Global Attraction for the One-Dimensional Heat Equation with Nonlinear Time-Dependent Boundary Conditions

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1. Introduction

Consider the mixed initial-boundary value problem

$$\text{(I)}\qquad \begin{aligned} &u_t = u_{xx}, && \text{for } 0<x<1,\ t>s && (1)\\ &u_x(i,t) = (-1)^i f_i(u(i,t),t), && \text{for } i=0,1,\ \ t>s && (2)\\ &u(x,s) = \psi(x), && \text{for } 0\leqq x\leqq 1, && (3)\end{aligned}$$

where $\psi \in C([0,1])$ and the functions f_0 and f_1 satisfy

$$v f_i(v,t) \geqq 0 \quad \text{for } |v| \geqq a, \quad t \geqq s, \quad i=0,1 \tag{4}$$

for some positive constant $a = a(s)$.

In [2] we showed that if the functions f_i are independent of t and twice continuously differentiable, then any solution u of problem I converges in $C^1([0,1])$ to an equilibrium solution as $t \to \infty$, each equilibrium solution being supposed isolated. In this paper we prove analogous results for the case when the functions $f_i(.,t)$ "stabilize" as $t \to \infty$, that is, tend to functions $\bar{f}_i(.)$ which do not depend on t. In the catalyst particle problem discussed in [1, 2] this happens when the concentration of the reactant in the bath containing the particle tends to a uniform value as $t \to \infty$.

The stabilization of the functions f_i is assumed to hold in the sense that, for any $\rho > 0$,

$$\int_t^{t+1} \sup_{|v| \leqq \rho} |f_i(v,s) - \bar{f}_i(v)|\, ds \to 0, \quad \text{as } t \to \infty, \quad i=0,1. \tag{5}$$

Under some additional assumptions on the functions f_i problem I generates an asymptotically dynamical system on $C^1([0,1])$ in the sense of DAFERMOS [4]. Our proof of global attraction in $C^1([0,1])$ is based on a new form of the invariance principle for asymptotically dynamical systems in which we require only the *limiting* autonomous system to possess a Lyapunov function. This result is stated and proved in Section 2.

2. An Invariance Principle for Asymptotically Dynamical Systems

Let X be a metric space with metric d. Following DAFERMOS [4–6] we define a *process* on X to be a family of operators $U(t, s): X \to X$, defined for $t \in \mathbb{R}^+$, $s \in \mathbb{R}$, and satisfying:
(a) $U(0, s) = $ identity for $s \in \mathbb{R}$;
(b) $U(t+\tau, s) = U(t, s+\tau)\, U(\tau, s)$ for $t, \tau \in \mathbb{R}^+$, $s \in \mathbb{R}$;
(c) for any fixed $s_0 \in \mathbb{R}$ the maps $(t, \psi) \mapsto U(t, s)\psi$, with parameter $s \in [s_0, \infty)$, are equicontinuous on $(0, \infty) \times X$ (i.e. given $\varepsilon > 0$, $t > 0$, $\psi \in X$, there exists a $\delta > 0$ such that $d(U(t, s)\psi, U(\bar{t}, s)\bar{\psi}) < \varepsilon$ whenever $|t - \bar{t}| + d(\psi, \bar{\psi}) < \delta$ and $s \geqq s_0$)*.

In the special case when the operators $U(t, s) \overset{\text{def}}{=} T(t)$ are independent of s, the process is called a *semigroup* of continuous operators.

If $U(.,.)$ is a process and $\psi \in X$, $s \in \mathbb{R}$, we define the positive orbit

$$\mathcal{O}^+(\psi, s) = \bigcup_{t \geqq 0} U(t, s)\psi,$$

and the *ω-limit set*

$\Omega(\psi, s) = \{\phi \in X$: there exists a sequence $\{t_n\} \subset \mathbb{R}^+$, $t_n \to \infty$ as $n \to \infty$, such that $U(t_n, s)\psi \to \phi$ in $X\}$.

The process $U(\cdot, \cdot)$ is called an *asymptotically dynamical system* if there exists a semigroup $T(\cdot)$ such that for any fixed $\psi \in X$, $t \in \mathbb{R}^+$, we have $U(t, s)\psi \to T(t)\psi$ in X as $s \to \infty$. A subset A of X is *positively invariant* if $T(\tau)A \subset A$ for all $\tau \in \mathbb{R}^+$. A point $\phi \in X$ is a *rest point* if $T(\tau)\phi = \phi$ for all $\tau \in \mathbb{R}^+$.

From now on we suppose that $U(\cdot, \cdot)$ is an asymptotically dynamical system with corresponding semigroup $T(\cdot)$. We also suppose that
(d) there are only finitely many rest points in any compact subset of X;
(e) there exists a continuous function $V: X \to \mathbb{R}$ such that
  (*i*) $V(T(\tau)\psi) \leqq V(\psi)$ for any $\psi \in X$, $\tau \in \mathbb{R}^+$;
  (*ii*) if $V(T(\tau)\phi) = V(\phi)$ for all $\tau \in \mathbb{R}^+$, then ϕ is a rest point.

Theorem 1. *Let $\psi \in X$, $s \in \mathbb{R}$ be such that $\mathcal{O}^+(\psi, s)$ is precompact. Then $\Omega(\psi, s)$ consists of a single rest point.*

Proof. Since $\mathcal{O}^+(\psi, s)$ is precompact, $\Omega(\psi, s)$ is nonempty, compact, positively invariant and connected. The first three of these properties follow from the method used by DAFERMOS [4–6]. The positive invariance is also a consequence of the following stronger assertion, which we need later.

Proposition. *If $\phi \in \Omega(\psi, s)$ and $U(t_n, s)\psi \to \phi$ in X for some sequence $t_n \to \infty$, then $U(t_n + \tau, s)\psi \to T(\tau)\phi$ in X as $n \to \infty$, uniformly for τ in any compact subset of $\mathbb{R}^+$.*

Suppose the proposition is false. Then there exists an $\varepsilon > 0$ and a sequence $\{\tau_n\} \subset \mathbb{R}^+$ with $\tau_n \to \tau$ as $n \to \infty$ such that $d(U(t_n + \tau_n, s)\psi, T(\tau)\phi) > \varepsilon$ for all n. Since $\mathcal{O}^+(\psi, s)$ is precompact there exists a subsequence $\{t_\mu\}$ of $\{t_n\}$ and an element $\chi \in \Omega(\psi, s)$ with $U(t_\mu - 1, s)\psi \to \chi$ in X as $\mu \to \infty$. Then

* This equicontinuity property differs from that used by DAFERMOS.

$$\begin{aligned}d\big(U(t_\mu+\tau_\mu, s)\psi, T(\tau)\phi\big) &= d\big(U(\tau_\mu+1, s+t_\mu-1)\,U(t_\mu-1, s)\psi,\ T(\tau)\phi\big)\\ &\leqq d\big(U(\tau_\mu+1, s+t_\mu-1)\,U(t_\mu-1, s)\psi,\ U(\tau+1, s+t_\mu-1)\,\chi\big)\\ &\quad + d\big(U(\tau+1, s+t_\mu-1)\,\chi,\ U(\tau+1, s+t_\mu-1)\,U(t_\mu-1, s)\psi\big)\\ &\quad + d\big(U(\tau, s+t_\mu)\,U(t_\mu, s)\psi,\ U(\tau, s+t_\mu)\phi\big)\\ &\quad + d\big(U(\tau, s+t_\mu)\phi,\ T(\tau)\phi\big),\end{aligned}$$

so that by (c), $d\big(U(t_\mu+\tau_\mu, s)\psi, T(\tau)\phi\big)\to 0$ as $\mu\to\infty$. This contradiction proves the proposition.

The connectedness of $\Omega(\psi, s)$ follows from continuity of the map $t\mapsto U(t, s)\psi$. Let

$$\alpha=\liminf_{t\to\infty} V\big(U(t, s)\psi\big), \qquad \beta=\limsup_{t\to\infty} V\big(U(t, s)\psi\big).$$

Since $\mathcal{O}^+(\psi, s)$ is precompact and V is continuous it follows that $-\infty<\alpha\leqq\beta<\infty$. Suppose for contradiction that $\alpha<\beta$. Choose $\gamma_1, \gamma_2\in\mathbb{R}$ so that $\alpha<\gamma_1<\gamma_2<\beta$. By (c) and the continuity of V the map $t\mapsto V\big(U(t, s)\psi\big)$ is continuous on $(0, \infty)$. Therefore there exist sequences $\{t_n\}$, $\{r_n\}$, $t_n<r_n$, $t_n\to\infty$ as $n\to\infty$, such that $V\big(U(t_n, s)\psi\big)=\gamma_1$, $V\big(U(r_n, s)\psi\big)=\gamma_2$, and $\gamma_1\leqq V\big(U(\tau, s)\psi\big)\leqq\gamma_2$ for all $\tau\in[t_n, r_n]$.

Since $\mathcal{O}^+(\psi, s)$ is precompact we can suppose that $U(t_n, s)\psi\to\phi$ in X for some $\phi\in\Omega(\psi, s)$. Using the proposition, we see that $V\big(U(t_n+\tau, s)\psi\big)\to V\big(T(\tau)\phi\big)$ as $n\to\infty$, uniformly in any compact subset of $\mathbb{R}^+$. Thus, if $\tau_n\in[0, r_n-t_n]$ is a bounded sequence with $\tau_n\to\tau$ as $n\to\infty$, then

$$0\leqq \lim_{n\to\infty}\big[V\big(U(t_n+\tau_n, s)\psi\big)-V\big(U(t_n, s)\psi\big)\big]=V\big(T(\tau)\phi\big)-V(\phi)\leqq 0. \qquad (6)$$

Since $\gamma_1<\gamma_2$ this implies that the sequence $\{r_n-t_n\}$ cannot be bounded. Thus, without loss of generality, we may suppose that $r_n-t_n\to\infty$ as $n\to\infty$. But then (6) and (e, ii) together imply that ϕ is a rest point with $V(\phi)=\gamma_1$. Since $\gamma_1\in(\alpha, \beta)$ is arbitrary, this contradicts (d). Hence $\alpha=\beta$, so that $V\big(U(t, s)\psi\big)\to\alpha$ as $t\to\infty$. Thus $V(\phi)=\alpha$ for all $\phi\in\Omega(\psi, s)$. Using the positive invariance of $\Omega(\psi, s)$ and the properties (d) and (e, ii) we find that $\Omega(\psi, s)$ consists of finitely many rest points. The desired result now follows from the fact that $\Omega(\psi, s)$ is connected.

3. The System (I) as an Asymptotically Dynamical System

We make the following hypotheses on the boundary functions $f_i(v, t)$, $i=0, 1$:

(H1) $f_i\in C(\mathbb{R}\times\mathbb{R})$;

(H2) for each $t\in\mathbb{R}$, $f_i(v, t)$ is twice continuously differentiable with respect to $v\in\mathbb{R}$, and for each $\rho>0$, $s\in\mathbb{R}$ there are constants $M_j=M_j(\rho, s)>0$, $j=0, 1, 2$, such that for all $|v|\leqq\rho$, $t\geqq s$,*

$$|f_i(v, t)|\leqq M_0, \qquad |f_i'(v, t)|\leqq M_1, \qquad |f_i''(v, t)|\leqq M_2; \qquad (7)$$

(H3) f_i satisfies (4);

(H4) there exist real valued functions $\bar{f}_i\in C^2(\mathbb{R})$ such that for each $\rho>0$ condition (5) holds.

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* Primes denote differentiation with respect to v. The arguments of the constants M_j will often be omitted where this will cause no confusion.

Remarks. It follows from (H2)–(H4) that there exists a constant $\bar{a}>0$ such that

$$v\bar{f}_i(v)\geqq 0 \quad \text{for } |v|\geqq \bar{a},\ i=0,1, \tag{8}$$

and that for each $\rho>0$ there exist constants $\bar{M}_j=\bar{M}_j(\rho)>0$, $j=0,1,2$, such that for all $|v|\leqq\rho$

$$|\bar{f}_i(v)|\leqq \bar{M}_0, \quad |\bar{f}_i'(v)|\leqq \bar{M}_1, \quad |\bar{f}_i''(v)|\leqq \bar{M}_2. \tag{9}$$

Note also that if $1\leqq\gamma<\infty$, then by (H2) and the bounded convergence theorem (5) is equivalent to

$$\int_t^{t+1} \sup_{|v|\leqq\rho} |f_i(v,s)-\bar{f}_i(v)|^\gamma ds \to 0 \quad \text{as } t\to\infty, \quad i=0,1. \tag{10}$$

For $s\in\mathbb{R}$, write

$$Q(s)=\{(x,t): 0<x<1,\ t>s\}, \qquad S(s)=\{(x,t): x\in\{0,1\},\ t>s\}.$$

The function $u=u(x,t;\psi,s)$ is said to be a solution of problem I if

$$u\in C(\overline{Q(s)}), \quad u_x\in C(Q(s)\cup S(s)), \quad u_t\in C(Q(s)), \quad u_{xx}\in C(Q(s))$$

and (1)–(3) hold. The usual norms in the spaces $C([0,1])$ and $C^1([0,1])$ are denoted by $\|\cdot\|$ and $\|\cdot\|_1$ respectively. The following theorem can be proved by exactly the methods of [1, 2]:

Theorem 2. *For any $\psi\in C([0,1])$ and $s\in\mathbb{R}$ there exists a unique solution*

$$u(x,t)=u(x,t;\psi,s)$$

of problem (I). This solution satisfies the integral equation

$$u(x,t)=\int_0^1 G(x,\xi,t-s)\psi(\xi)d\xi-\sum_{i=0}^1 \int_s^t G(x,i,t-\tau)f_i(u(i,\tau),\tau)d\tau \tag{11}$$

in $Q(s)$, where $G(x,\xi,t)$ denotes the Green function for the heat equation in $Q(0)$ with zero Neumann data. Furthermore

$$|u(x,t)|\leqq \max\{\|\psi\|, a(s)\} \quad \textit{for } (x,t)\in\overline{Q(s)} \tag{12}$$

and for any $\delta>0$ there exists a constant $K=K(\psi,s,\delta)$ such that

$$|u_x(x,t)|\leqq K \quad \textit{for } (x,t)\in\overline{Q(s+\delta)}. \tag{13}$$

For $\psi\in C([0,1])$, $s\in\mathbb{R}$, $t\in\mathbb{R}^+$, define

$$(U(t,s)\psi)(x)=u(x,t+s;\psi,s). \tag{14}$$

By Theorem 2 $U(t,s)$ maps $C([0,1])$ into $C^1([0,1])$ and satisfies conditions (a) and (b) in the definition of a process. We next prove that $U(t,s)$ satisfies a strengthened equicontinuity property.

Lemma 1. *Given any $\varepsilon>0$, $\psi\in C([0,1])$, $t>0$, $s_0\in\mathbb{R}$, there exists a $\delta>0$ such that*

$$\|U(t,s)\psi-U(\bar{t},s)\bar{\psi}\|_1<\varepsilon$$

whenever $|t-\bar{t}|+\|\psi-\bar{\psi}\|<\delta$ and $s\geqq s_0$.

Proof. Let $\varepsilon>0$, $t>0$, $s_0\in\mathbb{R}$, and let $\psi, \bar{\psi}\in C([0,1])$. Write

$$v_s(x,\tau)=u(x,\tau+s;\psi,s), \qquad \bar{v}_s(x,\tau)=u(x,\tau+s;\bar{\psi},s).$$

Then by (11)

$$v_s(x,\tau)-\bar{v}_s(x,\tau)=\int_0^1 G(x,\xi,\tau)\,[\psi(\xi)-\bar{\psi}(\xi)]\,d\xi \qquad (15)$$
$$-\sum_{i=0}^1 \int_0^\tau G(x,i,\tau-\sigma)\,[f_i(v_s(i,\sigma),\sigma+s)-f_i(\bar{v}_s(i,\sigma),\sigma+s)]\,d\sigma.$$

Let $m_s(\tau)=\|v_s(\cdot,\tau)-\bar{v}_s(\cdot,\tau)\|$. Then if $s\geqq s_0$ and $\|\psi-\bar{\psi}\|<1$,

$$m_s(\tau)\leqq\|\psi-\bar{\psi}\|+M_1\sum_{i=0}^1\int_0^\tau\|G(\cdot,i,\tau-\sigma)\|\,m_s(\sigma)\,d\sigma,$$

where $M_1=M_1(\max\{\|\psi\|+1, a(s_0)\}, s_0)$. But

$$|G(x,\xi,\tau)|\leqq\omega(\tau)\overset{\text{def}}{=}G(0,0,\tau) \qquad (16)$$

for $x,\xi\in[0,1]$ and $\tau>0$. It was shown in [2] that $\omega(\tau)$ is continuous for $\tau>0$ and satisfies $\omega(\tau)\sim(\pi\tau)^{-\frac{1}{2}}$ as $\tau\to0+$. Therefore if $\gamma>2$ and $\gamma'=\gamma/(\gamma+1)$ then

$$\max_{i=0,1}\int_0^\tau\|G(\cdot,i,\tau-\sigma)\|^{\gamma'}\,d\sigma\overset{\text{def}}{=}k(\tau)<\infty. \qquad (17)$$

Hence

$$m_s(\tau)\leqq\|\psi-\bar{\psi}\|+2M_1\{k(\tau)\}^{1/\gamma'}\left(\int_0^\tau\{m_s(\sigma)\}^\gamma d\sigma\right)^{1/\gamma},$$

so that

$$\{m_s(\tau)\}^\gamma\leqq2^{\gamma-1}\|\psi-\bar{\psi}\|^\gamma+2^{2\gamma-1}M_1^\gamma\{k(\tau)\}^{\gamma/\gamma'}\int_0^\tau\{m_s(\sigma)\}^\gamma d\sigma. \qquad (18)$$

Applying Gronwall's inequality we deduce that for $\bar{t}\in[0,3t/2]$, say, there exists a constant $C>0$ such that

$$m_s(\bar{t})\leqq C\|\psi-\bar{\psi}\|. \qquad (19)$$

From (15) and (19) we obtain

$$|v_{sx}(x,\bar{t})-\bar{v}_{sx}(x,\bar{t})|\leqq\Big\{\int_0^1|G_x(x,\xi,\bar{t})|\,d\xi$$
$$+M_1C\sum_{i=0}^1\int_0^{\bar{t}}|G_x(x,i,\sigma)|\,d\sigma\Big\}\,\|\psi-\bar{\psi}\|. \qquad (20)$$

Since both integrals in (20) are bounded for $\bar{t}\in[t/2,3t/2]$, it follows from (19) and (20) that there exists a $\delta_1>0$ such that

$$\|U(\bar{t},s)\psi-U(\bar{t},s)\bar{\psi}\|_1<\tfrac{1}{2}\varepsilon \qquad (21)$$

whenever $\|\psi-\bar{\psi}\|<\delta_1$, $\bar{t}\in[t/2,3t/2]$, and $s\geqq s_0$.

From (11) there results

$$v_s(x,\bar{t})-v_s(x,t)=\int_0^1[G(x,\xi,\bar{t})-G(x,\xi,t)]\,\psi(\xi)\,d\xi$$
$$-\sum_{i=0}^1\Big\{\int_0^t[G(x,i,\bar{t}-\sigma)-G(x,i,t-\sigma)]f(v_s(i,\sigma),\sigma+s)\,d\sigma$$
$$+\int_t^{\bar{t}}G(x,i,\bar{t}-\sigma)f(v_s(i,\sigma),\sigma+s)\,d\sigma\Big\}.$$

It follows that, for $s \geqq s_0$ and $\bar{t} \geqq t$,

$$\begin{aligned}\|v_s(\cdot, \bar{t}) - v_s(\cdot, t)\|_1 \leqq & \left\|\int_0^1 [G(\cdot, \xi, \bar{t}) - G(\cdot, \xi, t)]\psi(\xi)\,d\xi\right\|_1 \\ & + M_0 \sum_{i=0}^1 \left\{\int_0^t \|G(\cdot, i, t-\sigma) - G(\cdot, i, \bar{t}-\sigma)\|_1 \,d\sigma \right. \\ & \left. + \int_0^{\bar{t}-t} \|G(\cdot, i, \sigma)\|_1 \,d\sigma\right\},\end{aligned} \tag{22}$$

where $M_0 = M_0\left(\max\{\|\psi\|, a(s_0)\}, s_0\right)$. By estimating the integrals in (22) by means of the explicit representations for G, see [2], it can be shown that $\|v^s(\cdot, \bar{t}) - v_s(\cdot, t)\|_1 \to 0$ as $\bar{t} \to t+$, uniformly for $s \geqq s_0$. A similar argument applies if $\bar{t} \to t-$. Thus there exists a $\delta_2 > 0$ such that

$$\|U(\bar{t}, s)\psi - U(t, s)\psi\|_1 < \tfrac{1}{2}\varepsilon \tag{23}$$

whenever $|\bar{t} - t| < \delta_2$ and $s \geqq s_0$. Let $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}t\}$. The result then follows by combining (21) and (23).

Lemma 2. *$U(\cdot,\cdot)$ is an asymptotically dynamical system on $C^1([0, 1])$ with corresponding semigroup $T(\cdot)$ defined by*

$$(T(t)\psi)(x) = w(x, t; \psi),$$

where w is the solution of the autonomous system

$$\text{(I}')\qquad \begin{aligned} & w_t = w_{xx} && 0 < x < 1, \quad t > 0 \\ & w_x(i, t) = (-1)^i \bar{f}_i(w(i, t)) && i = 0, 1, \quad t > 0 \\ & w(x, 0) = \psi(x) && 0 \leqq x \leqq 1. \end{aligned}$$

Furthermore, if $\psi \in C([0, 1])$ and $t > 0$ then $U(t, s)\psi \to T(t)\psi$ in $C^1([0, 1])$ as $s \to \infty$.

Proof. That $T(\cdot)$ is a semigroup on $C^1([0, 1])$ can be proved as in [1, 2]. Let $\psi \in C([0, 1])$ and write

$$v_s(x, t) = u(x, t+s; \psi, s), \qquad w(x, t) = w(x, t; \psi),$$
$$z_s(t) = \|v_s(\cdot, t) - w(\cdot, t)\|.$$

We first show that, for any fixed $T > 0$, $z_s(t) \to 0$ as $s \to \infty$ uniformly for $t \in [0, T]$.

From (11) and the corresponding autonomous equation we have

$$z_s(t) \leqq \sum_{i=0}^1 \int_0^t \|G(\cdot, i, t-\tau)\| \, |f_i(v_s(i, \tau), \tau+s) - \bar{f}_i(w(i, \tau))| \, d\tau.$$

Therefore

$$z_s(t) \leqq \phi_s(t) + \sum_{i=0}^1 \int_0^t \omega(t-\tau) |\bar{f}_i(v_s(i, \tau)) - \bar{f}_i(w(i, \tau))| \, d\tau, \tag{24}$$

where

$$\phi_s(t) = \sum_{i=0}^1 \int_0^t \omega(t-\tau) |f_i(v_s(i, \tau), \tau+s) - \bar{f}_i(v_s(i, \tau))| \, d\tau,$$

and ω is defined in (16).

Note that if $\gamma > 2$,

$$\phi_s(t) \leqq \sum_{i=0}^1 \left(\int_0^t \{\omega(\tau)\}^{\gamma'} d\tau\right)^{1/\gamma'} \left(\int_s^{s+t} |f_i(v_s(i, \tau-s), \tau) - \bar{f}_i(v_s(i, \tau-s))|^\gamma \, d\tau\right)^{1/\gamma},$$

so that by (10) and (12), for any $T>0$,

$$\phi_s(t)\to 0 \quad \text{as } s\to\infty \tag{25}$$

uniformly for $t\in[0, T]$.

From (9) and (24)

$$z_s(t)\leqq \phi_s(t)+2\bar{M}_1 \int_0^t \omega(t-\tau)z_s(\tau)\,d\tau.$$

Hence if $\gamma>2$

$$\{z_s(t)\}^\gamma\leqq 2^{\gamma-1}\{\phi_s(t)\}^\gamma+2^{2\gamma-1}\bar{M}_1^\gamma\left(\int_0^t\{\omega(\tau)\}^{\gamma'}d\tau\right)^{\gamma/\gamma'}\int_0^t\{z_s(\tau)\}^\gamma d\tau.$$

Thus there exist constants $A(s)$ and B such that, for $0\leqq t\leqq T$,

$$\{z_s(t)\}^\gamma\leqq A(s)+B\int_0^t\{z_s(\tau)\}^\gamma d\tau,$$

where, by (25), $A(s)\to 0$ as $s\to\infty$. By Gronwall's inequality

$$\{z_s(t)\}^\gamma\leqq A(s)e^{Bt}.$$

Hence $z_s(t)\to 0$ as $s\to\infty$, uniformly for $t\in[0, T]$, as required.

Now let $t>0$ be fixed. Then

$$\begin{aligned}|v_{sx}(x, t)-w_x(x, t)|&\leqq\textstyle\sum_{i=0}^1\int_0^t|G_x(x, i, t-\tau)|\,|f_i(v_s(i, \tau), \tau+s)-\bar{f}_i(v_s(i, \tau))|\,d\tau\\&\quad+\textstyle\sum_{i=0}^1\int_0^t|G_x(x, i, t-\tau)|\,|\bar{f}_i(v_s(i, \tau))-\bar{f}_i(w(i, \tau))|\,d\tau\\&=I_1+I_2.\end{aligned} \tag{26}$$

Given $\varepsilon>0$, let $\delta>0$ be such that, for all $x\in[0, 1]$, $i=0, 1$,

$$\int_{t-\delta}^t|G_x(x, i, t-\tau)|\,d\tau<\frac{\varepsilon}{8M_0}. \tag{27}$$

It is easy to check that there exists a constant $C_1>0$ such that

$$|G_x(x, i, t-\tau)|<C_1 \tag{28}$$

for all $x\in[0, 1]$, $\tau\in[0, t-\delta]$, $i=0, 1$. Also, by (5) there exists an s_1 such that for $s\geqq s_1$, $i=0, 1$,

$$\int_0^t|f_i(v_s(i, \tau), \tau+s)-\bar{f}_i(v_s(i, \tau))|\,d\tau<\frac{\varepsilon}{4C_1}. \tag{29}$$

Combining (7), (27), (28) and (29) we see that for $s\geqq s_1$, $x\in[0, 1]$,

$$|I_1|<4M_0\frac{\varepsilon}{8M_0}+2C_1\frac{\varepsilon}{4C_1}=\varepsilon.$$

Therefore $I_1\to 0$ as $s\to\infty$, uniformly for $x\in[0, 1]$. But the same holds for I_2, since

$$|I_2|\leqq\bar{M}_1\left(\max_{[0,T]}z_s(\tau)\right)\textstyle\sum_{i=0}^1\int_0^t|G_x(x, i, t-\tau)|\,d\tau.$$

and the integrals on the right hand side are bounded (see [2]).

Thus we have shown that

$$\|U(t, s)\psi - T(t)\psi\|_1 \to 0 \qquad \text{as } s\to\infty$$

which completes the proof.

4. Global Attraction

We can now prove our main result.

Theorem 3. *Suppose that the equilibrium solutions of the autonomous problem* (I') *are isolated in* $C([0, 1])$. *Let* $\psi \in C([0, 1])$ *and* $s \in \mathbb{R}$. *Then*

$$\lim_{t\to\infty} \|u(\cdot, t; \psi, s) - v(\cdot)\|_1 = 0$$

for some equilibrium solution v *of problem* I'.

Proof. Since, for $t>0$, $U(t, s)$ maps $C([0, 1])$ into $C^1([0, 1])$, we may without loss of generality suppose that $\psi \in C^1([0, 1])$. Since by Lemma 2, $U(\cdot,\cdot)$ is an asymptotically dynamical system on $C^1([0, 1])$, and since hypothesis (e) of Theorem 1 was established in [2], we need only show that $\mathcal{O}^+(\psi, s)$ is precompact in $C^1([0, 1])$. But by (13) and the Arzela-Ascoli theorem $\mathcal{O}^+(\psi, s)$ is precompact in $C([0, 1])$. Therefore if $t_n \to \infty$ there exists a subsequence $\{t_\mu\}$ of $\{t_n\}$ and an element $\phi \in C([0, 1])$ such that $U(t_\mu - 1, s)\psi \to \phi$ in $C([0, 1])$ as $t_\mu \to \infty$. But

$$U(t_\mu, s)\psi = U(1, t_\mu - 1 + s)\, U(t_\mu - 1, s)\psi$$

so that, by Lemmas 1 and 2, $U(t_\mu, s)\psi \to T(1)\phi$ in $C^1([0, 1])$. This completes the proof of the theorem.

5. A Related Problem

As another application of the invariance pinciple for asymptotically dynamical systems which we proved in Section 2, we consider the problem

$$\text{(II)} \qquad \begin{aligned} &u_t = u_{xx} + f(x, t, u), && 0<x<1, \quad t>s \\ &u_x(0, t) = u_x(1, t) = 0, && t>s \\ &u(x, 0) = \psi(x), && 0\leqq x\leqq 1, \end{aligned}$$

where $\psi \in C([0, 1])$ and f satisfies the hypotheses

(*i*) $f \in C^1([0, 1] \times \mathbb{R} \times \mathbb{R})$;

(*ii*) for each $x \in [0, 1]$ and $t \in \mathbb{R}$, $f(x, t, v)$ is twice continuously differentiable with respect to $v \in \mathbb{R}$ and for each $\rho > 0$, $s \in \mathbb{R}$, there are constants $M_j = M_j(\rho, s)$ $j = 0, 1, 2$, such that for all $|v| \leqq \rho$, $t \geqq s$

$$\|f(\cdot, t, v)\| \leqq M_0, \qquad \|f'(\cdot, t, v)\| \leqq M_1, \qquad \|f''(\cdot, t, v)\| \leqq M_2;$$

(*iii*) for each $s \in \mathbb{R}$ there exists a constant $a(s) \in \mathbb{R}$ such that

$$\sup_{x\in(0,1)} v f(x, t, v) \leqq 0 \qquad \text{for} \qquad |v| \geqq a,\ t \geqq s.$$

Suppose there exists a function $\bar{f} \in C^1([0, 1] \times \mathbb{R})$ which for each $x \in [0, 1]$ is twice continuously differentiable with respect to the second argument, such that for each $\rho > 0$,

$$\int_t^{t+1} \sup_{|v| \le \rho} \|f(\cdot, \tau, v) - \bar{f}(\cdot, v)\| d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

Then the limiting autonomous system for (II) is

$$\text{(II')} \qquad \begin{aligned} &w_t = w_{xx} + \bar{f}(x, w), && 0 < x < 1, \quad t > 0 \\ &w_x(0, t) = w_x(1, t) = 0, && t > 0 \\ &w(x, 0) = \psi(x), && 0 \leqq x \leqq 1. \end{aligned}$$

Then it is possible to prove the following result.

Theorem 4. *Suppose that the equilibrium solutions of the autonomous problem (II') are isolated in* $C([0, 1])$. *Let* $\psi \in C([0, 1])$ *and* $s \in \mathbb{R}$. *Then the solution* $u = u(x, t; \psi, s)$ *of (II) satisfies*

$$\lim_{t \to \infty} \|u(\cdot, t; \psi, s) - v(\cdot)\|_1 = 0$$

for some equilibrium solution v *of (II').*

The proof follows very closely that of Theorem 3. The relevant results for the autonomous problem (II') may be found in [3]. We shall omit the details.

References

1. Aronson, D.G. & L.A. Peletier, Global stability of symmetric and asymmetric concentration profiles in catalyst particles, Arch. Rational Mech. Anal. **54**, 175–204 (1974).
2. Ball, J.M. & L.A. Peletier, Stabilization of concentration profiles in catalyst particles, J. Diff. Equations **20**, 356–368 (1976).
3. Chafee, N., Asymptotic behaviour for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, J. Diff. Equations **18**, 111–134 (1975).
4. Dafermos, C.M., An invariance principle for compact processes, J. Diff. Equations **9**, 239–252 (1971).
5. Dafermos, C.M., Uniform processes and semicontinuous Liapunov functionals, J. Diff. Equations 401–415 (1972).
6. Dafermos, C.M., Semiflows associated with compact and uniform processes, Math. Systems Theory **8**, 142–149 (1974).

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