REMARKS ON BLOW-UP AND NONEXISTENCE THEOREMS FOR NONLINEAR EVOLUTION EQUATIONS

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1. Introduction

OVER the last 20 years a large literature has developed concerning evolution equations which for certain initial data possess solutions that do not exist for all time. The bulk of this literature relates to problems arising from partial differential equations. To establish nonexistence it is customary to argue by contradiction. One supposes that for given u_0 and t_0 a solution $u(t)$ with $u(t_0) = u_0$ exists for all times $t \geq t_0$; typically u takes values in some Banach space X and we will assume that this is the case. A function $\rho : X \to \mathbb{R}$ is then constructed, and by use of differential inequalities it is shown that $\lim_{t \to t_1} \rho(u(t)) = \infty$ for some $t_1 \in (t_0, \infty)$. This usually leads immediately to a contradiction. It follows that if $u : [t_0, t_{max}) \to X$ is a maximally defined solution satisfying $u(t_0) = u_0$ then $t_{max} < \infty$.

The above argument, which possesses several variants, while it is quite correct as a proof both of nonexistence and of the fact that $t_{max} \leq t_1$, does not by itself establish that nonexistence occurs by 'blow-up', that is

$$\lim_{t \to t_{max}} \rho(u(t)) = \infty, \tag{1.1}$$

since it may happen that $t_{max} < t_1$. This observation puts into question the claims in a number of papers (see the references in Sections 3 and 4) that solutions of certain partial differential equations blow up in finite time. The examination of the validity of these claims is the purpose of this paper.

That some care is necessary in the interpretation of formal blow-up arguments is illustrated by an example of a backward nonlinear heat equation with Dirichlet boundary conditions discussed in Section 2. For this example an argument of the type described in the first paragraph correctly proves nonexistence, but

$$\lim_{t \to t_{max}} \rho(u(t))$$

exists and is finite.

The methods used in this paper to establish finite time blow-up of solutions in certain cases are based on continuation theorems for ordinary differential equations in Banach space. Several such theorems in a variety of contexts are given in **[2]**, **[7]**, **[9]**, **[10]**, **[21]**, **[26]**. The simplest type of continuation theorem says that if $t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}} \rho_1(u(t)) = \infty, \tag{1.2}$$

where $\rho_1 : X \to \mathbf{R}$ is some function (typically a norm). One may thus combine a nonexistence argument of the type described in the first paragraph with a continuation theorem to prove blow-up in the sense of (1.2). In most examples (1.2) turns out to be a weaker assertion than (1.1) (for instance, ρ may be an L^2-norm and ρ_1 the norm in some Sobolev space). One is thus led to ask whether in fact (1.1) holds, or at least some stronger property than (1.2). In other words, does the formal blow-up argument give the right answer? This question is investigated in Section 3 for the parabolic problem

$$\begin{aligned} &u_t = \Delta u + |u|^{\gamma-1} u, \qquad x \in \Omega, \\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{1.3}$$

and in Section 4 for the hyperbolic problem

$$\begin{aligned} &u_{tt} = \Delta u + |u|^{\gamma-1} u, \qquad x \in \Omega, \\ &u|_{\partial\Omega} = 0 . \end{aligned} \tag{1.4}$$

In (1.3) and (1.4) $\gamma > 1$ is a constant and Ω is a bounded open subset of $\mathbf{R}^n$. For certain initial data blow-up is established of solutions to both these problems in various norms depending on the value of γ.

For general information and references on nonexistence theorems proved by blow-up methods the reader is referred to Payne **[19]** and Straughan **[23]**.

Notations. The norms in the spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ are denoted by $\| \|_p$, $\| \|_{1,p}$ respectively. C denotes a generic constant.

2. An example of nonexistence without blow-up

Before giving the example we describe some preliminaries needed both in this section and in Section 3.

Let X be a real Banach space and let A be the generator of a holomorphic semigroup $T(t)$ of bounded linear operators on X. Suppose that $\|T(t)\| \leq M$ for some constant $M > 0$ and all $t \in \mathbf{R}^+$, and that A^{-1} is a bounded linear operator defined on all of X. Under these hypotheses the fractional powers $(-A)^\alpha$ can be defined for $0 \leq \alpha < 1$ (cf Henry [**7**], Pazy [**20**]) and $(-A)^\alpha$ is a closed linear operator with domain $D((-A)^\alpha) \stackrel{\text{def}}{=} X_\alpha$ dense in X. X_α is a Banach space under the norm $\|u\|_{(\alpha)} = \|(-A)^\alpha u\|$. Let $f: X_\alpha \to X$ be *locally Lipschitz*, i.e. for each bounded subset U of X_α there exists a constant C_U with

$$\|f(u) - f(v)\| \leq C_U \|u - v\|_{(\alpha)}$$

for all u, $v \in U$. Consider the equation

$$\dot{u} = Au + f(u). \tag{2.1}$$

DEFINITION. A *solution* of (2.1) on an interval $[0, t_1)$ is a function $u \in C([0, t_1); X_\alpha) \cap C^1((0, t_1); X)$ such that $u(t) \in D(A)$ and satisfies (2.1) for each $t \in (0, t_1)$.

The following proposition holds (cf Henry [**7**], Pazy [**20**]):

PROPOSITION 2.1. *Let* $u_0 \in X_\alpha$. *There exists* $t_1 > 0$ *and a unique solution* u *of* (2.1) *on* $[0, t_1)$ *with* $u(0) = u_0$.

Example. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$. Let $\phi \in L^2(\Omega)$. Consider the initial boundary value problem for $u = u(x, t)$,

$$u_t = \Delta u - \left\{\int_\Omega u^2 \, dx\right\} u, \qquad x \in \Omega, \ t > 0, \tag{2.2}$$

$$u = 0, \qquad x \in \partial\Omega, \ t > 0, \tag{2.3}$$

$$u(x, 0) = \phi(x), \qquad x \in \Omega. \tag{2.4}$$

Let $X = L^2(\Omega)$, $D(A) = \{v \in W_0^{1,2}(\Omega) : \Delta v \in L^2(\Omega)\}$, $A = \Delta$. It is well known that A satisfies the hypotheses listed above. Define $f: X \to X$ by $f(v) = -\|v\|_X^2 v$. f is locally Lipschitz, so that by Proposition 2.1 a unique solution u exists on some interval $[0, t_1)$, $t_1 > 0$. Multiplying (2.2) by u_t and integrating over Ω we see that u is bounded in $W_0^{1,2}(\Omega)$ for large t, so that u is defined for all $t \in \mathbb{R}^+$ (see Theorem 3.1 below). We write $u(t) = u(\cdot, t)$.

Consider the backward problem corresponding to (2.2)-(2.4) with initial data $\psi \in X$, namely

$$v_t = -\Delta v + \left\{\int_\Omega v^2 \, dx\right\} v, \qquad x \in \Omega, \ t > 0, \tag{2.5}$$

$$v = 0, \qquad x \in \partial\Omega, \ t > 0, \tag{2.6}$$

$$v(x, 0) = \psi(x), \qquad x \in \Omega. \tag{2.7}$$

By a *solution* of (2.5)-(2.7) on an interval $[0, \tau_1]$, $\tau_1 > 0$, we mean a function $v \in C([0, \tau_1]; X)$, $v(0) = \psi$, such that $u(t) \overset{\text{def}}{=} v(\tau_1 - t)$ is a solution of (2.2)-(2.3) on $[0, \tau_1)$. Suppose now that $\phi \in X \backslash D(A)$, let u be the solution of (2.2)–(2.4), and let $\psi = u(1)$. Clearly $v(t) \overset{\text{def}}{=} u(1-t)$ is a solution of (2.5)–(2.7) on $[0, 1]$. Furthermore v cannot be extended to a solution on any larger interval $[0, \tau_1]$, $\tau_1 > 1$, since otherwise by the smoothing properties of the forward equation ϕ would belong to $D(A)$, which is not the case. Let $\rho(\cdot) = \|\cdot\|_X^2$. Clearly

$$\lim_{t \to 1} \rho(v(t))$$

exists and equals $\|\phi\|_X^2$. Nevertheless one may give an alternative proof of nonexistence of global solutions to (2.5)–(2.7) by means of a 'blow-up' argument. Suppose $\psi \neq 0$ in $L^2(\Omega)$ and assume that a solution $v(t)$ of (2.5)–(2.7) exists and is defined for all $t \in \mathbb{R}^+$. Let $F(t) \overset{\text{def}}{=} \rho(v(t))$.

Then

$$\dot{F}(t) = 2[\|\nabla v(t)\|_X^2 + F^2(t)] \geq 2F^2(t).$$

Hence

$$F(t) \geq \frac{1}{F^{-1}(0) - 2t},$$

so that

$$\lim_{t \to F^{-1}(0)/2} \rho(v(t)) = \infty,$$

which is a contradiction.

Remark. A similar phenomenon occurs for the backward problem

$$v_t = -\Delta v + |v|^{\gamma-1} v, \qquad x \in \Omega,\ t > 0,$$

$$v|_{\partial\Omega} = 0,$$

where $\gamma > 1$ is such that the corresponding forward problem is well behaved (see Section 3).

3. Parabolic equations

We use the following continuation theorem:

THEOREM 3.1 *Let the hypotheses of Proposition* 2.1 *hold. Then u may be extended to a maximal interval of existence* $[0, t_{max})$. *If* $t_{max} < \infty$ *then*

$$\overline{\lim_{t \to t_{max}}} \int_0^t (t-\tau)^{-\alpha} \|f(u(\tau))\| \, d\tau = \infty, \tag{3.1}$$

and

$$\lim_{t \to t_{max}} \|u(t)\|_{(\alpha)} = \infty. \tag{3.2}$$

Proof. The existence of a maximally defined solution follows in the usual way from Zorn's lemma. Let $t_{max} < \infty$. For each $t \in [0, t_{max})$ u satisfies the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(u(s)) \, ds. \tag{3.3}$$

Following Henry [7] we first show that

$$\overline{\lim_{t \to t_{max}}} \|u(t)\|_{(\alpha)} = \infty. \tag{3.4}$$

If (3.4) does not hold then $\|u(t)\|_{(\alpha)} \leq C$ for all $t \in [0, t_{max})$. If $\alpha < \beta < 1$ then from (3.3) and the estimate $\|(-A)^\beta T(t)\| \leq Ct^{-\beta}$, $t > 0$, it follows that

$$\|u(t)\|_{(\beta)} \leq Ct^{-\beta} \|u_0\| + C \int_0^t (t-s)^{-\beta} \, ds \cdot \sup_{\tau \in [0, t_{max})} \|f(u(\tau))\|,$$

so that $\|u(t)\|_{(\beta)}$ is bounded as $t \to t_{max}$. For $0 \leq \tau < t < t_{max}$,

$$u(t) = T(t-\tau)u(t) + \int_\tau^t T(t-s) f(u(s)) \, ds. \tag{3.5}$$

Using the estimate $\|(-A)^{\alpha-\beta} (T(t) - I)\| \leq Ct^{\beta-\alpha}$, $t > 0$, we obtain

$$\|u(t) - u(\tau)\|_{(\alpha)} \leq C(t-\tau)^{\beta-\alpha} \|u(\tau)\|_{(\beta)} + C \int_\tau^t (t-s)^{-\alpha} \, ds \leq C(t-\tau)^{\beta-\alpha}.$$

Thus $\lim_{t \to t_{max}} u(t)$ exists in X_α, contradicting the maximality of t_{max}.

Now suppose that (3.2) does not hold. By (3.4) there exist numbers $r > 0$, $d > 0$, with d arbitrarily large, and sequences $\tau_n \to t_{max}$, $t_n \to t_{max}$ as

$n \to \infty$ with $\tau_n < t_n < \tau_{n+1}$ such that $\|u(\tau_n)\|_{(\alpha)} = r$, $\|u(t_n)\|_{(\alpha)} = r+d$ and $\|u(t)\|_{(\alpha)} \leq r+d$ for $t \in [\tau_n, t_n]$. Thus, using (3.5),

$$d \leq \|u(t_n) - u(\tau_n)\|_{(\alpha)} \leq C + C(d)(t_n - \tau_n)^{1-\alpha},$$

so that for large enough d, $t_n - \tau_n \geq k > 0$, contradicting $t_{\max} < \infty$.
Finally, if (3.1) were false, then by (3.3) we would have $\|u(t)\|_{(\alpha)} \leq C$ for $t \in [0, t_{\max})$, contradicting (3.2).

Remarks. An alternative proof of (3.2) is to show that one can get local existence to (3.3) on a time interval independent of u_0 in any bounded subset of X_α. The proof given above has the advantage that it extends immediately to cases when solutions for given initial data are not unique. The result (3.2) is a consequence of work of Kielhöfer [9], who omits the proof that (3.4) implies (3.2).

We use Theorem 3.1 to prove blow-up results for the initial boundary value problem

$$u_t = \Delta u + |u|^{\gamma-1} u, \qquad x \in \Omega, \tag{3.6}$$
$$u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0(x),$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $u_0 \in W_0^{1,2}(\Omega)$, and $\gamma > 1$ is a constant. This problem has been studied by Kaplan [8], Fujita [3], [4] and Levine [13], [17], but the results in these papers concerning blow-up are open to the objections made in the introduction.

Define the energy functional $E : W^{1,2}(\Omega) \cap L^{\gamma+1}(\Omega) \to \mathbb{R}$ by

$$E(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{\gamma+1}|u|^{\gamma+1}\right] dx. \tag{3.7}$$

THEOREM 3.2. *If $n = 1$ or 2 let $\gamma > 1$ be arbitrary. If $n \geq 3$ let $1 < \gamma \leq n/n-2$. Let $u_0 \in W_0^{1,2}(\Omega)$. Then there exists a unique solution u to* (3.6) *defined on a maximal interval of existence $[0, t_{\max})$ and satisfying $u \in C([0, t_{\max}); W_0^{1,2}(\Omega))$, $u \in C^1((0, t_{\max}); L^2(\Omega))$, $u(t) \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ for all $t \in (0, t_{\max})$. If $E(u_0) \leq 0$ and $u_0 \neq 0$ (such u_0 exist since $\gamma > 1$) then $t_{\max} < \infty$ and*

$$\lim_{t \to t_{\max}} \int_\Omega |u|^{\gamma+1}(t)\, dx = \infty.$$

Proof. Let $X = L^2(\Omega)$, and define $A = \Delta$, $D(A) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Then $X_{1/2} = W_0^{1,2}(\Omega)$. Let $f(u) = |u|^{\gamma-1}u$. By the Sobolev imbedding theorems and our hypotheses on γ it follows that $f : X_{1/2} \to X$ and is locally Lipschitz. By Proposition 2.1 and Theorem 3.1 there exists a maximally defined solution u. The energy inequality

$$E(u(t)) \leq E(u_0), \qquad t \in [0, t_{max}) \tag{3.8}$$

follows by multiplying (3.6) by u_t. Let $F(t) = \int_\Omega u^2 \, dx$. Then using (3.7), (3.8) we obtain

$$\dot{F}(t) = 2 \int_\Omega \left[|u|^{\gamma+1} - |\nabla u|^2 \right] dx$$

$$\geq -4E(u_0) + k \int_\Omega |u|^{\gamma+1} \, dx, \tag{3.9}$$

where $k = 2(\gamma-1)/(\gamma+1)$. If $E(u_0) \leq 0$ it follows that $\dot{F} \geq k_1 F^{(\gamma+1)/2}$, where $k_1 = km(\Omega)^{(\gamma-1)/2}$ and m denotes n-dimensional Lebesgue measure. Therefore

$$F(t)^{(\gamma-1)/2} \geq \left[F(0)^{-(\gamma-1)/2} - \frac{\gamma-1}{2} k_1 t \right]^{-1}$$

for $t \in [0, t_{max})$, so that if $u_0 \neq 0$

$$t_{max} \leq \frac{2}{k_1(\gamma-1)} F(0)^{-(\gamma-1)/2} < \infty.$$

By Theorem 3.1 and the Poincaré inequality,

$$\lim_{t \to t_{max}} \int_\Omega |\nabla u|^2(t) \, dx = \infty.$$

The result follows from (3.8).

Note that the formal blow-up argument used (rigorously) in the proof suggests that the stronger result

$$\lim_{t \to t_{max}} \int_\Omega |u|^2(t) \, dx = \infty \tag{3.10}$$

holds. The next theorem establishes (3.10) under stronger conditions on γ. It would be interesting to know if these conditions are essential or represent a deficiency of the method.

THEOREM 3.3. *Let* $1<\gamma<\min(3,1+4/n)$ *if* $n\leq 4$, $1<\gamma\leq n/n-2$ *if* $n>4$. *Let* $u_0\in W_0^{1,2}(\Omega)$, $u_0\neq 0$, $E(u_0)\leq 0$. *Then the solution u whose existence is proved in Theorem* 3.2 *satisfies* (3.10).

Proof. Choose α and p with $\max(1,2(\gamma+1)/(\gamma^2+1))<p<2/(\gamma-1)$, $(\gamma-1)n/4<\alpha<1$. Let $X=L^p(\Omega)$, $A=\Delta$, $D(A)=W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$ and let $f(u)=|u|^{\gamma-1}u$. We have that $\|f(u)\|_p\leq\|u\|_q\|u\|_2^{\gamma-1}$, where $1/q=1/p-(\gamma-1)/2$. Since $1/q>1/p-2\alpha/n$ it follows (cf Henry [7]) that $X_\alpha\subset L^q$ with continuous injection. Suppose that $\|u(t)\|_2\leq C$ for $t\in[0,t_{\max})$. From (3.3) we obtain

$$\|u(t)\|_{(\alpha)}\leq Ct^{-\alpha}+C_1\int_0^t(t-\tau)^{-\alpha}\|u(t)\|_{(\alpha)}\,d\tau.$$

Applying a version of Gronwall's lemma (cf [7]) we find that $\|u(t)\|_{(\alpha)}$ is bounded as $t\to t_{\max}$. Since $1/(\gamma+1)>1/p-2\alpha/n$, this implies also that $\|u(t)\|_{\gamma+1}$ is bounded, which contradicts Theorem 3.2. The result follows since $\|u(t)\|_2$ is increasing.

Remark. Another way to obtain (3.10) is to estimate $\int_0^{t_{\max}}\int_\Omega|u|^{\gamma+1}dx\,dt$ by (3.1), and then use (3.9). However this seems to work for $n\leq 4$ only if $1<\gamma<1+2/n$.

4. Hyperbolic equations

As an example of blow-up of solutions for a hyperbolic partial differential equation we treat the problem

$$u_{tt}=\Delta u+|u|^{\gamma-1}u,\qquad x\in\Omega, \tag{4.1}$$

$$u|_{\partial\Omega}=0,\qquad u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x),$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and where $\gamma>1$ is a constant. This problem has been considered by Glassey [6], Levine [14]–[17] and Tsutsumi [25], but the arguments in these papers establish nonexistence, rather than blow-up, of solutions.

We use the following local existence and continuation theorem. For the existence part of the proof see Segal [22], Reed [21], von Wahl [26]. The assertions (4.3) and (4.4) are proved in a similar way to the analogous statements in Theorem 3.1 (see also Ball [2] Theorem 5.9). The assertion (4.4) is a special case of von Wahl [26].

THEOREM 4.1. *Let X be a real Banach space. Let A be the generator of a strongly continuous semigroup $T(t)$ of bounded linear operators on X. Let $f: X \to X$ be locally Lipschitz, i.e. for each bounded subset U of X there exists a constant C_U such that*

$$\|f(u)-f(v)\| \leq C_U \|u-v\|$$

for all $u, v \in U$. Let $\phi \in X$. *Then there exists a unique maximally defined solution $u \in C([0, t_{\max}); X)$, $t_{\max} > 0$, of the integral equation*

$$u(t) = T(t)\phi + \int_0^t T(t-s) f(u(s))\, ds, \qquad t \in [0, t_{\max}). \tag{4.2}$$

Furthermore if $t_{\max} < \infty$ then

$$\int_0^{t_{\max}} \|f(u(s))\|\, ds = \infty, \tag{4.3}$$

and

$$\lim_{t \to t_{\max}} \|u(t)\| = \infty. \tag{4.4}$$

Remark. It is proved in Ball [1] that solutions of (4.2) are weak solutions in a natural sense of the equation

$$\dot{u} = Au + f(u). \tag{4.5}$$

Let $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$. Let $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ with $D(A) = W_0^{1,2}(\Omega) \times (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega))$. It is well knoen that A generates a strongly continuous group $T(t)$ of bounded linear operators on X. Let $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ |u|^{\gamma-1} u \end{pmatrix}$. We write (4.1) in the form

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = A\begin{pmatrix} u \\ u_t \end{pmatrix} + f\begin{pmatrix} u \\ u_t \end{pmatrix}. \tag{4.6}$$

By the imbedding theorems the conditions

$$\gamma > 1 \text{ arbitrary if } n = 1, 2; \qquad 1 < \gamma \leq \frac{n}{n-2} \text{ if } n \geq 3 \tag{4.7}$$

imply that $f: X \to X$ and is locally Lipschitz. Let $\phi = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$. It is easily verified using [1] that a solution u of (4.2) satisfies for any $\psi \in W_0^{1,2}(\Omega)$ the equation

$$\frac{d}{dt}(u_t, \psi) + (\nabla u, \nabla \psi) - (|u|^{\gamma-1}u, \psi) = 0 \tag{4.8}$$

almost everywhere on its interval of existence. In (4.8) (,) denotes the inner product in $L^2(\Omega)$. Clearly $u(\cdot, 0) = u_0$, $u_t(\cdot, 0) = u_1$. One can also show (cf Reed [21], Ball [2]) that any solution u of (4.2) satisfies the energy equation

$$E(u(\cdot, t), u_t(\cdot, t)) = E(u_0, u_1), \tag{4.9}$$

where $E : X \to \mathbb{R}$ is defined by

$$E(w, v) = \int_\Omega \left[\frac{1}{2}|v|^2 + \frac{1}{2}|\nabla w|^2 - \frac{1}{\gamma+1}|w|^{\gamma+1}\right] dx.$$

THEOREM 4.2. *Let* γ *satisfy* (4.7). *Then there exists a unique maximally defined solution* $\begin{pmatrix} u \\ u_t \end{pmatrix} \in C([0, t_{max}); X)$, $t_{max} > 0$, *of* (4.2). *If* $E_0 \overset{\text{def}}{=} E(u_0(\cdot), u_1(\cdot)) \leq 0$, *or if* $E_0 = 0$ *and* $(u_0, u_1) > 0$, *then* $t_{max} < \infty$ *and*

$$\lim_{t \to t_{max}} \|u(t)\|_{\gamma+1} = \lim_{t \to t_{max}} [\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2]^{1/2} = \infty. \tag{4.10}$$

Proof. The existence of u follows directly from Theorem 4.1. To complete the proof it suffices by Theorem 4.1 (cf (4.4)) and (4.9) to show that $t_{max} < \infty$. Let $F(t) = \|u(t)\|_2^2$. Then $\dot{F} = 2(u, u_t)$ and

$$\ddot{F} = 2\int_\Omega [|u_t|^2 - |\nabla u|^2 + |u|^{\gamma+1}]\, dx$$

(This formal calculation is easily justified.) Substituting for $\int_\Omega |\nabla u|^2\, dx$ from (4.9) we obtain

$$\ddot{F} \geq \frac{2(\gamma-1)}{\gamma+1}\int_\Omega |u|^{\gamma+1}\, dx - 4E_0 \geq k\, F^{(\gamma+1)/2} - 4E_0, \tag{4.11}$$

where $k > 0$ is a constant. Suppose $t_{max} = \infty$. If $E_0 < 0$ then by (4.11) $\dot{F}(t) > 0$ eventually, so that we may without loss of generality suppose that $(u_0, u_1) > 0$. Since $\ddot{F}(t) \geq 0$ it follows that $\dot{F}(t)$ and $F(t)$ are nondecreasing non-negative functions of t. Hence

$$\frac{1}{2}\dot{F}^2(t) \geq \frac{2k}{\gamma+3} F(t)^{(\gamma+3)/2} + C,$$

where $c \geq 0$ is a constant, so that

$$\int_{F(0)}^{F(t)} \frac{\mathrm{d}F}{[c+(2kF^{(\gamma+3)/2}/(\gamma+3))]^{1/2}} \geq t \tag{4.12}$$

for all $t \in [0, t_{max})$. But as $F(0) > 0$ the integral is bounded above by a constant. This is a contradiction.

Remark. Similar results for the case $E_0 > 0$, with extra restrictions on u_0, u_1, could probably be established by adapting arguments in Knops, Levine & Payne [**11**], Straughan [**24**].

As in the example in Section 3, the conclusion (4.10) is weaker than that suggested by the nonexistence argument based on (4.12), namely

$$\lim_{t \to t_{max}} \|u(t)\|_2 = \infty. \tag{4.13}$$

I have not been able to find conditions under which (4.13) holds, but the next theorem gives conditions under which the derivative of $\|u(t)\|_2^2$ blows up.

THEOREM 4.3. *Let γ satisfy*

$$1 < \gamma \leq 1 + \frac{4}{n} \qquad \text{if } n \leq 3,$$

$$1 < \gamma \leq \frac{n}{n-2} \qquad \text{if } n \geq 4.$$

Let u_0, u_1 satisfy $E_0 < 0$, or $E_0 = 0$ and $(u_0, u_1) > 0$. Then

$$\lim_{t \to t_{max}} (u, u_t)(t) = \infty. \tag{4.14}$$

The proof requires a lemma.

LEMMA 4.4 *If $n \leq 3$ and $2 < \gamma \leq 1 + \frac{4}{n}$ then*

$$\|u\|_{2\gamma} \leq \beta \|u\|_{\gamma+1}^{1-2/\gamma(\gamma-1)} \|\nabla u\|_2^{2/\gamma(\gamma-1)} \tag{4.15}$$

for all $u \in W_0^{1,2}(\Omega)$, where $\beta = \beta(\gamma, \Omega, n)$ is a constant.

Proof. We use a special case of an interpolation inequality due to Gagliardo

[5] and Nirenberg [18]; a detailed proof of this special case can be found in Ladyzenskaja, Solonnikov & Ural'ceva [12]. Let $m \geq 1$ be an integer. Then for all $u \in W_0^{1,m}(\Omega)$,

$$\|u\|_q \leq \delta \|u\|_r^{1-a} \|\nabla u\|_m^a, \tag{4.16}$$

where

$$a = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{m} + \frac{1}{r}\right)^{-1},$$

where $\delta = \delta(q, r, m, n)$ is a constant, and where q and r satisfy

(i) if $n = 1$, $\quad 1 \leq r < \infty, \quad r \leq q \leq \infty$,
(ii) if $n = m > 1$, $1 \leq r \leq q < \infty$,
(iii) if $n > m$, either $1 \leq r \leq q \leq \dfrac{mn}{n-m}$ or $1 \leq \dfrac{mn}{n-m} \leq q \leq r < \infty$.

Set $r = \gamma + 1$, $a = 2/\gamma(\gamma - 1)$. If $n = 1$ and $2 < \gamma \leq 3$ set $m = 1$, then $q = \gamma(\gamma-1)/(\gamma-2) \geq 2\gamma \geq \gamma + 1$, so that (4.15) follows from (4.16). Otherwise let $m = 2$; then $q = \gamma(\gamma-1)/(\gamma - 1 - 2/n) \geq 2\gamma$, $q < \infty$ if $n = 2$, $q \leq 2n/(n-2)$ if $n = 3$, so that (4.15) again follows from (4.16).

Proof of Theorem 4.3. Since $t_{\max} < \infty$, it follows from Theorem 4.1 (cf (4.3)) that

$$\int_0^{t_{\max}} \|u(t)\|_{2\gamma}^\gamma \, dt = \infty. \tag{4.17}$$

Also, from (4.9) and the fact that $E_0 \leq 0$,

$$\|\nabla u(t)\|_2^2 \leq \frac{2}{\gamma + 1} \|u(t)\|_{\gamma+1}^{\gamma+1}. \tag{4.18}$$

If n and γ satisfy the conditions of Lemma 4.4 then it follows from (4.18) that

$$\begin{aligned} \|u(t)\|_{2\gamma}^\gamma &\leq C \|u(t)\|_{\gamma+1}^{(\gamma-2)(\gamma+1)/(\gamma-1)} \|\nabla u(t)\|_2^{2/(\gamma-1)} \\ &\leq C \|u(t)\|_{\gamma+1}^{\gamma+1}. \end{aligned}$$

Otherwise $\gamma \leq 2$, $\gamma \leq n/(n-2)$ if $n \geq 4$, and thus the Poincaré inequality

$$\|u(t)\|_{2\gamma} \leq C \|\nabla u(t)\|_2$$

holds. Hence by (4.18)

$$\|u(t)\|_{2\gamma}^\gamma \leq C \|u(t)\|_{\gamma+1}^{\gamma(\gamma+1)/2} \leq C(1 + \|u(t)\|_{\gamma+1}^{\gamma+1}).$$

Therefore by (4.17) we have in all cases

$$\int_0^{t_{max}} \int_\Omega |u|^{\gamma+1}\, dx\, dt = \infty.$$

The result follows from (4.11).

Remark. One can prove stronger results for the 'averaged' version of (4.1)

$$u_{tt} = \Delta u + \|u\|_2^{\gamma-1} u, \qquad x \in \Omega,$$
$$u \mid_{\partial\Omega} = 0,$$

where $\gamma > 1$ is arbitrary. The corresponding energy function is $\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{\gamma+1}\|u\|_2^{\gamma+1}$. The same proof as in Theorem 4.2 then shows that when $t_{max} < \infty$

$$\lim_{t \to t_{max}} \|u(t)\|_2 = \infty.$$

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