Nonharmonic Fourier Series and the Stabilization of Distributed Semi-Linear Control Systems*

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1. Introduction

In this paper we continue our discussion, begun in [5], of feedback stabilization of distributed semilinear control systems. We restrict attention to systems of the "hyperbolic" form

$$\ddot{u}(t)+Au(t)+p(t)\,B(u(t))=0\,. \tag{1.1}$$

Here A is a densely defined positive selfadjoint linear operator on a real Hilbert space H with inner product $(\ ,\ )$, B is a locally Lipschitz map from $D(A^{1/2})$ (endowed with the graph norm) into H, and $p(t)$ is a real-valued control.

The finite-dimensional stabilization problem $H=\mathbb{R}^n$ has been considered in the recent papers of Jurdjevic and Quinn [20] and Slemrod [24]. In the case when (1.1) is bilinear, i.e., B linear, these papers give simple criteria for feedback stabilization. Specifically, it is a consequence of both [20] and [24] that if the only solution of the uncontrolled system

$$\ddot{y}(t)+Ay(t)=0\,,$$

which satisfies

$$(B(y(t)),\,\dot{y}(t))=0\quad\text{for all}\quad t\in\mathbb{R}^+\,,$$

is $y(t)\equiv 0$, then $p(t)=(B(u(t)),\,\dot{u}(t))$ is a stabilizing feedback control for (1.1), i.e., all solutions $u(t)$ of (1.1) tend to zero as $t\to\infty$.

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In [5] we attempted to generalize the above result to the case when H is infinite-dimensional. The main tool of our analysis was a recent extension by Ball [4] of the well-known LaSalle invariance principle for ordinary differential equations. In order to apply the theory in [4] and hence generalize the above sufficiency condition we were forced to make the additional continuity assumption

$$\text{(C)} \qquad \psi_n \rightharpoonup \psi \quad \text{in} \quad D(A^{1/2}) \Rightarrow B(\psi_n) \to B(\psi) \quad \text{in} \quad H.$$

As we noted in [5], condition (C) is too restrictive to apply to certain examples in structural mechanics. Consider for example the problem of stabilizing a vibrating beam by choosing the axial force as a feedback control. A simple model of this situation consists of the equation

$$u_{tt} + u_{xxxx} + p(t)u_{xx} = 0\,, \qquad 0 < x < 1\,, \tag{1.2}$$

where u denotes the transverse displacement of the beam and $p(t)$ is the axial force. For simplicity, assume the beam has clamped ends, $u = u_x = 0$ at $x = 0, 1$. In this case, $H = L^2(0, 1)$, $D(A^{1/2}) = H_0^2(0, 1)$ (see e.g. [1] for a discussion of Sobolev spaces) and $B = d^2/dx^2$. The important point is that while B is a bounded linear operator from $D(A^{1/2})$ into H (and hence is certainly locally Lipschitz) it is not compact and therefore does not satisfy condition (C). Of course, if we consider instead an equation of the form

$$u_{tt} + u_{xxxx} + p(t)u = 0\,, \qquad 0 < x < 1\,, \tag{1.3}$$

with similar boundary conditions, we see that $B = I$ (the identity on H) does satisfy (C). It is for this reason that the applications given in [5] were to problems of the type (1.3).

The aim of this paper is to present a theory of feedback stabilization for (1.1) which does not require condition (C) and hence will be applicable to problems of the form (1.2). We shall always make the choice $p(t) = (B(u(t)), \dot{u}(t))$ for the feedback control, so that the problem of feedback stabilization is reduced to showing that all solutions of the autonomous equation

$$\ddot{u}(t) + Au(t) + (B(u(t)), \dot{u}(t))B(u(t)) = 0 \tag{1.4}$$

tend to zero as $t \to \infty$. The damping in (1.4) induced by the feedback control can be very weak; in fact $(B(u(t)), \dot{u}(t))$ may vanish identically on certain finite time intervals. Consequently, many of the usual techniques for studying the asymptotic behavior of nonlinear evolution equations are inapplicable (for details see Remark 3.1). Our analysis is based, as in [5], on ideas of elementary topological dynamics and in particular on the notion of a weak ω-limit set. In addition we make use of the special structure of (1.4), which allows us to apply some delicate theorems on almost periodic functions and nonharmonic Fourier series. One of these results (Theorem 2.2), which appears to be new, gives sufficient conditions for the Fourier coefficients of a sequence of almost periodic functions, each having the same Fourier exponents, to converge, given that the functions converge on compact intervals of $\mathbb{R}$. Similar theorems have been used by Russell [23] for other purposes in linear control theory.

The paper is divided into five sections. Section 2 presents the above-mentioned results on nonharmonic Fourier series. Section 3 discusses the stabilization problem and gives the main result (Theorem 3.2) on feedback stabilizability of (1.1). In Section 4 we discuss a number of applications to hyperbolic initial boundary value problems. In each application it is necessary (and not always trivial) to satisfy the genericity conditions on the eigenvalues of A that form part of the hypotheses of Theorem 3.2. In the first three examples we prove weak feedback stabilizability of (1.2) under progressively more difficult boundary conditions. Next we discuss the Timoshenko beam equations, a coupled set of wave equations for the motion of an inextensible elastic beam that are in some ways preferable to (1.2) in that they may be obtained by linearization of a fully nonlinear theory.

The last two applications are to a linear and a semilinear wave equation, respectively. In our analysis of the semilinear equation we give a partial answer to the following question:

For what functions F is $u \equiv 0$ the only solution of

$$u_{tt} - u_{xx} = 0, \qquad 0 < x < 1,$$

$$u = 0 \quad \text{at} \quad x = 0, \qquad u + \alpha u_x = 0 \quad \text{at} \quad x = 1,$$

which satisfies

$$\int_0^1 F(u(x, t))\, dx = \text{constant?}$$

Our result (Theorem 4.7) is that for all but countably many α we may take for F any nonconstant polynomial. The question for $\alpha = 0$ was raised in [5] and remains unsolved. Finally, in Section 5 we collect some miscellaneous remarks connected with variants and possible extensions of our results; in particular, we exhibit some curious inequalities for solutions of linear evolution equations.

2. Nonharmonic Fourier Series

To any (uniformly) almost periodic function $f(t)$ one may associate a unique Fourier series

$$f(t) \sim \sum_{k=-\infty}^{\infty} a_k \exp\{i\mu_k t\},$$

where the a_k are complex constants given by

$$a_k = M(f(t)\exp\{-i\mu_k t\}) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\exp\{-i\mu_k t\}\,dt,$$

and where the exponents μ_k are real and distinct (cf Bohr [7], Besicovitch [6]). We shall be concerned with variants of the following question: Let

$$f_r(t) \sim \sum_{k=-\infty}^{\infty} a_k^{(r)} \exp\{i\mu_k t\}$$

be a sequence of almost periodic functions having the *same* exponents μ_k, and suppose that $f_r \to 0$ in $L^2(0, T)$ for some given $T>0$. Does $a_k^{(r)} \to 0$ as $r\to\infty$ for each k?

The answer to this question depends on the set of exponents μ_k. Note that a positive answer would imply in particular that an almost periodic function f with exponents μ_k is determined everywhere by its values on $(0, T)$. In some cases we will have to make the stronger hypothesis that the functions f_r are uniformly bounded and tend to zero in $L^1(0, T)$ for every $T>0$, and will be able to conclude only that $a_k^{(r)} \to 0$ for *some* integers k. In our applications the fixed set of exponents μ_k can be quite complicated; for example, it may possess an unbounded set of cluster points. However, we shall begin by considering a simpler situation, in which the μ_k possess an "asymptotic gap" of length γ. In this case the answer to our question is positive provided that $T>2\pi/\gamma$. The proof of the theorem is essentially the same as that given by Gaposhkin [13] for a similar result.

THEOREM 2.1. *Let $\mu_{-k} = \mu_k$ and suppose that*

$$\varliminf_{k\to\infty} (\mu_{k+1} - \mu_k) \geqq \gamma > 0. \tag{2.1}$$

For any $T>2\pi/\gamma$ there exist constants $C_i = C_i(T) > 0$, $i = 1, 2$, such that

$$C_2 \sum_{k=-\infty}^{\infty} |a_k|^2 \geqq \frac{1}{T}\int_0^T |f(t)|^2\,dt \geqq C_1 \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{2.2}$$

for every almost periodic function

$$f(t) \sim \sum_{k=-\infty}^{\infty} a_k \exp\{i\mu_k t\}.$$

Proof: Let $T > 2\pi/\gamma$. Without loss of generality we may suppose that $\mu_k < \mu_{k+1}$ for $k \geqq 0$. The result is known to be true (Ingham [19]) if (2.1) is replaced by the stronger condition that

$$\mu_{k+1} - \mu_k \geqq \gamma > 0 , \qquad k = 1, 2, \cdots .$$

Making a small adjustment to γ, it follows that there exists a positive integer M such that

$$K_2 \sum_{|k|>M} |a_k|^2 \geqq \frac{1}{T} \int_0^T |f(t)|^2 \, dt \geqq K_1 \sum_{|k|>M} |a_k|^2 \tag{2.3}$$

for every almost periodic function

$$f(t) \sim \sum_{|k|>M} a_k \exp\{i\mu_k t\} ,$$

where K_1, K_2 are positive constants.

If

$$f(t) \sim \sum_{k=-\infty}^{\infty} a_k \exp\{i\mu_k t\} ,$$

then

$$f(t) = \sum_{|k| \leq M} a_k \exp\{i\mu_k t\} + h(t) ,$$

where

$$h(t) \sim \sum_{|k|>M} a_k \exp\{i\mu_k t\} .$$

Applying (2.3) to $h(t)$ we easily obtain the left-hand inequality of (2.2). We prove the right-hand inequality by contradiction. If it is false, there exists a sequence

$$f_r(t) \sim \sum_{k=-\infty}^{\infty} a_k^{(r)} \exp\{i\mu_k t\} , \qquad r = 1, 2, \cdots ,$$

of almost periodic functions such that $\sum_{k=-\infty}^{\infty} |a_k^{(r)}|^2 = 1$ and $f_r \to 0$ in $L^2(0, T)$. By extracting a subsequence if necessary, we may suppose that $\{a_k^{(r)}\} \rightharpoonup \{a_k\}$ in l^2 as $r \to \infty$. Writing

$$f_r(t) = \sum_{|k| \leq M} a_k^{(r)} \exp\{i\mu_k t\} + g_r(t) ,$$

it follows that

$$\sum_{|k|\leq M} a_k^{(r)} \exp\{i\mu_k t\} \to \sum_{|k|\leq M} a_k \exp\{i\mu_k t\} \quad \text{in} \quad L^2(0, T),$$

and hence that $g_r \to g$ in $L^2(0, T)$, where

$$g(t) \stackrel{\text{def}}{=} - \sum_{|k|\leq M} a_k \exp\{i\mu_k t\}.$$

Applying (2.3) we see that $\{a_k^{(r)}\}$ is a Cauchy sequence in l^2, and therefore $\{a_k^{(r)}\} \to \{a_k\}$ strongly in l^2. A standard calculation using (2.3) now shows that

$$g(t) = \sum_{|k|\leq M} a_k \exp\{i\mu_k t\} \quad \text{in} \quad L^2(0, T). \tag{2.4}$$

Since g is analytic, we may apply a technique of Zygmund ([26], Vol I, page 206) to conclude that $a_k = 0$ for all $|k| > M$, and hence $a_k = 0$ for all k. This contradicts $\sum_{k=-\infty}^{\infty} |a_k|^2 = 1$.

Note that if $\mu_{k+1} - \mu_k \to \infty$ as $k \to \infty$, then we may take T arbitrarily small in the above result, so that, in particular, f is determined everywhere by the values it takes on any arbitrarily small interval; this type of unique continuation property has been studied in the case of lacunary Fourier series by Zygmund [26] and others. In the case $\mu_{-k} = -\mu_k$, $\mu_{k+1} - \mu_k \to \gamma > 0$ as $k \to \infty$, sufficient conditions have been given for $\{\exp\{i\mu_k t\}\}$ to form a basis of $L^2(0, 2\pi/\gamma)$; for a survey of these results see Higgins [18]. If, finally, $\mu_{k+1} - \mu_k \to 0$ as $k \to \infty$, then there is no $T > 0$ such that either of the inequalities in (2.1) hold. To see this, consider the functions

$$f_N^K(t) \stackrel{\text{def}}{=} \sum_{k=N}^{N+K} \exp\{i\mu_k t\}, \qquad f_N(t) \stackrel{\text{def}}{=} \exp\{i\mu_{N+1} t\} - \exp\{i\mu_N t\},$$

which satisfy

$$\lim_{K\to\infty} \lim_{N\to\infty} \frac{1}{TK} \int_0^T |f_N^K(t)|^2 \, dt = \infty, \qquad \lim_{N\to\infty} \frac{1}{2T} \int_0^T |f_N(t)|^2 \, dt = 0,$$

respectively.

We now turn to the case when the fixed countable set of exponents μ_k is arbitrary.

THEOREM 2.2. *Let*

$$f_r(t) \sim \sum_{k=-\infty}^{\infty} a_k^{(r)} \exp\{i\mu_k t\}$$

be a uniformly bounded sequence of almost periodic functions, let f be a locally integrable function, and suppose that $f_r \to f$ *in* $L^1(0, T)$ *for every* $T > 0$.* *Let* μ_j *be an isolated exponent; i.e.,* $\inf_{k \neq j} |\mu_j - \mu_k| > 0$. *Then*

$$a_j \stackrel{\text{def}}{=} M(f(t)\exp\{-i\mu_j t\})$$

exists and $a_j^{(r)} \to a_j$ *as* $r \to \infty$.

Proof: We have that

$$f_r(t)\exp\{-\mu_j t\} = a_j^{(r)} + g_r(t),$$

where

$$g_r(t) \sim \sum_{k \neq j} a_k^{(r)} \exp\{i(\mu_k - \mu_j)t\}.$$

Note that since the sequence f_r is bounded, so is $|a_j^{(r)}|$, by Parseval's equality. Hence g_r is a uniformly bounded sequence of almost periodic functions. Since $\inf_{k \neq j} |\mu_j - \mu_k| > 0$, a result of Levitan (cf. Fink [12], Theorem 4.12) implies that

$$\left| \int_0^T g_r(t)\, dt \right| \leqq C \sup_{t \in \mathbf{R}} |g_r(t)|,$$

where C is a constant which does not depend on r or T. Therefore,

$$\left| \frac{1}{T} \int_0^T f_r(t)\exp\{-i\mu_j t\}\, dt - a_j^{(r)} \right| \leqq \frac{C}{T}.$$

Let $a_j^{(r_k)}$ be any convergent subsequence of $a_j^{(r)}$ with limit a. Letting $k \to \infty$ we obtain

$$\left| \frac{1}{T} \int_0^T f(t)\exp\{-i\mu_j t\}\, dt - a \right| \leqq \frac{C}{T}.$$

The result follows by letting $T \to \infty$.

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* Actually, weak convergence in $L^1(0, T)$ suffices.

If μ_j is not isolated, the conclusion of Theorem 2.2 is trivially false, since if $\mu_{k_r} \to \mu_j$, then $\exp\{i\mu_{k_r}t\} \to \exp\{i\mu_j t\}$ uniformly on compact intervals. Nor may we omit the hypothesis that f_r be uniformly bounded; this follows by consideration of the case when the set of exponents μ_k is bounded. For definiteness suppose that $\mu_k \to 0$ as $|k| \to \infty$, $\mu_k \neq 0$. We claim that for any $T>0$ the sequence $\{\exp\{-i\mu_k t\}\}_{k\neq 0}$ is complete in $C([0, T])$.

Proof of the claim: Let S be the closure of $\{\exp\{i\mu_k t\}\}_{k\neq 0}$ in $C([0, T])$. Then

$$1 = \lim_{k\to\infty} \exp\{i\mu_k t\} \in S, \qquad t = \lim_{k\to\infty} \frac{\exp\{i\mu_k t\} - 1}{i\mu_k} \in S,$$

$$t^2 = \lim_{k\to\infty} \frac{\exp\{i\mu_k t\} - 1 - i\mu_k t}{\frac{1}{2}(i\mu_k)^2} \in S,$$

etc. Hence $S = C([0, T])$ by the Weierstrass approximation theorem.

It follows from the completeness that there exists a sequence

$$f_r(t) = \sum_{k\neq 0} a_k^{(r)} \exp\{i\mu_k t\}$$

of finite linear combinations of $\{\exp\{i\mu_k t\}\}_{k\neq 0}$ such that $f_r(t) \to \exp\{i\mu_0 t\}$ uniformly on $[0, T]$. By extracting a diagonal subsequence we may suppose that $f_r(t) \to \exp\{i\mu_0 t\}$ uniformly on $[0, T]$ for every $T>0$. But $a_0^{(r)} = 0$, $a_0 = 1$, so that the conclusion of Theorem 2.2 fails for the isolated exponent μ_0. We remark that in the case when $\mu_k \to 0$ as $|k| \to \infty$, even though there is no inequality of the form (2.1), and even though every infinite subsequence of the $\exp\{i\mu_k t\}$ spans $C([0, T])$, any almost periodic function $f(t) \sim \sum_{k=-\infty}^{\infty} a_k \exp\{i\mu_k t\}$ is uniquely determined by its values on any arbitrarily small interval. This is true because any such f may be extended into the complex plane as an entire function (cf Fink [12], Theorem 4.8).

3. Semilinear and Bilinear Stabilization Problems

Let H be a real Hilbert space with inner product $(\ ,\ )$ and norm $|\cdot|$. We consider the question of feedback stabilization for the second-order equation

$$\ddot{u} + Au + p(t)\, B(u) = 0. \tag{3.1}$$

Here A is a densely defined positive selfadjoint linear operator on H such that A^{-1} is everywhere defined and compact. We suppose that the eigenvalues λ_n^2, $n = 1, 2, \cdots$, of A, $0 < \lambda_1 < \lambda_2 < \cdots$, are simple, and we write

$\Lambda = \{\lambda_n\}_{n=1}^{\infty}$. We denote the normalized eigenfunction of A corresponding to λ_n^2 by ϕ_n. Let $H_A = D(A^{1/2})$. H_A forms a Hilbert space under the inner product

$$\langle v, w\rangle_A = (A^{1/2}v, A^{1/2}w).$$

We denote by $\|\cdot\|_A$ the norm in H_A and assume that $B: H_A \to H$ is locally Lipschitz.

The real-valued function $p(t)$ is a control, and we consider the problem of choosing it in such a way that all solutions $u(t)$ of (3.1) converge to zero, in an appropriate sense, as $t \to \infty$. A natural choice for $p(t)$ is

$$p(t) = (B(u(t)), \dot{u}(t)),$$

since then, formally,

$$E(t) - E(0) = -\int_0^t (B(u), \dot{u})^2\, dt, \tag{3.2}$$

where $E(t) \overset{\text{def}}{=} \frac{1}{2}(|\dot{u}(t)|^2 + \|u(t)\|_A^2)$, so that the linearized energy function does not increase. This suggests that solutions will decay under appropriate conditions on B. Therefore we consider the problem of proving that solutions of the autonomous equation

$$\ddot{u} + Au + (B(u), \dot{u})B(u) = 0 \tag{3.3}$$

decay to zero.

Let $X = H_A \times H$. X forms a Hilbert space under the inner product

$$\langle\{y_1, z_1\}, \{y_2, z_2\}\rangle_X = \langle y_1, y_2\rangle_A + (z_1, z_2).$$

We write (3.3) in the form

$$\dot{w} = \mathscr{A}w + f(w), \tag{3.4}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathscr{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \text{and} \quad f(w) = \begin{pmatrix} 0 \\ -(B(u), v)B(u) \end{pmatrix}.$$

Our assumptions imply that $\mathscr{A}$ generates a C^0 group $e^{\mathscr{A}t}$ of linear isometries on X, and that $f: X \to X$ is locally Lipschitz. Let $\phi = \binom{u_0}{u_1} \in X$. It can be shown (cf, Balakrishnan [2], Ball [3]) that, with an appropriate definition of weak solution, a function $w \in C([0, T]; X)$, $T>0$, is a weak solution of (3.4) with initial data $w(0)=\phi$ if and only if w satisfies the variation of constants formula

$$w(t) = e^{\mathscr{A}t}\phi + \int_0^t e^{\mathscr{A}(t-s)} f(w(s))\, ds \tag{3.5}$$

on $[0, T]$. A standard argument shows that for given ϕ there exists a unique solution w of (3.5) defined on some maximal interval $[0, t_{\max})$, $t_{\max}>0$. The corresponding $u(t)$ is a weak solution of (3.3) satisfying the initial conditions $u(0)=u_0$, $\dot{u}(0)=u_1$. It can be shown that u satisfies the energy equation (3.2). In particular, $\|w(t)\|_X$ is bounded on $[0, t_{\max})$, so that, by a standard continuation argument, $t_{\max}=\infty$. Writing $w(t)=T(t)\phi$, it follows that the operators $\{T(t)\}$, $t \geqq 0$, form a semigroup, i.e., $T(0)=\text{identity}$, $T(t+s)=T(t)T(s)$ for all $s, t \geqq 0$. The map $\{t, \phi\} \mapsto T(t)\phi$ from $[0, \infty] \times X$ to X is continuous. For more details of these assertions see Ball and Slemrod [5].

We define the weak ω-limit set of ϕ to be

$$\omega_W(\phi) = \{\psi \in X: \text{ there exists a sequence } t_n \to \infty \text{ as } n \to \infty \text{ such that } T(t_n)\phi \rightharpoonup \psi \text{ in } X\}.$$

The following result is an immediate consequence of Ball and Slemrod [5], Theorem 3.1.

THEOREM 3.1. *Let B satisfy the continuity condition*

$$\psi_n \rightharpoonup \psi \text{ in } H_A \text{ implies } B(\psi_n) \to B(\psi) \text{ in } H. \tag{C}$$

Suppose also that if $\binom{y}{\dot{y}} = e^{\mathscr{A}t}\psi$ *with*

$$(B(y(t)), \dot{y}(t)) = 0 \quad \text{for all} \quad t \geqq 0,$$

then $\psi = 0$.

Then every weak solution $w = \binom{u}{\dot{u}}$ *of (3.3) converges weakly to zero in X as* $t \to \infty$.

The proof of Theorem 3.1 makes repeated use of condition (C) to show that (a) f is sequentially weakly continuous, (b) $T(t): X \to X$ is sequentially weakly continuous for each $t \geqq 0$, and (c) if $\psi \in \omega_w(\phi)$, then $(B(y(t)), \dot{y}(t)) = 0$ for all $t \geqq 0$, where $\binom{y}{\dot{y}} = e^{\mathscr{A}t}\psi$.

From now on we assume that B is a bounded linear operator. In this case condition (C) is equivalent to compactness of B, and as discussed in the introduction this is a restrictive condition for applications.

First note that a necessary condition for all solutions of (3.3) to converge weakly to zero is that

(H1) $$(B\phi_k, \phi_k) \neq 0 \quad \text{for} \quad k = 1, 2, \cdots,$$

since if $(B\phi_k, \phi_k) = 0$ for some k, then $u(t) = \sin \lambda_k t\, \phi_k$ is a non-zero periodic solution of (3.3), and no such solution can converge weakly to zero.

The following genericity condition will prove important:

(H2) $\lambda_m \pm \lambda_n \neq 2\lambda_k$ unless $m = n = k$ and the + sign is taken, or both $(B\phi_m, \phi_n)$ and $(B\phi_n, \phi_m)$ are zero.

Note that (H2) is trivially satisfied if $(B\phi_n, \phi_m) = 0$ for $m \neq n$. A strengthened version of (H2) is

(H2+) For each k,
$$\inf |2\lambda_k - \lambda_m \pm \lambda_n| > 0,$$

where the infimum is taken over $m \neq n$ with $(B\phi_m, \phi_n)$ and $(B\phi_n, \phi_m)$ not both zero.

THEOREM 3.2. *Suppose that either*

(i) $B = K + S$, *where* $K: H_A \to H$ *and* $S: H_A \to H$ *are bounded linear operators with* K *compact and* S *symmetric (i.e.,* $(Su, v) = (u, Sv)$ *for all* $u, v \in H_A$*), and* (H1), (H2) *hold,*

or

(ii) (H1) *and* (H2+) *hold.*

Then every solution $w = \binom{u}{\dot{u}}$ *of* (3.3) *converges weakly to zero in* X *as* $t \to \infty$.

LEMMA 3.1. *Let* $\phi \in X$, $t_n \to \infty$, *and*

$$e^{\mathcal{A}t} T(t_n)\phi = \begin{pmatrix} y_n(t) \\ \dot{y}_n(t) \end{pmatrix}.$$

Then, for every $T > 0$,

(3.6) $$\lim_{n\to\infty} \sup_{t\in[0,T]} \|T(t+t_n)\phi - e^{\mathcal{A}t}T(t_n)\phi\|_X = 0,$$

and

(3.7) $$\lim_{n\to\infty} \int_0^T (By_n(t), \dot{y}_n(t))^2\, dt = 0.$$

Proof: We know that

$$T(t+t_n)\phi = e^{\mathscr{A}t}T(t_n)\phi + \int_0^t e^{\mathscr{A}(t-s)} f(T(t_n+s)\phi)\, ds .$$

Let

$$T(t+t_n)\phi = \begin{pmatrix} u_n(t) \\ \dot{u}_n(t) \end{pmatrix}.$$

By (3.2), $\|u_n(t)\|_A^2 + |\dot{u}_n(t)|^2 \leqq$ const., and

$$\lim_{n\to\infty} \int_0^T (Bu_n(t), \dot{u}_n(t))^2\, dt = 0 . \tag{3.8}$$

Therefore, for $t \in [0, T]$.

$$\left\| \int_0^t e^{\mathscr{A}(t-s)} f(T(t_n+s)\phi)\, ds \right\| \leqq \int_0^T \|f(T(t_n+s)\phi)\|\, ds$$

$$\leqq \text{const. } T^{1/2}\left(\int_0^T (Bu_n, \dot{u}_n)^2\, dt \right)^{1/2},$$

and hence (3.6) holds. It follows from (3.6) that

$$(By_n(t), \dot{y}_n(t))^2 = (Bu_n(t), \dot{u}_n(t))^2 + k_n(t) ,$$

where $\lim_{n\to\infty} \sup_{t\in[0,T]} |k_n(t)| = 0$. Using (3.8) we obtain (3.7).

LEMMA 3.2. *Let* (H2) *be satisfied. Let*

$$\psi \in X, \qquad e^{\mathscr{A}t}\psi = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}.$$

Then $(By(t), \dot{y}(t))$ *is an almost periodic function of t, and if*

$$\psi = \sum_{m=1}^{\infty} \begin{pmatrix} b_m \\ -\lambda_m c_m \end{pmatrix} \phi_m , \qquad a_m = b_m + ic_m ,$$

then

$$(By(t), \dot{y}(t)) \sim \sum_{k=1}^{\infty} i\lambda_k (B\phi_k, \phi_k)[a_k^2 \exp\{2i\lambda_k t\} - \bar{a}_k^2 \exp\{-2i\lambda_k t\}] + \sum_{\nu \in N \cup -N} d_\nu e^{i\nu t},$$

where $N = \{\lambda_m \pm \lambda_n : m \neq n$ *and* $(B\phi_m, \phi_n), (B\phi_n, \phi_m)$ *not both zero*$\}$.

Proof: Let

$$\psi_K = \sum_{m=1}^{K} \begin{pmatrix} b_m \\ -\lambda_m c_m \end{pmatrix} \phi_m .$$

Then

$$e^{\mathscr{A}t}\psi_K = \begin{pmatrix} y_K(t) \\ \dot{y}_K(t) \end{pmatrix},$$

where

$$y_K(t) = \sum_{m=1}^{K} (a_m \exp\{i\lambda_m t\} + \bar{a}_m \exp\{-i\lambda_m t\})\phi_m .$$

Since $e^{\mathscr{A}t}$ is an isometry for each t, $e^{\mathscr{A}t}\psi_K \to e^{\mathscr{A}t}\psi$ in X as $K \to \infty$, uniformly for $t \in \mathbb{R}$. It follows that $(By_K(t), \dot{y}_K(t)) \to (By(t), \dot{y}(t))$ as $K \to \infty$, uniformly for $t \in \mathbb{R}$. Now

$$\begin{aligned}(By_K(t), \dot{y}_K(t)) &= \sum_{m,n=1}^{K} i\lambda_n (B\phi_m, \phi_n)(a_m \exp\{i\lambda_m t\} + \bar{a}_m \exp\{-i\lambda_m t\}) \\ &\qquad \times (a_n \exp\{i\lambda_n t\} - \bar{a}_n \exp\{-i\lambda_n t\}) \\ &= \sum_{k=1}^{K} i\lambda_k (B\phi_k, \phi_k)(a_k^2 \exp\{2i\lambda_k t\} - \bar{a}_k^2 \exp\{-2i\lambda_k t\}) \\ &\qquad + \sum_{\nu \in N \cup -N} d_\nu^K e^{i\nu t},\end{aligned}$$

where the second sum is finite. Hence $(By(t), \dot{y}(t))$ is the uniform limit of finite trigonometric polynomials, and is consequently almost periodic. Also, for real λ, since $N \cap 2\Lambda = \phi$,

$$M((By_K(t), \dot{y}_K(t))e^{-i\lambda t}) = \begin{cases} i\lambda_k (B\phi_k, \phi_k) a_k^2 & \text{if} \quad \lambda = 2\lambda_k , \\ -i\lambda_k (B\phi_k, \phi_k) \bar{a}_k^2 & \text{if} \quad \lambda = -2\lambda_k , \\ 0 & \text{if} \quad \lambda, -\lambda \notin 2\Lambda \cup N . \end{cases}$$

Therefore,

$$M((By(t), \dot{y}(t))e^{-i\lambda t}) = \begin{cases} i\lambda_k (B\phi_k, \phi_k) a_k^2 & \text{if} \quad \lambda = 2\lambda_k , \\ -i\lambda_k (B\phi_k, \phi_k) \bar{a}_k^2 & \text{if} \quad \lambda = -2\lambda_k , \\ 0 & \text{if} \quad \lambda, -\lambda \notin 2\Lambda \cup N . \end{cases}$$

The result follows.

Proof of Theorem 3.2: Let $\phi \in X$. Since $\|T(t)\phi\|_X \leqq \text{const.}$ for all $t \geqq 0$, it follows that $\omega_X(\phi)$ is nonempty. Let $\psi \in \omega_X(\phi)$, so that there exists a sequence $t_n \to \infty$ such that $T(t_n)\phi \rightharpoonup \psi$ in X. By Lemma 3.1,

$$T(t+t_n)\phi = \begin{pmatrix} u_n(t) \\ \dot{u}_n(t) \end{pmatrix} \rightharpoonup e^{\mathscr{A}t}\psi$$

for each $t \geqq 0$, so that $\omega_X(\phi)$ is positively invariant under $e^{\mathscr{A}t}$. Let

$$e^{\mathscr{A}t}\psi = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}.$$

In case (i) we have, by (3.2),

$$\lim_{n\to\infty} \int_0^t (Bu_n(\tau), \dot{u}_n(\tau))\, d\tau = 0\,,$$

for any $t > 0$. But

$$\int_0^t (Bu_n, \dot{u}_n)\, d\tau = \tfrac{1}{2}[(Su_n(t), u_n(t)) - (Su_n(0), u_n(0))] + \int_0^t (Ku_n(\tau), \dot{u}_n(\tau))\, d\tau\,.$$

Passing to the limit, using the compactness of K and the fact that H_A is compactly embedded in H, we obtain

$$\tfrac{1}{2}[(Sy(t), y(t)) - (Sy(0), y(0))] + \int_0^t (Ky, \dot{y})\, d\tau = 0\,.$$

Differentiating with respect to t, we get

$$(By(t), \dot{y}(t)) = 0 \quad \text{for all} \quad t \geqq 0\,.$$

By Lemma (3.2) and (H1), $a_k = 0$ for all k, and hence $\psi = 0$. Hence $T(t)\phi \rightharpoonup 0$ in X as $t \to \infty$.

In case (ii), let

$$e^{\mathscr{A}t}T(t_n)\phi = \begin{pmatrix} y_n(t) \\ \dot{y}_n(t) \end{pmatrix}.$$

By Lemma 3.1, $(By_n(t), \dot{y}_n(t)) \to 0$ in $L^1(0, T)$ for each $T > 0$. Let

$$\begin{pmatrix} y_n(0) \\ \dot{y}_n(0) \end{pmatrix} = \sum_{m=1}^{\infty} \begin{pmatrix} b_m^{(n)} \\ -\lambda_m c_m^{(n)} \end{pmatrix}, \qquad a_m^{(n)} = b_m^{(n)} + ic_m^{(n)}\,.$$

Since $(By_n(t), \dot{y}_n(t))$ is a uniformly bounded sequence of almost periodic functions, and since $\inf_{\nu\in N}|2\lambda_k-\nu|>0$ for each k, it follows from Theorem 2.2, Lemma 3.2 and (H1) that

$$a_k^{(n)}\to 0 \quad \text{as} \quad n\to\infty, \qquad\qquad k=1,2,\cdots.$$

Thus $\binom{y_n(0)}{\dot{y}_n(0)}\rightharpoonup 0$ in X, and so $\psi=0$. Hence $T(t)\phi\rightharpoonup 0$ in X as $t\to\infty$.

Remark 3.1. We note that some other techniques commonly used in studying stability of nonlinear evolution equations are inapplicable to (3.4), specifically:

Linearization: The linearization of (3.4) is $\dot{w}(t)=\mathscr{A}w(t)$. Since the eigenvalues of $\mathscr{A}$ are purely imaginary, the linearization of (3.4) gives no information on the behavior of the full nonlinear equation.

Center Manifolds: The center manifold theorem and its generalizations to infinite dimensions (see Marsden and McCracken [21]) would require the operator $\mathscr{A}$ to have all but a finite number of eigenvalues strictly in the left half of the complex plane. As noted above this is not the case in our problem.

Contraction Semigroups: The theory of asymptotic behavior of contraction semigroups is particularly powerful for studying nonlinear evolution equations of the form $\dot{w}(t)=\mathscr{D}w(t)$, where $-\mathscr{D}$ is a maximal monotone operator on a Hilbert space (see Dafermos and Slemrod [11], Haraux [15], Pazy [22]). Unfortunately, for system (3.4) $-\mathscr{A}-f$ is not monotone on X.

4. Applications

EXAMPLE 4.1. Vibrating beam with hinged ends. Let $\Omega\subset\mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$ and consider the system

$$u_{tt}+\Delta^2u+p(t)\Delta u=0\,, \qquad x\in\Omega\,, \tag{4.1}$$

$$u=\Delta u=0\,, \qquad x\in\partial\Omega\,. \tag{4.2}$$

If $N=1$ and $\Omega=(0,1)$, then (4.1)–(4.2) is a standard model for the transverse deflection $u(x,t)$ of a beam with hinged ends, and in this case $p(t)$ denotes the axial load on the beam.

THEOREM 4.1. *Let the eigenvalues λ_n of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$ be simple. Set*

$$p(t)=\int_\Omega u_t\,\Delta u\,dx\,.$$

Then, for all initial data $\{u_0, \dot{u}_0\} \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega) = X$, (4.1)–(4.2) *possesses a unique weak solution* $\{u, \dot{u}\} \in C([0,\infty); X)$, *and* $\{u, \dot{u}\} \rightharpoonup \{0, 0\}$ *in* X *as* $t \to \infty$.

Remark 4.1. The condition that the eigenvalues λ_n be simple (which is automatically satisfied if $N = 1$, $\Omega = (0, 1)$) is a condition on Ω which is *necessary* for feedback stabilizability of (4.1)–(4.2). This may be proved by the same argument as in [5], Theorem 4.1.

Proof of Theorem 4.1: Set $A = \Delta^2$, $B = \Delta$, $H = L^2(\Omega)$, $D(A) = \{u \in H^4(\Omega): u, \Delta u \in H_0^1(\Omega)\}$, $D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$. The eigenvalues of A are λ_n^2 and the corresponding eigenfunctions $\phi_n(x)$ are the same as for $-\Delta$. Clearly, $(B\phi_n, \phi_n) \neq 0$, $(B\phi_m, \phi_n) = 0$ for $m \neq n$, so that (H1) and (H2) hold. Since B is symmetric, Theorem 3.2 (i) applies and the result follows.

EXAMPLE 4.2. Vibrating beam with clamped ends. Consider the system

$$u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, \qquad 0 < x < 1, \tag{4.3}$$

$$u = u_x = 0 \quad \text{at} \quad x = 0, 1. \tag{4.4}$$

THEOREM 4.2. *Set*

$$p(t) = \int_0^1 u_t u_{xx}\, dx.$$

Then for all initial data $\{u_0, \dot{u}_0\} \in H_0^2(0, 1) \times L^2(0, 1) = X$, (4.3)–(4.4) *possesses a unique weak solution* $\{u, \dot{u}\} \in C([0,\infty); X)$ *and* $\{u, \dot{u}\} \rightharpoonup \{0, 0\}$ *in* X *as* $t \to \infty$.

Proof: Let $A = d^4/dx^4$, $B = d^2/dx^2$, $H = L^2(0, 1)$, $D(A) = H^4(0, 1) \cap H_0^2(0, 1)$, $D(A^{1/2}) = H_0^2(0, 1)$. The eigenvalues of A are λ_n^2, where $\lambda_n = \mu_n^2$ and $\cosh \mu_n \cos \mu_n = 1$. The corresponding eigenfunctions, not normalized, (cf. Courant and Hilbert [10]) are

$$\phi_n(x) = (\sin \mu_n - \sinh \mu_n)(\cos \mu_n x - \cosh \mu_n x) - (\cos \mu_n - \cosh \mu_n) \times (\sin \mu_n x - \sinh \mu_n x).$$

Integration by parts shows that $(B\phi_n, \phi_n) = -|\phi_{nx}|^2 \neq 0$. Since B is symmetric, Theorem 3.2 (i) will apply if (H2) holds. Unlike Example 4.1, we do *not* have $(B\phi_n, \phi_m) = 0$ for $m \neq n$, so we apply the following

PROPOSITION 4.1. *For* μ_n *the positive roots of* $\cosh \mu_n \cos \mu_n = 1$, *the equation* $\mu_n^2 \pm \mu_m^2 = 2\mu_k^2$ *can only be satisfied if* $m = n = k$ *and the* + *sign is taken.*

Proof: The proof involves obtaining accurate asymptotic estimates for the μ_n. We omit the details because they are almost identical to those of the proof of Proposition 4.3 below. The proof of Proposition 4.3 is in fact slightly harder since in that case the lowest root $\mu_1 < \pi$ has to be handled separately, whereas here a simple computation shows that $\operatorname{sech} \mu > \cos \mu$ for $0 < \mu \leqq \frac{3}{2}\pi$, so that $\mu_1 > \frac{3}{2}\pi$. This larger value of μ_1 enables one to use the asymptotic estimates for all the μ_n including μ_1.

Proposition 4.1 shows that (H2) is satisfied, which completes the proof of Theorem 4.2.

EXAMPLE 4.3. Beam with "follower" load. We again consider (4.3), but allow one end of the rod to be free and loaded tangentially with load $p(t)$. The other end is assumed clamped. This problem is particularly interesting since it is analogous to problems in mechanics where a thrust is applied to a structure at a free end, e.g. hosepipe, flexible missile. The boundary conditions associated with this problem are

$$(4.5) \qquad u = u_x = 0 \quad \text{at} \quad x = 0, \qquad u_{xx} = u_{xxx} = 0 \quad \text{at} \quad x = 1 .$$

A, B and H are as in Example 4.2, but now $D(A) = \{u \in H^4(0, 1)\colon u = u_x = 0$ at $x = 0$, $u_{xx} = u_{xxx} = 0$ at $x = 1\}$ and $D(A^{1/2}) = \{u \in H^2(0, 1)\colon u = u_x = 0$ at $x = 0\}$. The eigenfunctions of A not normalized, are

$$(4.6) \quad \phi_n(x) = (\cos \mu_n + \cosh \mu_n)(\cos \mu_n x - \cosh \mu_n x) + (\sin \mu_n - \sinh \mu_n) \times (\sin \mu_n x - \sinh \mu_n x) ,$$

where $\lambda_n^2 = \mu_n^4$ are the corresponding eigenvalues and μ_n satisfies $\cosh \mu_n \cos \mu_n = -1$ (cf. Bolotin [8]). Note that, in this example, B is *not* of the form $K + S$ with K compact and S symmetric.

THEOREM 4.3. *Set*

$$p(t) = \int_0^1 u_t u_{xx} \, dx .$$

Then for all initial data $\{u_0, \dot{u}_0\} \in D(A^{1/2}) \times L^2(0, 1) = X$, (4.3)–(4.5) *possesses a unique weak solution* $\{u, \dot{u}\} \in C([0, \infty); X)$ *and* $\{u, \dot{u}\} \rightharpoonup \{0, 0\}$ *in* X *as* $t \to \infty$.

Proof: The proof is an immediate consequence of Theorem 3.2 (ii) and the following two propositions, which verify conditions (H1) and (H2+), respectively.

PROPOSITION 4.2. $(\phi_n, \phi_{nxx}) \neq 0$, $n = 1, 2, \cdots$.

Proof: For brevity write $\mu_n = \mu$, $\phi_n = \phi$. A direct computation using (4.6) yields

$$\frac{2}{\mu}(\phi, \phi_{xx}) = (\sin\mu - \sinh\mu)^2(\sinh\mu\cosh\mu + \sin\mu\cos\mu - 2\mu)$$
$$+ (\cos\mu + \cosh\mu)^2(\sinh\mu\cosh\mu - \sin\mu\cos\mu)$$
$$+ 2(\sin\mu - \sinh\mu)(\cos\mu + \cosh\mu)(\sinh^2\mu - \sin^2\mu).$$

Since $(\cos\mu + \cosh\mu)^2 = \sinh^2\mu - \sin^2\mu$, it follows that if $(\phi, \phi_{xx}) = 0$, we have

$$\mu(\sin\mu - \sinh\mu) = \sinh^2\mu\cos\mu - \sin^2\mu\cosh\mu$$
$$= -2(\cosh\mu + \cos\mu).$$

But $\sin\mu = \pm\tanh\mu$, and hence

$$\mu = \frac{2\sinh\mu}{\cosh\mu \mp 1}.$$

A numerical computation shows that $\mu_1 \neq \dfrac{2\sinh\mu_1}{\cosh\mu_1 \mp 1}$. Also $\mu_n > 10$ if $n > 1$, so that

$$10 < \frac{2\sinh\mu_n}{\cosh\mu_n \mp 1} \leqq \frac{2\cosh\mu_n}{\cosh\mu_n \mp 1} = 2 \pm \frac{2}{\cosh\mu_n \mp 1} \leqq 2 + \frac{2}{\cosh(10) - 1} < 10,$$

which is a contradiction.

PROPOSITION 4.3. *For μ_n the positive roots of $\cosh\mu_n\cos\mu_n = -1$, and for each $k = 1, 2, \cdots$,*

$$\inf|2\mu_k^2 - \mu_m^2 \mp \mu_n^2| > 0,$$

where the infimum is taken over m, n not both equal to k.

Proof: The roots $0 < \mu_1 < \mu_2 < \cdots$ of $\cosh\mu\cos\mu = -1$ are given by the μ coordinates of the intersections of the curves $\operatorname{sech}\mu$ and $-\cos\mu$. It follows that μ_n has the form

$$\mu_n = (n - \tfrac{1}{2})\pi + \delta_n, \qquad n = 1, 2, \cdots,$$

where $|\delta_n| < \frac{1}{2}\pi$. It is clear graphically that $|\delta_n|$ is a decreasing function of n. Computed values of μ_1, δ_1, δ_2 are $\mu_1 = 1.8751\cdots$, $\delta_1 = 0.3043\cdots$, $\delta_2 = -0.0182\cdots$. Thus we have the preliminary estimate

LEMMA 4.1. $|\delta_n| < 0.02$, $n > 1$.

Next we refine this estimate.

LEMMA 4.2. $\delta_n = 2c_n(-1)^{n-1}\exp\{-(n-\frac{1}{2})\pi\}$, $n > 1$, *where* $0{\cdot}97 < c_n < 1{\cdot}03$.

Proof: Let $f(\mu) = \cosh\mu\cos\mu$, so that $f'(\mu) = \sinh\mu\cos\mu - \sin\mu\cosh\mu$, $f''(\mu) = -2\sinh\mu\sin\mu$. Let $n > 1$. By Taylor's theorem,

$$-1 = f(\mu_n) = 0 + \delta_n f'((n-\tfrac{1}{2})\pi) + \tfrac{1}{2}\delta_n^2 f''((n-\tfrac{1}{2})\pi + \xi_n),$$

where $|\xi_n| < 0{\cdot}02$ by Lemma 4.1. Thus

$$(4.7)\qquad -1 = \delta_n(-1)^n\cosh((n-\tfrac{1}{2})\pi) + \delta_n^2(-1)^n\sinh[(n-\tfrac{1}{2})\pi + \xi_n]\cos\xi_n.$$

Furthermore, $|\cos\xi_n - 1| \leqq \frac{1}{2}\xi_n^2$, so that

$$(4.8)\qquad |\cos\xi_n - 1| < 0.0002.$$

Now,

$$|\exp\{\xi_n\} - 1| \leqq |\xi_n|\left(1 + \frac{|\xi_n|}{2} + \frac{|\xi_n|^2}{2^2} + \cdots\right) = \frac{|\xi_n|}{1 - \frac{1}{2}|\xi_n|} \leqq \frac{0.02}{0.99} < 0.021.$$

Hence

$$(4.9)\qquad \begin{aligned} &|2\exp\{-(n-\tfrac{1}{2})\pi\}\sinh[(n-\tfrac{1}{2})\pi + \xi_n] - 1| \\ &\qquad = |\exp\{\xi_n\} - 1 - \exp\{-(2n-1)\pi\}\exp\{-\xi_n\}| \\ &\qquad < 0.021 + e^{-3\pi}e^{0.02} < 0.022. \end{aligned}$$

Also

$$(4.10)\quad |2\exp\{-(n-\tfrac{1}{2})\pi\}\cosh((n-\tfrac{1}{2})\pi) - 1| = \exp\{-(2n-1)\pi\} \leqq e^{-3\pi} < 0.0001.$$

Substituting (4.8)–(4.10) in (4.7) we obtain

$$2(-1)^{n-1}\exp\{-(n-\tfrac{1}{2})\pi\} = \delta_n(1+\alpha_n) + \delta_n^2(1+\beta_n)(1+\gamma_n),$$

where $|\alpha_n|<0.0001$, $|\beta_n|<0.022$, $|\gamma_n|<0.0002$. Now $(1+\beta_n)(1+\gamma_n)=1+\sigma_n$, where $|\sigma_n|<0.023$. Therefore,

$$2(-1)^{n-1}\exp\{-(n-\tfrac{1}{2})\pi\}=\delta_n(1+\alpha_n+\delta_n(1+\delta_n))=\delta_n(1+\rho_n),$$

where $|\rho_n|<0.0001+0.02\times 1.023<0.021$. The result follows.

Since $\mu_{n+1}^2-\mu_n^2\sim 2n\pi^2$ as $n\to\infty$, it suffices for the proof of the proposition to show that the equation

$$\mu_n^2\pm\mu_m^2=2\mu_k^2 \tag{4.11}$$

has no solution with $m\neq n$. Suppose for the sake of contradiction that it has such a solution. By Lemma 4.2, to first order, (4.11) becomes

$$(2n-1)^2\pm(2m-1)^2=2(2k-1)^2. \tag{4.12}$$

This diophantine equation has an infinity of nontrivial solutions which may be written down by the procedure given in Hardy and Wright [16], pp. 241–243. Two such solutions are $1^2+7^2=2\cdot 5^2$, $7^2+17^2=2\cdot 13^2$. All such solutions of (4.12) are with the + sign, for otherwise the left-hand side would be divisible by 4. Because of these considerations we split the proof into two cases.

Case 1. *m, n, k do not satisfy* (4.12). Substitution into (4.11) yields

$$[(n-\tfrac{1}{2})\pi+\delta_n]^2\pm[(m-\tfrac{1}{2})\pi+\delta_m]^2=2[(k-\tfrac{1}{2})\pi+\delta_k]^2,$$

so that

$$\begin{aligned}1&\leqq|(2n-1)^2\pm(2m-1)^2-2(2k-1)^2|\\&=\frac{4}{\pi^2}|2\delta_k^2\mp\delta_m^2-\delta_n^2+4(k-\tfrac{1}{2})\pi\delta_k\mp 2(m-\tfrac{1}{2})\pi\delta_m-2(n-\tfrac{1}{2})\pi\delta_n|.\end{aligned} \tag{4.13}$$

By Lemma 4.2,

$$|(r-\tfrac{1}{2})\pi\delta_r|\leqq 2.06(r-\tfrac{1}{2})\pi\exp\{-(r-\tfrac{1}{2})\pi\},\qquad r>1.$$

Let $g(x)=xe^{-x}$, so that $g'(x)=(1-x)e^{-x}\leqq 0$ if $x\geqq 1$. Thus for $r>1$

$$|(r-\tfrac{1}{2})\pi\delta_r|\leqq 2.06g\left(\frac{3\pi}{2}\right)=2.06\times\frac{3\pi}{2}e^{-3\pi/2}<0.1. \tag{4.14}$$

Now note that at most one of n,m and k can be one, since otherwise we would have $k=m=1$ and $\mu_n=\sqrt{3}\mu_1=3.2477\cdots\in(\mu_1,\mu_2)$. Hence by (4.13), (4.14) we obtain

$$1\leqq\frac{4}{\pi^2}[2(0.31)^2+(0.02)^2+2\pi\times 0.31+2\times 0.1]<0.95\,,$$

which is a contradiction.

Case 2. *m, n, k satisfy* $(2n-1)^2+(2m-1)^2=2(2k-1)^2$. We may assume that $n<k<m$. Substitution into (4.11) yields

$$(4.15)\qquad (2n-1)\pi\delta_n+(2m-1)\pi\delta_m-2(2k-1)\pi\delta_k=2\delta_k^2-\delta_m^2-\delta_n^2\,.$$

If $n=1$, our previous estimates and (4.15) show that

$$0.3<\delta_1\leqq\frac{1}{\pi}(4\times 0.1+2(0.31)^2)=0.188\cdots,$$

which is impossible. Thus we suppose that $n>1$. By Lemma 4.2 and (4.15),

$$(4.16)\qquad \begin{aligned}(-1)^{n-1}c_n&=(-1)^m c_m\frac{g((m-\frac{1}{2})\pi)}{g((n-\frac{1}{2})\pi)}\\&\quad+2(-1)^{k-1}c_k\frac{g((k-\frac{1}{2})\pi)}{g((n-\frac{1}{2})\pi)}+\frac{(2\delta_k^2-\delta_m^2-\delta_n^2)}{4g((n-\frac{1}{2})\pi)}\,.\end{aligned}$$

Let $r\geqq 1$ and for fixed $n>1$ consider

$$\theta(r)\overset{\text{def}}{=}\frac{g((n-\frac{1}{2}+r)\pi)}{g((n-\frac{1}{2})\pi)}=\left(1+\frac{r}{n-\frac{1}{2}}\right)e^{-r\pi}\,.$$

Since

$$\theta'(r)=\frac{\pi}{n-\frac{1}{2}}\left[\frac{1}{\pi}-(n-\tfrac{1}{2})-r\right]e^{-r\pi}<0\,,$$

it follows that

$$\theta(r)\leqq\theta(1)=\left(1+\frac{1}{n-\frac{1}{2}}\right)e^{-\pi}\leqq\tfrac{5}{3}e^{-\pi}\,.$$

But

$$\begin{aligned}&|\tfrac{1}{4}[2\delta_k^2-\delta_m^2-\delta_n^2]/g((n-\tfrac{1}{2})\pi)|\\&=\frac{1}{(n-\frac{1}{2})\pi}|2c_k^2\exp\{(n-2k+\tfrac{1}{2})\pi\}-c_m^2\exp\{(n-2m+\tfrac{1}{2})\pi\}-c_n^2\exp\{-(n-\tfrac{1}{2})\pi\}|\\&\leqq\frac{(1.03)^2}{\frac{3}{2}\pi}[2e^{-7\pi/2}+e^{-11\pi/2}+e^{-3\pi/2}]\overset{\text{def}}{=}\Delta\end{aligned}$$

since $n+1\leqq k\leqq m-1$. Substituting the above into (4.16) yields

$$0.97<|c_n|<3\times 1.03\times\tfrac{5}{3}e^{-\pi}+\Delta<0.5\,.$$

This is a contradiction.

EXAMPLE 4.4. Timoshenko beam. We now treat a linearized model for the plane motion of a uniform beam that is somewhat more satisfactory than that discussed in Examples 4.1–4.3 above. The model consists of the coupled set of wave equations

$$\begin{aligned}\rho v_{tt}&=Nv_{xx}-(p(t)+\alpha+N)\theta_x\,,\\ d\theta_{tt}&=M\theta_{xx}-(p(t)+\alpha+N)(\theta-v_x)\,.\end{aligned}\tag{4.17}$$

Here $v(x,t)$ denotes the transverse displacement of the center line of the beam, while $\theta(x,t)$ denotes the angle which the normal to the cross section makes with the x-axis. The positive constants ρ, d, N, M are related to the material properties and to the cross section of the beam. The beam is subjected to a time-dependent axial force $p(t)+\alpha$, where α is a constant. Equations (4.17) are known as the Timoshenko beam equations. A derivation in the case $p(t)+\alpha=0$ can be found in Timoshenko and Young [25]. The equations may alternatively be derived by linearizing a fully nonlinear one-director theory of the plane motion of an inextensible elastic rod about the time dependent trivial solution in which the rod is straight and subjected to the axial force $p(t)+\alpha$.* For a derivation of a similar set of equations see also Green, Knops and Laws [14].

We assume that the ends of the beam are simply supported, so that

$$v=\theta_x=0\quad\text{at}\quad x=0,1\,.\tag{4.18}$$

We will be interested in stabilizing (4.17) by means of an axial force $p(t)+\alpha$ which tends in a suitable sense to α as $t\to\infty$; i.e., $p(t)\to 0$. Because in this

* We are indebted to Stuart Antman for discussions on this point.

model the beam may suffer instabilities both in compression and tension, we shall suppose that $|\alpha|$ is sufficiently small; specifically,

$$-N<\alpha<\tfrac{1}{2}(-N+(N^2+4MN\pi^2)^{1/2}). \tag{4.19}$$

Let $H=L^2(0,1)\times L^2(0,1)$ with inner product

$$\left(\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} \hat{v} \\ \hat{\theta} \end{pmatrix}\right)_H = \int_0^1 [\rho v(x)\hat{v}(x)+d\theta(x)\hat{\theta}(x)]\,dx.$$

Let

$$A\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} -\dfrac{N}{\rho} v_{xx} + \left(\dfrac{\alpha+N}{\rho}\right)\theta_x \\ -\dfrac{M}{d}\theta_{xx} + \left(\dfrac{\alpha+N}{d}\right)(\theta - v_x) \end{pmatrix},$$

$$B\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\rho}\theta_x \\ \dfrac{1}{d}(\theta - v_x) \end{pmatrix},$$

$$D(A)=\left\{\begin{pmatrix} v \\ \theta \end{pmatrix} \in H^2(0,1)\cap H_0^1(0,1)\times H^2(0,1): \theta_x = 0 \quad \text{at} \quad x=0,1\right\}.$$

Note that B is symmetric.

LEMMA 4.3. *A is a positive selfadjoint operation on H with an everywhere defined compact inverse.*

Proof: We have

$$\left(A\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} v \\ \theta \end{pmatrix}\right)_H = N|v_x|^2 + M|\theta_x|^2 + (\alpha+N)(|\theta|^2+2(\theta_x, v)).$$

Consider the problem of minimizing $N|v_x|^2+M|\theta_x|^2$ in $H_0^1(0,1)\times H^1(0,1)$ subject to the constraint $|\theta|^2+2(\theta_x, v)=\pm 1$. By use of the Poincaré inequality and the constraint we see that any minimizing sequence is bounded. A standard argument shows that the minimum value is $\nu^\pm$, where $\nu^\pm$ is the least non-negative value of ν for which a nontrivial solution of

$$\begin{aligned} &N v_{xx} \pm \nu\theta_x = 0, \\ &M\theta_{xx} \pm \nu(\theta - v_x) = 0, \end{aligned} \tag{4.20}$$

exists satisfying the constraint and boundary conditions. Since the solutions of (4.20), which are equilibrium solutions corresponding to (4.17), have the form $v = C\sin n\pi x$, $\theta = \cos n\pi x$, it follows that $\nu^+ = 0$, $\nu^- = \frac{1}{2}(N + (N^2 + 4MN\pi^2)^{1/2})$. Hence A is positive provided $0 < \alpha + N < \nu^-$, which holds by (4.19). A simple calculation shows that A is selfadjoint, while A^{-1} is compact by standard elliptic theory.

Note that $H_A = D(A^{1/2}) = H_0^1(0,1) \times H^1(0,1)$, and

$$\left\langle \begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} \hat{v} \\ \hat{\theta} \end{pmatrix} \right\rangle_A = N(v_x, \hat{v}_x) + M(\theta_x, \hat{\theta}_x) + (\alpha + N)[(\theta_x, \hat{v}) + (\theta - v_x, \hat{\theta})].$$

The eigenvalues $\lambda_n^{\pm 2}$ of A are given by

$$\lambda_n^{\pm 2} = \frac{1}{2\rho d}[(\rho M + dN)n^2\pi^2 + \rho(\alpha + N)$$
$$\pm((\rho M - dN)^2 n^4\pi^4 + 2n^2\pi^2\rho(\alpha + N)(\rho M + dN + 2d\alpha) + \rho^2(\alpha + N)^2)^{1/2}],$$
$$n = 0, 1, 2, \cdots.$$

Note that $\lambda_0^{+2} = \lambda_0^{-2} = (\alpha + N)/d$. The corresponding eigenfunctions are

$$\phi_n^{\pm} = \begin{pmatrix} C_n^{\pm} \sin n\pi x \\ \cos n\pi x \end{pmatrix}, \qquad \text{where } C_n^{\pm} = \frac{(\alpha + N)n\pi}{Nn^2\pi^2 - \rho\lambda_n^{\pm 2}}.$$

There are no other eigenvalues or eigenfunctions since the $\phi_n^{\pm}$ span H.

THEOREM 4.4. *Let*

$$p(t) = \left(B\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} v_t \\ \theta_t \end{pmatrix} \right)_H = (\theta_x, v_t) + (\theta - v_x, \theta_t).$$

Let α satisfy (4.19). *There exists a countable set K of real numbers such that for all initial data*

$$\left\{ \begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix}, \begin{pmatrix} \dot{v}_0 \\ \dot{\theta}_0 \end{pmatrix} \right\} \in H_A \times H = X,$$

(4.17)–(4.18) *possesses a unique weak solution*

$$\left\{ \begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} v_t \\ \theta_t \end{pmatrix} \right\} \in C([0, \infty); X),$$

and if $\alpha \notin K$, then

$$\left\{\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} v_t \\ \theta_t \end{pmatrix}\right\} \rightarrow \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \quad \text{in} \quad X \quad \text{as} \quad t \rightarrow \infty .$$

Proof: We apply Theorem 3.2 (i). It remains to verify (H1) and (H2). To verify (H1) we compute

$$(B\phi_n^{\pm}, \phi_n^{\pm})_H = 1 - 2n\pi C_n^{\pm},$$

which is zero if and only if $n \neq 0$, $C_n^{\pm} = 1/2n\pi$. Substituting for $\lambda_n^{\pm 2}$ and using the fact that $\alpha + N \neq 0$, we obtain

$$\alpha = \frac{2n^2\pi^2(dN - \rho M)}{\rho + 4dn^2\pi^2} - N \tag{4.21}$$

which is a contradiction if α does not equal one of this countable sets of numbers. (Note that if $dN > \rho M$, then, for all large enough n, α given by (4.21) satisfies (4.19). Consequently, for these values of α, (4.17)–(4.18) is not weakly stabilizable with our choice of $p(t)$.) To check (H2), first note that if $n \neq m$, then $(B\phi_n^+, \phi_m^-)_H = (B\phi_n^-, \phi_m^+)_H = 0$. In general, $(B\phi_n^+, \phi_n^-)_H \neq 0$. Thus we must prove that if $n \neq 0$

$$\lambda_n^+ \pm \lambda_n^- \neq 2\lambda_k^{\pm}, \qquad \lambda_n^+ \pm \lambda_n^- \neq 2\lambda_k^{\mp}.$$

Suppose for the sake of contradiction that $\lambda_n^+ \pm \lambda_n^- = 2\lambda_k^+$. Substituting for the λ's and eliminating the square roots by successive squaring, we obtain a quartic equation for α. The coefficient of α^4 is $(3\rho + 4d\pi^2(n^2 + 4h^2))^2 + 16n^2\pi^2\rho d$, which is nonzero. For each n, k this is a contradiction for all but finitely many values of α. We obtain the same contradiction if $\lambda_n^+ \pm \lambda_n^- = 2\lambda_k^{\mp}$. This completes the proof.

EXAMPLE 4.5. A wave equation. Consider the wave equation

$$u_{tt} - u_{xx} + p(t)u_x = 0, \qquad 0 < x < 1, \tag{4.22}$$

with boundary conditions

$$u = 0 \quad \text{at} \quad x = 0, \qquad u + \alpha u_x = 0 \quad \text{at} \quad x = 1, \tag{4.23}$$

where $\alpha > 0$ is a constant.

THEOREM 4.5. *Let* $p(t)=\int_0^1 u_t u_x\,dx$ *and* $X=\{\{u,v\}\in H^1(0,1)\times L^2(0,1): u(0)=0\}$. *Then, for all initial data* $\{u_0,\dot u_0\}\in X$, (4.22)–(4.23) *possesses a unique weak solution* $\{u,\dot u\}\in C([0,\infty);X)$ *and* $\{u,\dot u\}\rightharpoonup\{0,0\}$ *in* X *as* $t\to\infty$.

Proof: Set $A=-d^2/dx^2$, $B=d/dx$, $H=L^2(0,1)$, $D(A)=\{u\in H^2(0,1); u(0)=u(1)+\alpha u_x(1)=0\}$, $H_A=D(A^{1/2})=\{u\in H^1(0,1); u(0)=0\}$, $\|u\|_A^2=|u_x|^2+\alpha^{-1}u(1)^2$. The eigenfunctions of A are $\phi_n=\sin\lambda_n x$, where λ_n^2 are the associated eigenvalues and λ_n satisfies

$$\tan\lambda_n+\alpha\lambda_n=0\,.$$

Since $(B\phi_n,\phi_n)=\frac{1}{2}\sin^2\lambda_n$, and since $r^2\pi^2$ is not an eigenvalue, (H1) is satisfied. Since B is not symmetric, we must show that (H2+) holds.

PROPOSITION 4.4. *Let* $0<\lambda_1<\lambda_2<\cdots$ *denote the positive roots of* $\tan\lambda+\alpha\lambda=0$. *Then, for each* k,

$$\inf_{\substack{m,n\\ m\neq n}}|2\lambda_k-\lambda_m\pm\lambda_n|>0\,.$$

Proof: Since $\lambda_n=(n-\frac{1}{2})\pi+o(1)$, and since $r^2\pi^2$ is not an eigenvalue, it suffices to show that the equation

$$\lambda_m\pm\lambda_n=2\lambda_k \tag{4.24}$$

has no solution with $m\neq n$. If m,n,k satisfy (4.24), then taking tan of both sides gives

$$\frac{\tan\lambda_m\pm\tan\lambda_n}{1\mp\tan\lambda_m\tan\lambda_n}=\frac{2\tan\lambda_k}{1-\tan^2\lambda_k}\,.$$

Applying the definition of the λ's it follows that

$$\pm\lambda_m\lambda_n=\lambda_k^2\,.$$

In conjunction with (4.24) this implies that $\lambda_m=\lambda_n=\lambda_k$, which is a contradiction. The proof of the theorem is now completed by applying Theorem 3.2 (ii).

EXAMPLE 4.6. Nonlinear wave equation. As a new application of our earlier result, Theorem 3.1, we consider the nonlinear wave equation

$$u_{tt}-u_{xx}+p(t)f(u)=0\,, \tag{4.25}$$

where $f(u)$ is a nonzero real polynomial in u, with boundary conditions

$$u=0 \quad \text{at} \quad x=0\,, \qquad u+\alpha u_x=0 \quad \text{at} \quad x=1\,, \tag{4.26}$$

where $\alpha>0$ is a constant.

The definitions of H, A, $D(A)$ and the eigenvalues and eigenfunctions of A are the same as in Example 4.5. In this case, however, $B=f$ is nonlinear. Let $\psi_n \rightharpoonup \psi$ in H_A. By the Sobolev imbedding theorem and the continuity of f, $f(\psi_n)\to f(\psi)$ in $H=L^2(0,1)$. Hence condition (C) is satisfied.

THEOREM 4.6. *There exists a countable set* $A_0\subset(0,\infty)$ *such that, for any initial data* $\{u_0, \dot u_0\}\in H_A\times H=X$, *and for any nonzero polynomial* f, *there exists a unique weak solution* $\{u,\dot u\}\in C([0,\infty); X)$ *to* (4.25)–(4.26) *with* $p(t)=(f(u), u_t)$, *and if* $\alpha\notin A_0$, *then* $\{u,\dot u\}\rightharpoonup\{0,0\}$ *in* X *as* $t\to\infty$.

The proof of Theorem 4.6 is an immediate consequence of Theorem 3.1 and the following theorem applied to $F(u)=\int^u f(s)\,ds$.

THEOREM 4.7. *There exists a countable set* $A_1\subset\mathbb{R}$ *such that if* F *is any nonconstant real polynomial and if* $\alpha\in\mathbb{R}\backslash A_1$, *then there is no nonzero solution* u *of*

$$\begin{gathered} u_{tt}-u_{xx}=0\,,\\ u=0 \quad at \quad x=0\,, \qquad u+\alpha u_x=0 \quad at \quad x=1\,, \end{gathered} \tag{4.27}$$

satisfying

$$\int_0^1 F(u(x,t))\,dx=\text{constant}\,,$$

for all $t\geqq 0$.

Remark 4.2. The set A_1 contains 0, since then, if $u(x,t)=\sin 2\pi t\sin 2\pi x$,

$$\int_0^1 u(x,t)^{2m+1}\,dx\equiv 0\,.$$

As far as we know, Theorem 4.7 may be valid with $A_1=\{0\}$.

Proof of Theorem 4.7: In this proof, $\lambda_n=\lambda_n(\alpha)$ denotes the unique root of $\tan\lambda+\alpha\lambda=0$ lying in the interval $(n-\frac{1}{2})\pi<\lambda<(n+\frac{1}{2})\pi$; note that $\lambda_{-n}=-\lambda_n$. The general solution of (4.27) is

$$u(x,t)=\sum_{n=1}^{\infty}(a_n\exp\{i\lambda_n t\}+\bar a_n\exp\{-i\lambda_n t\})\sin\lambda_n x\,.$$

Let $F(u)=\sum_{j=0}^{p} c_j u^j$ with $p \geqq 1$, $a_p \neq 0$. An argument like that in the proof of Lemma 3.5 shows that $\int_0^1 F(u(x,t))\,dx$ is an almost periodic function of t and that the corresponding Fourier series may be obtained by formal multiplication. We consider the coefficient of $\exp\{ip\lambda_n t\}$ in this Fourier series. This coefficient is equal to $c_p a_n^p \int_0^1 \sin^p \lambda_n x\,dx$ provided that there are no contributions from off-diagonal products; i.e., provided that if

$$p\lambda_n = \lambda_{r_1} + \cdots + \lambda_{r_s},$$

with $1 \leqq s \leqq p$ and the r_i integers, then $s=p$ and $\lambda_{r_j}=\lambda_n$ for each j. We shall show for fixed $s, p, n, r_1, \cdots, r_s$ that this is true for all but countably many α. The result then follows by taking for A_1 the union of the exceptional α values over $s, p, n, r_1, \cdots, r_s$ and applying the uniqueness theorem for the Fourier expansions of almost periodic functions to deduce that $c_p a_n^p \int_0^1 \sin^p \lambda_n x\,dx = 0$. Since a graphical argument shows that $\int_0^1 \sin^p \lambda_n x\,dx \neq 0$, this gives $a_n = 0$ for all n, and hence $u \equiv 0$. For each s, p with $1 \leqq s \leqq p$, and for each $r_1, \cdots, r_s$ with either $s < p$ or some $r_j \neq n$, let

$$g(\alpha) = \lambda_{r_1}(\alpha) + \cdots + \lambda_{r_s}(\alpha) - p\lambda_n(\alpha).$$

We have to show that g has at most countably many roots. First note that since $\tan \lambda_j(\alpha) + \alpha\lambda_j(\alpha) = 0$, it follows by differentiation that

$$(1+\alpha+\alpha^2\lambda_j^2)\lambda_j' + \lambda_j = 0, \qquad \lambda_j'(0) = -\lambda_j(0). \tag{4.28}$$

If g has uncountably many roots, then it has a finite cluster point of roots. By the analytic implicit function theorem, $\lambda_j(\alpha)$ is analytic in a neighbourhood of the real axis. Hence g is analytic there, and thus by the identity theorem $g(\alpha) \equiv 0$. We prove by induction that

$$g_m(\alpha) \equiv \lambda_{r_1}^{2m+1}(\alpha) + \cdots + \lambda_{r_s}^{2m+1}(\alpha) - p\lambda_n^{2m+1}(\alpha) = 0$$

for all α. This is true for $m=0$. Assume it is true for m. Multiply (4.28) by λ_j^{2m}. Thus

$$(1+\alpha)\left(\frac{\lambda_j^{2m+1}}{2m+1}\right)' + \alpha^2\left(\frac{\lambda_j^{2m+3}}{2m+3}\right)' + \lambda_j^{2m+1} = 0.$$

Hence

$$\frac{(1+\alpha)}{2m+1} g_m'(\alpha) + \frac{\alpha^2}{2m+3} g_{m+1}'(\alpha) + g_m(\alpha) = 0,$$

so that $g'_{m+1}(\alpha) \equiv 0$ and $g_{m+1}(\alpha) = \text{constant}$. But

$$g'_{m+1}(0) = (2m+3)[\lambda_{r_1}^{2m+2}(0)\lambda'_{r_1}(0) + \cdots + \lambda_{r_s}^{2m+2}(0)\lambda'_{r_s}(0) - p\lambda_n^{2m+2}(0)\lambda'_n(0)]$$
$$= -(2m+3)g_{m+1}(0) = 0 .$$

Therefore, $g_{m+1}(\alpha) = 0$. This completes the induction. We deduce in particular that

$$\frac{g_m(0)}{\pi^{2m+1}} = r_1^{2m+1} + \cdots + r_s^{2m+1} - pn^{2m+1}, \qquad m = 0, 1, 2, \cdots .$$

Let $P(t)$ be a polynomial with only odd powers; then it follows that

$$P(r_1) + \cdots + P(r_s) = pP(n) .$$

By the Weierstrass approximation theorem, if h is any odd continuous function,

$$h(r_1) + \cdots + h(r_s) = ph(n) .$$

Choose h such that $h(n) = 1$, $h(-n) = -1$, $h(t) = 0$ if $|t-n| \geqq \frac{1}{2}$ or $|t+n| \geqq \frac{1}{2}$. Then

$$h(r_1) + \cdots + h(r_s) = p ,$$

and so $s = p$ and $r_j = n$ for $1 \leqq j \leqq p$. This is a contradiction.

5. Concluding Remarks

As the reader will have observed, the principal difficulty in applying Theorem 3.2 to particular examples is the verification of the genericity condition (H2) or (H2+), which requires precise information about the eigenvalues λ_n^2 of A. There are several interesting generalizations of the examples in Section 4 to which our methods presumably apply, but which we have omitted because of the difficulty of checking the genericity condition. For example, consider the problem of stabilizing the equation

(5.1) $$u_{tt} + u_{xxxx} + P(t)u_{xx} = 0$$

under either of the boundary conditions (4.4), (4.5), subject to the additional requirement that in some sense $P(t) \rightarrow p_0$ as $t \rightarrow \infty$, where the constant p_0 is less than the first critical load. (For information about the critical load in the case of the boundary conditions (4.5) see Carr and Malhardeen [9].) If $p(t)=P(t)-p_0$, then the system has the form (3.1) with $A=d^4/dx^4+p_0\,d^2/dx^2$. The analogues of Propositions 4.1, 4.3 are now harder to prove on account of the complexity of the characteristic equation for the eigenvalues of A. In the case of the follower load problem (5.1), (4.5) there is the additional complication that A is no longer selfadjoint, although it can be made selfadjoint by a suitable change of coordinates (cf [9]).

The problem of verifying (H2), and especially (H2+), in applications to partial differential equations increases with the number of space dimensions. In particular, if $\lambda_{n+1}-\lambda_n\to 0$ as $n\to\infty$, then (H2+) cannot hold. Consider, for example, the case $H=L^2(\Omega)$, $A=-\Delta$, $D(A)=\{u\in H_0^1(\Omega):\Delta u\in L^2(\Omega)\}$, with Ω a bounded open subset of $\mathbb{R}^2$. If Ω is the unit square, then it is known that

$$N(t)=\sum_{\lambda_n^2<t}1=\frac{t}{4\pi}-\frac{t^{1/2}}{\pi}+O(t^{1/3}).$$

(For references see Hejhal [17]). Thus, for any $\varepsilon>0$, $N((t+\varepsilon)^2)-N(t^2)\to\infty$ as $t\to\infty$, so that $\lambda_{n+1}-\lambda_n\to 0$ as $n\to\infty$ and (H2+) does not hold. Known asymptotic estimates suggest, but do not appear to prove, that the same is in general true for elliptic operators in more general domains Ω and for dimensions greater than 2.

We did not make explicit use in this paper of Theorem 2.1. However, it may be used in certain cases to give an alternative proof of weak stabilizability. Suppose, under the general assumptions of Section 3, that B is symmetric and $(B\phi_n,\phi_m)=0$ for $m\neq n$. Suppose further that

$$\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)\geqq\gamma>0.$$

Let

$$u(t)=\sum_{n=1}^{\infty}(a_n\exp\{i\lambda_n t\}+\bar{a}_n\exp\{-i\lambda_n t\})\phi_n$$

be any solution of $\ddot{u}+Au=0$. By Theorem 2.1,

$$\int_0^T(Bu,\dot{u})^2\,dt\geqq C\sum_{n=1}^{\infty}\lambda_n^2(B\phi_n,\phi_n)^2|a_n|^4$$

for some positive constant $C=C(T)$, provided $T>\pi/\gamma$. Therefore,

$$\sup_{t\in\mathbb{R}}|u(t)|^4\leqq\left(2\sum_{n=1}^{\infty}|a_n|\right)^4\leqq C_1\left[\sum_{n=1}^{\infty}|\lambda_n(B\phi_n,\phi_n)|^{-2/3}\right]^3\int_0^T(Bu,\dot{u})^2\,dt,\tag{5.2}$$

and similarly,

$$\left(\int_0^T |u(t)|^2\,dt\right)^2 \leqq C_2\left[\sum_{n=1}^{\infty} |\lambda_n (B\phi_n, \phi_n)|^{-2}\right] \int_0^T (Bu, \dot{u})^2\,dt. \tag{5.3}$$

If the sum in square brackets is known to be finite, either of these inequalities can be used in conjunction with Lemma 3.1 and the invariance of the weak ω-limit set under $e^{\mathscr{A}t}$ to show that solutions of (3.3) tend weakly to zero as $t \to \infty$. The crucial point is that one may pass to the limit in the left-hand sides of (5.2), (5.3) using weak convergence. For the interest of the reader we make explicit these *a priori* inequalities in two special cases.

EXAMPLE 5.1. Consider the system

$$\begin{aligned} u_{tt} - u_{xx} &= 0\,, & 0<x<1\,,\\ u = 0 \quad \text{at} \quad x &= 0,1\,. \end{aligned} \tag{5.4}$$

For every $T>1$ there exists a constant $K(T)>0$ such that for all solutions u of (5.4)

$$\int_0^T \left(\int_0^1 u^2\,dx\right)^2 dt \leqq K \int_0^T \left(\int_0^1 u u_t\,dx\right)^2 dt. \tag{5.5}$$

In fact one may take $T=1$ in (5.5).

EXAMPLE 5.2. Consider the system

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0\,, & 0<x<1\,,\\ u = u_{xx} = 0 \quad \text{at} \quad x &= 0,1\,. \end{aligned} \tag{5.6}$$

From (5.2) we deduce that for every $T>0$, however small, there exists a constant $K(T)>0$ such that for all solutions u of (5.6)

$$\sup_{t\in\mathbf{R}} \left(\int_0^1 u^2\,dx\right)^2 \leqq K \int_0^T \left(\int_0^1 u_{xx} u_t\,dx\right)^2 dt. \tag{5.7}$$

A similar argument shows that the inequality

$$\sup_{t\in\mathbf{R}} \left(\int_0^1 u_x^2\,dx\right)^2 \leqq K_1(T) \int_0^T \left(\int_0^1 u_{xx} u_t\,dx\right)^2 dt$$

holds. But the inequality

$$\int_0^T \left(\int_0^1 u_{xx}^2\,dx\right)^2 dt \leqq K_2(T) \int_0^T \left(\int_0^1 u_{xx}u_t\,dx\right)^2 dt$$

is false. Were such an inequality true, then one could use Lemma 3.1 to prove strong convergence to zero in Example 4.1.

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