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Continuity Properties of Nonlinear Semigroups

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1. Introduction

Let X be a topological space. By a *semigroup* on X we mean a family of maps $\{T(t)\}t \geqslant 0$ from X to itself satisfying (i) $T(0) =$ identity, and (ii) $T(s)\, T(t) = T(s + t)$ for all $s, t \geqslant 0$. It is usual to require that the maps $\{T(t)\}t \geqslant 0$ satisfy certain properties related to continuity. In this paper we discuss several situations in which, by virtue of the semigroup properties, these continuity requirements in fact imply stronger ones. When X is a Banach space and $\{T(t)\}t \geqslant 0$ a semigroup of bounded linear maps from X to X such results are well known, e.g., if for a fixed $x \in X$ the map $t \mapsto T(t)x$ is strongly Lebesgue measurable on $(0, \infty)$, then it is strongly continuous on $(0, \infty)$.

In a recent paper [6] Chernoff and Marsden considered the case of a semigroup defined on a metric space M. They proved that for such a semigroup the conditions (i) $T(t)\colon M \to M$ is continuous for each $t \geqslant 0$, (ii) $t \mapsto T(t)x$ is continuous on $(0, \infty)$ for fixed $x \in M$, imply that the map $(t, x) \mapsto T(t)x$ is continuous on $(0, \infty) \times M$. Furthermore, they showed that the same conclusion holds if M is separable and if (ii) is replaced by the condition (ii)′ $t \mapsto T(t)x$ is Borel measurable for fixed $x \in M$. We extend their techniques to prove a number of analogous results for semigroups defined on certain function spaces. Our methods involve extensive use of the Baire category theorem.

A particular case that we treat, of importance for applications, is when X is an infinite-dimensional Banach space endowed with the weak topology (this space is not metrizable). We prove (Corollary 3.4) that, if X is reflexive, sequential weak continuity of the maps $T(t)$, $t \mapsto T(t)x$ implies sequential weak continuity of the map $(t, x) \mapsto T(t)x$ on $(0, \infty) \times X$. If X is an arbitrary Banach space we prove

(Theorem 5.2) that (a) strong (i.e., norm) continuity of $T(t)$ for each $t \geqslant 0$, and (b) weak Baire continuity of the map $t \mapsto T(t)x$ for each $x \in X$, implies strong continuity of $(t, x) \mapsto T(t)x$ on $(0, \infty) \times X$. In particular, the same conclusion holds if (b) is replaced by (b)′ weak continuity from the right at $t = 0$ of the map $t \mapsto T(t)x$ for fixed $x \in X$. This last result is proved in two distinct, though related, ways; firstly by application of Chernoff and Marsden's Borel measurability arguments, and secondly by a simpler method, based on lower semicontinuity, which appears to be new even for linear semigroups. The second proof is physically suggestive, in that an adaptation of it gives conditions under which one or more "energy" functionals are continuous along orbits of the semigroup (which may in fact be defined on any topological space). Under the same hypotheses strong continuity at $t = 0$ is shown to hold for certain special semigroups, provided X is uniformly convex, generalizing a result of Crandall and Pazy [7] for semigroups of nonexpansions on uniformly convex spaces. We remark that in [7] a proof, due to Phillips, is given, that for a semigroup of nonexpansions on an arbitrary Banach space, strong Lebesgue measurability of the map $t \mapsto T(t)x$ implies strong continuity of the same map on $(0, \infty)$.

Our results include those of Chernoff and Marsden that we have mentioned and have an application to semigroups on Tychonov spaces. The sequential weak continuity properties of semigroups that we consider turn up in applications to partial differential equations arising from, e.g., continuum mechanics [see e.g. 3, 4]. For the relationship between weak continuity and sequential weak continuity for maps and semigroups defined on a Banach space see [5]. Throughout this paper subscripts denote infinite sequences rather than nets. The symbols $\rightharpoonup$, $\rightharpoonup^*$ denote weak and weak* convergence respectively. $\mathbb{R}$ denotes the real numbers. $\mathscr{C}$ denotes "the complement of".

2. Semigroups on Function Spaces

Definition 2.1. A semigroup $\{T(t)\}t \geqslant 0$ defined on a topological space (X, τ) is *separately (sequentially) τ continuous* on $(0, \infty) \times X$ if $x_n \xrightarrow{X} x$, $t_n \to t > 0$, imply $T(t_n)x \xrightarrow{X} T(t)x$ and $T(t)x_n \xrightarrow{X} T(t)x$. It is *jointly (sequentially) τ continuous* on $(0, \infty) \times X$ if $x_n \xrightarrow{X} x$, $t_n \to t > 0$ imply $T(t_n)x_n \xrightarrow{X} T(t)x$.

When only one topology τ is under consideration we abbreviate the above to "separately continuous" and "jointly continuous".

Let E be a set and Y a metric space with metric d. Let $A \subseteq Y^E = \{f\colon E \to Y\}$. Throughout we suppose A to have the product topology, that is the topology of pointwise convergence on E. Let $\{T(t)\}t \geqslant 0$ be a semigroup on A that is separately continuous on $(0, \infty) \times A$.

THEOREM 2.1. *Let $x \in E$ and $f_n \xrightarrow{A} f$. Then there exists a dense G_δ subset G of $(0, \infty)$ such that if $t_0 \in G$ and $t_n \to t_0$ then*

$$T(t_n)f_n(x) \xrightarrow{Y} T(t_0)f(x).$$

Proof. For $0 < a < b < \infty$ and $\epsilon > 0$ define

$$S_{n,\epsilon} = \{t \in [a, b]\colon r \geqslant n \text{ implies } d(T(t)f_r(x), T(t)f(x)) \leqslant \epsilon\}.$$

$S_{n,\epsilon}$ is clearly closed and $[a, b] = \bigcup_{n=1}^{\infty} S_{n,\epsilon}$. By Baire's category theorem some $S_{n,\epsilon}$ contains an open interval. Since we may apply this argument to any such closed interval $[a, b]$, there exists a dense open subset S_ϵ of $(0, \infty)$ such that $t_0 \in S_\epsilon$ implies that there is an open neighbourhood $N(t_0)$ of t_0 and some m with $d(T(t)f_r(x), T(t)f(x)) \leqslant \epsilon$ for each $t \in N(t_0)$ and $r \geqslant m$. Then $G \equiv \bigcup_{i=1}^{\infty} S_{1/i}$ is a dense G_δ set and has the required properties. ∎

COROLLARY 2.2 (Chernoff and Marsden). *Let $\{S(t)\}t \geqslant 0$ be a semigroup on a metric space Y which is separately continuous on $(0, \infty) \times Y$. Then $\{S(t)\}$ is jointly continuous on $(0, \infty) \times Y$.*

Proof. Let $E = \{x_0\}$ be a singleton and let $y_n \xrightarrow{Y} y$. By the theorem, applied to the semigroup induced on Y^E by $\{S(t)\}$, there exists a dense G_δ subset G of$(0, \infty)$ such that $t_n \to t_0 \in G$ implies $T(t_n)\,y_n \xrightarrow{Y} T(t_0)\,y$. Let $t_1 > 0$ and choose $t_0 \in G$, $0 < t_0 < t_1$. Then if $\tau_n \to t_1$, $T(\tau_n)\,y_n = T(t_1 - t_0)\,T(t_0 - t_1 + \tau_n)\,y_n \xrightarrow{Y} T(t_1 - t_0)\,T(t_0)\,y = T(t_1)\,y$. ∎

THEOREM 2.3. *Under the hypotheses of Theorem 2.1 suppose that $A_1 \subseteq A$ and that $T([a, b])A_1$ is a relatively sequentially compact subset of A for any $0 < a < b < \infty$. Let $\{f_n\} \subseteq A_1$, $f_n \xrightarrow{A} f \in A$, $t_n \to t_0 > 0$. Then*

$$T(t_n)f_n \xrightarrow{A} T(t_0)f.$$

Proof. Choose t_1 with $0 < t_1 < t_0$. We can without loss of generality suppose that $T(t_n + t_1 - t_0)f_n \xrightarrow{A} \chi$, say. Then $T(t_n)f_n \xrightarrow{A} T(t_0 - t_1)\chi \equiv z$.

Let $x \in E$ and let G be the dense subset of $(0, \infty)$ corresponding to $\{f_n\}$ and f whose existence was proved in Theorem 2.1. There exists $\{\tau_\mu\} \subseteq G$, $\tau_\mu \to t_0$ as $\mu \to \infty$. Define $f_{n,\mu} = T(\tau_\mu + t_n - t_0)f_n$. Then

$$f_{n,\mu} = T(\tau_\mu - t_1)\, T(t_n - t_0 + t_1) f_n .$$

Hence

$$\lim_{\mu\to\infty} \lim_{n\to\infty} f_{n,\mu} = \lim_{\mu\to\infty} T(\tau_\mu - t_1)\chi = T(t_0 - t_1)\chi = z.$$

But since $\tau_\mu + t_n - t_0 \to \tau_\mu$ as $n \to \infty$ we have also that

$$\lim_{\mu\to\infty} \lim_{n\to\infty} f_{n,\mu}(x) = \lim_{\mu\to\infty} T(\tau_\mu) f(x) = T(t_0) f(x).$$

Hence $z(x) = T(t_0)f(x)$ for all $x \in E$, which completes the proof. ∎

3. Separate and Joint Continuity for Semigroups on Banach Spaces and Tychonov Spaces

It follows from Corollary 2.2 that for a semigroup defined on a subset of a Banach space, separate strong continuity implies joint strong continuity. In this section we discuss the case when the norm topology is replaced by the weak or weak* topology.

Definition 3.1. A Banach space X is said to be of *type A* if the closed unit ball of X^* is sequentially weak* compact.

Separable and reflexive Banach spaces are both of type A, in the former case by the Banach–Alaoglu theorem and the metrizability of the closed unit ball in X^* [8, p. 426], and in the latter case by the Eberlein–Smulyan theorem. Not all Banach spaces are of type A [13, p. 298].

Let X be a Banach space of type A and let $U \subseteq X^*$ be sequentially weak* closed. Let $\{T(t)\}t \geqslant 0$ be a semigroup on U which is separately weak* continuous on $(0, \infty) \times U$.

Lemma 3.2. *Let $V \subseteq U$ be bounded. Then if $0 < a < b < \infty$ the set $T([a, b])V$ is bounded.*

Proof. For each $t \geqslant 0$, $T(t)$ maps sequentially weak* compact sets to sequentially weak* compact sets, and thus bounded sets to bounded sets. Choose α, β with $0 < \alpha < \beta < a$. For $n = 1, 2, \ldots$, define $E_n = \{t \in [\alpha, \beta] : \| T(t)x^* \| \leqslant n \text{ for all } x^* \in V\}$. Since closed balls in X^* are weak* closed, E_n is closed. By the above $\bigcup_{n=1}^\infty E_n = [\alpha, \beta]$. Hence for some interval $(\gamma, \gamma + \delta) \subseteq (\alpha, \beta)$, $T([\gamma, \gamma + \delta])V$ is bounded.

Hence each of the sets

$$T(\delta)\, T([\gamma + r\delta, \gamma + (r+1)\delta])V$$
$$= T([\gamma + (r+1)\delta, \gamma + (r+2)\delta])V, r = 0, 1, 2, \ldots,$$

is bounded, and therefore $T([a, b])V$ is. ∎

THEOREM 3.3. *$\{T(t)\}t \geqslant 0$ is jointly weak* continuous on* $(0, \infty) \times U$.

Proof. In Theorem 2.3 set $E = X$, $Y = \mathbb{R}$, $A = U$. Then $\{T(t)\}t \geqslant 0$ is separately continuous on $(0, \infty) \times A$ with respect to the product topology on A. Let $x_n^* \rightharpoonup^* x^*$ in U. Define $A_1 = \{x^*, x_n^*, n = 1, 2, \ldots,\}$. Then A_1 is bounded, and by Lemma 3.2 $T([a, b])A_1$ is bounded for $0 < a < b < \infty$. Since X is of type A, $T([a, b])A_1$ is a relatively sequentially compact subset of A. Let $t_n \rightarrow t_0 > 0$. Then by Theorem 2.3 $T(t_n)x_n^* \rightharpoonup^* T(t_0)x^*$, which completes the proof. ∎

COROLLARY 3.4. *Let $\{S(t)\}t \geqslant 0$ be a semigroup on a sequentially weakly closed subset W of a reflexive Banach space X. If $\{S(t)\}t \geqslant 0$ is separately weakly continuous on* $(0, \infty) \times W$ *then it is jointly weakly continuous on* $(0, \infty) \times W$.

Remarks. 3.1. If X is separable Theorem 3.3 follows immediately from Corollary 2.2 and Lemma 3.2.

3.2. For an application of joint weak continuity to a stability problem for a nonlinear partial integro-differential equation see [4, Theorem 9.2].

3.3. In [3] the invariance principle for semigroups on reflexive separable Banach spaces with the weak topology was established under supposedly weaker continuity requirements than assumed by Slemrod [12]. Corollary 3.4 shows that these continuity requirements are basically the same.

3.4. If X is not reflexive Corollary 3.4 may be false, even when $W = X$ and X is separable. Indeed let $\{S(t)\}t \geqslant 0$ be the semigroup of *linear* maps on $C(S)$, where S is the circumference of the unit circle, defined by $(S(t)f)(s) = f(t + s)$, where the argument of f is taken modulo 2π. [$C(S)$ is the space of continuous real-valued functions on S with the supremum norm.] Recall that a sequence $\{f_n\}$ converges weakly to f in $C(S)$ if and only if $\{f_n\}$ is bounded and $f_n \rightarrow f$ pointwise on S [8, p. 265]. It follows that $\{S(t)\}t \geqslant 0$ is separately weakly continuous on $(0, \infty) \times C(S)$ (and, in addition, each $S(t)$ is strongly continuous). For small enough $\epsilon > 0$ define f_n by

$f_n(\epsilon) = f_n(\epsilon + 2/n) = 0$, $f_n(\epsilon + 1/n) = 1$, f_n linear in the intervals $[\epsilon, \epsilon + 1/n]$, $[\epsilon + 1/n, \epsilon + 2/n]$ and f_n zero elsewhere. Then $f_n \rightharpoonup 0$ in $C(S)$. But $(S(\epsilon + 1/n)f_n)(0) = 1$, while $(S(\epsilon)0)(0) = 0$. Thus $S(\epsilon + 1/n) f_n \not\rightharpoonup S(\epsilon)0$, which shows that $\{S(t)\}t \geqslant 0$ is not jointly weakly continuous on $(0, \infty) \times C(S)$.

THEOREM 3.4. *Let E be a Tychonov space (i.e., T_1 and completely regular). Let $\{S(t)\}t \geqslant 0$ be a separately continuous semigroup defined on a subset F of E. Suppose that for any $0 < a < b < \infty$ and any sequentially compact subset F_1 of F, the set $T([a, b])\ F_1$ is relatively sequentially compact in F (this will hold, in particular, if F is sequentially compact). Then $\{S(t)\}t \geqslant 0$ is jointly continuous on $(0, \infty) \times F$.*

Proof. Note that in the proof of Theorem 3, the fact that X is of type A and Lemma 3.2 are used only to show that $T([a, b])A_1$ is relatively sequentially compact. With this remark in mind the result follows immediately from the observation that E is homeomorphic to a subset of the closed unit ball in $C(E)$ endowed with the weak* topology, the homeomorphism being given by the map $x \mapsto \hat{x}$ for $x \in E$, where $\hat{x}(f) = f(x)$ for each $f \in C(E)$. [13, p. 103]. ∎

4. MEASURABILITY

In this section we outline the information on almost open sets and weak and strong measurability that we require. For background material we refer the reader to the discussions in Kuratowski [10] and Hille and Phillips [9].

Let E and F be topological spaces. Let $B(E)$ denote the Borel sets in E, the σ-algebra generated by the open sets in E. Define

$$S(E) = \{A \subseteq E : A = U \mathbin{\Delta} N \text{ for some open } U \subseteq E \text{ and first category set } N \subseteq E\}$$

It is easy to check [10, p. 88] that $S(E)$ is a σ-algebra. Elements of $S(E)$ are said to be *almost open*, or to have the *Baire property*. Clearly $B(E) \subseteq S(E)$; note, however, that $S(\mathbb{R}^n)$ neither includes nor is included in $L(\mathbb{R}^n)$, the Lebesgue measurable sets in $\mathbb{R}^n$ [10, p. 525]. Let R be a σ-algebra of subsets of E. A function $f: E \to F$ is said to be *R-measurable* if and only if $f^{-1}(V) \in R$ for each open $V \subseteq F$. Functions $f: E \to F$ which are $S(E)$-measurable are said to be *Baire continuous* (or *almost open* or to have the *Baire property*). Borel measurable functions are obviously Baire continuous.

We need the following theorem, used implicitly by Chernoff and Marsden.

THEOREM 4.1. *Let F be second countable. Let $f: E \to F$ be Baire continuous. Then f is continuous when restricted to the complement of a first category set M.*

A proof of this theorem for E, F metric (the argument is unchanged for our case) is given by Kuratowski [10, p. 400]. If E is a Baire space then the complement of M is clearly dense in E.

Now consider the case when $F = X$ is a Banach space. Let R be a σ-algebra of subsets of E. A function $f: E \to X$ is said to be *weakly R-measurable* if and only if $x^* \circ f$ is an R-measurable map from E to $\mathbb{R}$ for every $x^* \in X^*$. f is *separably-valued* if and only if $f(E)$ is separable. We now prove a version of Pettis' theorem [9, p. 72].

THEOREM 4.2. *Let R be a σ-algebra of subsets of E. Let $f: E \to X$. Then*

(i) *If f is R-measurable then f is weakly R-measurable,*

(ii) *If f is weakly R-measurable and separably-valued then f is R-measurable.*

Proof. Part (i) is trivial since $(x^* \circ f)^{-1}(U) = f^{-1}(x^{*-1}(U))$ for any $x^* \in X^*$ and open $U \subseteq \mathbb{R}$. To prove (ii) we may assume that X is separable. Then X possesses a countable determining set $\{x_n^*\} \subseteq X^*$; i.e., $\| x \| = \sup_n | x_n^*(x)|$ for each $x \in X$ [9, p. 34]. Let $x_0 \in X$ and define $B_r(x_0) = \{x \in X: \| x - x_0 \| \leqslant r\}$. Then

$$B_r(x_0) = \{x \in X : | x_n^*(x - x_0)| \leqslant r \quad n = 1, 2, \ldots\}.$$

Therefore

$$f^{-1}(B_r(x_0)) = \bigcap_{n=1}^{\infty} (x_n^* \circ f)^{-1}\{t \in \mathbb{R} : | t - x_n^*(x_0)| \leqslant r\},$$

which belongs to R. As X is separable it follows that $f^{-1}(V) \in R$ for any open $V \subseteq X$. ∎

Remarks. 4.1. Suppose $R = S(E)$. Then (ii) may be proved using the method in Hille and Phillips [9, Theorem 3.5.3, p. 72] and the result of Kuratowski [10, p. 401], with the stronger conclusion that f is the uniform limit of a sequence of countably-valued functions. Note, however, that if f is Baire continuous there need not exist a first category set $M \subseteq E$ such that $f(\mathscr{C}M)$ is separable. For example, let X be of uncountable dimension and let f be the identity map, which is Baire continuous. Then $\mathscr{C}M$ would be nowhere dense and hence $X = M \cup \mathscr{C}M$ would be a countable union of nowhere dense sets, which is impossible. Thus with our definition of R-measurability Hille and Phillips' Theorem 3.5.3 does not have a direct analogue.

4.2. One alternative way of defining weak R-measurability of $f: E \to X$ is to require that $f^{-1}(U) \in R$ for any weakly open subset of X. This kind of measurability is implied by R-measurability and implies weak R-measurability, and thus in a separable space all three concepts are equivalent.

5. Strong and Weak Continuity

The following theorem is due essentially to Chernoff and Marsden, who assume that X is a separable metric space.

THEOREM 5.1. *Let $\{T(t)\}t \geqslant 0$ be a semigroup on a second countable topological space X. Suppose that for each $t \geqslant 0$ the map $T(t)$: $X \to X$ is sequentially continuous, and that for each $x \in X$ the map $t \mapsto T(t)x$ is Baire continuous. Then the map $t \mapsto T(t)x$ is continuous on* $(0, \infty)$.

Proof. By Theorem 4.1 f is continuous when restricted to a dense G_δ subset G of $(0, \infty)$. Let $t_0 > 0$ and $t_n \to t_0$. We can suppose that $t_n > \frac{1}{2}t_0$ for each n. Define $S_n = \{s \in (0, \infty): s + t_n - \frac{1}{2}t_0 \in G\}$, which is a dense G_δ subset of $(0, \infty)$. Then $S \equiv \bigcup_{n=0}^{\infty} S_n$ is a dense subset of $(0, \infty)$. Choose $s \in S$ with $0 < s < \frac{1}{2}t_0$. Then for $x \in X$, $T(t_n)x = T(\frac{1}{2}t_0 - s)\, T(s + t_n - \frac{1}{2}t_0)x \xrightarrow{X} T(t_0)x$. ∎

The next theorem generalizes to nonlinear semigroups the Corollary to Theorem 10.2.3 in Hille and Phillips [9, p. 306].

THEOREM 5.2. *Let $\{T(t)\}t \geqslant 0$ be a semigroup on a subset W of a Banach space X. Suppose that for each $t \geqslant 0$ the map $T(t)$: $W \to W$ is strongly continuous, and that for each $x \in W$ the map $t \mapsto T(t)x$ is weakly continuous from the right on* $(0, \infty)$. *Then the map $(t, x) \mapsto T(t)x$ is sequentially continuous on* $(0, \infty) \times W$ *with respect to the strong topology.*

Proof. Let $x \in W$. Define X_1 to be the closed linear subspace of X spanned by $\{T(q)x: q > 0 \text{ rational}\}$. Then X_1 is separable and $T(t)x \in X_1$ for each $t > 0$. Let $E = (0, \infty)$ and define $\theta: E \to X_1$ by $\theta(t) = T(t)x$. Since θ is weakly continuous from the right it is weakly $B(E)$-measurable, and hence weakly $S(E)$-measurable. By Theorem 4.2 θ is $S(E)$-measurable, and by Theorem 5.1 θ is strongly continuous on $(0, \infty)$. The result now follows from Corollary 2.2. ▌

Remarks. 5.1. The map $t \mapsto T(t)x$ will, of course, be weakly continuous from the right on $(0, \infty)$ if it is weakly continuous from the right at $t = 0$.

5.2. The proof shows that the map $t \mapsto T(t)x$ will be strongly continuous on $(0, \infty)$ under the weaker continuity requirements that $t \mapsto T(t)x$ is weakly continuous from the right on $(0, \infty)$ and that for each $t \geqslant 0$, $T(t)$ is strongly continuous when restricted to orbits of the semigroup.

5.3. A similar proof to that of the theorem, based on Corollary 3.4, shows that for a semigroup $\{S(t)\}t \geqslant 0$ defined on a sequentially weakly closed subset W of a reflexive Banach space X, weak continuity from the right of the maps $t \mapsto S(t)x$, $t \in (0, \infty)$, together with sequential weak continuity of $S(t)$ for each $t \geqslant 0$, implies joint sequential weak continuity of the map $(t, x) \mapsto S(t)x$ on $(0, \infty) \times W$. Weak continuity of $t \mapsto S(t)x$ on $(0, \infty)$ follows under the weaker assumptions (a) $t \mapsto S(t)x$ is weakly continuous from the right on $(0, \infty)$ and (b) for each $t \geqslant 0$ $T(t)x_n \rightharpoonup T(t)x$ whenever $x_n \to x$ strongly.

5.4. In applications sequential weak continuity of $t \mapsto T(t)x$ on $(0, \infty)$ can sometimes be established using a result of Lions and Magenes [11, p. 297].

We now consider under what conditions $t \mapsto T(t)x$ is strongly continuous from the right at $t = 0$. First note that this will happen under the conditions of Theorem 5.2 if $W = \bigcup_{\tau>0} T(\tau)W$; for in this case if $t_n \to 0+$ and $x = T(\tau)y \in W$ then $T(t_n)x = T(t_n + \tau)x \to T(\tau)y = x$.

DEFINITION 5.1. A semigroup $\{T(t)\}t \geqslant 0$ on a subset W of a Banach space X is *nonexpansive at* $t = 0$ if $t_n \to 0+$ implies

$$\limsup_{n\to\infty} \| T(t_n)x - T(t_n)y \| \leqslant \| x - y \| \qquad \text{for all} \quad x, y \in W.$$

Clearly a semigroup of nonexpansions is nonexpansive at $t = 0$. The following theorem, together with Theorem 5.2, therefore generalizes a result of Crandall and Pazy [7] for a semigroup of nonexpansions defined on a closed convex subset W of a uniformly convex space. We impose no conditions on W and give a simpler proof.

THEOREM 5.3. *Let $\{T(t)\}t \geqslant 0$ be a semigroup on a subset W of a uniformly convex Banach space X such that*

(a) *$\{T(t)\}t \geqslant 0$ is nonexpansive at $t = 0$,*

(b) *for each $x \in W$ the map $t \mapsto T(t)x$ is weakly continuous from the right at $t = 0$, and*

(c) *for each $t \geqslant 0$ $T(t)\colon W \to W$ is strongly continuous. Then $t \mapsto T(t)x$ is strongly continuous from the right at $t = 0$.*

Proof. Let $x \in W$ and $t_n \to 0+$. Then $T(1)x \in W$. Therefore

$$\limsup_{n\to\infty} \| T(t_n)\, x - T(t_n)\, T(1)\, x \| \leqslant \| x - T(1)\, x \|. \tag{5.1}$$

But by (b),

$$\| x - T(1)\, x \| \leqslant \liminf_{n\to\infty} \| T(t_n)\, x - T(t_n)\, T(1)\, x \|. \tag{5.2}$$

Combining (5.1) and (5.2) we have that $\| T(t_n)x - T(t_n + 1)x \| \to \| x - T(1)x \|$. By uniform convexity $T(t_n)x - T(t_n + 1)x \to x - T(1)x$ strongly. But by Theorem 5.2 and Remark 5.1 $T(t_n + 1)x \to T(1)x$ strongly. Hence $T(t_n)x \to x$ strongly. ∎

6. ORBIT CONTINUITY OF SEMICONTINUOUS FUNCTIONS

Let $\{T(t)\}t \geqslant 0$ be a semigroup on a topological space S. Associated with S we consider a family $\mathscr{V}$ of at most countably many functions $V\colon S \to R$ each of which is *sequentially lower semicontinuous*, i.e., $x_n \xrightarrow{S} x$ implies $V(x) \leqslant \liminf_{n\to\infty} V(x_n)$—this property ensures that if V is bounded below on a sequentially compact subset E of S then a minimizer exists for V in E. An interesting special case is when $\mathscr{V}$ consists of a single function $V = V(x)$ which measures the energy of a state x of a physical system modelled by the semigroup $\{T(t)\}t \geqslant 0$, and in this case a minimizer of V on a subset E of S may represent a "stable" equilibrium state. The existence of such equilibrium states for problems in one and two-dimensional nonlinear elasticity has been established in this way by Antman [1, 2], the spaces used being various reflexive Banach spaces endowed with the weak topology. In this section we give conditions under which each $V \in \mathscr{V}$ is continuous along orbits of the semigroup, so that "shocks" do not occur. As an application we give a simple proof of Theorem 5.2.

Our results are based on the following lemma.

LEMMA 6.1. *Let f be a real-valued function on* $(0, \infty)$ *which is lower semicontinuous from the right. Then f is continuous at all points of a dense G_δ subset G_1 of* $(0, \infty)$.

Proof. Let $0 < a < b < \infty$ and $\epsilon > 0$. For each positive integer n the set $E_n = \{t \in [a, b]: f(t) \leqslant n\}$ is closed from the right, and $[a, b] = \bigcup_{n=1}^{\infty} E_n$. Hence some E_n contains an open interval (α, β). Let $\sup_{t\in(\alpha,\beta)} f(t) = k$. There exists $t_0 \in (\alpha, \beta)$ with $f(t_0) \geqslant k - \epsilon$. Then there exists $\delta > 0$ with $t_0 + \delta \leqslant \beta$ such that $s \in (t_0, t_0 + \delta)$ implies $f(t_0) \leqslant f(s) + \epsilon$ and thus $|f(s) - f(t_0)| \leqslant \epsilon$. Thus $s_1, s_2 \in (t_0, t_0 + \delta)$ implies $|f(s_1) - f(s_2)| \leqslant 2\epsilon$. Applying the same argument to every $[a, b] \subseteq (0, \infty)$ leads, as in the proof of Theorem 2.1, to the required result. ∎

Remark 6.1. This proof is due to Dr. A. M. Davie and is reproduced here with his kind permission. Another proof follows simply from Theorem 4.1, and it is interesting to compare the two results.

Now let $x \in S$, and for each $V \in \mathscr{V}$ define $\theta_V: (0, \infty) \to \mathbb{R}$ by $\theta_V(t) = V(T(t)x)$.

LEMMA 6.2. *If the map $t \mapsto T(t)x$ is continuous from the right on* $(0, \infty)$ *then there is a dense G_δ subset G_2 of* $(0, \infty)$ *at the points of which each θ_V is continuous.*

Proof. Each θ_V is lower semicontinuous from the right on $(0, \infty)$. The result follows from Lemma 6.1 and the fact that $\mathscr{V}$ is at most countable. ∎

We now reprove Theorem 5.2.

Second proof of Theorem 5.2. With the notation of the original proof let $S = X_1$ endowed with the weak topology. Let $\{y_i\}$ be dense in X_1 and let $\mathscr{V} = \{V_i\}$, where V_i is defined by $V_i(x) = \| x - y_i \|$ and is sequentially weakly lower semicontinuous. Note that if $z \in X_1$, $\{z_n\} \subseteq X_1$ and $\lim_{n\to\infty} V_i(z_n) = V_i(z)$ for all i, then $z_n \xrightarrow{X_1} z$ strongly. Let G_2 be as in Lemma 6.2 and let $t_n \to t_0 \in G_2$. Then letting $z_n = T(t_n)x$ we see that $T(t_n)x \xrightarrow{X_1} T(t_0)x$ strongly. The result follows from the strong continuity of $T(t)$ for $t \geqslant 0$. ∎

Reverting to the hypotheses made at the beginning of this section, we make the following definition:

DEFINITION 6.1. The semigroup $\{T(t)\}t \geqslant 0$ is said to be $\mathscr{V}$*-continuous* if the conditions $x_n \xrightarrow{S} x$, $V(x_n) \to V(x)$ for all $V \in \mathscr{V}$, imply that $V(T(t)x_n) \to V(T(t)x)$ for all $V \in \mathscr{V}$ and $t \geqslant 0$.

The proof of the following theorem is immediate from Lemma 6.2 and is thus omitted.

THEOREM 6.3. *Let $\{T(t)\}t \geqslant 0$ be $\mathscr{V}$-continuous. Let $x \in X$. If the map $t \mapsto T(t)x$ is continuous (resp. continuous from the right) on $(0, \infty)$ then for each $V \in \mathscr{V}$ the map $t \mapsto V(T(t)x)$ is continuous (resp. continuous from the right) on $(0, \infty)$.*

Remarks 6.2. By using the methods of Section 5 it can be shown that Theorem 6.3 holds when the semicontinuity conditions on $\mathscr{V}$ are replaced by the condition that the map $t \mapsto V(T(t)x)$ is Baire continuous for each $V \in \mathscr{V}$. Under this hypothesis, and if S is a subset of a Banach space endowed with the weak topology, the maps $t \mapsto V(T(t)x)$ will, in fact, be continuous on $(0, \infty)$ if the map $t \mapsto T(t)x$ is continuous from the right on $(0, \infty)$.

6.3. Let S be a subset of a Banach space X endowed with the weak topology, and suppose that $\mathscr{V}$ consists of a single strongly continuous function V. Then it may happen in applications that $x_n \xrightarrow{S} x$, $V(x_n) \to V(x)$, imply that $x_n \xrightarrow{S} x$ strongly. (e.g., let X be uniformly convex and $V(\cdot) \equiv \|\cdot\|$.) In such cases $\mathscr{V}$-continuity is implied by weak continuity of $t \mapsto T(t)x$ on $(0, \infty)$ and strong continuity of $T(t)$, $t \geqslant 0$.

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