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Feedback Stabilization of Distributed Semilinear Control Systems

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Abstract. This paper considers feedback stabilization for the semilinear control system $\dot{u}(t) = Au(t) + v(t)B(u(t))$. Here A is the infinitesimal generator of a linear C^0 semigroup of contractions on a Hilbert space H and $B: H \to H$ is a nonlinear operator. A sufficient condition for feedback stabilization is given and applications to hyperbolic boundary value problems are presented.

Introduction

This paper considers the question of feedback stabilizability for the semilinear control system

$$\dot{u}(t) = Au(t) + v(t)B(u(t)). \qquad (\mathcal{P})$$

Here A is the infinitesimal generator of a linear C^0 semigroup of contractions e^{At} on a real Hilbert space H with inner product $\langle\ ,\ \rangle$, so that A is *dissipative*, i.e.

$$\langle A\psi, \psi \rangle \leqslant 0 \qquad \text{for all } \psi \in D(A).$$

B is a (possibly nonlinear) operator from H into H and $v(t)$ is a real valued control.

An important special case of $(\mathcal{P})$ is when e^{At} is a group of isometries (so that $\langle A\psi, \psi\rangle = 0$ for all $\psi \in D(A)$) and B is a bounded linear operator. This *bilinear control problem* has been considered in the case $H = \mathbb{R}^n$ by Jurdjevic and

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Quinn [10] and Slemrod [14], who showed that the condition

$$\langle e^{At}\psi, B(e^{At}\psi)\rangle = 0 \qquad \text{for all } t \in \mathbb{R}^+ \;\Rightarrow\; \psi = 0 \tag{$\mathcal{C}$}$$

is sufficient for stabilizability of ($\mathcal{P}$). We generalize this result to the infinite-dimensional case with A dissipative under the assumption that B is sequentially continuous from H_w (H endowed with its weak topology) to H. In the bilinear control problem this is equivalent to assuming that B is compact. Note that in the case when e^{At} is a group of isometries the equation ($\mathcal{P}$) with $v(t)\equiv 0$ is undamped.

The paper is divided into four sections. Sections 1 and 2 provide background material on nonlinear semigroups and nonlinear evolution equations respectively. In particular, we derive in Section 2 a simplified version of an invariance principle originally presented in Ball [2]. In Section 3 we apply the invariance principle to the stabilization problem and show that under our assumptions condition ($\mathcal{C}$) implies weak stabilizability of ($\mathcal{P}$). Section 4 provides an application to certain hyperbolic boundary value problems—as an example of the type of result we obtain, consider the system

$$y_{tt} - \Delta y + v(t)y = 0, \quad x \in \Omega, \qquad t \in \mathbb{R}^+,$$

$$y|_{\partial\Omega} = 0, \qquad t \in \mathbb{R}^t,$$

where $y = y(x,t)$ and Ω is a bounded open subset of $\mathbb{R}^n$. We prove that this system is weakly stabilizable in $H_0^1(\Omega)\times L^2(\Omega)$ if and only if Ω is such that all eigenvalues of $-\Delta$ with Dirichlet boundary conditions are simple.

In order to simplify the proofs it will be assumed throughout that H is separable. The necessary techniques for dealing with nonseparable H may be found in [2].

1. Preliminary Results on Nonlinear Semigroups

Definitions. Let H be a real Hilbert space. A (generally nonlinear) *semigroup* $T(t)$ on H is a family of continuous maps $T(t): H \to H, t \in \mathbb{R}^+$, satisfying (i) $T(0) =$ identity, (ii) $T(t+s) = T(t)T(s)$, for all $t,s \in \mathbb{R}^+$.

For $\phi \in H$ define the *positive orbit through* ϕ by $\mathcal{O}^+(\phi) = \cup_{t\in\mathbb{R}^+} T(t)\phi$. The *$\omega$-limit set of* ϕ is the (possibly empty) set given by $\omega(\phi) = \{\psi \in H$: there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $T(t_n)\phi \to \psi$ as $n \to \infty\}$. The *weak ω-limit set of* ϕ is the (possibly empty) set given by $\omega_w(\phi) = \{\psi \in H$: there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $T(t_n)\phi \rightharpoonup \psi$ as $n \to \infty\}$.

A subset C of H is said to be *positively invariant* if $T(t)C \subset C$ for all $t \in \mathbb{R}^+$, and *invariant* if $T(t)C = C$ for all $t \in \mathbb{R}^+$.

Theorem 1.1. (i) *If $\mathcal{O}^+(\phi)$ is precompact then $\omega(\phi)$ is a nonempty, invariant set in H.* (ii) *If each $T(t)$ is sequentially weakly continuous on H (i.e. $T(t)\phi_n \rightharpoonup T(t)\phi$ if $\phi_n \rightharpoonup \phi$), then $\mathcal{O}^+(\phi)$ bounded implies $\omega_w(\phi)$ is a nonempty, invariant set in H.*

Proof.

(i) The proof is a direct consequence of Prop. 2.2 in Dafermos [6].

(ii) Since $\mathcal{O}^+(\phi)$ belongs to a sequentially weakly compact set in H, $\omega_w(\phi)$ is non-empty. Furthermore, since H is separable this weakly compact set may be regarded as a compact set in a metric space with a metric induced by the weak topology (see Dunford and Schwartz [8]). The result again follows from Prop. 2.2 in Dafermos [6].

Remark 1.1. In the study of nonlinear semigroups of "parabolic" type and nonlinear contraction semigroups of "hyperbolic" type sufficient conditions have been given for $\mathcal{O}^+(\phi)$ to be precompact and hence $\omega(\phi)$ to be nonempty (see Henry [9], Pazy [11] and Dafermos and Slemrod [7]). Unfortunately these results do not apply to the problems considered here. For this reason our main conceptual tool in studying asymptotic behaviour of $(\mathcal{P})$ is the weak ω-limit set.

2. Preliminary Results on Nonlinear Evolution Equations

Consider the initial value problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(u(t),t), \\ u(t_0) &= u_0, \end{aligned} \qquad (\mathcal{N})$$

where A is the infinitesimal generator of a C^0 semigroup e^{At} on a real Hilbert space H with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$, $f: H \times \mathbb{R} \to H$ is a given function, and $u_0 \in H$ is a given initial datum.

Definition. *Let $t_1 > t_0$. A function $u \in C([t_0,t_1]; H)$ is a* weak solution *of $(\mathcal{N})$ on $[t_0,t_1]$ if $u(t_0) = u_0$, $f(u(\cdot),\cdot) \in L^1(t_0,t_1;H)$ and if for each $w \in D(A^*)$ the function $\langle u(t), w\rangle$ is absolutely continuous on $[t_0,t_1]$ and satisfies*

$$\frac{d}{dt}\langle u(t), w\rangle = \langle u(t), A^*w\rangle + \langle f(u(t),t), w\rangle$$

for almost all $t \in [t_0,t_1]$.

Theorem 2.1 (cf. Balakrishnan [1], Ball [3]). *Let $t_1 > t_0$. A function $u: [t_0,t_1] \to H$ is a weak solution of $(\mathcal{N})$ on $[t_0,t_1]$ if and only if $f(u(\cdot),\cdot) \in L^1(t_0,t_1;H)$ and u satisfies the variation of constants formula*

$$u(t) = e^{A(t-t_0)}u_0 + \int_{t_0}^{t} e^{A(t-s)} f(u(s),s)\,ds$$

for all $t \in [t_0,t_1]$.

Remark 2.1. Functions u satisfying the variation of constants formula are often called "mild solutions" of $(\mathcal{N})$.

The following elementary existence and uniqueness result for $(\mathcal{N})$ is sufficient for our purposes (see Segal [13] or Pazy [12]).

Theorem 2.2. *Let $f: H \times \mathbb{R} \to H$ be continuous in t and locally Lipschitz in u. Then for each $u_0 \in H$ $(\mathfrak{N})$ has a unique weak solution u defined on a maximal interval of existence $[t_0, t_{max})$, $t_{max} > t_0$, $u \in C([t_0, t_{max}); H)$. Moreover, if $u_n \in C([t_0, t_1]; H)$ are weak solutions of $(\mathfrak{N})$ such that $u_n(0) \to u_0$ as $n \to \infty, t_1 > t_0$, then $u_n \to u$ in $C([t_0, t_1]; H)$ as $n \to \infty$, where u is the unique weak solution of $(\mathfrak{N})$ satisfying $u(0) = u_0$. Furthermore, for any weak solution u with $t_{max} < \infty$ there holds*

$$\lim_{t \nearrow t_{max}} \|u(t)\| = \infty.$$

Theorem 2.2 provides information on continuity with respect to initial conditions in the norm topology of weak solutions of $(\mathfrak{N})$. The following theorem provides similar information in the weak topology of H. For simplicity we consider only the autonomous case $f(t,u) = f(u)$, $t_0 = 0$.

Theorem 2.3. *Let $f: H \to H$ be sequentially weakly continuous ($\psi_n \rightharpoonup \psi$ implies $f(\psi_n) \rightharpoonup f(\psi)$). Let $(\mathfrak{N})$ possess a unique weak solution $u(t; u_0)$ on $[0, T]$ for each $u_0 \in H$. Furthermore suppose $\|u(t; u_0)\| \leqslant$ const. if $t \in [0, T]$ and u_0 is restricted to a bounded subset of H. Then $u_{0n} \rightharpoonup u_0$ implies $u(t; u_{0n}) \rightharpoonup u(t; u_0)$ for every $t \in [0, T]$.*

Proof. Let $u_n(t) = u(t; u_{0n})$ and $u(t) = u(t; u_0)$. Since $\{u_{0n}\}$ is bounded so is $\{u_n(t)\}$ for all n, $t \in [0, T]$. Also, f maps bounded sets to bounded sets, so that $\|f(u_n(t))\| \leqslant$ const. for all n, $t \in [0, T]$. Let $t_n \searrow t$ in $[0, T]$. For $w \in H$ let

$$a_r = \sup_{\substack{\|\phi\| \leqslant 1 \\ 0 \leqslant s \leqslant t}} |\langle [e^{A(t-s)} - e^{A(t_r - s)}]\phi, w\rangle|.$$

We claim that $a_r \to 0$ as $r \to \infty$. If not there exist sequences $\{\phi_\mu\}, \{s_\mu\}$ such that $\phi_\mu \rightharpoonup \phi, s_\mu \to s$ in $[0, t], t_\mu \to t$, and a number $\varepsilon > 0$ with

$$|\langle [e^{A(t-s_\mu)} - e^{A(t_\mu - s_\mu)}]\phi_\mu, w\rangle| \geqslant \varepsilon.$$

But the map $(t, \phi) \mapsto e^{At}\phi$ is jointly sequentially weakly continuous on $\mathbb{R}^+ \times H$ (see Ball [2]) so that

$$e^{A(t-s_\mu)}\phi_\mu \rightharpoonup e^{A(t-s)}\phi,$$
$$e^{A(t_\mu - s_\mu)}\phi_\mu \rightharpoonup e^{A(t-s)}\phi,$$

and hence $a_r \to 0$.

We have by the variation of constants formula that

$$\begin{aligned} |\langle u_n(t_r) - u_n(t), w\rangle| &\leqslant |\langle [e^{At_r} - e^{At}]u_{0n}, w\rangle| \\ &\quad + \int_0^t |\langle [e^{A(t_r - \tau)} - e^{A(t-\tau)}] f(u_n(\tau)), w\rangle| d\tau \\ &\quad + \int_t^{t_r} |\langle e^{A(t_r - \tau)} f(u_n(\tau)), w\rangle| d\tau \\ &\leqslant \text{const.}_1 a_r + \text{const.}_2 |t_r - t|. \end{aligned}$$

Hence $\langle u_n(t_r)-u_n(t),w\rangle \to 0$ uniformly as $r\to\infty$. A similar argument shows that for $t_r \nearrow t$, $\langle u_n(t_r)-u_n(t),w\rangle \to 0$ uniformly as $r\to\infty$. Thus $\{u_n(t)\}$ is equicontinuous in $C([0,T];H_w)$. Furthermore, since $\{u_n(t)\}$ is uniformly bounded in n for all $t\in[0,T]$, we may view $\{u_n(t)\}$ as belonging to a bounded set in H endowed with the metrized weak topology. Hence we can apply the Ascoli-Arzela theorem for metric spaces to conclude that there exists $\tilde{u}\in C([0,T];H_w)$ and a subsequence $\{u_\nu(t)\}$ so that $u_\nu(t)\rightharpoonup \tilde{u}(t)$ uniformly on $[0,T]$ as $\nu\to\infty$. But

$$u_\nu(t) = e^{At}u_{0\nu} + \int_0^t e^{A(t-s)}f(u_\nu(s))ds,$$

and hence, for any $w\in H$,

$$\langle u_\nu(t),w\rangle = \langle e^{At}u_{0\nu},w\rangle + \int_0^t \langle e^{A(t-s)}f(u_\nu(s)),w\rangle ds.$$

We may now take the limit as $\nu\to\infty$ and employ the sequential weak continuity of f and the dominated convergence theorem to conclude that

$$\langle \tilde{u}(t),w\rangle = \langle e^{At}u_0,w\rangle + \int_0^t \langle e^{A(t-s)}f(\tilde{u}(s)),w\rangle ds.$$

Since this equality holds for all $w\in H$,

$$\tilde{u}(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(\tilde{u}(s))ds.$$

By uniqueness of solutions to $(\mathcal{N})$ we must have $u(t)=\tilde{u}(t)$ on $[0,T]$.

To conclude the proof assume that $u_{0n}\rightharpoonup u_0$ and that $\{u_n(t)\}$ does not converge to $u(t)$ in $C([0,T];H_w)$. We may assume that $\{u_n(t)\}$ lies outside a fixed neighbourhood of $u(t)$ in $C([0,T];H_w)$. By the above argument $\{u_n(t)\}$ possesses a subsequence converging to $u(t)$. This contradiction completes the proof.

The next result characterizes the asymptotic behaviour of solutions to $(\mathcal{N})$ in an important special case. We apply it to the stabilization problem in the next section.

Theorem 2.4. *Let A generate a linear C^0 semigroup e^{At} of contractions on H. Let $f: H\to H$ satisfy*

(i) *f is locally Lipschitz,*

(ii) $\psi_n \rightharpoonup \psi \Rightarrow f(\psi_n)\to f(\psi)$,

(iii) $\langle f(\psi),\psi\rangle \leqslant 0$ for all $\psi\in H$.

Then $(\mathcal{N})$ possesses a unique weak solution $u(t;u_0)$ on $\mathbb{R}^+$ for each $u_0\in H$. Furthermore $T(t)u_0=u(t;u_0)$ defines a semigroup on H, $\omega_w(u_0)$ is a nonempty invariant set for each $u_0\in H$, and for each $\psi\in\omega_w(u_0)$

$$\langle T(t)\psi, f(T(t)\psi)\rangle = 0 \qquad \text{for all } t\in\mathbb{R}^+.$$

If, in addition, the only solution to the above equation is $\psi=0$, then $u(t;u_0)\rightharpoonup 0$ as $t\to\infty$.

Proof. From Theorem 2.1 we know $(\mathfrak{N})$ possesses a unique local weak solution $u(t)=u(t;u_0)$ for each $u_0\in H$. Since A is dissipative a simple approximation argument (cf. Ball [2 Lemma 5.5]) shows that

$$\|u(t)\|^2-\|u_0\|^2\leqslant 2\int_0^t\langle f(u(s)),u(s)\rangle ds\leqslant 0, \tag{$\mathcal{E}$}$$

and hence, again using Theorem 2.1, $u(t;u_0)$ exists for all $t\in\mathbb{R}^+$. Also $(\mathcal{E})$ and Theorem 2.3 imply that $u(t;\cdot):H\to H$ is sequentially weakly continuous. Clearly $T(t)u_0=u(t;u_0)$ defines a semigroup, and by Theorem 1.1 the weak ω-limit set $\omega_w(u_0)$ is nonempty and invariant. Let $\psi\in\omega_w(u_0)$. Then there exists a sequence $t_n\to\infty$ such that $T(t_n)u_0\rightharpoonup\psi$ as $n\to\infty$. By $(\mathcal{E})$

$$\lim_{n\to\infty}\int_{t_n}^{t_n+t}\langle f(T(s)u_0),T(s)u_0\rangle ds$$
$$=\lim_{n\to\infty}\int_0^t\langle f(T(s)T(t_n)u_0),T(s)T(t_n)u_0\rangle ds=0$$

for each $t\in\mathbb{R}^+$. By Theorem 2.3 and hypothesis (ii)

$$\lim_{n\to\infty}\langle f(T(s)T(t_n)u_0),T(s)T(t_n)u_0\rangle=\langle f(T(s)\psi),T(s)\psi\rangle$$

for each $s\in[0,t]$. Hence by the dominated convergence theorem

$$\int_0^t\langle f(T(s)\psi),T(s)\psi\rangle ds=0.$$

Since f is continuous this implies that

$$\langle f(T(t)\psi),T(t)\psi\rangle=0\qquad\text{for all } t\in\mathbb{R}^+,$$

as required.

3. The Stabilization Problem

Let $(\mathcal{P})$ be as given in the introduction, and let B be locally Lipschitz.

Definition. *System* $(\mathcal{P})$ *is* stabilizable (weakly stabilizable) *if there exists a continuous feedback control* $v:H\to\mathbb{R}$ *such that* $(\mathcal{P})$ *with* $v(t)=v(u(t))$ *satisfies the properties*

(i) *For each* u_0 *there exists a unique weak solution* $u(t;u_0)$ *defined for all* $t\in\mathbb{R}^+$, *of* $(\mathcal{P})$.

(ii) $\{0\}$ *is a stable equilibrium of* $(\mathcal{P})$.

(iii) $u(t;u_0)\to 0$ $(u(t;u_0)\rightharpoonup 0)$ *as* $t\to\infty$ *for all* $u_0\in H$.

The natural approach to the stabilization problem is to formally differentiate $\|u(t)\|^2$ along trajectories of $(\mathcal{P})$, obtaining thus

$$\frac{d}{dt}\|u(t)\|^2=2\langle Au(t),u(t)\rangle+2v(t)\langle u(t),B(u(t))\rangle.$$

An obvious choice of feedback control (though not the only one) is

$$v(u) = -\langle u, B(u)\rangle,$$

since this control yields the "dissipating energy inequality"

$$\frac{d}{dt}\|u(t)\|^2 \leqslant -2\langle u(t), B(u(t))\rangle^2.$$

For this choice of $v(u)$ our feedback control system becomes

$$\dot{u}(t) = Au(t) - \langle u(t), B(u(t))\rangle B(u(t)). \qquad (\mathcal{F})$$

While we would like to be able to treat the general case of continuous $B: H \to H$, our results unfortunately apply only to the case when $B: H_W \to H$ is sequentially continuous.

Theorem 3.1. *If $B: H_w \to H$ is sequentially continuous and*

$$\langle e^{At}\psi, B(e^{At}\psi)\rangle = 0 \text{ for all } t \in \mathbb{R}^+ \quad \Rightarrow \quad \psi = 0, \qquad (\mathcal{C})$$

then $(\mathcal{F})$ is weakly stabilizable.

Proof. Set $f(u) = -\langle u, B(u)\rangle B(u)$. (i) Since B maps bounded sets to bounded sets, it is easily verified that f is locally Lipschitz. (ii) Since $B: H_W \to H$ is sequentially continuous, $\psi_n \rightharpoonup \psi$ implies $f(\psi_n) \to f(\psi)$. (iii) Clearly $\langle f(\psi), \psi\rangle \leqslant 0$ for all $\psi \in H$. Thus f satisfies the hypotheses of Theorem 2.4. Let $u_0 \in H, \psi \in \omega_w(u_0)$. By Theorem 2.4

$$\langle T(t)\psi, f(T(t)\psi)\rangle = 0 \qquad \text{for all } t \in \mathbb{R}^+.$$

Hence $\langle T(t)\psi, B(T(t)\psi)\rangle = 0$ for all $t \in \mathbb{R}^+$, so that $f(T(t)\psi) = 0$ for all $t \in \mathbb{R}^+$. By the variation of constants formula $T(t)\psi = e^{At}\psi$. Hence $(\mathcal{C})$ implies that $\psi = 0$.

4. Applications to Hyperbolic Problems

Let V be a real Hilbert space with inner product $\langle\ ,\ \rangle_V$. Let P be a densely defined positive self-adjoint linear operator on V such that P^{-1} is everywhere defined and compact. Let $V_P = D(P^{\frac{1}{2}})$. V_P forms a Hilbert space under the inner product

$$\langle w_1, w_2\rangle_P = \langle P^{\frac{1}{2}} w_1, P^{\frac{1}{2}} w_2\rangle_V.$$

Consider the abstract wave equation

$$\ddot{y} + Py + v(t)y = 0, \qquad (\mathcal{Q})$$

where $v(t)$ is a real valued control.

To write $(\mathcal{Q})$ in the form $(\mathcal{P})$ we set

$$u = (y,z), H = V_P \times V, \langle (y_1,z_1),(y_2,z_2)\rangle_H = \langle y_1,y_2\rangle_P + \langle z_1,z_2\rangle_V,$$
$$A = \begin{pmatrix} 0 & I \\ -P & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}, D(A) = D(P)\times V_P.$$

A is skew-adjoint, and the compactness of the injection $V_P \to V$ implies that $B: H \to H$ is compact.

Theorem 4.1. *System $(\mathcal{Q})$ is weakly stabilizable if and only if all eigenvalues λ_m of P are simple.*

Proof. Suppose that the eigenvalues $\{\lambda_m\}$ are simple, and let $\{\phi_m\}$ denote the corresponding eigenfunctions normalized so that $\|\phi_m\|_V = 1$ for all $m = 1,2,\cdots$. We apply Theorem 3.1; the feedback control is given by

$$\begin{aligned} v(t) &= -\langle u(t), Bu(t)\rangle_H \\ &= \langle y(t), \dot{y}(t)\rangle_V. \end{aligned}$$

To check whether $(\mathcal{C})$ is satisfied we expand $\psi \in H$ in terms of the complete set of eigenfunctions of A, i.e.

$$\psi = \sum_{m=1}^{\infty} \begin{pmatrix} c_m \\ \sqrt{\lambda_m}\, d_m \end{pmatrix} \phi_m.$$

Separation of variables yields

$$e^{At}\psi = \sum_{m=1}^{\infty} \begin{bmatrix} c_m \cos\sqrt{\lambda_m}\, t + d_m \sin\sqrt{\lambda_m}\, t \\ -\sqrt{\lambda_m}\, c_m \sin\sqrt{\lambda_m}\, t + \sqrt{\lambda_m}\, d_m \cos\sqrt{\lambda_m}\, t \end{bmatrix} \phi_m,$$

and we easily see that

$$\langle e^{At}\psi, Be^{At}\psi\rangle_H = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\left[\tfrac{1}{2}(c_m^2 - d_m^2)\sin 2\sqrt{\lambda_m}\, t - c_m d_m \cos 2\sqrt{\lambda_m}\, t\right].$$

From the uniqueness of the Fourier series expansion for almost periodic functions (cf. Besicovitch [4]) we deduce that $\langle e^{At}\psi, Be^{At}\psi\rangle_H = 0$ for all $t \in \mathbb{R}^+$ implies $c_m^2 - d_m^2 = 0, c_m d_m = 0, m = 1,2,\cdots$, i.e. $c_m = d_m = 0$ for $m = 1,2,\cdots$ and $\psi = 0$. Hence $(\mathcal{C})$ holds, so that by Theorem 3.1 system $(\mathcal{Q})$ is weakly stabilizable.

Conversely, let λ be an eigenvalue of P with two linearly independent eigenfunctions ϕ and ϕ^*. Let $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\phi + \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}\phi^*$ with $\alpha^*\beta \neq \alpha\beta^*$. Then the solution y of $(\mathcal{Q})$ satisfying $(y(0), \dot{y}(0)) = \psi$ is given by

$$y(t) = w(t)\phi + w^*(t)\phi^*,$$

where

$$\ddot{w} + \lambda w + v(t)w = 0,$$
$$\ddot{w}^* + \lambda w^* + v(t)w^* = 0,$$

and $w(0)=\alpha, \dot{w}(0)=\beta, w^*(0)=\alpha^*, \dot{w}^*(0)=\beta^*$. Eliminating $v(t)$ we obtain

$$\dot{w}(t)w^*(t) - w(t)\dot{w}^*(t) = \alpha^*\beta - \alpha\beta^* \neq 0.$$

Hence $(y(t),\dot{y}(t)) \not\to (0,0)$ as $t\to\infty$ for any control $v(t)$.

Example 1 (Wave equation). Let Ω be a bounded open subset of $\mathbb{R}^n$, and consider the system

$$\begin{aligned} &y_{tt} - \Delta y + v(t)y = 0, \qquad x\in\Omega, t\in\mathbb{R}^+,\\ &y|_{\partial\Omega} = 0, \end{aligned} \tag{1}$$

where $y=y(x,t)$. This system has the form $(\mathscr{Q})$ if we set

$$V = L^2(\Omega), D(P) = \{w\in V: -\Delta w\in V\}, P = -\Delta,$$

so that $V_P = H_0^1(\Omega)$. Hence (1) is weakly stabilizable in $H_0^1(\Omega)\times L^2(\Omega)$ if and only if the eigenvalues λ_m of $-\Delta$ with Dirichlet boundary conditions are simple. This is a condition on Ω, which holds, for example, if $n=1$ and Ω is an open interval. If $n=2$ and Ω is a disc then the condition does not hold, while if Ω is a rectangle with sides a,b then the condition holds if and only if a/b is irrational (cf Courant-Hilbert [5]).

Example 2 (Beam equation). Consider the equation

$$y_{tt} + y_{xxxx} + v(t)y = 0, \qquad 0 < x < 1, t\in\mathbb{R}^+,$$

with boundary conditions either

$$y = y_x = 0 \text{ at } x = 0,1 \text{ (clamped ends)} \tag{2}$$

or

$$y = y_{xx} = 0 \text{ at } x = 0,1 \text{ (simple supported ends)}$$

This system has the form $(\mathscr{Q})$ if we set

$$V = L^2(0,1), P = \frac{d^4}{dx^4},$$

$$D(P) = \{y\in V: y_{xxxx}\in V, y \text{ satisfies boundary conditions}\}.$$

In the clamped (simply supported) case $V_P = H_0^2(0,1)$ ($V_P = H^2(0,1)\cap H_0^1(0,1)$). It is well known (cf. Courant-Hilbert [5]) that in both cases the eigenvalues of P are simple. Hence (2) is weakly stabilizable in $H = V_P\times L^2(0,1)$.

Remark 4.1. We have been unable to determine whether in Examples 1 (with λ_m simple) and 2 above, and with our choice of feedback control

$$v(t) = \langle y, y_t \rangle_{L^2}, \tag{3}$$

all solutions (y, y_t) converge *strongly* to $(0,0)$ in $V_P \times V$ as $t \to \infty$.

Remark 4.2. It is important to notice that the equality in condition $(\mathcal{C})$ must hold for *all* $t \in \mathbb{R}^+$, and not merely for all sufficiently small t; this fact was used crucially in the proof of Theorem 4.1. Indeed for any $0 < \tau < 1$ there are nonzero solutions y of the one-dimensional wave equation

$$\begin{aligned} y_{tt} &= y_{xx}, \qquad 0 < x < 1, \\ y &= 0 \text{ at } x = 0, \quad 1, \end{aligned}$$

satisfying

$$\langle y, y_t \rangle_{L^2(0,1)} = \frac{d}{dt} \int_0^1 y^2 dx = 0 \qquad \text{for all } t \in [0, \tau].$$

(Consider a unidirectional pulse with small compact support.) If $v(t)$ is given by (3) then such solutions satisfy (1) on $[0, \tau]$; this illustrates the subtle nature of the damping induced by the feedback control.

Remark 4.3. Consider the problem

$$\begin{aligned} y_{tt} - y_{xx} + v(t) f(y) &= 0, \qquad 0 < x < 1, \\ y &= 0 \quad \text{at } x = 0, 1, \end{aligned}$$

where $f \in C^1(\mathbb{R})$ is nonlinear. Let $F(y) \overset{\text{def}}{=} \int^y f(s)ds$. It is easily seen using Theorem 2.4 that with the choice of feedback control

$$v(t) = \int_0^1 f(y) y_t dx,$$

all solutions (y, y_t) of (4) converge weakly in $H_0^1(0,1) \times L^2(0,1)$ to the set S consisting of the initial data of all solutions w of

$$\begin{aligned} w_{tt} &= w_{xx}, \\ w &= 0 \quad \text{at } x = 0, 1, \end{aligned}$$

satisfying

$$\int_0^1 F(w(x,t)) dx = \text{constant}, \qquad \text{for all } t \in \mathbb{R}^+.$$

However it seems to be a difficult problem to find conditions on f guaranteeing that $S = (0,0)$, so that (4) is weakly stabilizable.

Remark 4.4. An interesting example arising in mechanics is that of stabilizing a vibrating beam by choosing the axial load as a feedback control. A simple model of this situation consists of the equation

$$y_{tt} + y_{xxxx} + v(t)y_{xx} = 0, \qquad 0 < x < 1,$$

where $v(t)$ is the axial load and $y(x,t)$ the transverse displacement, with either clamped or simply supported boundary conditions. Let $H, \langle \cdot, \cdot \rangle_H$ and A be defined as for Example 2. The relevant point is that the operator $B: H \to H$ given by

$$B = \begin{bmatrix} 0 & 0 \\ \dfrac{-d^2}{dx^2} & 0 \end{bmatrix}$$

is bounded, but not compact. Hence our theory does not apply.

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