\Omega [f(\nabla u) + h(\det \nabla u)]\, dx$ is swlsc on $W^{1,\alpha}(\Omega; \mathbb{R}^n)$ we necessarily have

(i) if $1 \leqslant \alpha < n$, $h =$ constant (Corollary 3.2 + Lemma 4.3 + Theorem 4.5(ii)),

(ii) if $n \leqslant \alpha < 2n$, h is convex (Corollary 3.4 + Lemma 4.3),

(iii) if $\alpha > 2n$, no information on h (Remark 4.4).

5. $W^{1,p}$-Quasiconvexity as a Necessary Condition for the Existence of Minimizers

Let $1 \leqslant p \leqslant \infty$, let $\Omega \subset \mathbb{R}^m$ be bounded open, and let $g: M^{n \times m} \to \overline{\mathbb{R}}$ be Borel measurable and bounded below. We examine the consequences of lack of $W^{1,p}$-quasiconvexity of g for the existence of minimizers for functionals of the form

$$J_\Omega(u) = \int_\Omega [g(\nabla u(x)) + \Psi(x, u(x))]\, dx, \tag{5.1}$$

where $\Psi: \Omega \times \mathbb{R}^n \to \mathbb{R}$.

Theorem 5.1. *Suppose that* meas $\partial\Omega = 0$. *Let* $A \in M^{n \times m}$ *and suppose that* g *is not* $W^{1,p}$*-quasiconvex at* A. *Let* $\Psi(x, u) = \phi(|u - Ax|^2)$, *where* $\phi: \mathbb{R} \to \mathbb{R}$ *is continuous, bounded and satisfies* $\phi(0) = 0$, $\phi(t) > 0$ *if* $t \neq 0$. *Let*

$$X_A = \{u: u - Ax \in W_0^{1,p}(\Omega; \mathbb{R}^n)\}.$$

Then J_Ω *does not attain a minimum on* X_A.

Corollary 5.2. *Let* meas $\partial\Omega = 0$, $A \in M^{n \times m}$. *If* J_Ω *attains a minimum on* X_A *for all smooth nonnegative* Ψ *then* g *is* $W^{1,p}$*-quasiconvex at* A.

Proof of Theorem 5.1. Let $I_\Omega(u) = \int_\Omega g(\nabla u(x))\, dx$, $\lambda = \inf_{u \in X_A} I_\Omega(u)$. Since g is not $W^{1,p}$-quasiconvex at A we have $\lambda < \infty$. We claim that

$$\inf_{u \in X_A} J_\Omega(u) = \lambda. \tag{5.2}$$

To prove (5.2) we use a method similar to that of the proof of Proposition 2.3. Let $\varepsilon > 0$ and $w = z + Ax \in X_A$ satisfy

$$I_\Omega(w) \leqslant \lambda + \varepsilon.$$

By the Vitali covering theorem, given j there exists a finite or countable disjoint sequence $a_i + \varepsilon_i \bar{\Omega}$ of subsets of Ω, where $a_i \in \mathbb{R}^m$, $0 < \varepsilon_i \leqslant 1/j$, such that $\operatorname{meas}(\Omega \setminus \bigcup_i (a_i + \varepsilon_i \bar{\Omega})) = 0$. (The reader content to prove the theorem when $\Omega = (0, 1)^m$ can avoid the use of Vitali's theorem by writing instead Ω as the union of j^m m-cubes of side $1/j$ and a set of measure zero.) Since $\operatorname{meas} \partial\Omega = 0$ we have that $\sum_i \varepsilon_i^m = 1$. Define

$$\begin{aligned} u_j(x) &= Ax + \varepsilon_i z\left(\frac{x - a_i}{\varepsilon_i}\right) &&\text{if} \quad x \in a_i + \varepsilon_i \bar{\Omega}, \\ &= Ax &&\text{otherwise.} \end{aligned}$$

Then $u_j \in X_A$, and

$$\begin{aligned} J_\Omega(u_j) &= \sum_i \int_{a_i + \varepsilon_i \Omega} g\left(A + \nabla z\left(\frac{x - a_i}{\varepsilon_i}\right)\right) dx + \int_\Omega \phi(|u_j - Ax|^2)\, dx \\ &= \left(\sum_i \varepsilon_i^m\right) \int_\Omega g(A + \nabla z(y))\, dy + \int_\Omega \phi(|u_j - Ax|^2)\, dx \\ &= I_\Omega(w) + \int_\Omega \phi(|u_j - Ax|^2)\, dx. \end{aligned}$$

But if $1 \leqslant p < \infty$

$$\begin{aligned} \int_\Omega |u_j - Ax|^p\, dx &= \sum_i \varepsilon_i^p \int_{a_i + \varepsilon_i \Omega} \left| z\left(\frac{x - a_i}{\varepsilon_i}\right)\right|^p dx \\ &= \sum_i \varepsilon_i^{m+p} \int_\Omega |z(y)|^p\, dy \\ &\leqslant Cj^{-p}. \end{aligned}$$

Thus, by the dominated convergence theorem,

$$\lim_{j \to \infty} \int_\Omega \phi(|u_j - Ax|^2)\, dx = 0.$$

and hence

$$\inf_{u \in X_A} J_\Omega(u) \leqslant \lambda + 2\varepsilon,$$

giving (5.2).

Suppose for contradiction that $J_\Omega(u_0) = \inf_{u \in X_A} J_\Omega(u)$ for some $u_0 \in X_A$. Then

$$\lambda = I_\Omega(u_0) + \int_\Omega \phi(|u_0 - Ax|^2)\,dx \geqslant \lambda + \int_\Omega \phi(|u_0 - Ax|^2)\,dx,$$

so that $u_0(x) = Ax$ a.e. in Ω and $I_\Omega(Ax) = \inf_{u \in X_A} I_\Omega(u)$. Hence g is $W^{1,p}$-quasiconvex at A, contradicting the hypothesis. ∎

Remarks 5.3. (1) Let $Q \subset \mathbb{R}^m$ be an m-cube, let $A \in M^{n \times m}$, and define

$$Y_A = \{u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^n)\colon u - Ax\ Q\text{-periodic}\}.$$

Suppose that (cf. (3.1))

$$\inf_{u \in Y_A} I_Q(u) < I_Q(Ax).$$

Then, if Ψ is defined as in Theorem 5.1, J_Q does not attain a minimum on Y_A. The proof is similar.

(2) If Ω is bounded open with meas $\partial\Omega = 0$ then

$$\inf_{u \in X_A} I_\Omega(u) = \hat{g}(A) \text{ meas } \Omega$$

for some $\hat{g}\colon M^{n \times m} \to \bar{\mathbb{R}}$. To prove this let $H_\Omega(A) = \inf_{u \in X_A} I_\Omega(u)$, and let Ω_1, Ω_2 be bounded open with meas $\partial\Omega_1 =$ meas $\partial\Omega_2 = 0$. By Vitali's theorem there exists a finite or countable disjoint sequence $a_i + \varepsilon_i \bar{\Omega}_2$ of subsets of Ω_1, where $a_i \in \mathbb{R}^m$, $\varepsilon_i > 0$ such that $\text{meas}(\Omega_1 \setminus \bigcup_i (a_i + \varepsilon_i \bar{\Omega}_2)) = 0$.

Given $\varepsilon > 0$ let $w = z + Ax \in X_A$ satisfy

$$I_{\Omega_2}(w) \leqslant H_{\Omega_2}(A) + \varepsilon.$$

Define $u(x) = Ax + \varepsilon_i z((x - a_i)/\varepsilon_i)$ if $x \in a_i + \varepsilon_i \bar{\Omega}_2$, $= 0$ otherwise. Then

$$H_{\Omega_1}(A) \leqslant I_{\Omega_1}(u) = \sum_i \varepsilon_i^m \cdot I_{\Omega_2}(w) = \frac{\text{meas } \Omega_1}{\text{meas } \Omega_2} I_{\Omega_2}(w)$$

and so

$$\frac{1}{\text{meas } \Omega_1} H_{\Omega_1}(A) = \frac{1}{\text{meas } \Omega_2} H_{\Omega_2}(A)$$

as required.

A similar result to this is given by Dacorogna [8, Theorem 5; 9, p. 88] for the case when $p=\infty$ and g is continuous with $g(A)\leqslant C(1+|A|^r)$ for all $A\in M^{n\times m}$ and some $r\geqslant 1$. In this case, as he shows, $\hat{g}$ is the $W^{1,\infty}$-quasiconvex envelope of g, namely, the largest $W^{1,\infty}$-quasiconvex function less than or equal to g. For other relevant results see Acerbi and Fusco [3, Section III].

6. Existence and Nonexistence of Minimizers in Elasticity

Consider a homogeneous elastic body occupying in a reference configuration a bounded domain $\Omega\subset\mathbb{R}^n$. We suppose that $\partial\Omega$ is strongly Lipschitz and that $\partial\Omega_1$ is a measurable subset of $\partial\Omega$ with positive $(n-1)$-dimensional measure. We consider a mixed displacement zero traction boundary value problem in which the deformation $u:\Omega\to\mathbb{R}^n$ is required to satisfy

$$u(x)=\bar{u}(x), \qquad \text{a.e.} \quad x\in\partial\Omega_1 \tag{6.1}$$

while the remainder $u(\partial\Omega\backslash\partial\Omega_1)$ of the boundary is traction-free. The total energy is given by the functional

$$J(u)\stackrel{\text{def}}{=}\int_\Omega |g(\nabla u(x))+\Psi(x,u(x))|\,dx. \tag{6.2}$$

In (6.2) $g:M^{n\times n}\to\overline{\mathbb{R}}$ is the stored-energy function of the material, and $\Psi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is the body force potential per unit volume. Corresponding to (6.1), (6.2) we consider the set

$$\mathscr{A}=\{u\in W^{1,1}(\Omega;\mathbb{R}^n)\text{: (6.1) holds and } J(u)<\infty\}$$

of admissible functions.

Theorem 6.1. *Let $n=3$, and let g, Ψ satisfy the following hypotheses.*

(H1) *g is continuous,*

(H2) $g(A)=\infty$ *if and only if* $\det A\leqslant 0$,

(H3) *g is polyconvex, i.e., there exists a convex function $G:M^{3\times 3}\times M^{3\times 3}\times(0,\infty)\to\mathbb{R}$ such that $g(A)=G(A, \operatorname{adj} A, \det A)$ for all $A\in M^{3\times 3}_+$, where $M^{n\times n}_+\stackrel{\text{def}}{=}\{A\in M^{n\times n}: \det A>0\}$,*

(H4) $g(A)\geqslant K_1+K(|A|^p+|\operatorname{adj} A|^q)$ *for all* $A\in M^{3\times 3}_+$, $\qquad$ (6.3)

where $K>0$, K_1 are constants, $p\geqslant 2$ and $q\geqslant p/(p-1)$,

(H5) *Ψ is continuous and bounded below.*

Let $\mathscr{A}$ *be nonempty. Then J attains its absolute minimum on* $\mathscr{A}$, *and the minimizer u satisfies* $\det \nabla u(x) > 0$ *a.e.* $x \in \Omega$.

Remark 6.2. For the case of a homogeneous material the theorem is a slight refinement of earlier results of Ball [4, Theorem 7.7; 5, Theorem 4.1] and Ball, Currie, and Olver [7, Theorem 6.2], the difference being that there is no term depending on $\det A$ on the right-hand side of (6.3). It is not hard to prove a similar theorem valid for nonhomogeneous materials, for which $g = g(x, A)$.

Proof of Theorem 6.1. *Step* 1. Let $L = M^{3\times 3} \times M^{3\times 3} \times \mathbb{R} \cong \mathbb{R}^{19}$, $L^+ = M^{3\times 3} \times M^{3\times 3} \times (0, \infty)$, and define $\Delta: M^{3\times 3} \to L$ by $\Delta(A) = (A, \operatorname{adj} A, \det A)$. We suppose without loss of generality that $K_1 \geqslant 0$. Let $\tilde{G}: L \to \overline{\mathbb{R}}$ be the greatest convex function such that $g(A) = \tilde{G}(\Delta(A))$ for all $A \in M^{3\times 3}_+$. Since the function

$$\bar{G}(H) = G(H) \qquad \text{for} \quad H \in L^+,$$
$$= +\infty \qquad \text{for} \quad H \in L \backslash L^+,$$

is convex it follows from (H3) that $\tilde{G}$ exists and $\tilde{G}(H) = +\infty$ for all $H \in L \backslash L^+$. Since $\operatorname{co} \Delta(M^{3\times 3}_+) = L^+$ (cf. Ball [4, Theorem 4.3]) we have that $0 \leqslant \tilde{G}(H) < \infty$ for all $H \in L^+$, and hence that $\tilde{G}$ is continuous on L^+.

Given $M > 0$ define $\theta_M(A) = (g(A) - M)/\det A$. Since $K_1 \geqslant 0$ we have $\theta_M(A) \geqslant -M$ if $\det A \geqslant 1$. Let A_j be a minimizing sequence for θ_M on $S = \{A \in M^{3\times 3}: 0 < \det A \leqslant 1\}$. By (H4) A_j is bounded, and so a subsequence A_μ converges to A_0, say. By (H1) and (H2), $\det A_0 > 0$. Therefore $\theta_M(A) \geqslant \theta_M(A_0)$ on S. Hence there is a constant $m = m(M)$ such that

$$\det A \leqslant C\,|\operatorname{adj} A|\,|A| \leqslant C(g(A) + 1),$$

and so $\theta_M(A) \geqslant C$ if $\det A \geqslant 1$, say. Let A_j be a minimizing sequence for θ_M on $S = \{A \in M^{3\times 3}: 0 < \det A \leqslant 1\}$. By (H4) A_j is bounded, and so a subsequence A_μ converges to A_0, say. By (H1) and (H2), $\det A_0 > 0$. Therefore $\theta_M(A) \geqslant \theta_M(A_0)$ on S. Hence there is a constant $m = m(M)$ such that

$$g(A) \geqslant M - m \det A \qquad \text{for all } A \in M^{3\times 3}_+.$$

Thus $\theta(A, B, \delta) = M - m\delta$ is an affine function such that $\theta(\Delta(A)) \leqslant g(A)$ for all $A \in M^{3\times 3}_+$. Hence, using the maximality of $\tilde{G}$,

$$\tilde{G}(A, B, \delta) \geqslant M - m\delta \qquad \text{for all} \quad (A, B, \delta) \in L$$

and since M is arbitrary $\tilde{G}(A_k, B_k, \delta_k) \to \infty$ as $(A_k, B_k, \delta_k) \to (A, B, 0)$. Therefore $\tilde{G}$ is continuous on the whole of L and $g(A) = \tilde{G}(\Delta(A))$ for all $A \in M^{3\times 3}$.

Step 2. Let $\{u_j\}$ be a minimizing sequence for J in $\mathscr{A}$. Then by (H4), (H5) there exists a subsequence $\{u_\mu\}$ such that

$$u_\mu \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^3), \qquad u_\mu \to u \text{ a.e.},$$

and

$$\operatorname{adj} \nabla u_\mu \rightharpoonup \chi \quad \text{in} \quad L^q(\Omega; \mathbb{R}^9).$$

By Ball [5, p. 218] we have that $\chi = \operatorname{adj} \nabla u$, $\det \nabla u_\mu \to \det \nabla u$ in $\mathscr{D}'(\Omega)$. Since $|\det \nabla u_\mu(x)| \leqslant C\,|\operatorname{adj} \nabla u_\mu(x)|\,|\nabla u_\mu(x)|$ we have that

$$\int_\Omega |\det \nabla u_\mu(x)|\,dx \leqslant C \left(\int_\Omega |\nabla u_\mu(x)|^p\,dx\right)^{1/p} \left(\int_\Omega |\operatorname{adj} \nabla u_\mu(x)|^q\,dx\right)^{1/q} \leqslant C,$$

and therefore $\det \nabla u_\mu \rightharpoonup * \det \nabla u$ in the sense of measures. Thus

$$(\nabla u_\mu, \operatorname{adj} \nabla u_\mu, \det \nabla u_\mu) \rightharpoonup * (\nabla u, \operatorname{adj} \nabla u, \det \nabla u)$$

in the sense of measures, and hence by Proposition A.3

$$\begin{aligned}\int_\Omega g(\nabla u(x))\,dx &= \int_\Omega \tilde{G}(\nabla u(x), \operatorname{adj} \nabla u(x), \det \nabla u(x))\,dx \\ &\leqslant \liminf_{\mu \to \infty} \int_\Omega \tilde{G}(\nabla u_\mu(x), \operatorname{adj} \nabla u_\mu(x), \det \nabla u_\mu(x))\,dx \\ &= \liminf_{\mu \to \infty} \int_\Omega g(\nabla u_\mu(x))\,dx.\end{aligned}$$

Since by Fatou's lemma,

$$\int_\Omega \Psi(x, u(x))\,dx \leqslant \liminf_{\mu \to \infty} \int_\Omega \Psi(x, u_\mu(x))\,dx$$

it follows that

$$J(u) \leqslant \liminf_{\mu \to \infty} J(u_\mu) = \inf_{\mathscr{A}} J.$$

But by trace theory u satisfies (6.1), and thus $u \in \mathscr{A}$. By (H2) $\det \nabla u(x) > 0$ a.e. $x \in \Omega$. ∎

We leave to the reader the routine extension of Theorem 6.1 to more general conservative boundary value problems and to arbitrary n. When $n = 2$, for example, hypotheses (H3) and (H4) can be replaced by

(H3′) there exists a convex function $G: M^{2\times 2} \times (0, \infty) \to \mathbb{R}$ such that $g(A) = G(A, \det A)$ for all $A \in M^{2\times 2}_+$,

(H4′) $g(A) \geqslant K_1 + K\,|A|^2$ for all $A \in M^{2\times 2}_+$, where $K > 0$, K_1 are constants,

and the theorem remains valid.

We now discuss the implications for elasticity of the results in Sections 4 and 5. Comparing (5.1) and (6.2) we see that in the case $\partial\Omega = \partial\Omega_1$, $\bar{u}(x) = Ax$, Corollary 5.2 shows that a necessary condition for $J(u)$ to attain its minimum on $\mathscr{A}$ for all smooth nonnegative body force potentials Ψ is that the stored-energy function g be $W^{1,1}$-quasiconvex at A. (Of course the Ψ in Theorem 6.1 is not very realistic physically.) Consider, as a critical example, the isotropic stored-energy function (of a type used to model natural rubber)

$$g(A) = \mu(v_1^\alpha + v_2^\alpha + v_3^\alpha) + h(\det A), \tag{6.4}$$

where $\mu > 0$, $\alpha \geqslant 1$, $v_i = v_i(A)$ denotes the ith principal stretch (i.e., the ith eigenvalue of $\sqrt{A^{\mathrm{T}}A}$) and where $h\colon \mathbb{R} \to \bar{\mathbb{R}}$ is convex, continuous, bounded below and such that $h(t) = \infty$ if and only if $t \leqslant 0$. Since $v_1^\alpha + v_2^\alpha + v_3^\alpha$ is a convex function of A (cf. Ball [4, Theorem 5.1]) it is easily seen that g satisfies hypotheses (H1)–(H4) of Theorem 6.1 if and only if $\alpha \geqslant 3$. If $1 \leqslant \alpha < 3$ and $\lim_{t\to\infty} h(t)/t = \infty$ then by Theorem 4.5(ii), g is not $W^{1,1}$-quasiconvex at $\lambda\mathbf{1}$ for $\lambda > 0$ sufficiently large, and thus $J(u)$ does not always attain a minimum. Since in this case g satisfies (H1)–(H3) the growth condition (H4) in Theorem 6.1 is optimal. (For other remarks concerning $W^{1,\infty}$-quasiconvexity and the existence of minimizers see Ball [4, p. 351].) That g is not $W^{1,1}$-quasiconvex at $\lambda\mathbf{1}$ for $\lambda > 0$ sufficiently large corresponds to the fact that a solid ball $B = \{|x| < 1\}$ made of material with stored-energy function g and subjected to the radial boundary displacement $u(x)|_{\partial B} = \lambda x$ can reduce its energy by cavitation, i.e., by forming a hole in its interior. The phenomenon of cavitation in nonlinear elasticity is discussed at length in Ball [6]; particular attention is paid to the critical value λ_{crit} of λ, and the corresponding critical load, at which cavitation occurs, and the results are related to experimental work on the internal rupture of rubber.

An interesting conclusion can be drawn concerning the existence theorems involving use of the distributional determinant $\operatorname{Det} \nabla u$ (see (7.7) for the definition) that are proved in Ball [4, Theorem 7.6], Ball, Currie, and Olver [7, p. 166]. In the case of the stored-energy function (6.4) these theorems assert that if $9/4 < \alpha < 3$ and $\lim_{t\to\infty} h(t)/t = \infty$ then

$$\tilde{J}(u) \stackrel{\text{def}}{=} \int_\Omega \left[\sum_{i=1}^{3} v_i(\nabla u(x))^\alpha + h(\operatorname{Det} \nabla u(x)) + \Psi(x, u(x)) \right] dx$$

attains an absolute minimum on the set

$$\tilde{X}_A = \{u\colon u - Ax \in W_0^{1,\alpha}(\Omega; \mathbb{R}^3),\ \operatorname{Det} \nabla u(x) \in L^1(\Omega)\}.$$

The point to note is that the assumption that $\operatorname{Det} \nabla u(x)$ is a function, and not just a distribution, acts as a constraint preventing cavitation. This can be seen in a simple example by verifying that if $A = \lambda \mathbf{1}$ and $u(x) = (r(R)/R)x$ is a radial mapping in $\tilde{X}_A$ with $r(R)$ smooth for $R > 0$ then $r(0) = 0$.

The results and comments in this section apply with appropriate modifications to incompressible elasticity, in which the deformation u is required to satisfy the pointwise constraint

$$\det \nabla u(x) = 1 \qquad \text{a.e.} \quad x \in \Omega. \tag{6.5}$$

The constraint (6.5) is conveniently incorporated into our framework by requiring that the stored-energy function $g: M^{3\times 3} \to \bar{\mathbb{R}}$ satisfy $g(A) = \infty$ if and only if $\det A \neq 1$; note that such g are Borel measurable but not continuous. If, for example,

$$\begin{aligned} g(A) &= \mu(v_1^\alpha + v_2^\alpha + v_3^\alpha) \qquad && \text{if} \quad \det A = v_1 v_2 v_3 = 1, \\ &= +\infty && \text{otherwise,} \end{aligned}$$

where $\mu > 0$, $\alpha \geqslant 1$, then g is $W^{1,p}$-quasiconvex if and only if $\max(\alpha, p) \geqslant 3$. In fact, if $\max(\alpha, p) \geqslant 3$ then g is $W^{1,p}$-quasiconvex by Theorem 4.5(i). If $\max(\alpha, p) < 3$ and $\lambda > 1$ let $u(x) = (r(R)/R)x$ be the radial mapping of $B = \{|x| < 1\}$ defined by

$$r(R) = (R^3 + \lambda^3 - 1)^{1/3}.$$

Then $u \in W^{1,p}(B; \mathbb{R}^3)$ and $u(x)|_{\partial B} = \lambda x$ but $g(\lambda \mathbf{1}) = \infty$, $\int_B g(\nabla u(x))\, dx < \infty$, so that g is not $W^{1,p}$-quasiconvex.

7. Counterexamples Concerning Weak Continuity of Jacobians

Let $n > 1$ and let $\Omega \subset \mathbb{R}^n$ be open. In the preceding sections we used the fact that the mapping $u \mapsto \det \nabla u$ is sequentially continuous from $W^{1,n}(\Omega; \mathbb{R}^n)$ endowed with the weak topology to $L^1(\Omega)$ endowed with the topology of measures. By means of two counterexamples of a different nature we now show that $u \mapsto \det \nabla u$ is *not* sequentially weakly continuous from $W^{1,n}(\Omega; \mathbb{R}^n)$ to $L^1(\Omega)$, that is, $u_j \rightharpoonup u$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ does not imply that $\det \nabla u_j \rightharpoonup \det \nabla u$ in $L^1(\Omega)$. This should be contrasted with the fact that if h is convex and lower semicontinuous then $\int_\Omega h(\det \nabla u(x))\, dx$ is swlsc on $W^{1,n}(\Omega; \mathbb{R}^n)$ (Theorem 4.1(ii)), and the fact that, for example, the functional $u \mapsto \int_\Omega a(x)\, |\det \nabla u(x)|\, dx$ is swlsc on $W^{1,n}(\Omega; \mathbb{R}^n)$ for all $a \in L^\infty(\Omega)$, $a \geqslant 0$ (see Acerbi and Fusco [3, Theorem 2.4]).

Counterexample 7.1. Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$, $n > 1$. Consider for $j = 1, 2, \ldots$, the radial mappings

$$u_j(x) = \frac{r_j(R)}{R}\, x, \qquad R = |x|, \tag{7.1}$$

where

$$\begin{aligned} r_j(R) &= jR && \text{if} \quad 0 \leqslant R \leqslant 1/j, \\ &= 2 - jR && \text{if} \quad 1/j \leqslant R \leqslant 2/j, \\ &= 0 && \text{if} \quad 2/j \leqslant R \leqslant 1. \end{aligned} \tag{7.2}$$

We recall that a radial mapping $u(x) = (r(R)/R)x$ belongs to $W^{1,n}(B; \mathbb{R}^n)$ if and only if r is absolutely continuous on $(0, 1)$ and

$$N(r) \stackrel{\text{def}}{=} \left(\int_0^1 R^{n-1} \left[|r'(R)|^n + \left| \frac{r(R)}{R} \right|^n \right] dR \right)^{1/n} < \infty, \tag{7.3}$$

(see Ball [6, p. 566]). Furthermore there are constants $C_1 > 0$, $C_2 > 0$ such that $C_1 N(r) \leqslant \|u\|_{W^{1,n}(B; \mathbb{R}^n)} \leqslant C_2 N(r)$. But

$$N^n(r_j) = \frac{2}{n} + \frac{2^n - 1}{n} + \int_1^2 S^{-1}(2 - S)^n \, dS,$$

and therefore $u_j \rightharpoonup 0$ in $W^{1,n}(B; \mathbb{R}^n)$ as $j \to \infty$. But

$$\det \nabla u_j = \left(\frac{r_j(R)}{R} \right)^{n-1} r_j'(R) \qquad \text{a.e. in } B,$$

and so if $B_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$, $0 < 2/j < \delta < 1$,

$$\begin{aligned} \int_{B_\delta} |\det \nabla u_j| \, dx &= C \int_0^\delta R^{n-1} \left| \frac{r_j(R)}{R} \right|^{n-1} |r_j'(R)| \, dR \\ &= C \int_0^{2/j} |r_j(R)|^{n-1} |r_j'(R)| \, dR = \frac{2C}{n}. \end{aligned}$$

This shows that $\det \nabla u_j$ is not equi-integrable in B, and therefore $\det \nabla u_j$ does not converge weakly in $L^1(B)$.

Remark 7.2. Let $n > 1$. Consider the space $\mathrm{Rad}^{1,n}(B)$ of radial mappings $u(x) = (r(R)/R)x$ belonging to $W^{1,n}(B; \mathbb{R}^n)$, with norm $N(r)$ given by (7.3). Supposing that $0 < \bar{R} < R < \eta$ and using Hölder's inequality with exponents n and $n/(n-1)$ we have

$$\begin{aligned} |r^n(R) - r^n(\bar{R})| &\leqslant \int_{\bar{R}}^R n \, |r^{n-1}(R)| \, |r'(R)| \, dR \\ &\leqslant n \left(\int_0^\eta |r'(R)|^n R^{n-1} \, dR \right)^{1/n} \left(\int_0^\eta |r(R)|^n \frac{1}{R} \, dR \right)^{(n-1)/n}. \end{aligned} \tag{7.4}$$

The right-hand side of (7.4) tends to zero when $\eta \to 0$ since $|r'(R)|^n R^{n-1}$ and $|r(R)/R|^n R^{n-1}$ belong to $L^1(0, 1)$. Hence $r^n(R)$ is Cauchy as $R \to 0$ and therefore tends to a limit, which can only be zero since $|r(R)|^n$ is integrable. Therefore the elements $u \in \mathrm{Rad}^{1,n}(B)$ are continuous on B and satisfy $u(0) = 0$. Using (7.4) with $\bar{R} = 0$ and $\eta = 1$ we see that

$$|r^n(R)| \leqslant C \|u\|^n_{\mathrm{Rad}^{1,n}(B)} \qquad \text{for all} \quad R \in [0, 1],$$

which shows that the imbedding of $\mathrm{Rad}^{1,n}(B)$ in $C(\bar{B}; \mathbb{R}^n)$ is continuous. In addition it is easily verified that the imbedding

$$\mathrm{Rad}^{1,n}(B) \subset C(\bar{A}_\delta; \mathbb{R}^n), \qquad A_\delta = \{x: \delta < |x| < 1\},$$

is compact for all $\delta > 0$. However, the example (7.1), (7.2) (for which $|u_j(x)| = 1$ if $|x| = 1/j$) shows that the imbedding $\mathrm{Rad}^{1,n}(B) \subset C(\bar{B}; \mathbb{R}^n)$ *is not compact*.

COUNTEREXAMPLE 7.3. We now give an example of a different type showing that $u \longmapsto \det \nabla u$ is not sequentially weakly continuous from $W^{1,n}(\Omega; \mathbb{R}^n) \to L^1(\Omega)$. It is an adaptation of an idea of L. Tartar (see Murat [15, p. 252] for a related example). Let $n = 2$ and $Q = (-1, 1)^2$. Define functions $u_j: Q \to \mathbb{R}^2$ for $j = 1, 2,\dots$, by

$$u_j(x, y) = j^{-1/2}(1 - |y|)^j\,(\sin jx, \cos jx).$$

Since

$$\|u_j\|_{L^\infty(Q; \mathbb{R}^2)} \leqslant j^{-1/2},$$

$$\|\nabla u_j\|^2_{L^2(Q; M^{2\times 2})} = 2j \int_{-1}^{1} \int_0^1 [(1-y)^{2j} + (1-y)^{2(j-1)}]\, dy\, dx$$

$$= 4j \left(\frac{1}{2j+1} + \frac{1}{2j-1} \right) < 8,$$

we have that $u_j \rightharpoonup 0$ in $W^{1,2}(Q; \mathbb{R}^2)$. But if $0 < a < 1$

$$\int_{-a}^{a} \int_0^a \det \nabla u_j\, dy\, dx = a[(1-a)^{2j} - 1] \to -a$$

as $j \to \infty$. Hence $\det \nabla u_j \not\rightharpoonup 0$ in $L^1_{\mathrm{loc}}(Q)$.

In the next example we show that $u \longmapsto \det \nabla u$ has even worse continuity properties with respect to weak convergence in $W^{1,p}(\Omega; \mathbb{R}^n)$ for $p < n$. In fact we exhibit a sequence $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ for all $p < n$ such that $\det \nabla u_j(x) = 0$ a.e. in Ω but $\det \nabla u(x) = 1$ a.e. in Ω.

COUNTEREXAMPLE 7.4. Let $n > 1$. For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ denote $\|x\| = \max_{1 \leqslant k \leqslant n} |x^k|$ and $Q = \{x \in \mathbb{R}^n : \|x\| < 1\} = (-1, 1)^n$. We let

$$u(x) = \frac{x}{\|x\|} - x \qquad \text{for all} \quad x \in Q.$$

This function is continuous except at zero and vanishes on ∂Q. We extend u as a Q-periodic function to the whole of $\mathbb{R}^n$. In $Q \setminus \{0\}$ $u = (u^1, \ldots, u^n)$ has distributional derivatives

$$\begin{aligned} \frac{\partial u^i}{\partial x^k}(x) &= \frac{\delta^i_k}{\|x\|} - \frac{x^i}{\|x\|^2} \frac{\partial \|x\|}{\partial x^k} - \delta^i_k, \\ \frac{\partial \|x\|}{\partial x^k} &= \operatorname{sign} x^k \qquad \text{if} \quad \|x\| = |x^k|, \\ &= 0 \qquad\qquad \text{if} \quad \|x\| = |x^i|,\ i \neq k. \end{aligned} \tag{7.5}$$

We note that

$$\left| \frac{\partial u^i}{\partial x^k}(x) \right| \leqslant \frac{2}{\|x\|} + 1, \tag{7.6}$$

and show that (7.5) defines the derivatives of u in the sense of distributions in Q (and not just in $Q \setminus \{0\}$). In fact, if $\phi \in \mathscr{D}(Q)$ and $\varepsilon > 0$,

$$\begin{aligned} \left\langle \frac{\partial u^i}{\partial x^k}, \phi \right\rangle &= -\int_Q u^i \frac{\partial \phi}{\partial x^k}\, dx = -\int_{Q \setminus \varepsilon Q} u^i \frac{\partial \phi}{\partial x^k}\, dx + r_\varepsilon \\ &= \int_{Q \setminus \varepsilon Q} \left(\frac{\delta^i_k}{\|x\|} - \frac{x^i}{\|x\|^2} \frac{\partial \|x\|}{\partial x^k} - \delta^i_k \right) \phi\, dx + \int_{\partial(\varepsilon Q)} u^i \frac{x^k}{\|x\|} \phi\, dS + r_\varepsilon \\ &= \int_Q \left(\frac{\delta^i_k}{\|x\|} - \frac{x^i}{\|x\|^2} \frac{\partial \|x\|}{\partial x^k} - \delta^i_k \right) \phi\, dx + r_\varepsilon, \end{aligned}$$

where r_ε denotes terms tending to zero as $\varepsilon \to 0$. Finally, since by (7.6)

$$\left\| \frac{\partial u^i}{\partial x^k} \right\|^p_{L^p(Q)} \leqslant \int_Q \left(\frac{2}{\|x\|} + 1 \right)^p dx \leqslant C \int_0^1 \left(\frac{2}{\rho} + 1 \right)^p \rho^{n-1}\, d\rho$$

it follows that $u \in W^{1,p}(Q; \mathbb{R}^n)$ for all p, $1 \leqslant p < n$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let for $j = 1, 2, \ldots$,

$$u_j(x) = x + j^{-1} u(jx) \qquad \text{for all} \quad x \in \Omega.$$

By Corollary A.2 $u_j \rightharpoonup x$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ for $1 \leqslant p < n$; furthermore $u_j \to x$ strongly in $L^\infty(\Omega; \mathbb{R}^n)$. But given j, for almost all $x \in \Omega$ we have that

$\nabla u_j(x) = \mathbf{1} + \nabla u(jx) = \nabla(y/\|y\|)$ where $y \in Q \setminus \{0\}$. Hence $\det \nabla u_j(x) = 0$ a.e. $x \in \Omega$, which completes the example.

We recall the definition of the distributional determinant $\operatorname{Det} \nabla u$ introduced in Ball [4]. If $u: \Omega \to \mathbb{R}^n$, $n > 1$, then

$$\operatorname{Det} \nabla u \stackrel{\text{def}}{=} \sum_{k=1}^{n} (-1)^{k+1} \frac{\partial}{\partial x^k} \left(u^1 \frac{\partial(u^2, \ldots, u^n)}{\partial(x^1, \ldots, \hat{x}^k, \ldots, x^n)} \right), \tag{7.7}$$

where $\partial(u^2, \ldots, u^n)/\partial(x^1, \ldots, \hat{x}^k, \ldots, x^n)$ is the determinant of the matrix $\partial u^i/\partial x^j$ $(2 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n,\ j \neq k)$. We record the following facts concerning $\operatorname{Det} \nabla u$ (for proofs see Ball, Currie, and Olver [7, p. 166]).

(1) $\operatorname{Det} \nabla u$ is well defined as an element of $\mathscr{D}'(\Omega)$ if $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $p \geqslant n^2/(n+1)$,

(2) if $p > n^2/(n+1)$ then $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ implies that $\operatorname{Det} \nabla u_j \to \operatorname{Det} \nabla u$ in $\mathscr{D}'(\Omega)$,

(3) $\operatorname{Det} \nabla u = \det \nabla u$ if $u \in W^{1,n}(\Omega; \mathbb{R}^n)$, or more generally if $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, $\operatorname{adj} \nabla u \in L^{p'}(\Omega; \mathbb{R}^n)$, where $p \geqslant n-1$ and $(1/p) + (1/p') = 1$ (the last statement is proved for $n = 3$ in Ball [5, p. 217]).

These properties are consistent with Counterexample 7.4; in fact, $\operatorname{Det} \nabla u_j$ is not then zero, but consists of a sum of Dirac masses placed at the centres of a grating of $\mathbb{R}^n$ by n-cubes of side $2/j$.

APPENDIX: AUXILIARY RESULTS ON WEAK CONVERGENCE

In this appendix we prove some auxiliary results concerning weak convergence that are used frequently in the paper. We begin with a lemma which, though often quoted, especially for $p > 1$, is not easy to find proved completely in the literature.

LEMMA A.1. *Let $Q \subset \mathbb{R}^m$ be an m-cube. Let $1 \leqslant p \leqslant \infty$ and let $f \in L^p_{\text{loc}}(\mathbb{R}^m)$ be Q-periodic. Define $f_j(x) = f(jx)$. Then as $j \to \infty$*

$$f_j \rightharpoonup \frac{1}{\operatorname{meas} Q} \int_Q f(y)\, dy \qquad (\rightharpoonup * \text{ if } p = \infty)$$

in $L^p(\Omega)$ for any bounded open subset $\Omega \subset \mathbb{R}^m$.

Proof. We suppose without loss of generality that $Q = (0, 1)^m$; in this case a function f is Q-periodic provided $f(x + e_i) = f(x)$ for all x and $i = 1, \ldots, m$, where $\{e_i\}$ denotes the standard basis of $\mathbb{R}^m$.

Suppose first that $f \in L^\infty(\mathbb{R}^m)$. Then f_j is bounded in $L^\infty(\Omega)$ and so contains a subsequence $f_\mu \rightharpoonup * \theta$ in $L^\infty(\Omega)$. We show that

$$\theta(x) = \int_Q f(y)\, dy = \text{const.} \qquad \text{a.e. } x \in \Omega. \tag{A.1}$$

In fact, let $Q_1 \subset \Omega$ be an m-cube with edges parallel to the axes and of side a. If $\mu a > 1$ then μQ_1 is a union of $([\mu a - 1])^m$ unit cubes of the fundamental lattice and a set E_μ with meas $E_\mu \leqslant ([\mu a + 2])^m - ([\mu a - 1])^m$, where [ ] denotes "integer part." Thus

$$\left| \int_{Q_1} \left(f_\mu(x) - \int_Q f(y)\, dy \right) dx \right| = \left| \mu^{-m} \int_{\mu Q_1} \left(f(z) - \int_Q f(y)\, dy \right) dz \right|$$
$$\leqslant C\mu^{-m} \text{ meas } E_\mu,$$

which tends to zero as $\mu \to \infty$. Thus $\int_{Q_1} (\theta(x) - \int_Q f(y)\, dy)\, dx = 0$ for all Q_1 and (A.1) follows by a density argument. Since θ is uniquely determined by f it follows that the whole sequence $f_j \rightharpoonup * \int_Q f(y)\, dy$ in $L^\infty(\Omega)$.

Next suppose that $f \in L^p_{\text{loc}}(\mathbb{R}^m)$ with $1 \leqslant p < \infty$. Let ρ be a mollifier, i.e., $\rho \in C_0^\infty(\mathbb{R}^m)$, $\rho \geqslant 0$, $\int_{\mathbb{R}^m} \rho(x)\, dx = 1$. For $\varepsilon > 0$ let $\rho_\varepsilon(x) =^{\text{def}} \varepsilon^{-m}\rho(x/\varepsilon)$, and let $f^\varepsilon = \rho_\varepsilon * f$. Then $f^\varepsilon \to f$ in $L^p(Q)$ and clearly f^ε is Q-periodic. We claim that $f_j^\varepsilon \to f_j$ in $L^p(\Omega)$ as $\varepsilon \to 0$, uniformly in j. We have that

$$\int_\Omega |f_j^\varepsilon(x) - f_j(x)|^p\, dx = j^{-m} \int_{j\Omega} |f^\varepsilon(y) - f(y)|^p\, dy$$
$$\leqslant j^{-m} \sum_k \int_{Q_k} |f^\varepsilon(y) - f(y)|^p\, dy,$$

where the sum $\sum_k$ is over cubes $Q_k = a_k + Q$ of the fundamental lattice, the number of these cubes being less than Cj^m. But

$$\int_{Q_k} |f^\varepsilon(y) - f(y)|^p\, dy = \int_Q |f^\varepsilon(y) - f(y)|^p\, dy$$

since f^ε, f are Q-periodic. Therefore $\|f_j^\varepsilon - f_j\|_{L^p(\Omega)} \leqslant C\, \|f^\varepsilon - f\|_{L^p(Q)}$, which proves the claim. Since f^ε is smooth we know that $f_j^\varepsilon \rightharpoonup * \int_Q f^\varepsilon(y)\, dy$ in $L^\infty(\Omega)$ as $j \to \infty$. Writing $f_j = f_j^\varepsilon + f_j - f_j^\varepsilon$ we deduce that $f_j \rightharpoonup \int_Q f(y)\, dy$ in $L^p(\Omega)$ as required. ∎

Corollary A.2. *Let $Q \subset \mathbb{R}^m$ be an m-cube. Let $1 \leqslant p \leqslant \infty$, let $v \in W^{1,p}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^n)$ be such that ∇v is Q-periodic, and define $v_j(x) = j^{-1}v(jx)$. Then*

$$v_j \rightharpoonup \frac{1}{\text{meas } Q} \left(\int_Q \nabla v(y)\, dy \right) x \qquad (\rightharpoonup * \text{ if } p = \infty)$$

in $W^{1,p}(\Omega; \mathbb{R}^n)$ for any bounded open subset $\Omega \subset \mathbb{R}^m$.

Proof. Let $Q = (0, 1)^m$ and consider the function $z_i(x) = v(x + e_i) - v(x)$. Since $\nabla z_i(x) = \nabla v(x + e_i) - \nabla v(x) = 0$, $z_i(x)$ is actually independent of x: let $A \in M^{n \times m}$ be defined by $Ae_i = z_i(x)$ and w by $w(x) = v(x) - Ax$. Then

$$w(x + e_i) - w(x) = v(x + e_i) - v(x) - Ae_i = 0.$$

Hence w is Q-periodic. But

$$v_j(x) = Ax + \frac{1}{j} w(jx),$$

$$\nabla v_j(x) = A + \nabla w(jx).$$

Therefore $v_j \to Ax$ in $L^p(\Omega; \mathbb{R}^n)$, and by the lemma $\nabla v_j \rightharpoonup \int_Q (A + \nabla w(y))\, dy = \int_Q \nabla v(y)\, dy$ in $L^p(\Omega; M^{n \times n})$ ($\rightharpoonup *$ if $p = \infty$), which completes the proof. ∎

The final result is a lower semicontinuity theorem.

PROPOSITION A.3. *Let $\Omega \subset \mathbb{R}^m$ be bounded open, and let $H: \mathbb{R}^s \to \bar{\mathbb{R}}$ be convex, lower semicontinuous and bounded below. Let $\theta_j, \theta \in L^1(\Omega; \mathbb{R}^s)$ with $\theta_j \rightharpoonup * \theta$ in the sense of measures (i.e., $\int_\Omega \theta_j \phi\, dx \to \int_\Omega \theta\phi\, dx$ for all continuous functions $\phi: \Omega \to \mathbb{R}$ with compact support in Ω). Then*

$$\int_\Omega H(\theta(x))\, dx \leqslant \liminf_{j \to \infty} \int_\Omega H(\theta_j(x))\, dx. \qquad \text{(A.2)}$$

Remark A.4. This proposition was proved by Reshetnyak [17, p. 805] for the case when H depends also on x, but takes only finite values. In our case we are able to give a slightly simpler proof, but still based on Reshetnyak's idea. (Note added in proof; see also Marcellini [21].)

Proof of Proposition A.3. We suppose first that $H(z) = \max_{1 \leqslant i \leqslant M} (\alpha_i + \langle \beta_i, z \rangle)$ is piecewise affine, convex and nonnegative. For each $z \in \mathbb{R}^s$ we choose an element $A(z) \in \partial H(z)$. There exists a sequence $v_k \to \theta$ a.e. in Ω and strongly in $L^1(\Omega; \mathbb{R}^s)$ with the property that v_k is constant on each open s-cube $Q_{k,m} = k^{-1}[m + (0, 1)^s]$, $m \in \mathbb{Z}^s$. Also, there is a sequence $\phi_k \in \mathscr{D}(\Omega)$ such that $0 \leqslant \phi_k \leqslant 1$, $\phi_k = 0$ in a neighborhood of each $\partial Q_{k,m}$ and $\phi_k \to 1$ a.e. in Ω. By the convexity of H,

$$H(\theta_j(x)) \geqslant H(v_k(x)) + \langle A(v_k(x)), \theta_j(x) - v_k(x) \rangle$$
$$\text{a.e. } x \in \Omega.$$

We multiply this inequality by $\phi_k(x)$ and integrate over Ω to obtain

$$\int_\Omega H(\theta_j(x))\,dx \geqslant \int_\Omega \phi_k(x)\,H(\theta_j(x))\,dx$$

$$\geqslant \int_\Omega \phi_k(x)\,H(v_k(x))\,dx + \int_\Omega \phi_k(x)\langle A(v_k(x)), \theta_j(x) - v_k(x)\rangle\,dx.$$

Fix k and let $j \to \infty$. Since $\phi_k(x)\,A(v_k(x))$ is continuous with compact support in Ω we obtain

$$\liminf_{j\to\infty} \int_\Omega H(\theta_j(x))\,dx \geqslant \int_\Omega \phi_k(x)\,H(v_k(x))\,dx$$

$$+ \int_\Omega \phi_k(x)\langle A(v_k(x)), \theta(x) - v_k(x)\rangle\,dx.$$

Let $k \to \infty$. By Fatou's lemma $\liminf_{k\to\infty} \int_\Omega \phi_k(x)\,H(v_k(x))\,dx \geqslant \int_\Omega H(\theta(x))\,dx$. Since $A(\cdot)$ is uniformly bounded a simple estimate shows that

$$\lim_{k\to\infty} \int_\Omega \phi_k(x)\langle A(v_k(x)), \theta(x) - v_k(x)\rangle\,dx = 0.$$

Hence we obtain (A.2).

Now let $H\colon \mathbb{R}^s \to \overline{\mathbb{R}}$ be convex, lower semicontinuous and bounded below. Any such function is the supremum of a countable family of affine functions. Assuming without loss of generality that $H \geqslant 0$ it follows that H can be written as the limit of an increasing sequence H_l of piecewise affine, convex, nonnegative functions. For each l

$$\int_\Omega H_l(\theta(x))\,dx \leqslant \liminf_{j\to\infty} \int_\Omega H_l(\theta_j(x))\,dx \leqslant \liminf_{j\to\infty} \int_\Omega H(\theta_j(x))\,dx.$$

Letting $l \to \infty$ we deduce from the monotone convergence theorem that

$$\int_\Omega H(\theta(x))\,dx \leqslant \liminf_{j\to\infty} \int_\Omega H(\theta_j(x))\,dx$$

as required. ∎

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