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On the Asymptotic Behavior of Generalized Processes, with Applications to Nonlinear Evolution Equations

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1. INTRODUCTION

The invariance principle, introduced by LaSalle [40] and subsequently generalized by Hale [34], gives information on the structure of ω-limit sets in dynamical systems possessing a Liapunov function, and the principle and related methods have been used to determine the asymptotic behavior of solutions to a wide variety of evolution equations (see Refs. [4, 10, 17–19, 23–29, 34, 48, 51, 55, 56]). The principle has been extended by Dafermos [24] to compact processes, a special class of nonautonomous systems, including, in particular, dynamical and asymptotically dynamical systems, periodic, almost periodic, asymptotically periodic, and asymptotically almost periodic processes. In this paper we describe and apply some modified versions of the invariance principle for a class of nonautonomous systems which we call *generalized processes*. A generalized process is a natural extension of the concept of a process to evolutionary systems whose solutions for given initial data are not, or are not known to be, unique. Aside from treating nonuniqueness, this paper significantly weakens two hypotheses which are customarily made in connection with the invariance principle, namely, that the Liapunov function V be (i) continuous with respect to convergence in the phase space, and (ii) nonincreasing along solution paths.

The need for weakening (i) may be seen from the problem of proving that all weak solutions $w(x, t)$ of

$$\begin{aligned} w_{tt} + w_t - \Delta_w + w^3 &= 0, \qquad x \in (\Omega),\ t > 0, \\ w(\cdot, t)|_{\partial\Omega} &= 0, \qquad t > 0, \end{aligned} \tag{1.1}$$

tend to zero as $t \to \infty$, where Ω is a bounded open subset of $\mathscr{R}^3$ with boundary $\partial\Omega$. A suitable phase space for (1.1) is

$$X = W_0^{1,2}(\Omega) \times L^2(\Omega).$$

For $\phi = \{w_0, w_1\} \in X$ let $T(t)\phi = \{w(t), w_t(t)\}$, where w is the weak solution of (1.1) satisfying $\{w(0), w_t(0)\} = \{w_0, w_1\}$. A natural Liapunov function $V : X \to \mathscr{R}$

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is given by $V(w, v) = \int_\Omega [|\nabla w|^2 + v^2 + \frac{1}{2}w^4]\, dx$, in terms of which the energy equation becomes

$$V(\phi) - V(T(t)\phi) = 2 \int_0^t \| w_t \|_{L^2}^2 (\tau)\, d\tau. \tag{1.2}$$

It is not obvious that the positive orbit $\mathcal{O}^+(\phi) = \bigcup_{t \in \mathscr{R}^+} T(t)\phi$ is precompact in X for each $\phi \in X$. Since precompactness of $\mathcal{O}^+(\phi)$ is essential for the existence of a nonempty ω-limit set it is tempting to give X the weak topology, since then it follows from (1.2) that $\| T(t)\phi \|_X$ is bounded for $t \in \mathscr{R}^+$, so that $\mathcal{O}^+(\phi)$ is sequentially weakly precompact. Unfortunately, however, V is not sequentially weakly continuous, and hence the standard invariance principle arguments break down. This difficulty was overcome in a similar problem in [4], but in the present paper a different remedy is adopted which is more amenable to abstract generalization; in place of continuity conditions on V itself we substitute lower semicontinuity or related conditions on the change in V in fixed time along solution paths. For example, it is easily seen from (1.2) that the function $\phi \to V(\phi) - V(T(t)\phi)$ is sequentially weakly lower semicontinuous on X for each $t \in \mathscr{R}^+$. A less obvious property of V is that if $\phi_n \xrightharpoonup{X} \phi$, and if $V(\phi_n) - V(T(t)\phi_n) \to 0$ for some $t > 0$, then $V(\phi_n) \to V(\phi)$. These two properties are special cases of the general conditions discussed in this paper. In the case of (1.1) our results imply that $T(t)\phi$ tends to zero (strongly) in X as $t \to \infty$ for any $\phi \in X$. In other examples we obtain convergence to some one of a number of steady-state solutions. It should be noted that Dafermos [25] has proved an interesting invariance principle for *uniform* compact processes on a metric space under the assumption that V itself be lower semicontinuous; simple examples for ordinary differential equations in $\mathscr{R}^2$ show, however, that this result does not extend to compact processes in general.

The need for weakening hypothesis (ii), that V be nonincreasing along solution paths, is illustrated by the problem of determining the asymptotic behavior of solutions of nonautonomous equations that in some sense become autonomous as $t \to \infty$. Under appropriate conditions such equations generate an asymptotically generalized flow (cf. Section 3) on a suitable function space. It may then happen that the autonomous equation possesses a Liapunov function, V, which is nonincreasing along solution paths of the autonomous equation, but which may increase along solution paths of the nonautonomous equation. In such cases the rate of increase of $V(t)$ for large t is not arbitrary, but is restricted by the requirement that the nonautonomous equation is asymptotically autonomous. Typically the following condition holds, that for any $s \in \mathscr{R}^+$

$$\varliminf_{t \to \infty} [V(t) - V(t + s)] \geq 0. \tag{1.3}$$

Under conditions of this type, and under hypotheses such as those discussed in the previous paragraph, our results have the following flavor: For the given solution there exists an interval I of real numbers such that for each $\gamma \in I$ there is an orbit in the ω-limit set on which V takes the constant value γ. In general, as is shown in Section 4 by examples of ordinary differential equations in $\mathscr{R}^2$, one cannot conclude that V is constant on every orbit in the ω-limit set. If, however, it is known that there are only finitely many orbits on which V is constant, or if certain other conditions hold, then stronger results may be obtained. Condition (1.3) was motivated by work of Ball and Peletier [11], who considered the asymptotic behavior of solutions of the one-dimensional heat equation with asymptotically autonomous nonlinear boundary conditions. In [11] an invariance principle was established for asymptotically dynamical systems defined on a metric space, possessing a continuous Liapunov function V, and with only a finite number of rest points. The main idea of the proof is used in this paper. Even for continuous V, however, the results presented here improve those in [11] by weakening other continuity requirements, by allowing for nonuniqueness of solutions, and by giving information when there are infinitely many rest points.

The plan of the paper is as follows. In Section 2 we prove the abstract results for nonlinear semigroups defined on a limit space. In Section 3 we combine devices of Dafermos [27] and Sell [50] to deduce corresponding results for generalized processes possessing an asymptotic hull. In Section 4 the results for asymptotically generalized flows are applied to ordinary differential equations in $\mathscr{R}^n$. Using work of Artstein [1] we give conditions under which every bounded solution tends to a rest point as $t \to \infty$. In Section 5 we prove analogous results for weak solutions of operator equations of the form

$$\dot{u} = Au + f(u, t), \tag{1.4}$$

where A is the generator of a strongly continuous semigroup $T(t)$ of bounded linear operators on a Banach space X, and where $f\colon X \times \mathscr{R} \to X$ is a nonlinear function which stabilizes as $t \to \infty$ to an autonomous function $\bar{f}\colon X \to X$ in the sense that

$$\lim_{t \to \infty} \int_t^{t+1} \sup_{u \in G} \| f(u, s) - \bar{f}(u) \| \, ds = 0 \tag{1.5}$$

for every bounded subset G of X. Use is made of a result (cf. Balakrishnan [57], Ball [8]) which establishes the equivalence between weak solutions of (1.4) and solutions of the integral equation

$$u(t) = T(t - t_0)\, u(t_0) + \int_{t_0}^{t} T(t - s)\, f(u(s), s)\, ds, \qquad t \geq t_0. \tag{1.6}$$

The discussion is divided into two cases.

In Subsection (a) we consider the case when $T(t)$ is compact for $t > 0$ and f is continuous in u. Under further natural hypotheses, and for the case when the autonomous equation

$$\dot{u} = Au + f(u) \tag{1.7}$$

possesses a continuous Liapunov function $V : X \to \mathscr{R}$, we determine the asymptotic behavior of weak solutions of (1.4). The necessary existence theory for (1.4) and (1.7) is due to Pazy [43], and some improvements of his results are described. The theory is applied to parabolic initial boundary value problems of the form

$$\begin{aligned} u_t &= \Delta u + g(u, t), \qquad x \in \Omega,\ t > s, \\ u\,|_{\partial\Omega} &= 0, \qquad u\,|_{t=s} \text{ prescribed}, \end{aligned} \tag{1.8}$$

where $\Omega \subseteq \mathscr{R}^n$ is a bounded open set, and where $g(u, t)$ stabilizes as $t \to \infty$ to an autonomous function $\bar{g}(u)$.

In Subsection (b) we consider the case when X is reflexive and f is sequentially weakly continuous with respect to u. In this case it is necessary for applications to consider Liapunov functions which are not sequentially weakly continuous, so that the full strength of the abstract theory is required. Under further hypotheses an existence and continuation theorem is proved for (1.4) using the Schauder–Tychonov fixed point theorem; the theorem extends similar results for the case of an ordinary differential equation in a Banach space ($A = 0$) due to Chow and Schuur [21], Fitzgibbon [33], and Knight [36]. Similar results to those in subsection (a) are proved concerning the asymptotic behavior of solutions of (1.4) when (1.7) possesses a Liapunov function. The theory is specialized further to an abstract damped nonlinear wave equation

$$\ddot{w} + Bw + F(w, \dot{w}, t) = 0, \tag{1.9}$$

where B is a densely defined positive self-adjoint operator on a real Hilbert space H with B^{-1} compact, and where F: $D(B^{1/2}) \times H \times \mathscr{R} \to H$. Fairly strong conditions are imposed on the asymptotic form of F as $t \to \infty$. Two special cases of (1.9) are discussed in detail. The first is the nonlinear hyperbolic initial boundary value problem

$$\begin{aligned} w_{tt} + a(w, t)\, w_t - \Delta w + \phi(w, t) &= 0, \qquad x \in \Omega,\ t > s, \\ w\,|_{\partial\Omega} = 0, \qquad w\,|_{t=s} &\text{ and } w_t\,|_{t=s} \text{ prescribed}. \end{aligned} \tag{1.10}$$

The second is an initial boundary value problem for a nonautonomous version of a rod equation discussed by Ball [3–5], namely,

$$\begin{aligned} w_{tt} + (\delta + d(t))\, w_t + \alpha w_{xxxx} - \Big(\beta + b(t) + k \int_0^l w_\xi(\xi, t)^2\, d\xi\Big)\, w_{xx} &= 0, \\ x \in \Omega,\ t &> s, \\ w = w_x = 0 \text{ at } x = 0, l, \qquad w\,|_{t=s} \text{ and } w_t\,|_{t=s} &\text{ prescribed}. \end{aligned} \tag{1.11}$$

The existence and continuation theorems for (1.4) have applications to proving rigorous blow-up theorems for certain nonlinear partial differential equations; these results will appear in [9].

2. Invariance Principles for Nonlinear Semigroups

In this section we shall be concerned with nonlinear semigroups defined on a set $\mathscr{X}$. We shall suppose that $\mathscr{X}$ forms a limit space (see below); this turns out to be more convenient, as well as more general, than assuming $\mathscr{X}$ to be a topological space. A similar point of view in a dynamical systems context has been adopted by LaSalle [40].

Definition. A set $\mathscr{X}$ forms a *limit space* if to each of certain infinite sequences $\{x_n : n = 1, 2,...\}$ in $\mathscr{X}$ (called convergent sequences) there corresponds at least one element x of $\mathscr{X}$, called a *limit* of x_n, so that the following conditions are satisfied. (For convenience we write $x_n \xrightarrow{\mathscr{X}} x$ if x is a limit of x_n.)

(i) If $x_n = x$ for all n, then $x_n \xrightarrow{\mathscr{X}} x$,

(ii) If $x_n \xrightarrow{\mathscr{X}} x$ and x_μ is a subsequence of x_n, then $x_\mu \xrightarrow{\mathscr{X}} x$.

Example. If $\mathscr{X}$ is a topological space then $\mathscr{X}$ forms a limit space in which convergence is the usual convergence of sequences in the topology of $\mathscr{X}$.

We now make a number of definitions. Each has a natural topological counterpart, but the reader is warned that if $\mathscr{X}$ is a topological space, then in general none of the terms defined below have their usual topological meanings.

Definitions. A limit space $\mathscr{X}$ is *Hausdorff* if each convergent sequence has precisely one limit.

If $\mathscr{X}$ is a limit space and $A \subseteq \mathscr{X}$ then A is *precompact* if any sequence in A has a subsequence converging to a point of $\mathscr{X}$; if $B \subseteq \mathscr{X}$ then the *closure* of B is defined by $\mathscr{C}\ell B = \{x \in \mathscr{X}$: there exists $\{x_n\} \subseteq B$ with $x_n \xrightarrow{\mathscr{X}} x\}$.

A map $f : \mathscr{X} \to \mathscr{Y}$ between limit spaces $\mathscr{X}, \mathscr{Y}$ is *continuous* if $x_n \xrightarrow{\mathscr{X}} x$ implies $f(x_n) \xrightarrow{\mathscr{Y}} f(x)$.

A real-valued function g defined on a limit space $\mathscr{X}$ is lower semicontinuous if $x_n \xrightarrow{\mathscr{X}} x$ implies $g(x) \leq \underline{\lim}_{n\to\infty} g(x_n)$.

Let $\mathscr{X}$ be a limit space. Let $T(\cdot)$ be a semigroup on $\mathscr{X}$, that is a family of continuous maps $T(t): \mathscr{X} \to \mathscr{X}$, $t \in \mathscr{R}^+$, satisfying (i) $T(0) =$ identity, (ii) $T(s + t) = T(s)\, T(t)$ for all $s, t \in \mathscr{R}^+$. Let $V: \mathscr{X} \to \mathscr{R}$.

For $\psi \in \mathscr{X}$ define the *positive orbit through* ψ by $\mathscr{O}^+(\psi) \overset{\text{def}}{=} \bigcup_{t\in\mathscr{R}^+} T(t)\psi$, the *$\omega$-limit set of* ψ by $\omega(\psi) = \{\phi \in \mathscr{X}$: there exists a sequence $t_n \to \infty$ such that $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi\}$, and the *$V$ ω-limit set of* ψ by $\omega_V(\psi) = \{\phi \in \mathscr{X}$: there exists a sequence $t_n \to \infty$ such that $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi$ and $V(T(t_n)\psi) \to V(\phi)\}$. Clearly $\omega_V(\psi) \subseteq \omega(\psi)$ for any $\psi \in \mathscr{X}$, with equality if V is continuous.

A subset A of $\mathscr{X}$ is said to be *positively invariant* if $T(t)A \subseteq A$ for all $t \in \mathscr{R}^+$, and *invariant* if $T(t)A = A$ for all $t \in \mathscr{R}^+$. An invariant set consisting of a single point is called a *rest point*.

Various forms of the following basic lemma are given in [13, 27, 34, 40]. The proof, though well known, is included for the convenience of the reader.

LEMMA 2.1. *$\omega(\psi)$ is positively invariant for each $\psi \in \mathscr{X}$. If $\mathscr{O}^+(\psi)$ is precompact then $\omega(\psi)$ is nonempty, and if in addition $\mathscr{X}$ is Hausdorff, then $\omega(\psi)$ is invariant.*

Proof. Let $\phi \in \omega(\psi)$, $t \in \mathscr{R}^+$. There exists $t_n \to \infty$ such that $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi$. It follows from the continuity of $T(t)$ that $T(t + t_n)\psi = T(t)\, T(t_n)\psi \xrightarrow{\mathscr{X}} T(t)\phi$, so that $T(t)\phi \in \omega(\psi)$. Hence $\omega(\psi)$ is positively invariant. Let $\mathscr{O}^+(\psi)$ be precompact and let $\mathscr{X}$ be Hausdorff. Clearly $\omega(\psi)$ is nonempty. Let $\phi \in \omega(\psi)$, $t_n \to \infty$, $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi$, $t \in \mathscr{R}^+$. By the precompactness of $\mathscr{O}^+(\psi)$ there exist a subsequence t_μ of t_n and an element $\chi \in \omega(\psi)$ such that $T(t_\mu - t)\psi \xrightarrow{\mathscr{X}} \chi$. Hence $T(t_\mu)\psi = T(t)\, T(t_\mu - t)\psi \xrightarrow{\mathscr{X}} T(t)\chi$. Thus $T(t)\chi = \phi$, so that $\omega(\psi)$ is invariant. ∎

Our first result is

THEOREM 2.2. *Let $\psi \in \mathscr{X}$ and let V satisfy the condition*

$$(\mathrm{A}_\psi)\quad \begin{cases} \text{For any sequence } t_n \to \infty \text{ with } T(t_n)\psi \xrightarrow{\mathscr{X}} \phi, \text{ and for any } t \in \mathscr{R}^+, \\ \qquad V(\phi) - V(T(t)\phi) \leqslant \varliminf_{n\to\infty}[V(T(t_n)\psi) - V(T(t_n + t)\psi)]. \end{cases}$$

Suppose that for each $\tau \in \mathscr{R}^+$

$$\varliminf_{t\to\infty}[V(T(t)\psi) - V(T(t + \tau)\psi)] \leqslant 0.$$

Then for each $\phi \in \omega(\psi)$ the function $V(T(t)\phi)$ is nondecreasing on $\mathscr{R}^+$.

Proof. Let $t_n \to \infty$, $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi$, $t \in \mathscr{R}^+$. Then $T(t_n + t)\psi \xrightarrow{\mathscr{X}} T(t)\phi$, so that by (A_ψ) we obtain

$$V(\phi) - V(T(t)\phi) \leqslant \varliminf_{n\to\infty}[V(T(t_n)\psi) - V(T(t_n + t)\psi)] \leqslant 0.$$

The result follows since $\omega(\psi)$ is positively invariant. ∎

Remark. Condition (A_ψ) is satisfied in particular (for all $\psi \in \mathscr{X}$) if for each $\tau \in \mathscr{R}^+$ the map $\phi \mapsto [V(\phi) - V(T(\tau)\phi)]$ is lower semicontinuous on $\mathscr{X}$. Under this assumption, however, it is not so easy to deduce Theorem 3.5 from Theorem 2.2.

For $\gamma \in \mathscr{R}$ let $M_\gamma = \{\phi \in \mathscr{X}\colon V(T(t)\phi) = \gamma \text{ for all } t \in \mathscr{R}^+\}$.

THEOREM 2.3. *Let $\psi \in \mathscr{X}$ and let V satisfy the condition*

$$(\mathrm{B}_\psi)\quad \begin{cases} \textit{If } t_n \to \infty,\ T(t_n)\psi \xrightarrow{\mathscr{X}} \varphi, \textit{ and } V(T(t_n)\psi) - V(T(t_n + \tau)\psi) \to 0 \textit{ uniformly} \\ \textit{for } \tau \textit{ in compact subsets of } \mathscr{R}^+, \textit{ then } V(T(t_n)\psi) \to V(\phi). \end{cases}$$

Let $\mathcal{O}^+(\psi)$ be precompact, let the map $t \mapsto V(T(t)\psi)$ be continuous on $(0, \infty)$, *and let $\alpha = \underline{\lim}_{t\to\infty} V(T(t)\psi)$, $\beta = \overline{\lim}_{t\to\infty} V(T(t)\psi)$.*

(i) *Suppose that*

$$\underline{\lim}_{t\to\infty}[V(T(t)\psi) - V(T(t+\tau)\psi)] \geqslant 0 \qquad \textit{for every} \quad \tau \in \mathscr{R}^+,$$

and let $\beta > -\infty$. Then $\infty > \beta \geqslant \alpha > -\infty$ and $\omega_V(\psi) \cap M_\gamma$ is nonempty for each $\gamma \in [\alpha, \beta]$.

(ii) *Suppose that*

$$\lim_{t\to\infty}[V(T(t)\psi) - V(T(t+\tau)\psi)] = 0 \qquad \textit{for every} \quad \tau \in \mathscr{R}^+.$$

Then $\infty > \beta \geqslant \alpha > -\infty$, $\omega_V(\psi) \cap M_\gamma$ is nonempty for each $\gamma \in [\alpha, \beta]$, and $\omega(\psi) = \omega_V(\psi) \subseteq \bigcup_{\gamma\in[\alpha,\beta]} M_\gamma$.

Remark. Condition (B_ψ) is satisfied in particular (for all $\psi \in \mathscr{X}$) if $\phi_n \xrightarrow{\mathscr{X}} \phi$ and $V(\phi_n) - V(T(\tau)\phi_n) \to 0$ uniformly for τ in compact subsets of $\mathscr{R}^+$ imply that $V(\phi_n) \to V(\phi)$.

In order to prove Theorem 2.3 we need the following lemma.

LEMMA 2.4. *Let f be a real-valued continuous function on* $(0, \infty)$.

(i) *Suppose*

$$\underline{\lim}_{t\to\infty}[f(t) - f(t+s)] = 0 \qquad \textit{for every} \quad s \in \mathscr{R}^+.$$

Let $\alpha = \underline{\lim}_{t\to\infty} f(t)$, $\beta = \overline{\lim}_{t\to\infty} f(t)$, and let $\beta > -\infty$. Let $\gamma \in [\alpha, \beta]$ (allowing $\gamma = \mp\infty$ if $\alpha = -\infty$ or $\beta = \infty$). Then there exists a sequence $t_n \to \infty$ such that $f(t_n + \tau) \to \gamma$ uniformly for τ in compact subsets of $\mathscr{R}^+$ with (in the case $\gamma = \pm\infty$) $f(t_n) - f(t_n + \tau) \to 0$ uniformly for τ in compact subsets of $\mathscr{R}^+$.

(ii) *Suppose*

$$\lim_{t\to\infty}[f(t) - f(t+s)] = 0 \qquad \textit{for every } s \in \mathscr{R}^+.$$

Then $f(t) - f(t+s) \to 0$ as $t \to \infty$ uniformly for s in compact subsets of $\mathscr{R}^+$.

Proof. (i) We first show that if $s_n \to s_0 > 0$ and $t_n \to \infty$, then

$$\underline{\lim}_{n\to\infty}[f(t_n) - f(t_n + s_n)] \geqslant 0. \tag{*}$$

(We remark that (*) does not hold in general if $s_0 = 0$.) Let $0 < a < b < \infty$. For positive integers m, n let $G_{m,n} = \{s \in [a, b]\colon f(t) - f(t+s) \geqslant -1/m$ for all $t \geqslant n\}$. Since f is continuous, $G_{m,n}$ is closed. Also, for each m we have by hypothesis [a, b] $= \bigcup_{n=1}^{\infty} G_{m,n}$. By the Baire category theorem some $G_{m,n}$ contains an open interval. Repeat this procedure for each $[a, b] \subseteq (0, \infty)$ and let G_m be the union of the corresponding open intervals. Clearly G_m is open and dense in $(0, \infty)$. Let $G = \bigcap_{m=1}^{\infty} G_m$. G is dense in $(0, \infty)$. Let $s_n \to s_0 \in G$. It is easy to see that (*) holds. Now let $s_0 > 0$ be arbitrary and choose $s_1 \in G$ with $s_1 < s_0$. Let $s_n \to s_0$ and $t_n \to \infty$. Then $t_n + s_0 - s_1 \to \infty$ and $s_n - s_0 + s_1 \to s_1$. Hence

$$\varliminf_{n\to\infty}[f(t_n + s_0 - s_1) - f(t_n + s_n)] \geqslant 0.$$

But by hypothesis

$$\varliminf_{n\to\infty}[f(t_n) - f(t_n + s_0 - s_1)] \geqslant 0,$$

so that (*) again follows. If $\alpha = \beta < \infty$ then any sequence $t_n \to \infty$ satisfies the conclusion of the lemma. If $\alpha = \beta = \infty$ let $t_n = 1 + \max\{t \in \mathscr{R}^+ : f(t) \leqslant n\}$. Suppose that $\tau_n \xrightarrow{\mathscr{R}^+} \tau$. By (*) $\varliminf_{n\to\infty} [f(t_n - 1) - f(t_n + \tau_n)] \geqslant 0$. But $f(t_n - 1) - f(t_n + \tau_n) \leqslant 0$, so that $f(t_n - 1) - f(t_n + \tau_n) \to 0$. In particular $f(t_n - 1) - f(t_n) \to 0$. Therefore $f(t_n) - f(t_n + \tau_n) \to 0$ as required.

Let $\alpha < \beta$ and let $\gamma \in (\alpha, \beta)$. Choose $\gamma_1 \in (\alpha, \gamma)$. There exists a sequence $s_m \to \infty$ with $f(s_m) = \gamma_1$ for each m. Let $r_m = \min\{t : t \geqslant s_m + 1 \text{ and } f(t) = \gamma\}$. The sequence $r_m - s_m \to \infty$, since otherwise there would be a subsequence $r_\mu - s_\mu \to l \geqslant 1$. By (*) we would then have $\varliminf_{\mu\to\infty} [f(s_\mu) - f(r_\mu)] \geqslant 0$, which contradicts the fact that $f(s_\mu) - f(r_\mu) = \gamma_1 - \gamma < 0$. Let $t_{n,m} = r_m - n - 1$. For fixed n and large enough m we have $t_{n,m} \geqslant s_m$ and $f(t_{n,m} + \tau) \leqslant \gamma$ for all $\tau \in [0, n]$. Further, by (*), for large enough m and all $\tau \in [0, n]$,

$$f(t_{n,m} + \tau) - f(r_m) \geqslant -1/n.$$

Hence there exists $m(n)$ such that

$$|f(t_{n,m(n)} + \tau) - \gamma| \leqslant 1/n \qquad \text{for all} \quad \tau \in [0, n],$$

and $t_n =^{\text{def}} t_{n,m(n)} \to \infty$ as $n \to \infty$. Clearly $\{t_n\}$ has the required properties.

Finally let $\alpha < \beta$ and $\gamma \in [\alpha, \beta]$. Choose $\gamma_n \in (\alpha, \beta)$ with $\gamma_n \to \gamma$. By the above, for each n there exists $t_n' \in \mathscr{R}^+$ with $|f(t_n' + \tau) - \gamma_n| \leqslant 1/n$ for all $\tau \in [0, n]$ and $t_n' \to \infty$. The sequence $\{t_n'\}$ has the required properties.

(ii) Let $t_n \to \infty$, $s_n \xrightarrow{\mathscr{R}^+} s$. Applying (*) to $\pm f$ we see that $f(t_n - 1) - f(t_n + s_n) \to 0$. Hence $f(t_n) - f(t_n + s_n) \to 0$. ∎

Proof of Theorem 2.3. (i) Let $f(t) = V(T(t)\psi)$, and let $\gamma \in [\alpha, \beta]$ with $\gamma \neq \pm\infty$. Let $\{t_n\}$ be a sequence with the properties given in Lemma 2.4(i). Since $C^+(\psi)$ is precompact there exist a subsequence $\{t_\mu\}$ of $\{t_n\}$ and an element $\phi \in \omega(\psi)$ with $T(t_\mu)\psi \xrightarrow{\mathscr{X}} \phi$. For any $\tau \in \mathscr{R}^+$, $T(t_\mu + \tau)\psi \xrightarrow{\mathscr{X}} T(\tau)\phi$. By the lemma $V(T(t_\mu + \tau)\psi) - V(T(t_\mu + \tau + \bar{\tau})\psi) \to 0$ uniformly for $\bar{\tau}$ in any compact subset of $\mathscr{R}^+$, so that by condition (B_ψ) we have $V(T(t_\mu + \tau)\psi) \to V(T(\tau)\phi) = \gamma$. Thus $\omega_V(\psi) \cap M_\gamma$ is nonempty. Suppose $\beta = +\infty$ ($\alpha = -\infty$). Let $\{t_n\}$ be the sequence in the lemma corresponding to $\gamma = +\infty$ ($\gamma = -\infty$). The same argument as above shows that there is an element $\phi \in \omega(\psi)$ with $V(T(t_n)\psi) \to V(\phi)$, which is a contradiction.

(ii) Let $\phi \in \omega(\psi)$ and let $t_n \to \infty$, $T(t_n)\psi \xrightarrow{\mathscr{X}} \phi$. By the given condition, Lemma 2.4(ii), and condition (B_ψ) we deduce that for every $\tau \in \mathscr{R}^+$, $V(T(t_n + \tau)\psi) \to V(\phi)$. Hence $\omega(\psi) = \omega_V(\psi) \subseteq \bigcup_{\gamma\in[\alpha,\beta]} M_\gamma$ and $\beta > -\infty$. The result now follows from part (i). ▌

One way in which Theorem 2.3 may be applied is the following. Suppose that $\mathscr{X}$ is a Hausdorff topological space and that only finitely many of the sets $M_\gamma \cap \mathscr{C}\ell\mathscr{C}^+(\psi)$, $\gamma \in [\alpha, \beta]$, are nonempty. Then under the hypotheses of Theorem 2.3(i) we deduce that $\alpha = \beta$, so that by Theorem 2.3(ii), $\omega(\psi) = \omega_V(\psi) \subseteq M_\alpha$. If, further, it is known that M_α consists only of a finite number of points, and if the map $t \to T(t)\psi$ is continuous on $\mathscr{R}^+$ (so that $\omega(\psi)$ is connected), then it follows that $\omega(\psi)$ consists of a single rest point ϕ, and that $T(t)\psi \xrightarrow{\mathscr{X}} \phi$, $V(T(t)\psi) \to V(\phi)$ as $t \to \infty$.

3. Generalized Processes

Let X be a limit space. For simplicity we suppose that X is Hausdorff. We denote by $X^{\mathscr{R}^+}$ the set of all maps $\phi: \mathscr{R}^+ \to X$ and give $X^{\mathscr{R}^+}$ the limit space structure of pointwise convergence, i.e., $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \phi$ if and only if $\phi_n(t) \xrightarrow{X} \phi(t)$ for all $t \in \mathscr{R}^+$. If $\phi \in X^{\mathscr{R}^+}$ and $\tau \in \mathscr{R}^+$ the *τ-translate* ϕ_τ of ϕ is defined by $\phi_\tau(t) = \phi(t + \tau)$ for all $t \in \mathscr{R}^+$. Let $A(X)$ denote the set of all subsets of $X^{\mathscr{R}^+}$.

Definition. A map $G: \mathscr{R} \to A(X)$ is a *generalized process* on X if the following properties are satisfied:

(i) If $s \in \mathscr{R}$, $\phi \in G(s)$, $\tau \in \mathscr{R}^+$, then $\phi_\tau \in G(s + \tau)$.

(ii) If $s \in \mathscr{R}$ and $\phi_n \in G(s)$ with $\phi_n(0)$ convergent then there exist $\phi \in G(s)$ and a subsequence ϕ_μ of ϕ_n such that $\phi_\mu \xrightarrow{X^{\mathscr{R}^+}} \phi$.

A function $\phi \in G(s)$ with $\phi(0) = x$ is called a *path* originating at (s, x). A generalized process G which is a constant (i.e., $G(s) = G(t)$ for all $s, t, \in \mathscr{R}$) is called a *generalized flow*. Let 2^X denote the set of all subsets of X. If G is a generalized process on X we may define a corresponding family of operators $U_G(t, s): 2^X \to 2^X$ by

$$U_G(t, s)E = \bigcup_{\substack{\phi\in G(s)\\ \phi(0)\in E}} \phi(t), \qquad E \subseteq X. \tag{3.1}$$

If $x \in X$ we abbreviate $U_G(t, s)\{x\}$ by $U_G(t, s)x$. The following result is easily proved.

THEOREM 3.1. *Let G be a generalized process on X, and let $U_G(\cdot, \cdot)$ be defined by* (3.1). *Then*

$$U_G(0, s)E \subseteq E \qquad \text{for all} \quad s \in \mathscr{R}, \quad E \subseteq X \tag{3.2}$$

and

$$U_G(t+\tau, s)E \subseteq U_G(t, s+\tau)U_G(\tau, s)E \quad \text{for all} \quad t, \tau \in \mathscr{R}^+, s \in \mathscr{R}, \quad E \subseteq X. \tag{3.3}$$

Note that equality need not hold in (3.3), since if $s \in \mathscr{R}$, $\tau \in \mathscr{R}^+$, $\phi \in G(s)$, $\psi \in G(s+\tau)$, and $\phi(\tau) = \psi(0)$, then the function $\bar{\phi}$ defined by $\bar{\phi}(t) = \phi(t)$ for $0 \leqslant t \leqslant \tau, \bar{\phi}(t) = \psi(t-\tau)$ for $t \geqslant \tau$ need not belong to $G(s)$. Thus the definition of a generalized process allows for a type of history dependence. Note also that it follows from the definition that if E is precompact then $U_G(t, s)E$ is precompact for each $t \in \mathscr{R}^+$, $s \in \mathscr{R}$, $x \in X$.

If G is a generalized process such that for each $s \in \mathscr{R}$, $x \in X$ there is precisely one path originating at (s, x) then G is called a *process*. If G is a process then for each $s \in \mathscr{R}$, $x \in X$, and $t \in \mathscr{R}^+$ the set $U_G(t, s)x$ consists of a single point, so that $U_G(t, s)$ induces a map from X into X. Furthermore equality holds in (3.2) and (3.3).

Our definition of a generalized process is but one of a number of ways to give an abstract framework for nonautonomous systems with possibly nonunique solutions. For a survey of other methods see Bushaw [16]. Of particular interest is the work of Barbashin [12] on autonomous systems, extended and developed by Bronstein [14], Budak [15], Minkevic [41], and Roxin [46, 47]. These authors consider a family of operators possessing properties similar to those of the $U_G(\cdot, \cdot)$, and then deduce from assumed continuity conditions the existence of suitably defined solution paths. Our approach, which takes as fundamental the solutions themselves, makes the application of the theory to examples more direct.

Let $\mathscr{G}$ be the set of all generalized processes on X. We define convergence of sequences in $\mathscr{G}$ as follows: $G_n \xrightarrow{\mathscr{G}} G$ if and only if for any subsequence G_m of G_n, for any $s \in \mathscr{R}$, and for any sequence $\phi_m \in G_m(s)$ such that $\phi_m(0)$ is convergent, there exist $\phi \in G(s)$ and a subsequence ϕ_μ of ϕ_m with $\phi_\mu \xrightarrow{X^{\mathscr{R}^+}} \phi$. It is clear that with this definition of convergence $\mathscr{G}$ forms a limit space. Note that in general $\mathscr{G}$ is not Hausdorff, since if $G_n \xrightarrow{\mathscr{G}} G$ and if $\bar{G} \in \mathscr{G}$ satisfies $\bar{G}(s) \supseteq G(s)$ for all $s \in \mathscr{R}$ then $G_n \xrightarrow{\mathscr{G}} \bar{G}$.

If $\sigma \in \mathscr{R}$ then the σ*-translate* of a generalized process G on X is the generalized process G_σ defined by

$$G_\sigma(s) = G(s+\sigma) \qquad \text{for all} \quad s \in \mathscr{R}.$$

DEFINITION (compare Dafermos [23]). A subset $\mathscr{H}$ of $\mathscr{G}$ is said to be a *hull* of the generalized process G if and only if the following properties are satisfied:

(1) Given any sequence σ_n in $\mathscr{R}^+$ there exist $G \in \mathscr{H}$ and a subsequence σ_μ of σ_n such that $G_{\sigma_\mu} \xrightarrow{\mathscr{G}} G$,

(2) $G \in \mathscr{H}$,

(3) $\mathscr{H}$ is translation invariant, i.e., if $G \in \mathscr{H}$, $\sigma \in \mathscr{R}$ then $G_\sigma \in \mathscr{H}$.

Remarks. If $\mathscr{H} \subseteq \mathscr{G}$ satisfies (1) and (2), then $\hat{\mathscr{H}} = \{G_\sigma : G \in \mathscr{H}, \sigma \in \mathscr{R}\}$ is a hull of G. A generalized process may have infinitely many hulls, or it may have none.

For the remainder of this section we suppose that G is a generalized process on X and that $\mathscr{H}$ is a hull for G. Let $\mathscr{X}[\mathscr{H}] = \{\{G, \phi\} \in \mathscr{H} \times X^{\mathscr{R}^+} : \phi \in G(s)$ for some $s \in \mathscr{R}\}$. We define convergence in $\mathscr{X}[\mathscr{H}]$ as follows. We say that $\{G_n, \phi_n\} \xrightarrow{\mathscr{X}[\mathscr{H}]} \{G, \phi\}$ if and only if there exists $s \in \mathscr{R}$ such that $\phi_n \in G_n(s)$, $\phi \in G(s)$, and $G_n \xrightarrow{\mathscr{G}} G$, $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \phi$. Clearly $\mathscr{X}[\mathscr{H}]$ forms a limit space with this convergence. Combining ideas of Dafermos [27] and Sell [50] we make the following:

DEFINITIONS. For $t \in \mathscr{R}^+$, $G \in \mathscr{H}$ let

$$S(t)G = G_t,$$

and for $t \in \mathscr{R}^+$, $\{G, \phi\} \in \mathscr{X}[\mathscr{H}]$ let

$$T(t)\{G, \phi\} = \{G_t, \phi_t\}.$$

It follows from the definitions that $S(\cdot)$ defines a semigroup, the *semigroup of translates*, on $\mathscr{H}$, and that $T(\cdot)$ defines a semigroup, the *semigroup associated with G and* $\mathscr{H}$, on $\mathscr{X}[\mathscr{H}]$. Note that the sequential continuity of $S(\cdot)$ and $T(\cdot)$ for $t \in \mathscr{R}^+$ is immediate.

Since $\mathscr{H}$ is a hull of G, the positive orbit through G of $S(\cdot)$ is precompact in $\mathscr{H}$. The ω-limit set is denoted $\mathscr{H}_\infty$ and is called the *asymptotic hull* of G in $\mathscr{H}$. Note that, by Lemma 2.1, $\mathscr{H}_\infty$ is positively invariant under $S(\cdot)$, and that if $\mathscr{H}$ is Hausdorff (with respect to convergence in $\mathscr{G}$) then $\mathscr{H}_\infty$ is invariant under $S(\cdot)$.

DEFINITIONS (compare Barbashin [12]). A subset A of X is said to be *positively quasi-invariant* if for any $x \in A$ there exist $G \in \mathscr{H}_\infty$ and a path $\phi \in G(0)$ such that $\phi(0) = x$ and $\phi(\mathscr{R}^+) \subseteq A$, and *quasi-invariant* if for any $x \in A$ there exist $G \in \mathscr{H}_\infty$ and a map $\psi: \mathscr{R} \to A$ such that $\psi(0) = x$ and $\psi_\sigma \in G(\sigma)$ for all $\sigma \in \mathscr{R}$, where $\psi_\sigma(t) \stackrel{\text{def}}{=} \psi(\sigma + t)$ for all $t \in \mathscr{R}^+$. If $\sigma \in \mathscr{R}, \phi: [\sigma, \infty) \to X$, the *$\Omega$-limit set* of ϕ is defined by $\Omega(\phi) = \{x \in X$: there exists a sequence $t_n \to \infty$ such that $\phi(t_n) \xrightarrow{X} x\}$.

If $E \subseteq X^{\mathscr{R}^+}$ let $E^0 = \{\phi(0) : \phi \in E\}$.

LEMMA 3.2. *Let $s \in \mathscr{R}$, $\phi \in G(s)$, and let $\omega(G, \phi)$ be the ω-limit set of $\{G, \phi\}$ with respect to $T(\cdot)$. Let $E = \{\psi \in X^{\mathscr{R}^+} : \{G, \psi\} \in \omega(G, \phi)$ for some $G \in \mathscr{H}_\infty\}$. Then $E^0 = \Omega(\phi)$.*

Proof. Clearly $E^0 \subseteq \Omega(\phi)$. Let $x \in \Omega(\phi)$, so that $\phi_{t_n}(0) \xrightarrow{X} x$ for some sequence $t_n \to \infty$. There exist a subsequence t_μ of t_n and $\{\bar{G}, \psi\} \in \mathscr{X}[\mathscr{H}]$ such that $T(t_\mu)\{G, \phi\} \xrightarrow{\mathscr{X}[\mathscr{H}]} \{G, \psi\}$. Hence $x = \psi(0)$ belongs to E^0, so that $\Omega(\phi) \subseteq E^0$. ∎

LEMMA 3.3. *Let $s \in \mathscr{R}$, $\phi \in G(s)$ and suppose that $\phi(\mathscr{R}^+)$ is precompact in X. Then the positive orbit $\mathscr{C}^+(\{G, \phi\})$ of the semigroup $T(\cdot)$ is precompact in $\mathscr{X}[\mathscr{H}]$.*

Proof. This follows directly from the definitions. ∎

The next theorem describes the invariance properties of Ω-limit sets.

THEOREM 3.4. *Let $s \in \mathscr{R}$ and $\phi \in G(s)$. Then $\Omega(\phi)$ is positively quasi-invariant. If $\phi(\mathscr{R}^+)$ is precompact then $\Omega(\phi)$ is nonempty, and if in addition $\mathscr{H}$ is Hausdorff then $\Omega(\phi)$ is quasi-invariant.*

Proof. Let $x \in \Omega(\phi)$. By Lemma 3.2 there exists $\{\bar{G}, \psi\} \in \omega(G, \phi)$ such that $\psi(0) = x$. By Lemma 2.1, $\omega(G, \phi)$ is positively invariant, so that, again by Lemma 3.2, $\psi(\mathscr{R}^+) \subseteq \Omega(\phi)$. Since $\psi \in \bar{G}_s(0)$ it follows that $\Omega(\phi)$ is positively quasi-invariant. The rest of the theorem follows similarly, using Lemmas 2.1, 3.2, and 3.3. ∎

Let $V: \mathscr{R} \times X \to \mathscr{R}$. We suppose that for each $\bar{G} \in \mathscr{H}$ there exists a function $V_{\bar{G}}: \mathscr{R} \times X \to \mathscr{R}$ such that for any sequence s_n in $\mathscr{R}$ that is bounded below and is such that $G_{s_n} \xrightarrow{\mathscr{G}} \bar{G}$, and for any $t \in \mathscr{R}$, $x \in X$,

$$\lim_{n \to \infty} V(s_n + t, x) = V_{\bar{G}}(t, x). \tag{3.4}$$

Note that $V_{G_\sigma}(t, x) = V(t + \sigma, x)$ for all $\sigma, t \in \mathscr{R}$, $x \in X$.

DEFINITION. We say that *condition* (C) is satisfied if and only if whenever $t_n \to \infty$, $G_{t_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $s \in \mathscr{R}$, $\phi_n \in G_{t_n}(s)$, and $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \bar{\phi} \in \bar{G}(s)$ then for each $t \in \mathscr{R}^+$

$$V_{\bar{G}}(s, \bar{\phi}(0)) - V_{\bar{G}}(s + t, \bar{\phi}(t)) \leqslant \varliminf_{n \to \infty} [V(t_n + s, \phi_n(0)) - V(t_n + s + t, \phi_n(t))].$$

THEOREM 3.5. *Let condition* (C) *be satisfied, let $s \in \mathscr{R}$, $\phi \in G(s)$ and suppose that for each $\tau \in \mathscr{R}^+$*

$$\varliminf_{t \to \infty} [V(t + s, \phi(t)) - V(t + \tau + s, \phi(t + \tau))] \leqslant 0. \tag{3.5}$$

Then for each $x \in \Omega(\phi)$ there exist $\bar{G} \in \mathscr{H}_\infty$ and a path $\bar{\phi} \in \bar{G}(0)$ such that $\bar{\phi}(0) = x$, $\bar{\phi}(\mathscr{R}^+) \subseteq \Omega(\phi)$, and $V_{\bar{G}}(t, \bar{\phi}(t))$ is nondecreasing on $\mathscr{R}^+$.

Proof. Let $\psi = \{G, \phi\}$. Apply Theorem 2.2 to $T(\cdot)$ and ψ with $\mathscr{X} = \mathscr{C}\ell\mathscr{C}^+(\psi)$ and with $J: \mathscr{X} \to \mathbb{R}$ defined by $J(\bar{G}, \bar{\phi}) = V_{\bar{G}}(s, \bar{\phi}(0))$.

So as to verify condition (A_ψ) let $t_n \to \infty$, $\{G_{t_n}, \phi_{t_n}\} \xrightarrow{\mathscr{X}[\mathscr{H}]} \{G, \bar{\phi}\}$. Then $\bar{\phi} \in G(s)$. Let $\tau \in \mathscr{R}^+$. By condition (C) we obtain

$$\begin{aligned} J(G, \bar{\phi}) - J(G_\tau, \bar{\phi}_\tau) &= V_{\bar{G}}(s, \bar{\phi}(0)) - V_{\bar{G}}(s + \tau, \bar{\phi}(\tau)) \\ &\leq \varliminf_{n \to \infty} [V(t_n + s, \phi_{t_n}(0)) - V(t_n + s + \tau, \phi_{t_n}(\tau))] \\ &= \varliminf_{n \to \infty} [J(G_{t_n}, \phi_{t_n}) - J(G_{t_n+\tau}, \phi_{t_n+\tau})] \end{aligned}$$

so that (A_ψ) is satisfied. Let $x \in \Omega(\phi)$. By Theorem 3.4 there exist $G \in \mathscr{H}_\infty$ and a path $\bar{\phi} \in \bar{G}(0)$ with $\bar{\phi}(0) = x$, $\bar{\phi}(\mathscr{R}^+) \subseteq \Omega(\phi)$. By Lemma 3.2, the positive invariance of $\omega(G, \phi)$, and Theorem 2.2 it follows that $V_{\bar{G}}(t, \bar{\phi}(t))$ is nonincreasing on $\mathscr{R}^+$. ∎

DEFINITIONS. For $s \in \mathscr{R}$, $G \in \mathscr{H}_\infty$, $\phi \in G(s)$ we define the *$V_{\bar{G}}\Omega$-limit set of ϕ* by $\Omega_{V,\bar{G}}(\phi) = \{x \in X$: there exists a sequence $t_n \to \infty$ such that $\phi(t_n) \xrightarrow{X} X$ and $V_{\bar{G}}(0, \phi(t_n)) \to V_{\bar{G}}(0, x)\}$. If $\gamma \in \mathscr{R}$ and $\bar{G} \in \mathscr{H}_\infty$ let $M_\gamma[\bar{G}] = \{x \in X$: there exists $\bar{\phi} \in \bar{G}(0)$ such that $\bar{\phi}(0) = x$ and $V_{\bar{G}}(t, \bar{\phi}(t)) = \gamma$ for all $t \in \mathscr{R}^+\}$.

DEFINITION. We say that *condition* (D) is satisfied if and only if whenever $t_n \to \infty$, $G_{t_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $s \in \mathscr{R}$, $\phi_n \in G_{t_n}(s)$, $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \bar{\phi} \in \bar{G}(s)$, and

$$V(t_n + s, \phi_n(0)) - V(t_n + s + t, \phi_n(t)) \to 0$$

uniformly for t in compact subsets of $\mathscr{R}^+$ then $V(t_n + s, \phi_n(0)) \to V_{\bar{G}}(s, \bar{\phi}(0))$.

THEOREM 3.6. Let condition (D) be satisfied, let $s \in \mathscr{R}$, $\phi \in G(s)$, let $\phi(\mathscr{R}^+)$ be precompact in X, and suppose that the map $t \mapsto V(t + s, \phi(t))$ is continuous on $(0, \infty)$. Let $\alpha = \varliminf_{t \to \infty} V(t + s, \phi(t))$, $\beta = \varlimsup_{t \to \infty} V(t + s, \phi(t))$.

(i) Suppose that for each $\tau \in \mathscr{R}^+$

$$\varliminf_{t \to \infty} [V(t + s, \phi(t)) - V(t + \tau + s, \phi(t + \tau))] \geq 0 \tag{3.6}$$

and let $\beta > -\infty$. Then $\infty > \beta \geq \alpha > -\infty$ and for each $\gamma \in [\alpha, \beta]$ there exists $\bar{G} \in \mathscr{H}_\infty$ with $\Omega_{V,\bar{G}}(\phi) \cap M_\gamma[\bar{G}]$ nonempty.

(ii) Suppose that for each $\tau \in \mathscr{R}^+$

$$\lim_{t \to \infty} [V(t + s, \phi(t)) - V(t + \tau + s, \phi(t + \tau))] = 0. \tag{3.7}$$

Then $\infty > \beta \geq \alpha > -\infty$, for each $\gamma \in [\alpha, \beta]$ there exists $\bar{G} \in \mathscr{H}_\infty$ with $\Omega_{V,\bar{G}}(\phi) \cap M_\gamma[\bar{G}]$ nonempty, and

$$\Omega(\phi) = \bigcup_{\bar{G} \in \mathscr{H}_\infty} \Omega_{V,\bar{G}}(\phi) \subseteq \bigcup_{\substack{\bar{G} \in \mathscr{H}_\infty \\ \gamma \in [\alpha, \beta]}} M_\gamma[\bar{G}].$$

Proof. Let J and ψ be defined as in the proof of Theorem 3.5. We apply Theorem 2.3 to $T(\cdot)$ and ψ. To show that condition (B_ψ) is satisfied let $t_n \to \infty$, $\{G_{t_n}, \phi_{t_n}\} \xrightarrow{\mathscr{X}_L \mathscr{H}_J} \{\bar{G}, \bar{\phi}\}$. Suppose that

$$V(t_n + s, \phi_{t_n}(0)) - V(t_n + s + t, \phi_{t_n}(t)) \to 0$$

uniformly for t in compact subsets of $\mathscr{R}^+$.

By condition D, $J(G_{t_n}, \phi_{t_n}) \to J(\bar{G}, \bar{\phi})$ as required. The result follows by Lemmas 3.2, 3.3 and Theorems 2.3, 3.4. ▌

In several important cases certain hypotheses of Theorems 3.5, 3.6 may be replaced by simpler ones. We collect these cases together in the next theorem.

THEOREM 3.7. (i) *If the limit in* (3.4) *holds uniformly for x in precompact subsets of X, and if $V_{\bar{G}}$ is continuous in x for each $\bar{G} \in \mathscr{H}_\infty$, then conditions* (C) *and* (D) *are satisfied.*

(ii) *If condition* (C) *is satisfied and if $V_{\bar{G}}(t + s, \phi(t))$ is nonincreasing in $t \in \mathscr{R}^+$ for any $s \in \mathscr{R}$, $\phi \in \bar{G}(s)$, then* (3.6) *follows from the general hypotheses of Theorem* 3.6. *In particular* (3.7) *may be replaced by* (3.5).

(iii) *Suppose that each $\bar{G} \in \mathscr{H}_\infty$ is a process. Suppose also that*

$$V(s_n + t + \tau, \phi_n(t)) - V_{\bar{G}}(t + \tau, U_{\bar{G}}(t, \tau)\phi_n(0)) \to 0 \tag{3.8}$$

whenever $s_n \to \infty$, $G_{s_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $t \in \mathscr{R}^+$, $\tau \in \mathscr{R}$, $\phi_n \in G_{s_n}(\tau)$, and $\phi_n(0)$ is convergent. (This is a stronger condition than (3.4).) *Then conditions* (C) *and* (D) *are implied by the following conditions* (C′) *and* (D′), *respectively.*

(C′) *For any $\bar{G} \in \mathscr{H}_\infty$ and any $t \in \mathscr{R}^+$ the function*

$$x \mapsto V_{\bar{G}}(0, x) - V_{\bar{G}}(t, U_{\bar{G}}(t, 0)x)$$

is lower semicontinuous on X.

(D′) *If $\bar{G} \in \mathscr{H}_\infty$, $x_n \xrightarrow{\lambda} x$, and if $V_{\bar{G}}(0, x_n) - V_{\bar{G}}(t, U_{\bar{G}}(t, 0)x_n) \to 0$ for all $t \in \mathscr{R}^+$, then $V_{\bar{G}}(0, x_n) \to V_{\bar{G}}(0, x)$.*

Proof. (i) This is trivial, since if $s_n \to \infty$, $G_{s_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $x_n \xrightarrow{X} x$, and $t \in \mathscr{R}$ then

$$\lim_{n\to\infty} V(s_n + t, x_n) = \lim_{n\to\infty} V_{\bar{G}}(t, x_n) = V_{\bar{G}}(t, x).$$

(ii) Let the general hypotheses of Theorem 3.6 be satisfied, let $\tau \in \mathscr{R}^+$, and let $t_n \to \infty$. Without loss of generality we may assume that $G_{t_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$ and $\phi_{t_n} \xrightarrow{X^{\mathscr{R}^+}} \bar{\phi} \in \bar{G}(s)$. Thus by (C)

$$\begin{aligned}
&\underline{\lim}_{n\to\infty}[V(t_n + s, \phi(t_n)) - V(t_n + \tau + s, \phi(t_n + \tau))] \\
&\qquad \geq V_{\bar{G}}(s, \bar{\phi}(0)) - V_{\bar{G}}(s + \tau, \bar{\phi}(\tau)),
\end{aligned}$$

which is nonnegative by assumption.

(iii) Let (C′) hold, let $t_n \to \infty$, $G_{t_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $s \in \mathscr{R}$, $\phi_n \in G_{t_n}(s)$, $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \bar{\phi} \in \bar{G}(s)$, and $t \in \mathscr{R}^+$. Then using (3.8) and (C′) we obtain

$$\begin{aligned} V_{\bar{G}}(s, \bar{\phi}(0)) - V_{\bar{G}}(s + t, \bar{\phi}(t)) &= V_{\bar{G}}(s, \bar{\phi}(0)) - V_{\bar{G}}(s + t, U_{\bar{G}}(t, s)\,\bar{\phi}(0)) \\ &= \varliminf_{n\to\infty}[V_{\bar{G}}(s, \phi_n(0)) - V_{\bar{G}}(s + t, U_{\bar{G}}(t, s)\,\phi_n(0))] \\ &= \varliminf_{n\to\infty}[V(t_n + s, \phi_n(0)) - V(t_n + s + t, \phi_n(t))] \end{aligned}$$

so that (C) is satisfied.

Let (D′) be satisfied, let $t_n \to \infty$, $G_{t_n} \xrightarrow{\mathscr{G}} \bar{G} \in \mathscr{H}_\infty$, $s \in \mathscr{R}$, $\phi_n \in G_{t_n}(s)$, $\phi_n \xrightarrow{X^{\mathscr{R}^+}} \bar{\phi} \in \bar{G}(s)$, and $V(t_n + s, \phi_n(0)) - V(t_n + s + t, \phi_n(t)) \to 0$ for all $t \in \mathscr{R}^+$. Then

$$\lim_{n\to\infty} V(t_n + s, \phi_n(0)) = \lim_{n\to\infty} V_{\bar{G}}(s, \phi_n(0)) = V_{\bar{G}}(s, \bar{\phi}(0)),$$ so that (D) holds. ∎

Remark. The advantage of conditions (C′) and (D′) is that they are expressed solely in terms of the limiting processes $\bar{G}$, and thus can be easier to verify.

Finally in this section we mention the special case of asymptotically generalized flows.

DEFINITION. A pair $\{G, \bar{G}\}$, where G is a generalized process on X and $\bar{G}$ is a generalized flow on X, is an *asymptotically generalized flow* if $\mathscr{H} = \{\bar{G}\} \cup \bigcup_{\sigma\in\mathscr{R}} G_\sigma$ is a hull of G and $\mathscr{H}_\infty = \{\bar{G}\}$.

If $\{G, \bar{G}\}$ is an asymptotically generalized flow then $G_{s_n} \xrightarrow{\mathscr{G}} \bar{G}$ for any sequence $s_n \to \infty$. Clearly $V_{\bar{G}}$ is independent of t, so that $V_{\bar{G}}: X \to \mathscr{R}$.

4. APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS IN $\mathscr{R}^n$

Consider the ordinary differential equation in $\mathscr{R}^n$

$$\dot{x} = f(x, t), \tag{4.1}$$

where $f: \mathscr{R}^n \times \mathscr{R} \to \mathscr{R}^n$ satisfies the Carathéodory conditions, i.e., f is continuous in x for each fixed t, measurable in t for each fixed x, and for each compact $K \subseteq \mathscr{R}^n$ there exists a locally integrable function m_K such that

$$|f(x, t)| \leqslant m_K(t) \tag{4.2}$$

for all $x \in K$. A *solution* of (4.1) on an interval $[a, b]$ is by definition a continuous function $x: [a, b] \to \mathscr{R}^n$ satisfying

$$x(t) = x(s) + \int_s^t f(x(\tau), \tau)\, d\tau \tag{4.3}$$

for all $s, t \in [a, b)$. We study the asymptotic behavior of solutions of (4.1) when, in a sense to be specified shortly, $f(\cdot, t)$ tends to a continuous function $\bar{f}\colon \mathscr{R}^n \to \mathscr{R}^n$ as $t \to \infty$. The associated autonomous equation is

$$\dot{x} = \bar{f}(x). \tag{4.4}$$

Solutions of (4.4) are defined in the same way as for (4.1).

We make the following further assumptions on f and $\bar{f}$.

(1) For every compact $K \subseteq \mathscr{R}^n$ there is a nondecreasing function $\mu_K\colon [0, \infty) \to [0, \infty]$, continuous at 0 with $\mu_K(0) = 0$, such that whenever $a, b \in \mathscr{R}$ and $u\colon [a, b] \to K$ is continuous $\int_a^b f(u(s), s)\, ds$ is defined, and

$$\left| \int_a^b f(u(s), s)\, ds \right| \leqslant \mu_K(b - a). \tag{4.5}$$

(2) For any $a, b \in \mathscr{R}$, any sequence $u_k \to u_0$ in $C([a, b])$, and any sequence $t_k \to \infty$,

$$\int_a^b f(u_k(s), s + t_k)\, ds \to \int_a^b \bar{f}(u_0(s))\, ds.$$

Assumptions (1) and (2) follow Artstein [1]. Let $X = \mathscr{R}^n$, $R > 0$. For $s \in \mathscr{R}$, $u\colon [s, \infty) \to X$, define $u_s \in X^{\mathscr{R}^+}$ by $u_s(t) = u(s + t)$ for all $t \in \mathscr{R}^+$. For $s \in \mathscr{R}$ let $G(s) = \{x_s \in X^{\mathscr{R}^+} : x : [s, \infty) \to X$ is a solution of (4.1) satisfying $|x(\sigma)| \leqslant R$ for all $\sigma \in [s, \infty)\}$. Let $\bar{G} = \{x \in X^{\mathscr{R}^+} : x$ is a solution of (4.4) satisfying $|x(\sigma)| \leqslant R$ for all $\sigma \in \mathscr{R}^+\}$.

LEMMA 4.1. *$\{G, \bar{G}\}$ is an asymptotically generalized flow on X.*

Proof. We first show that G is a generalized process. Property (i) of the definition is clearly satisfied. To prove (ii) let $s \in \mathscr{R}$ and let $x_n\colon [s, \infty) \to X$ be solutions of (4.1) satisfying $|x_n(\sigma)| \leqslant R$ for all $\sigma \in [s, \infty)$. Let $[a, b] \subseteq [s, \infty)$. By (4.3), assumption (1), and the Arzela–Ascoli theorem the functions x_n are precompact in $C([a, b])$. It follows by a diagonal argument, and by using (4.2) and the dominated convergence theorem, that there exists a subsequence x_μ of x_n, and a solution $x : [s, \infty) \to X$ of (4.1), such that $x_\mu \xrightarrow{X^{\mathscr{R}^+}} x$. This proves (ii).

A similar proof shows that $\bar{G}$ is a generalized flow. If $\sigma_n \to \sigma$ in $\mathscr{R}$ then another equicontinuity argument shows that $G_{\sigma_n} \xrightarrow{\mathscr{G}} G_\sigma$, while if $\sigma_n \to \infty$ then we deduce using (2) that $G_{\sigma_n} \xrightarrow{\mathscr{G}} \bar{G}$. This completes the proof. ∎

THEOREM 4.2. *Let $\bar{V}\colon \mathscr{R}^n \to \mathscr{R}$ be continuously differentiable and satisfy $\nabla \bar{V}(x) \cdot \bar{f}(x) \leqslant 0$ for all $x \in \mathscr{R}^n$. Let $s \in \mathscr{R}$ and let $x\colon [s, \infty) \to \mathscr{R}^n$ be a bounded solution of* (4.1). *Let $\alpha = \varliminf_{t\to\infty} \bar{V}(x(t))$, $\beta = \varlimsup_{t\to\infty} \bar{V}(x(t))$. For $\gamma \in \mathscr{R}$ let*

$M_\gamma = \{y \in \mathscr{R}^n$: There exists a solution $\bar{x}: \mathscr{R}^+ \to \mathscr{R}^n$ of (4.4) such that $\bar{x}(0) = y$ and $\bar{V}(\bar{x}(t)) = \gamma$ for all $t \in \mathscr{R}^+\}$. Then $\Omega(x) \cap M_\gamma$ is nonempty for each $\gamma \in [\alpha, \beta]$.

Proof. Choose $R > 0$ large enough so that $x \in G(s)$. We apply Theorem 3.6(i) with $V = V_{\bar{G}} = \bar{V}$ and $\phi = x$. By Theorem 3.7(i) conditions C and D are satisfied, and hence by Theorem 3.7(ii) so is (3.6). The result follows. ∎

COROLLARY 4.3. *Let $\bar{V}$ be as in Theorem 4.2. Suppose further that for each $\gamma \in \mathscr{R}$ the set M_γ is either empty or contains only rest points of (4.4). (i.e., zeros of g), and that each rest point of (4.4) is isolated in $\mathscr{R}^n$. Then every bounded solution $x: [s, \infty) \to \mathscr{R}^n$ of (4.1) converges to a rest point of (4.4) as $t \to \infty$.*

Proof. Only finitely many of the sets $M_\gamma \cap \{y \in \mathscr{R}^n: |y| \leqslant R\}$, $\gamma \in [\alpha, \beta]$ are nonempty. By Theorem 4.2, $\alpha = \beta$. By Theorem 3.7(ii), $\Omega(x)$ contains only rest points. Since $\Omega(x)$ is connected and each rest point is isolated the result follows. ∎

THEOREM 4.4. *Let $\bar{V}$ be as in Theorem 4.2. Let $V: \mathscr{R} \times \mathscr{R}^n \to \mathscr{R}$ satisfy*

$$\lim_{n\to\infty} V(s_n + t, x) = \bar{V}(x),$$

for any sequence $s_n \to \infty$ and for any $t \in \mathscr{R}^+$, the limit holding uniformly for x in any bounded set. Let $s \in \mathscr{R}$, let $x: [s, \infty) \to \mathscr{R}^n$ be a bounded solution of (4.1), let $\alpha = \underline{\lim}_{t\to\infty} V(t, x(t))$, $\beta = \overline{\lim}_{t\to\infty} V(t, x(t))$, and let

$$\underline{\lim}_{t\to\infty}[V(t, x(t)) - V(t + \tau, x(t + \tau))] \leqslant 0 \qquad \textit{for every} \quad \tau \in \mathscr{R}^+.$$

Then $\Omega(x) \cap M_\gamma$ is nonempty for every $\gamma \in [\alpha, \beta]$ and $\Omega(x) \subseteq \bigcup_{\gamma\in[\alpha,\beta]} M_\gamma$.

Proof. Apply Theorem 3.6(ii). ∎

The preceding results should be compared with those of Strauss and Yorke [52, 53] for the case when there is a single rest point.

Finally we give two illustrative examples.

EXAMPLES. (1) The number of rest points of (4.3) can be finite with every solution of (4.4) converging to some rest point, but a bounded solution of (4.1) may have nontrivial orbits in its Ω-limit set.

Consider the system

$$\begin{aligned} \dot{r} &= -r(r-1)^2, \\ \dot{\theta} &= \cos^2\theta + f(t), \end{aligned} \tag{4.6}$$

where $f: \mathscr{R} \to \mathscr{R}^+$ is continuous, $f(t) \to 0$ as $t \to \infty$, but $\int_0^\infty f(t)\,dt = \infty$, and where (r, θ) are plane polar coordinates. If one changes to Cartesian coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ it is easy to verify that (4.6) satisfies our general hypotheses, so that it generates an asymptotically dynamical system. The phase portrait of the limiting autonomous system

$$\begin{aligned} \dot r &= -r(r-1)^2, \\ \dot\theta &= \cos^2\theta \end{aligned} \tag{4.7}$$

is shown in Fig. 1. Every solution of (4.7) converges to one of the three rest points $(x_1, x_2) = (0, 0), (0, 1), (0, -1)$. However, it follows from (4.6) that

$$\theta(t) \geq \theta(0) + \int_0^t f(s)\,ds,$$

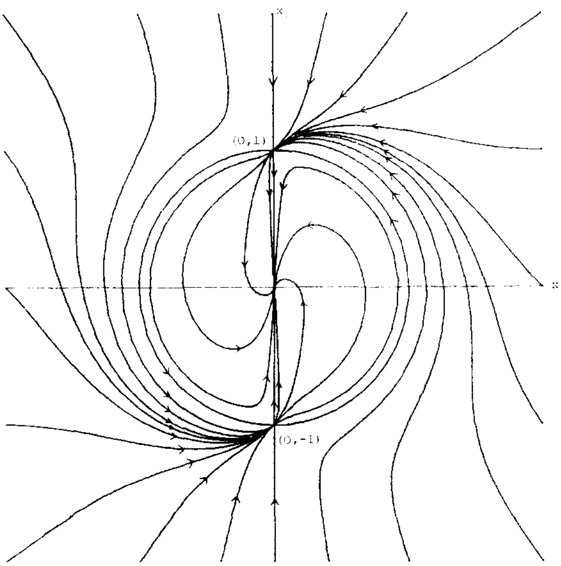

FIGURE 1

so that, for example, any solution with initial data on the unit circle has limit set the whole of the unit circle. In this example the autonomous system has no Liapunov function $\bar V$ for which the corresponding sets M_γ contain only rest points, so that Corollary 4.3 does not apply. Similar examples may be constructed using homoclinic orbits.

(2) Every solution of (4.4) can converge to some rest point, there can exist a Liapunov function $\bar V$ satisfying the hypotheses of Theorem 4.1, but a

bounded solution of (4.1) may have nontrivial orbits in its limit set. Here the trouble arises because the rest points may not be isolated. Consider the system

$$
\begin{aligned}
\dot{r} &= -r(r-1)^2, \\
\dot{\theta} &= f(t), && -\pi/2 \leqslant \theta \leqslant \pi/2, \\
&= \cos^2\theta \, f(t), && \pi/2 \leqslant \theta \leqslant 3\pi/2,
\end{aligned}
\tag{4.8}
$$

where f is as in the preceding example. In this case the limiting autonomous system has for rest points the origin and the right half of the unit circle, and every solution converges to one of these. Let $h(\theta)$ be a smooth 2π-periodic function satisfying $0 \leqslant h(\theta) < 1$ for all θ, $h' = 0$ for $\theta \in [\pi/2, 3\pi/2]$, and define

$$V(r, \theta) = r^2[1 - h(\theta)].$$

It is easy to check that V is continuously differentiable and is nonincreasing along solutions of the autonomous system, decreasing strictly unless the solution is a rest point. But, as in the preceding example, any solution of (4.8) with initial data on the unit circle has limit set the whole of the unit circle. Notice that Theorem 4.2 applies in this case. For discussion of a related point see Artstein [2, Sect. 8].

5. Applications to Nonlinear Evolution Equations in Banach Space

Let X be a real or complex Banach space with dual space X^*, and let A be the generator of a strongly continuous semigroup $T(\cdot)$ of bounded linear operators on X. It is well known that there exist constants $M > 0$, $\omega \in \mathscr{R}^+$ such that $\| T(t)\| \leqslant Me^{\omega t}$ for all $t \in \mathscr{R}^+$. Let A^* denote the adjoint of A and let $D(A) \subseteq X$, $D(A^*) \subseteq X^*$ denote the domains of A, A^*, respectively.

Consider the formal equation

$$\dot{u} = Au + f(u, t), \tag{5.1}$$

where $f: X \times \mathscr{R} \to X$ is a given function.

Definition. Let $t_1 > t_0$. A function $u \in C([t_0, t_1] : X)$ is a *weak solution* of (5.1) on $[t_0, t_1]$ if $f(u(\cdot), \cdot) \in L^1(t_0, t_1; X)$ and if for each $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[t_0, t_1]$ and satisfies

$$(d/dt)\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(u(t), t), v\rangle \tag{5.2}$$

for almost all $t \in [t_0, t_1]$.

The following result is an immediate consequence of Ball [8] (see also Balakrishnan [57]).

THEOREM 5.1. *Let $t_1 > t_0$. A function $u : [t_0, t_1] \to X$ is a weak solution of (5.1) on $[t_0, t_1]$ if and only if $f(u(\cdot), \cdot) \in L^1(t_0, t_1; X)$ and u satisfies the variation of constants formula*

$$u(t) = T(t - t_0)\, u(t_0) + \int_{t_0}^{t} T(t - s)\, f(u(s), s)\, ds \tag{5.3}$$

for all $t \in [t_0, t_1]$.

We consider the asymptotic behavior of weak solutions of (5.1) when, in a sense to be specified precisely, $f(\cdot, t)$ tends to a function $\bar{f}: X \to X$ as $t \to \infty$. The associated autonomous equation is

$$\dot{u} = Au + \bar{f}(u). \tag{5.4}$$

In Subsections (a) and (b) below we consider two different sets of hypotheses on X, A, f, and $\bar{f}$. The discussion in Subsection (a) applies mainly to "parabolic" problems; that of Subsection (b) is particularly suited to hyperbolic problems, but may be relevant in other situations also.

Subsection (a)

We make the following hypotheses:

(a_1) $T(t)$ is a compact operator for each $t > 0$.

(a_2) $f(u, \cdot)$ is strongly measurable for each $u \in X$, $f(\cdot, t)$ is continuous for almost all $t \in \mathscr{R}$, and for each bounded subset G of X there exists a locally integrable function m_G on $\mathscr{R}$ such that

$$\|f(u, t)\| \leqslant m_G(t) \qquad \text{for all } u \in G \text{ and almost all } t \in \mathscr{R}.$$

(a_3) For each $t_0 \in \mathscr{R}$

$$\lim_{s \to 0} \sup_{t \geq t_0} \int_t^{t+s} m_G(\tau)\, d\tau = 0.$$

(a_4) $\bar{f}$ is continuous and for each bounded subset G of X

$$\lim_{t \to \infty} \int_t^{t+1} \sup_{u \in G} \|f(u, s) - \bar{f}(u)\|\, ds = 0.$$

Remarks. (1) It follows from (a_3) and (a_4) that $\bar{f}$ maps bounded sets to bounded sets.

(2) It would be possible to develop the material in this situation under weaker assumptions generalizing (1) and (2) in Section 4, but for simplicity we have not taken this course.

Using just hypotheses (a_1) and (a_2) we obtain the following local existence and continuation theorem; an analogous result obviously holds for weak solutions of (5.4).

THEOREM 5.2. *Let $u_0 \in X$, $t_0 \in \mathscr{R}$. There exists a weak solution $u(t)$ to* (5.1) *satisfying $u(t_0) = u_0$ and defined on a maximal interval of existence $[t_0, t_{\max})$, where $t_{\max} > t_0$. For any such solution with $t_{\max} < \infty$ there holds*

$$\int_{t_0}^{t_{\max}} \| f(u(\tau), \tau)\| \, d\tau = \infty. \tag{5.5}$$

Proof. A routine adaptation of the proof of Pazy [43, Theorem 2.1], who assumed f to be continuous, shows that a continuous solution u of (5.3) exists on some interval $[t_0, t_1)$ with $t_1 > t_0$. By Theorem 5.1, u is a weak solution of (5.1) on $[t_0, t_1)$. By Zorn's lemma this weak solution may be extended to a weak solution, again denoted u, defined on a maximal interval of existence $[t_0, t_{\max})$. Finally, if $t_{\max} < \infty$ and (5.5) does not hold then for $t_0 \leqslant s \leqslant t < t_{\max}$ we have that

$$\begin{aligned} \| u(t) - u(s)\| \leqslant \| [T(t - t_0) - T(s - t_0)\, u_0] \| \\ + \int_{t_0}^{s} \| [T(t - \tau) - T(s - \tau)] f(u(\tau), \tau)\| \, d\tau \\ + \int_{s}^{t} \| T(t - \tau) f(u(\tau), \tau)\| \, d\tau. \end{aligned}$$

It follows from the dominated convergence theorem that $u(t_n)$ is a Cauchy sequence for any sequence t_n tending to $t_{\max}$ from below. Thus $\lim_{t \to t_{\max}, t < t_{\max}} u(t)$ exists, so that u may be continued into some interval $[t_0, t_2)$ with $t_2 > t_{\max}$, which is a contradiction. ∎

It follows from (a_2) and (5.5) that if $t_{\max} < \infty$ then

$$\overline{\lim_{t \to t_{\max}}} \| u(t)\| = \infty,$$

which is the result of Pazy [43, Theorem 3.1]. Actually one may replace $\overline{\lim}$ by lim (cf. [9]).

The following result improves Pazy [43, Theorem 4.1]. Observe that hypothesis (a_4) is not used in the proof.

LEMMA 5.3. *Let $t_0 \in \mathscr{R}$ and let $u: [t_0, \infty) \to X$ be a bounded weak solution of* (5.1). *Then u has precompact range.*

Proof. Let $\| u(t)\| \leqslant R$ for all $t \in [t_0, \infty)$ and let $B = \{x \in X: \| x \| \leqslant R\}$. If $\{t_n\}$ is a bounded sequence in $[t_0, \infty)$ then clearly $u(t_n)$ has a convergent subsequence. Let $t_n \to \infty$. Then

$$u(t_n) = T(1)\, u(t_n - 1) + \int_0^1 T(1 - s) f(u(t_n + s - 1), t_n + s - 1)\, ds.$$

Since the sequence $u(t_n - 1)$ is bounded, the sequence $T(1)\, u(t_n - 1)$ is precompact. It therefore suffices to show that the sequence

$$z_n \overset{\text{def}}{=} \int_0^1 T(1 - s) f(u(t_n + s - 1), t_n + s - 1)\, ds$$

is precompact. Let $0 < \delta < 1$. Then

$$z_n = T(\delta) y_n + r_n ,$$

where

$$y_n = \int_0^{1-\delta} T(1 - \delta - s) f(u(t_n + s - 1), t_n + s - 1)\, ds,$$

$$r_n = \int_{1-\delta}^1 T(1 - s) f(u(t_n + s - 1), t_n + s - 1)\, ds.$$

Clearly

$$\| r_n \| \leqslant Me^{\omega} \int_{t_n-\delta}^{t_n} m_B(\tau)\, d\tau.$$

Given $\epsilon > 0$, by (a_3) there exists $\delta > 0$ such that $\| r_n \| < \epsilon/2$. Since

$$\| y_n \| \leqslant Me^{\omega} \int_{t_n-1}^{t_n-\delta} m_B(\tau)\, d\tau < \infty,$$

the set $T(\delta) y_n$ is precompact and thus totally bounded, so that $T(\delta) y_n$ is covered by a finite number of open balls of radius $\epsilon/2$ and centers $x_1 ,\ldots, x_m$. Given n there exists i, $1 \leqslant i \leqslant m$, such that $\| T(\delta) y_n - x_i \| < \epsilon/2$. Hence

$$\| z_n - x_i \| \leqslant \| r_n \| + \| T(\delta) y_n - x_i \| < \epsilon.$$

Therefore $\{z_n\}$ is totally bounded and thus precompact. ▮

Let $R > 0$. For $s \in \mathscr{R}$, u: $[s, \infty) \to X$, define $u_s \in X^{\mathscr{R}^+}$ by $u_s(t) = u(s + t)$ for all $t \in \mathscr{R}^+$. For $s \in \mathscr{R}$ let $G(s) = \{u_s \in X^{\mathscr{R}^+} : u : [s, \infty) \to X$ is a weak solution of (5.1) satisfying $\| u(\sigma)\| \leqslant R$ for all $\sigma \in [s, \infty)\}$. Let $\bar{G} = \{u \in X^{\mathscr{R}^+}$: u is a weak solution of (5.4) satisfying $\| u(\sigma)\| \leqslant R$ for all $\sigma \in \mathscr{R}^+\}$. Let X have the limit space structure induced by the norm topology of X.

THEOREM 5.4. *$\{G, \bar{G}\}$ is an asymptotically generalized flow on X.*

Proof. Let $s \in \mathscr{R}$, let $u: [s, \infty) \to X$ be a bounded weak solution of (5.1), and let $\sigma_n \to \infty$. By Lemma 5.3 the sequence $u_{\sigma_n}(s)$ is precompact. Let $s_1 > s$. An argument similar to that in the proof of Pazy [43, Theorem 2.1], and using (a_3), shows that the functions u_{σ_n} are precompact in $C([s, s_1] : X)$. It follows by a diagonal argument that there exist a subsequence u_{σ_μ} of u_{σ_n} and a continuous function $u: [s, \infty) \to X$ such that $u_{\sigma_\mu} \to u$ uniformly on compact subsets of $[s, \infty)$. Let $t \in \mathscr{R}^+$. We have that

$$u_{\sigma_\mu}(s + t) = T(t)\, u_{\sigma_\mu}(s) + \int_0^t T(t - \tau) f(u_{\sigma_\mu}(s + \tau), \sigma_\mu + s + \tau)\, d\tau.$$

But

$$\begin{aligned}\Big\| \int_0^t & T(t - \tau)[f(u_{\sigma_\mu}(s + \tau), \sigma_\mu + s + \tau) - f(u(s + \tau))]\, d\tau \Big\| \\ &\leq Me^{\omega t} \int_0^t [\| f(u_{\sigma_\mu}(s + \tau), \sigma_\mu + s + \tau) - f(u_{\sigma_\mu}(s + \tau))\| \\ &\quad + \| f(u_{\sigma_\mu}(s + \tau)) - f(u(s + \tau))\|]\, d\tau,\end{aligned}$$

which tends to zero as $\mu \to \infty$ by (a_4) and the fact that f maps bounded sets to bounded sets. Hence u is a weak solution of (5.4).

Similar arguments show that G is a generalized process, that $\bar{G}$ is a generalized flow, and that if $\sigma_n \to \sigma$ in $\mathscr{R}^+$, then $G_{\sigma_n} \xrightarrow{\mathscr{G}} G_\sigma$. This completes the proof of the theorem. ∎

DEFINITION. A continuously Fréchet differentiable function $V: X \to \mathscr{R}$ is said to be a *Liapunov function* for (5.4) if and only if

$$\langle Au + f(u), V'(u) \rangle \leq 0 \qquad \text{for all} \quad u \in D(A).$$

LEMMA 5.5. *Let V be a Liapunov function for* (5.4) *and let $u : [0, \infty) \to X$ be a weak solution of* (5.4). *Then $V(u(t))$ is nonincreasing on* $[0, \infty)$.

Proof. It suffices to show that $V(u(t)) \leq V(u(0))$ for all $t \in \mathscr{R}^+$. Let $T > 0$, define $F(t) = f(u(t))$, and let $F_n \to F$ in $C([0, T] : X)$ with $F_n \in C^1([0, T]; X)$ for each $n = 1, 2,\ldots$. Let $v_{n0} \xrightarrow{X} u(0)$ with $v_{n0} \in D(A)$ for each n. Define $v_n \in C([0, T] : X)$ by

$$v_n(t) = T(t)\, v_{n0} + \int_0^t T(t - s)\, F_n(s)\, ds.$$

Then $v_n(t) \in D(A)$, $v_n \in C^1([0, T]; X)$, and $\dot{v}_n(t) = Av_n(t) + F_n(t)$ for all $t \in [0, T]$ (cf. Pazy [42]). Thus

$$\begin{aligned} V(v_n(t)) - V(v_{n0}) &= \int_0^t \langle Av_n(s) + F_n(s), V'(v_n(s)) \rangle\, ds \\ &\leq \int_0^t \langle F_n(s) - f(v_n(s)), V'(v_n(s)) \rangle\, ds. \end{aligned} \tag{5.6}$$

Define $z_n(t) = v_n(t) - u(t)$. Then

$$\| z_n(t)\| \leqslant Me^{\omega t} \| z_n(0)\| + \int_0^t Me^{\omega(t-s)} \| F_n(s) - F(s)\| \, ds,$$

so that $v_n \to u$ in $C([0, T]; X)$. In particular the set $\{v_n(s): s \in [0, T], n = 1, 2,...\}$ is precompact in X, so that there exists a constant $K > 0$ such that $\| V'(v_n(s))\|_{X^*} \leqslant K$ for all $s \in [0, T]$ and $n = 1, 2,...$. Thus passing to the limit in (5.6) we obtain $V(u(t)) \leqslant V(u(0))$ for all $t \in [0, T]$ as required. ∎

If $w \in X$ is a rest point of (5.4) then

$$\langle w, A^*v\rangle + \langle f(w), v\rangle = 0 \qquad \text{for all} \quad v \in D(A^*),$$

so that, by a lemma in Ball [8], $w \in D(A)$ and

$$Aw + f(w) = 0.$$

We can now write down results corresponding to Theorem 4.2 and Corollary 4.3 for ordinary differential equations in $\mathscr{R}^n$. The proofs are the same as for the ordinary differential equation case.

THEOREM 5.6. *Let V be a Liapunov function for* (5.4). *Let $s \in \mathscr{R}$ and let $u: [s, \infty) \to X$ be a bounded weak solution of* (5.1). *Let $\alpha = \underline{\lim}_{t\to\infty} V(u(t))$, $\beta = \overline{\lim}_{t\to\infty} V(u(t))$. For $\gamma \in \mathscr{R}$ let $M_\gamma = \{y \in X$: there exists a weak solution $\bar{u}: \mathscr{R}^+ \to X$ of* (5.4) *such that $\bar{u}(0) = y$ and $V(\bar{u}(t)) = \gamma$ for all $t \in \mathscr{R}^+\}$. Then $\Omega(u) \cap M_\gamma$ is nonempty for each $\gamma \in [\alpha, \beta]$.*

COROLLARY 5.7. *Let V be a Liapunov function for* (5.4). *Suppose further that for each $\gamma \in \mathscr{R}$ the set M_γ is either empty or contains only rest points of* (5.4), *and that each rest point of* (5.4) *is isolated in X. Then every bounded weak solution $u: [s, \infty) \to X$ of* (5.1) *converges to a rest point of* (5.4) *as $t \to \infty$.*

Remark. One may also easily prove a result corresponding to Theorem 4.4.

EXAMPLE. Let $\Omega \subseteq \mathscr{R}^n$ be a nonempty bounded open set with boundary $\partial\Omega$. Consider the initial boundary value problem

$$\begin{aligned} u_t &= \Delta u + g(u, t), \qquad x \in \Omega,\ t > s, \\ u\,|_{\partial\Omega} &= 0, \qquad u\,|_{t=s} \text{ prescribed}, \end{aligned} \tag{5.7}$$

and the corresponding autonomous problem

$$\begin{aligned} u_t &= \Delta u + \bar{g}(u), \qquad x \in \Omega,\ t > 0, \\ u\,|_{\partial\Omega} &= 0, \qquad u\,|_{t=0} \text{ prescribed}. \end{aligned} \tag{5.8}$$

We make the following assumptions on g and $\bar{g}$.

(1) $g(u, \cdot)$ and $g_u(u, \cdot)$ are measurable for each $u \in \mathscr{R}$, and for almost all $t \in \mathscr{R}$, $g(\cdot, t)$ is continuously differentiable.

(2) $g(0, t) = 0$ for almost all $t \in \mathscr{R}$, and $\bar{g}(0) = 0$.

(3) There exists a nonnegative locally integrable function $m(t)$ satisfying

$$\limsup_{s \to 0} \sup_{t \geqslant t_1} \int_t^{t+s} m(\tau)\, d\tau = 0 \tag{5.9}$$

for each $t_1 \in \mathscr{R}$, such that

$$\frac{|g(u, t)|}{1 + |u|} + |g_u(u, t)| \leqslant m(t) \tag{5.10}$$

for all $u \in \mathscr{R}$ and almost all $t \in \mathscr{R}$.

(4) $\bar{g}$ is continuously differentiable, and there exists a nonnegative locally integrable function $\eta(t)$ satisfying

$$\lim_{t \to \infty} \int_t^{t+1} \eta(s)\, ds = 0$$

such that

$$\frac{|g(u, t) - \bar{g}(u)|}{1 + |u|} + |g_u(u, t) - \bar{g}_u(u)| \leqslant \eta(t) \tag{5.11}$$

for all $u \in \mathscr{R}$ and almost all $t \in \mathscr{R}$. (If $n = 1$, then (5.10), (5.11) may be replaced by the inequalities

$$|g(u, t)| + |g_u(u, t)| \leqslant m(t)\, \rho(u), \tag{5.10'}$$

and

$$|g(u, t) - \bar{g}(u)| + |g_u(u, t) - \bar{g}_u(u)| \leqslant \eta(t)\, \rho(u), \tag{5.11'}$$

respectively, where ρ is a continuous function of u.)

Let X be the Sobolev space $W_0^{1,2}(\Omega)$. Define $D(A) = \{u \in X; \Delta u \in X\}$, $A = \Delta$. It is well known that A is the generator of a semigroup $T(\cdot)$ on X such that $T(t)$ is compact for $t > 0$. Define the functions f and $\bar{f}$ by $f(u, t)(x) = g(u(x), t)$ and $\bar{f}(u)(x) = \bar{g}(u(x))$, respectively. It follows from (1)–(3) that, after possible modification on a set of t measure zero, $f: X \times \mathscr{R} \to X$ and $\bar{f}: X \to X$. (If $n = 1$ one uses the fact that $W_0^{1,2}(\Omega)$ is continuously imbedded in $C(\bar{\Omega})$.) We claim that hypotheses (a_2)–(a_4) are satisfied. First note that by Ekeland and Témam [32, Chap. 8, Proposition 1.1] the functions $(t, x) \mapsto g(u(x), t)$ and $(t, x) \mapsto g_u(u(x), t) u_x(x)$ are measurable on $K \times \Omega$ for any compact subset K of $\mathscr{R}$ and any fixed $u \in X$. Let $\theta_1, \theta_2 \in L^2(\Omega)$. By (5.10) the functions $g(u(x), t)\, \theta_1(x)$ and $g_u(u(x), t)\, u_x(x)\, \theta_2(x)$ are integrable over $K \times \Omega$, so that by Fubini's theorem the function $t \to \int_\Omega [g(u(x), t)\, \theta_1(x) + g_u(u(x), t)\, u_x(x)\, \theta_2(x)]\, dx$ is measurable on K. Hence $f(u, \cdot)$ is weakly measurable, thus strongly measurable by Dunford and Schwartz [31, Theorem III.6.11]. That $f(\cdot, t)$ is continuous for almost all $t \in \mathscr{R}$ is a consequence of (5.10) and the Vitali convergence theorem. (For a similar result, see Krasnosel'skii [37, Theorem 2.3].) The other statements in (a_2)–(a_4) are easily checked using (1)–(4). We now write (5.7) in the form

$$\dot{u} = Au + f(u, t) \tag{5.12}$$

and apply the preceding theory. Note that since for all $v \in D(A)$, $\phi \in C_0^\infty(\Omega)$,

$$(\Delta v, \phi) = (v, \Delta\phi),$$

where $(\ ,\ )$ is the inner product in $L^2(\Omega)$, any weak solution u of (5.12) satisfies (5.7) in the sense of distributions.

Let

$$\bar{G}(v) = \int_0^v g(r)\, dr, \tag{5.13}$$

and define

$$V(u) = \int_\Omega [\tfrac{1}{2} \mid \nabla u(x)|^2 - \bar{G}(u(x))]\, dx. \tag{5.14}$$

Since by (5.10), (5.11), (5.13),

$$\begin{aligned} |\bar{G}(v)| &\leqslant C \left| \int_0^v [1 + | r |]\, dr \right| \\ &\leqslant C(| v | + \tfrac{1}{2} | v |^2), \end{aligned}$$

where C is a constant, it follows that $V\colon X \to \mathscr{R}$ and is continuously differentiable. Furthermore, if $u \in D(A)$ then

$$\langle Au + \bar{f}(u), V'(u)\rangle = -\int_\Omega | \Delta u(x) + \bar{g}(u(x))|^2\, dx \leqslant 0,$$

so that V is a Liapunov function for (5.4).

Next we show that if $\gamma \in \mathscr{R}$ then M_γ is either empty or consists only of rest points of (5.4). Let u be a weak solution of (5.4) satisfying $V(u(t)) = \gamma$ for all $t \in \mathscr{R}^+$. Suppose for a moment that $u(t) \in D(A)$ for all $t \in \mathscr{R}^+$. Then for $t \in \mathscr{R}^+$, $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} |(u(t), \phi) - (u(0), \phi)| &\leqslant \int_0^t |(\dot{u}(s), \phi)|\, ds \\ &\leqslant \left(\int_0^t \| \dot{u}(s)\|^2_{L^2(\Omega)}\, ds\right)^{1/2} \left(\int_0^t \| \phi \|^2_{L^2(\Omega)}\, ds\right)^{1/2} \\ &= [V(u(0)) - V(u(t))]^{1/2}\, t^{1/2}\, \| \phi \|_{L^2(\Omega)}\, . \end{aligned}$$

Hence

$$|(u(t), \phi) - (u(0), \phi)| \leqslant [V(u(0)) - V(u(t))]^{1/2} t^{1/2} \|\phi\|_{L^2(\Omega)} . \qquad (5.15)$$

But inequality (5.15) in fact holds for *any* weak solution of (5.4), as may be shown by a method analogous to that used to prove Lemma 5.5. Thus $u(t) = u(0)$ for all $t \in \mathscr{R}^+$, which is our assertion.

From Theorem 5.6 and Corollary 5.7 we obtain

THEOREM 5.8. *Let $s \in \mathscr{R}$, and let $u: [s, \infty) \to W_0^{1,2}(\Omega)$ be a weak solution of* (5.7) *that is bounded in norm. Let $\alpha = \underline{\lim}_{t\to\infty} V(u(t))$, $\beta = \overline{\lim}_{t\to\infty} V(u(t))$. Then for each $\gamma \in [\alpha, \beta]$ there exists a rest point v of* (5.8) *belonging to $\Omega(u)$ with $V(v) = \gamma$. If the rest points of* (5.8) *are isolated in $W_0^{1,2}(\Omega)$ then $u(t)$ converges to a unique rest point as $t \to \infty$.*

Results similar to the above may be proved for weak solutions of (5.7) with less smooth initial data by exploiting the fact that Δ generates a holomorphic semigroup on $L^2(\Omega)$, but the hypotheses on g and $\bar{g}$ required differ somewhat from (1)–(4); for an exposition of some of the techniques that would be required the reader is referred to Henry [35] and Pazy [42, 43]. There is also no difficulty in applying our method to the case when Δ is replaced in (5.7) by a strongly elliptic operator of order $2m$, $m > 1$, and g by the gradient of a function of the derivatives of u of order less than m.

We remark that sometimes it is possible to prove boundedness of a weak solution of (5.7) by use of maximum principle arguments.

Subsection (b)

We make the following hypotheses:

(b_1) X is reflexive.

(b_2) $f(u, \cdot)$ is strongly measurable for each $u \in X$, $f(\cdot, t)$ is sequentially weakly continuous for almost all $t \in \mathscr{R}$, and for each bounded subset G of X there exists a locally integrable function m_G on $\mathscr{R}$ such that

$$\|f(u, t)\| \leqslant m_G(t)$$

for all $u \in G$ and almost all $t \in \mathscr{R}$.

(b_3) For each $t_0 \in \mathscr{R}$

$$\lim_{s\to 0} \sup_{t \geqslant t_0} \int_t^{t+s} m_G(\tau)\, d\tau = 0.$$

(b_4) f is sequentially weakly continuous, and for each bounded subset G of X

$$\lim_{t\to\infty} \int_t^{t+1} \sup_{v\in G} \|f(u, s) - f(u)\|\, ds = 0.$$

Using just hypotheses (b_1) and (b_2) we can state the following local existence and continuation theorem (an analogous result holds for weak solutions of (5.4)).

THEOREM 5.9. *Let $u_0 \in X$, $t_0 \in \mathscr{R}$. There exists a weak solution $u(t)$ of* (5.1) *satisfying $u(t_0) = u_0$ and defined on a maximal interval of existence $[t_0, t_{\max})$, where $t_{\max} > t_0$. For any such solution with $t_{\max} < \infty$ there holds*

$$\int_{t_0}^{t_{\max}} \| f(u(\tau), \tau) \| \, d\tau = \infty. \tag{5.16}$$

Remark. It suffices for the theorem that f be defined on $B_r(u_0) \times [t_0, t_1]$ for some $r > 0$, $t_1 > t_0$, where $B_r(u_0) \stackrel{\text{def}}{=} \{v \in X : \| v - u_0 \| \leqslant r\}$.

In order to prove Theorem 5.9 we require two lemmas.

LEMMA 5.10. *Let G be a weakly compact subset of a Banach space Y, and let $F: Y \to Y$ be sequentially weakly continuous. Let $F|_G$ denote the restriction of F to G. Then $F|_G: G \to Y$ is weakly continuous.*

Proof. We use a method similar to that of Ball [6, Lemma 3.1]. Let $A \subseteq Y$ be weakly closed, and let y belong to the weak closure of $F^{-1}(A) \cap G$. There exists a sequence $\{y_n\} \subseteq F^{-1}(A) \cap G$ such that $y_n \xrightarrow{Y} y$ (see, for example, Wilansky [54, Theorem 13.4.2]). Since F is sequentially weakly continuous, $Fy_n \xrightarrow{Y} Fy$, so that $y \in F^{-1}(A) \cap G$. Hence $F^{-1}(A) \cap G$ is weakly closed, which proves the assertion, ∎

LEMMA 5.11. *The map $(t, x) \mapsto T(t)x$ is jointly sequentially weakly continuous on $\mathscr{R}^+ \times X$, i.e., if $t_n \xrightarrow{\mathscr{R}^+} t$ and $x_n \xrightarrow{X} x$ then $T(t_n)x_n \xrightarrow{X} T(t)x$.*

Proof. For each $t \in \mathscr{R}^+$ the map $T(t): X \to X$ is linear and continuous, thus sequentially weakly continuous by Dunford and Schwartz [31, p. 422]. Let $t_n \to t$ in $\mathbb{R}^+$, $x_n \rightharpoonup x$ in X. The sequence $T(t_n)x_n$ is bounded. Since X is reflexive there exist subsequences t_μ and x_μ such that $T(t_\mu)x_\mu \rightharpoonup y$ for some $y \in X$. But by Theorem 5.1 with $f = 0$,

$$\langle T(t_\mu)x_\mu - x_\mu, v \rangle = \int_0^{t_\mu} \langle T(s)x_\mu, A^*v \rangle \, ds$$

for all $v \in D(A^*)$. Passing to the limit we obtain $\langle y, v \rangle = \langle T(t)x, v \rangle$ for all $v \in D(A^*)$. Hence $y = T(t)x$. Thus the whole sequence $T(t_n)x_n$ converges weakly to $T(t)x$. ∎

Remark. One may also deduce Lemma 5.11 from general results for non-linear semigroups (cf. Ball [7, Corollary 3.4], Chernoff [20]).

Proof of Theorem 5.9. Let $r > 0$, let $m_r = m_{B_r(u_0)}$, and choose $t_1 > t_0$ such that (i) $\| T(t) \| \leqslant 2M$ for all $t \in [0, t_1 - t_0]$, (ii) $\| T(t)u_0 - u_0 \| \leqslant r/2$ for all $t \in [0, t_1 - t_0]$, and (iii) $\int_{t_0}^{t_1} m_r(t) \, dt \leqslant r/4M$. Let X_w denote X endowed with the weak topology, and let $Y = C([t_0, t_1]; X_w)$, the space of continuous functions from $[t_0, t_1]$ into X_w. Let $K = \{u \in Y: \|u(t) - u_0\| \leqslant r$ for all $t \in [t_0, t_1]\}$. For $u \in K$ and $t \in [t_0, t_1]$ define

$$(Pu)(t) = T(t - t_0)\, u_0 + \int_{t_0}^{t} T(t - s)\, f(u(s), s)\, ds. \tag{5.17}$$

If $u \in K$ then by Hille and Phillips [38, p. 73], u is strongly measurable. Therefore by a lemma in Knight [36], $f(u(\cdot), \cdot)$ is strongly measurable. It follows that if $t \in [t_0, t_1]$ then $T(t - s) f(u(s), s)$ is strongly measurable in s on $[t_0, t]$. But $\| T(t - s) f(u(s), s)\| \leqslant 2Mm_r(s)$, and thus $T(t - s) f(u(s), s)$ is integrable over $[t_0, t]$. Hence $(Pu)(t)$ is well defined.

We claim that for any $t \in [t_0, t_1]$, $\epsilon > 0$, $x^* \in X^*$, there exists an open neighborhood $N(t, x^*, \epsilon)$ of t in $[t_0, t_1]$ such that if $\tau \in N(t, x^*, \epsilon)$ then $|\langle (Pu)(t) - (Pu)(\tau), x^*\rangle| \leqslant \epsilon$ for all $u \in K$. To establish this it suffices to show that if $\tau_n \to t$ in $[t_0, t_1]$, then $|\langle (Pu)(t) - (Pu)(\tau_n), x^*\rangle| \leqslant \epsilon$ for all $u \in K$ and all large enough n. First we consider n for which $\tau_n > t$. For such n

$$\begin{aligned}|\langle (Pu)(t) - (Pu)(\tau_n), x^*\rangle| \leqslant{} & |\langle T(t - t_0)\, u_0 - T(\tau_n - t_0)\, u_0, x^*\rangle| \\ & + \left| \int_{t_0}^{t} \langle [T(t - s) - T(\tau_n - s)]\, f(u(s), s), x^*\rangle\, ds \right| \\ & + \left| \int_{t}^{\tau_n} \langle T(\tau_n - s)\, f(u(s), s), x^*\rangle\, ds \right|.\end{aligned}$$

The first term on the right-hand side tends to zero as $n \to \infty$, while the third term is bounded by

$$2M\, \|x^*\|_{X^*} \int_{t}^{\tau_n} m_r(s)\, ds,$$

which also tends to zero as $n \to \infty$. The second term is bounded by $a_n \int_{t_0}^{t} m_r(s)\, ds$, where

$$a_n \stackrel{\text{def}}{=} \sup_{\substack{\|v\| \leqslant 1 \\ 0 \leqslant s \leqslant t}} |\langle [T(t - s) - T(\tau_n - s)]v, x^*\rangle|.$$

But since the closed unit ball in X is weakly sequentially compact, it follows from Lemma 5.11 that $a_n \to 0$. Hence the second term tends to zero uniformly for $u \in K$. For n such that $\tau_n \leqslant t$ we argue similarly using the inequality

$$\begin{aligned}|\langle u(t) - u(\tau_n), x^*\rangle| \leqslant{} & |\langle T(t - t_0)\, u_0 - T(\tau_n - t_0)\, u_0, x^*\rangle| \\ & + \left| \int_{t_0}^{t} \langle T(t - s) - T(\tau_n - s)]\, f(u(s), s), x^*\rangle\, \chi_n(s)\, ds \right| \\ & + \left| \int_{\tau_n}^{t} \langle T(t - s)\, f(u(s), s), x^*\rangle\, ds \right|,\end{aligned}$$

where χ_n denotes the characteristic function of the interval $[t_0, \tau_n]$. This establishes the existence of the neighborhoods $N(t, x^*, \epsilon)$. In particular $Pu \in C([t_0, t_1]; X_w)$ for each $u \in K$. But if $u \in K$, $t \in [t_0, t_1]$, then

$$\|(Pu)(t) - u_0\| \leqslant \| T(t) u_0 - u_0 \| + 2M \int_{t_0}^{t_1} m_r(s)\, ds \leqslant r.$$

Thus $P: K \to K$.

Let $K_1 = \{u \in K: |\langle u(t) - u(\tau), x^* \rangle| \leqslant \epsilon$ whenever $t \in [t_0, t_1]$, $\epsilon > 0$, $x^* \in X^*$, and $\tau \in N(t, x^*, \epsilon)\}$. It is easily checked that K_1 is a nonempty, closed, convex subset of the product space $X_w^{[t_0, t_1]}$ with the product topology. Also K_1 is a subset of $B_r(u_0)^{[t_0, t_1]}$, and is thus compact by the reflexivity of X and Tychonov's theorem. We have already shown that $P: K_1 \to K_1$. To show that P is continuous, let u_δ be a net in K_1 converging to u. By Lemma 5.10, $f(u_\delta(s), s) \xrightarrow{X} f(u(s), s)$ for almost all $s \in [t_0, t_1]$. Thus if $t \in [t_0, t_1]$, $x^* \in X^*$, then

$$\langle T(t-s) f(u_\delta(s), s), x^* \rangle \to \langle T(t-s) f(u(s), s), x^* \rangle$$

for almost all $s \in [t_0, t]$. It follows from the dominated convergence theorem that $Pu_\delta \to Pu$ in K_1. Hence P is continuous. By the Schauder–Tychonov theorem (Dunford and Schwartz [31, Theorem V. 10.5]), P has a fixed point u in K_1. By Theorem 5.1, u is a weak solution of (5.1). Clearly $u(t_0) = u_0$. By Zorn's lemma u may be extended to a weak solution, again denoted u, defined on a maximal interval of existence $[t_0, t_{\max})$.

The continuation assertion of the theorem is proved in the same way as the corresponding statement in Theorem 5.2. ∎

For $t_1 > t_0$ let $C([t_0, t_1]; X_w)$ have the topology of uniform convergence on $[t_0, t_1]$ (cf. Wilansky [54, Sect. 13.2]). We shall need the following variant of the Arzela–Ascoli theorem.

LEMMA 5.12. *Let $S \subseteq C([t_0, t_1]; X_w)$ satisfy the properties*

(i) $\| u(t) \| \leqslant C$ *for some constant* C *and for all* $u \in S$, $t \in [t_0, t_1]$;

(ii) *for each* $x^* \in X^*$ *the maps* $\{\langle u(\cdot), x^* \rangle; u \in S\}$ *are equicontinuous in* $C([t_0, t_1])$.

Then S is sequentially precompact.

Proof. Let u_n be a sequence in S. Since X is reflexive it follows from (i) that for each $t \in [t_0, t_1]$ the set $\{u_n(t)\}$ is weakly sequentially precompact. A diagonal argument shows that there exists a subsequence u_μ of u_n such that for every rational $r \in [t_0, t_1]$ the sequence $u_\mu(r)$ is weakly convergent in X to a limit $u(r)$. From (ii) we deduce that for each $x^* \in X^*$ the function $\langle u(r), x^* \rangle$ is uniformly continuous on the set $\{r \in [t_0, t_1]; r \text{ rational}\}$. It is not hard to show that u may be extended to the whole of $[t_0, t_1]$ in such a way that $u \in C([t_0, t_1]; X_w)$, and that $u_\mu \to u$ in $C([t_0, t_1]; X_w)$. ∎

Let $R > 0$, and for $s \in \mathscr{R}$ let $G(s) = \{u_s \in X^{\mathscr{R}^+} : u : [s, \infty) \to X$ is a weak solution of (5.1) satisfying $\| u(\sigma)\| \leqslant R$ for all $\sigma \in [s, \infty)\}$. Let $\bar{G} = \{u \in X^{\mathscr{R}^+} : u$ is a weak solution of (5.1) satisfying $\| u(\sigma)\| \leqslant R$ for all $\sigma \in \mathscr{R}^+\}$. Let X have the limit space structure induced by weak convergence in X.

THEOREM 5.13. *$\{G, \bar{G}\}$ is an asymptotically generalized flow on X.*

Proof. Let $s \in \mathscr{R}$, let u: $[s, \infty) \to X$ be a weak solution of (5.1) satisfying $\| u(\sigma)\| \leqslant R$ for all $\sigma \in [s, \infty)$, and let $\sigma_n \to \infty$. Let $m_R = m_{B_R(0)}$. Let $s_1 > s$, $t \in [s, s_1]$ and suppose that $t_r \to t$ in $[s, s_1]$. For $x^* \in X^*$ let

$$a_r = \sup_{\substack{\|v\| \leqslant 1 \\ s \leqslant \tau \leqslant t}} |\langle [T(t_r - \tau) - T(t - \tau)]v, x^*\rangle|.$$

If $t_r \geqslant t$ then

$$\begin{aligned} &|\langle u_{\sigma_n}(t_r) - u_{\sigma_n}(t), x^*\rangle| \\ &\quad \leqslant |\langle [T(t_r - s) - T(t - s)]\, u_{\sigma_n}(s), x^*\rangle| \\ &\qquad + \int_s^t |\langle [T(t_r - \tau) - T(t - \tau)]\, f(u_{\sigma_n}(\tau), \sigma_n + \tau), x^*\rangle|\, d\tau \\ &\qquad + \int_t^{t_r} |\langle T(t_r - \tau)\, f(u_{\sigma_n}(\tau), \sigma_n + \tau), x^*\rangle|\, d\tau \\ &\leqslant a_r \left[R + \int_s^t m_R(\sigma_n + \tau)\, d\tau\right] \\ &\qquad + Me^{\omega(s_1 - s)} \| x^* \|_{X^*} \int_t^{t_r} m_R(\sigma_n + \tau)\, d\tau. \end{aligned}$$

But we showed in the proof of Theorem 5.9 that $a_r \to 0$ as $r \to \infty$. Thus by (b_3), $|\langle u_{\sigma_n}(t_r) - u_{\sigma_n}(t), x^*\rangle|$ tends to zero as $r \to \infty$ uniformly in n. Applying a similar argument for $t_r \leqslant t$ we deduce that the functions $\langle u_{\sigma_n}(\cdot), x^*\rangle$, $n = 1, 2,\ldots$, are equicontinuous in $C([s_0, s_1])$. It thus follows from Lemma 5.12 and a diagonal argument that there exist a subsequence u_{σ_μ} of u_{σ_n} and a weakly continuous function u: $[s, \infty) \to X$ such that $u_{\sigma_\mu} \to u$ in $C(K; X_w)$ for any compact subset K of $[s, \infty)$. For any $t \in \mathscr{R}^+$ and $x^* \in X^*$ we have that

$$\begin{aligned} \langle u_{\sigma_\mu}(s + t), x^*\rangle &= \langle T(t)\, u_{\sigma_\mu}(s), x^*\rangle \\ &\quad + \int_0^t \langle T(t - \tau)\, f(u_{\sigma_\mu}(s + \tau), \sigma_\mu + s + \tau), x^*\rangle\, d\tau. \end{aligned}$$

Using (b_4) it is easily shown that

$$\langle u(s+t), x^*\rangle = \langle T(t)\, u(s), x^*\rangle + \int_0^t \langle T(t-\tau)\, \bar{f}(u(s+\tau)), x^*\rangle \, d\tau.$$

Since $\int_0^t T(t-\tau)\bar{f}(u(s+\tau))\, d\tau$ exists (see the proof of Theorem 5.9), it follows that $u_s \in \bar{G}$.

The rest of the theorem is proved similarly. ∎

We make the following further hypothesis:

(b_5) There exists a continuously Frechet differentiable function $V\colon X \to \mathscr{R}$, and a continuous function $g\colon X \to \mathscr{R}^+$ such that

(i) $\langle Au + \bar{f}(u), V'(u)\rangle = -g(u)$ for all $u \in D(A)$,

(ii) $V'\colon X \to X^*$ maps bounded sets to bounded sets,

(iii) g is sequentially weakly lower semicontinuous.

THEOREM 5.14. *Let $t_0 \in \mathscr{R}$ and let $u\colon [t_0, \infty) \to X$ be a bounded weak solution of* (5.1) *satisfying* $\lim_{t\to\infty} \int_t^{t+1} g(u(\tau))\, d\tau = 0$. *Let $\Omega(u) = \{z \in X$: there exists a sequence $t_n \to \infty$ such that $u(t_n) \xrightarrow{X} z\}$. Then if $z \in \Omega(u)$ there exists a weak solution $\bar{u}$ of* (5.4) *with* $\bar{u}(0) = z$ *such that* $g(\bar{u}(t)) = 0$ *for all* $t \in \mathscr{R}^+$.

Proof. Suppose that $\| u(\sigma)\| \leqslant R$ for all $\sigma \in [t_0, \infty)$. Define $V(t, \cdot) = V_{\bar{G}}(\cdot) = V(\cdot)$ for all $t \in \mathscr{R}$. We first show that condition (C) of Section 3 is satisfied. Let $t_n \to \infty$, $s \in \mathscr{R}$, $u_n \in G_{t_n}(s)$, $u_n \xrightarrow{X^{\mathscr{R}^+}} \bar{u} \in \bar{G}$, and let $t \in \mathscr{R}^+$. By the same method as in Lemma 5.5 one can show that

$$V(u_n(t)) - V(u_n(0)) = -\int_0^t g(u_n(\tau))\, d\tau + \int_0^t \langle f(u_n(\tau), s + t_n + \tau) - \bar{f}(u_n(\tau), V'(u_n(\tau))\rangle \, d\tau.$$

By (b_4), ($b_5$ii) the second integral tends to zero as $n \to \infty$. Thus by ($b_5$iii) and Fatou's lemma

$$\begin{aligned}\varliminf_{n\to\infty}[V(u_n(0)) - V(u_n(t))] &= \varliminf_{n\to\infty} \int_0^t g\,(u_n(\tau))\, d\tau \\ &\geqslant \int_0^t g(\bar{u}(\tau))\, d\tau \\ &= V(\bar{u}(0)) - V(\bar{u}(t)),\end{aligned}$$

which is condition (C).

A similar argument shows that for each $\tau \in \mathscr{R}^+$

$$\varliminf_{t\to\infty}[V(u(t)) - V(u(t+\tau))] = 0.$$

The hypotheses of Theorem 3.5 are thus satisfied, and since $B_G(0)$ is sequentially weakly closed, we conclude that if $z \in \Omega(u)$ there exists a weak solution $\bar{u}$ of (5.4) satisfying $\bar{u}(0) = z$ and such that $\int_0^t g(\bar{u}(\tau))\, d\tau = 0$ for any $t > 0$. Since the map $\tau \mapsto g(\bar{u}(\tau))$ is continuous on $\mathscr{R}^+$ it follows that $g(\bar{u}(t)) = 0$ for all $t \in \mathscr{R}^+$. ∎

In order to apply Theorem 3.6 we need to make further hypotheses; those made in the following theorem, while they should not be thought of as in any way fundamental, will prove useful in the example considered later.

THEOREM 5.15. *Suppose that there exists a continuously Frechet differentiable function* $J: X \to \mathscr{R}$, *and a continuous function* $h: X \to \mathscr{R}$, *bounded below on bounded subsets of* X, *satisfying the properties*

(i) J *is sequentially weakly continuous, and* $J': X \to X^*$ *maps bounded sets to bounded sets,*

(ii) $\langle Au + f(u), J'(u)\rangle = h(u)$ *for all* $u \in D(A)$,

(iii) *Let* $z_n \xrightarrow{X} z$, $g(z_n) \to 0$. *Then* $h(z) \leq \underline{\lim}_{n\to\infty} h(z_n)$. *If, further,* $h(z_n) \to h(z)$, *then* $V(z_n) \to V(z)$.

Let $t_0 \in \mathscr{R}$, *let* $u: [t_0, \infty) \to X$ *be a bounded weak solution of* (5.1). *Let* $\alpha = \underline{\lim}_{t\to\infty} V(u(t))$, $\beta = \overline{\lim}_{t\to\infty} V(u(t))$. Define $\Omega_V(u) = \{z \in X$: *there exists a sequence* $t_n \to \infty$ *such that* $u(t_n) \xrightarrow{X} z$ *and* $V(u(t_n)) \to V(z)\}$, *and for* $\gamma \in \mathscr{R}$ *define* $M_\gamma = \{z \in X$: *there exists a weak solution* $\bar{u}$ *of* (5.4) *with* $\bar{u}(0) = z$ *such that* $V(\bar{u}(t)) = \gamma$, $g(\bar{u}(t)) = 0$ *for all* $t \in \mathscr{R}^+\}$. *Then* $\infty > \beta \geq \alpha > -\infty$ *and for each* $\gamma \in [\alpha, \beta]$ *the set* $\Omega_V(u) \cap M_\gamma$ *is nonempty.*

If, further, for each $\gamma \in [\alpha, \beta]$ *there are only finitely many elements of* M_γ *in the weak closure of the range of* u, *then* $\alpha = \beta$, *and for some* $z_0 \in M_\alpha$, $u(t) \xrightarrow{X} z_0$, *and* $V(u(t)) \to V(z_0)$ *as* $t \to \infty$.

Proof. Suppose that $\|u(\sigma)\| \leq R$ for all $\sigma \in [t_0, \infty)$. Define $V(t, \cdot) = V_{\bar{G}}(\cdot) = V(\cdot)$ for all $t \in \mathscr{R}$. We begin by showing that condition (D) is satisfied. Let $s \in \mathscr{R}$, $t_n \to \infty$, $u_n \in G_{t_n}(s)$, $u_n \xrightarrow{X^{\mathscr{R}^+}} \bar{u} \in \bar{G}$ and suppose that $V(u_n(0)) - V(u_n(t)) \to 0$ uniformly for t in compact subsets of $\mathscr{R}^+$. Suppose that $V(u_n(0)) \not\to V(\bar{u}(0))$. Without loss of generality we may assume that $|V(u_n(0)) - V(\bar{u}(0))| \geq \epsilon > 0$ for all n. Let $T > 0$. From the proof of Theorem 5.14 we see that $\lim_{n\to\infty} \int_0^T g(u_n(\tau))\, d\tau = 0$. Since $g \geq 0$ there exists a subsequence u_μ of u_n such that

$$g(u_\mu(\tau)) \to 0 = g(\bar{u}(\tau)) \qquad \text{for almost all} \quad \tau \in [0, T].$$

Similarly,

$$\begin{aligned}\lim_{\mu\to\infty} \int_0^T h(u_\mu(\tau))\, d\tau &= \lim_{\mu\to\infty} [J(u_\mu(T)) - J(u_\mu(0))] \\ &= J(\bar{u}(T)) - J(\bar{u}(0)) \\ &= \int_0^T h(\bar{u}(\tau))\, d\tau,\end{aligned}$$

where the second equality follows from the sequential weak continuity of J.

Let $S = \{\tau \in [0, T]: h(\bar{u}(\tau)) < \underline{\lim}_{\mu\to\infty} h(u_\mu(\tau))\}$. Since h is bounded below on bounded sets it follows from (iii) and Fatou's lemma that S has measure zero. Hence we may assume without loss of generality that for some $t_0 \in [0, T]$, $g(u_\mu(t_0)) \to 0$ and $h(u_\mu(t_0)) \to h(\bar{u}(t_0))$. Thus by (iii) we have $V(u_\mu(t_0)) \to V(\bar{u}(t_0))$. But $V(u_\mu(t_0)) - V(u_\mu(0)) \to 0$ and $V(\bar{u}(t_0)) = V(\bar{u}(0))$. Hence $V(u_\mu(0)) \to V(\bar{u}(0))$, which is a contradiction. Thus condition (D) is satisfied.

Since V and u are continuous, so is the map $t \mapsto V(u(t))$. The set $u([t_0, \infty))$ is clearly sequentially weakly precompact. Also, by Theorem 3.7(ii) and the proof of Theorem 5.14, (3.6) holds. By ($b_5$ii) and the representation

$$V(y) = V(0) + \int_0^1 \langle y, V'(\eta y)\rangle\, d\eta,$$

V maps bounded sets in X to bounded sets. Therefore $\beta > -\infty$. Thus by Theorem 3.6(i), $\infty > \beta \geqslant \alpha > -\infty$ and for each $\gamma \in [\alpha, \beta]$ the set $\Omega_V(u) \cap M_\gamma$ is nonempty.

The last statement of the theorem follows from the weak connectedness of $\Omega_V(u)$. ▮

EXAMPLE. Let H be a real Hilbert space with inner product $(\,,\,)$ and norm $\|\cdot\|$. Let B be a positive self-adjoint operator densely defined on H and such that B^{-1} is defined on all of H and is compact. Let H_B denote the domain of $B^{1/2}$. H_B is a Hilbert space with inner product $(\,,\,)_B$ and norm $\|\cdot\|_B$, where $\|v\|_B = \|B^{1/2}v\|$ for all $v \in H_B$. We identify H with its dual. It is clear that H_B is dense in H, that H is dense in $H_B{}^*$, and that both imbedding maps are compact. Let $X = H_B \times H$. Under the norm $\|\{w, v\}\|_X = (\|w\|_B^2 + \|v\|^2)^{1/2}$, X forms a Hilbert space.

Consider the abstract damped wave equation

$$\ddot{w} + Bw + F(w, \dot{w}, t) = 0, \tag{5.18}$$

and the corresponding autonomous equation

$$\ddot{w} + Bw + D(w, \dot{w}) + \bar{F}(w) = 0. \tag{5.19}$$

We make the following hypotheses on $F, \bar{F}$, and D.

(ei) $F: H_B \times H \times \mathscr{R} \to H$; for each $\{w, v\} \in X$ the map $F(w, v, \cdot)$ from $\mathscr{R}$ into H is strongly measurable, and for almost all $t \in \mathscr{R}$ the map $F(\cdot, \cdot, t)$ from X into H is sequentially weakly continuous;

(eii) for each bounded subset G of X there is a locally integrable function m_G on $\mathscr{R}$ such that

$$\|F(w, v, t)\| \leqslant m_G(t)$$

for all $\{w, v\} \in X$ and almost all $t \in \mathscr{R}$, and for each $t_0 \in \mathscr{R}$

$$\lim_{s\to 0} \sup_{t \geq t_0} \int_t^{t+s} m_G(\tau)\, d\tau = 0;$$

(eiii) the maps $D : X \to H$ and $\bar{F} : H_B \to H$ are sequentially weakly continuous, and for each bounded subset G of X

$$\lim_{t\to\infty} \int_t^{t+1} \sup_{\{w,v\}\in G} \| F(w, v, s) - D(w, v) - \bar{F}(w) \| \, ds = 0;$$

(eiv) there exists a continuously Fréchet differentiable function $\Phi: H_B \to \mathscr{R}$ such that $\Phi'(w) = \bar{F}(w)$ for all $w \in H_B$ (thus $\langle v, \Phi'(w)\rangle = (\bar{F}(w), v)$ for all $w, v \in H_B$);

(ev) the map $\{w, v\} \mapsto (D(w, v), v)$ is sequentially weakly lower semicontinuous on X; furthermore, for each bounded subset Γ of H_B there exists a strictly increasing continuous function $k_\Gamma: \mathscr{R}^+ \to \mathscr{R}^+$ with $k_\Gamma(0) = 0$, such that

$$\inf_{w\in\Gamma}(D(w, v), v) \geq k_\Gamma(\| v \|)$$

for all $v \in H$.

Before analyzing (5.18) and (5.19) we mention two special cases of these equations.

Special Case 1. Let Ω be a nonempty bounded open set in $\mathscr{R}^n$ with boundary $\partial\Omega$. Let $H = L^2(\Omega)$. Consider the problem

$$\begin{gathered} w_{tt} + a(w, t)\, w_t - \Delta w + \phi(w, t) = 0, \qquad x \in \Omega,\ t > s, \\ w\,|_{\partial\Omega} = 0, \qquad w\,|_{t=s} \text{ and } w_t|_{t=s} \text{ prescribed,} \end{gathered} \tag{5.20}$$

and the corresponding autonomous problem

$$\begin{gathered} w_{tt} + \bar{a}(w)\, w_t - \Delta w + \bar{\phi}(w) = 0, \qquad x \in \Omega,\ t > 0, \\ w\,|_{\partial\Omega} = 0, \qquad w\,|_{t=0} \text{ and } w_t\,|_{t=0} \text{ prescribed.} \end{gathered} \tag{5.21}$$

We suppose that

(i) $a: \mathscr{R} \times \mathscr{R} \to \mathscr{R}$; $a(w, \cdot)$ is measurable for each $w \in \mathscr{R}$, and $a(\cdot, t)$ is continuous for almost all $t \in \mathscr{R}$; $\bar{a}: \mathscr{R} \to \mathscr{R}$ is continuous and satisfies

$$\bar{a}(w) \geq \delta > 0$$

for all $w \in \mathscr{R}$, where δ is some constant;

(ii) $\phi: \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is measurable in t for each fixed w and continuous in w for almost all t; $\bar{\phi}: \mathscr{R} \to \mathscr{R}$ is continuous;

(iii) there exists a nonnegative locally integrable function $m(t)$ satisfying

$$\limsup_{s\to 0}\sup_{t>t_0}\int_t^{t+s} m(\tau)\,d\tau = 0$$

for each $t_0 \in \mathscr{R}$, such that if $n = 1$

$$|a(w,t)| + |\phi(w,t)| \leqslant m(t)\,\theta_1(w),$$

for all $w \in \mathscr{R}$ and almost all $t \in \mathscr{R}$, where $\theta_1: \mathscr{R} \to \mathscr{R}$ is continuous, and if $n > 1$

$$|a(w,t)| + \frac{|\phi(w,t)|}{1 + |w|^\gamma} \leqslant m(t),$$

for all $w \in \mathscr{R}$ and almost all $t \in \mathscr{R}$, where $1 \leqslant \gamma < \infty$ if $n = 2$, $1 \leqslant \gamma \leqslant n/(n-2)$ if $n > 2$

(iv) there exists a nonnegative locally integrable function $\eta(t)$ satisfying

$$\lim_{t\to\infty}\int_t^{t+1} \eta(\tau)\,d\tau = 0,$$

such that if $n = 1$,

$$|a(w,t) - \bar{a}(w)| + |\phi(w,t) - \bar{\phi}(w)| \leqslant \eta(t)\,\theta_2(w),$$

for all $w \in \mathscr{R}$ and almost all $t \in \mathscr{R}$, where $\theta_2: \mathscr{R} \to \mathscr{R}$ is continuous, and if $n > 1$,

$$|a(w,t) - \bar{a}(w)| + \frac{|\phi(w,t) - \bar{\phi}(w)|}{1 + |w|^\gamma} \leqslant \eta(t),$$

for all $w \in \mathscr{R}$ and almost all $t \in \mathscr{R}$, where $1 \leqslant \gamma < \infty$ if $n = 2$, $1 \leqslant \gamma \leqslant n/(n-2)$ if $n > 2$.

Let $D(B) = \{w \in W_0^{1,2}(\Omega): \Delta w \in L^2(\Omega)\}$, and let $B = -\Delta$. Then $H_B = W_0^{1,2}(\Omega)$. Put $F(w,v,t) = a(w(\cdot),t)\,v(\cdot) + \phi(w(\cdot),t)$, $D(w,v) = \bar{a}(w(\cdot))\,v(\cdot)$, $\bar{F}(w) = \bar{\phi}(w(\cdot))$, $\Phi(w) = \int_\Omega \int_0^{w(x)} \bar{\phi}(r)\,dr\,dx$. We claim that hypotheses (ei)–(ev) hold. The measurability assertions follow in the same way as for the example in Subsection (a). The fact that $(D(w,v),v)$ is sequentially weakly lower semicontinuous is a consequence, for example, of Ekeland and Témam [32, Chap. VIII, Theorem 2.1]. The other assertions are not hard to prove; as an example we show that $F(\cdot,\cdot,t)$ is sequentially weakly continuous from X into H for almost all $t \in \mathscr{R}$. Let $w_n \xrightarrow{W_0^{1,2}(\Omega)} W$, $v_n \xrightarrow{L^2(\Omega)} v$. Then by the imbedding theorems, Mazur's theorem, and (i)–(iii),

$$a(w_n(\cdot),t) \xrightarrow{L^2(\Omega)} a(w(\cdot),t),$$

and

$$\phi(w_n(\cdot),t) \xrightarrow{L^2(\Omega)} \phi(w(\cdot),t).$$

Hence $a(w_n(\cdot),t)v_n \xrightarrow{L^2(\Omega)} a(w(\cdot),t)v$, and the result follows.

Special Case 2. We consider a crude model for the transverse deflection w of an extensible elastic rod of length $l > 0$. This model has been studied in the autonomous case by Dickey [30] and Ball [3–5]. If the ends of the beam are clamped then the appropriate initial boundary value problem is

$$w_{tt} + (\delta + d(t))\, w_t + \alpha w_{xxxx} - \left(\beta + b(t) + k \int_0^l w_\xi(\xi, t)^2\, d\xi\right) w_{xx} = 0, \qquad x \in \Omega,\ t > s$$

$$w = w_x = 0 \text{ at } x = 0, l, \qquad w\,|_{t=s} \text{ and } w_t\,|_{t=s} \text{ prescribed.} \tag{5.22}$$

In (5.22), $\Omega = (0, l)$, and $\delta > 0$, $\alpha > 0$, β, $k > 0$ are constants. The measurable real-valued functions $d(\cdot)$ and $b(\cdot)$ represent perturbations to the damping coefficient and the axial load, respectively. We assume that

$$\lim_{t \to \infty} \int_t^{t+1} [|\, d(\tau)| + |\, b(\tau)|]\, d\tau = 0.$$

Let $H = L^2(\Omega)$, $D(B) = \{w \in W_0^{2,2}(\Omega){:}\ w_{xxxx} \in L^2(\Omega)\}$, $B = \alpha(d^4/dx^4)$. Then $H_B = W_0^{2,2}(\Omega)$. Put $F(w, v, t) = (\delta + d(t))v(\cdot) - (\beta + b(t) + k \| w_x \|^2) w_{xx}(\cdot)$, $D(w, v) = \delta v(\cdot)$, $\bar{F}(w) = -(\beta + k \| w_x \|^2) w_{xx}(\cdot)$, $\Phi(w) = (\beta/2) \| w_x \|^2 + (k/4) \| w_x \|^4$. It is easily verified that hypotheses (ei)–(ev) hold.

Returning now to the study of Eqs. (5.18) and (5.19), we proceed to verify that the hypotheses of Theorem 5.15 hold. Define $D(A) = D(B) \times H_B \subseteq X$, $A(\{w, v\}) = \{v, -Bw\}$. Standard results from linear semigroup theory show that A generates a strongly continuous group on X. Define $f{:}\ X \times \mathscr{R} \to X$ by $f(\{w, v\}, t) = \{0, -F(w, v, t)\}$. If $u = \{w, w_t\}$, then (5.18) has the form (5.1). Similarly, if $\bar{f}(\{w, v\}) = \{0, -D(w, v) - \bar{F}(w)\}$ then (5.19) has the form (5.4). It is clear from (ei)–(eiii) that hypotheses (b_1)–(b_4) are satisfied. Define $V{:}\ X \to \mathscr{R}$ and $g{:}\ X \to \mathscr{R}$ by

$$V(w, v) = \tfrac{1}{2}\|\{w, v\}\|_X^2 + \Phi(w), \tag{5.23}$$

$$g(w, v) = (D(w, v), v), \tag{5.24}$$

for $\{w, v\} \in X$. It is easily verified that V is continuously differentiable and that (b_5i) holds. Since D is sequentially weakly continuous it follows that g is continuous. The fact that V' maps bounded sets to bounded sets follows from the sequential weak continuity of $\bar{F}$. Hence (b_5) is satisfied.

Define $J{:}\ X \to \mathscr{R}$ and $h{:}\ X \to \mathscr{R}$ by

$$J(w, v) = -(w, v),$$
$$h(w, v) = \| w \|_B^2 + (D(w, v), w) + (\bar{F}(w), w) - \| v \|^2,$$

for $\{w, v\} \in X$. It is easily checked that J is continuously differentiable, that J' and h map bounded sets to bounded sets, and that $\langle Au + \bar{f}(u), J'(u)\rangle = h(u)$ for all $u \in D(A)$. Since the imbedding of H_B in H is compact, J is sequentially weakly continuous. Let $w_n \xrightarrow{H_B} w$, $v_n \xrightarrow{H} 0$. Then $w_n \xrightarrow{H} w$, $D(w_n, v_n) \xrightarrow{H} D(w, 0)$, $\bar{F}(w_n) \xrightarrow{H} \bar{F}(w)$, $\| w \|_B^2 \leqslant \underline{\lim}_{n\to\infty} \| w_n \|_B^2$. Hence $h(w, 0) \leqslant \underline{\lim}_{n\to\infty} h(w_n, v_n)$. Suppose further that $h(w_n, v_n) \to h(w, 0)$. Then $\| w_n \|_B \to \| w \|_B$. Therefore $w_n \xrightarrow{H_B} w$, and thus $V(w_n, v_n) \to V(w, 0)$. We have thus shown that conditions (i)–(iii) of Theorem 5.15 are satisfied. Applying the theorem we obtain

THEOREM 5.16. *Let $t_0 \in \mathscr{R}$, and let $\{w, w_t\}$: $[t_0, \infty) \to X$ be a weak solution of* (5.18) *that is bounded in norm. Let V be given by* (5.23), *let $\alpha = \underline{\lim}_{t\to\infty} V(w(t), w_t(t))$, and let $\beta = \overline{\lim}_{t\to\infty} V(w(t), w_t(t))$. Define $\Omega_s(w) = \{y \in H_B$: there exists a sequence $t_n \to \infty$ such that $\{w, w_t\}(t_n) \xrightarrow{X} \{y, 0\}\}$ and for $\gamma \in \mathscr{R}$ define $M_\gamma = \{\{y, 0\} \in X: y \in D(B), By + \bar{F}(y) = 0, V(y, 0) = \gamma\}$. Then $\infty > \beta \geqslant \alpha > -\infty$ and for each $\gamma \in [\alpha, \beta]$ the set $\Omega_s(w) \cap M_\gamma$ is nonempty.*

If, further, for each $\gamma \in [\alpha, \beta]$ there are only finitely many elements of M_γ in the weak closure of the range of $\{w, w_t\}$, then $\alpha = \beta$ and for some $\{y, 0\} \in M_\alpha$, $\{w(t), w_t(t)\} \xrightarrow{X} \{y, 0\}$ as $t \to \infty$.

Remarks. (1) If in special case 1 the function $a(w, t)$ depends also explicitly on $x \in \Omega$, and vanishes for x outside some compact subset of Ω, then (ev) does not hold. This is an example of "weak damping." For results in the linear case the reader is referred to Dafermos [22, 24, 28]. For decay estimates in the linear case see Rauch [44] for the case of strong damping and Russell [49] for weak damping.

(2) In special case 2 there are only finitely many equilibrium positions for the rod (cf. [4]), so that the last statement of Theorem 5.16 holds. An existence and uniqueness theory for (5.22) could be given using the fact that the corresponding $f(u, t)$ in (5.1) satisfies a Lipschitz condition with respect to u (cf. Ball [5], Reed [45]); in this case, however, a separate argument is required to prove the necessary weak continuous dependence results.

Finally we discuss an example which does not fit directly into the theory developed in this paper, but can be handled by similar methods. Let $\Omega, \bar{\phi}$ be as in special case 1, let $H = L^2(\Omega)$, $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$, and consider the autonomous problem

$$\begin{gathered} w_{tt} + (1 + \| \nabla w \|^2)\, w_t - \Delta w + \bar{\phi}(w) = 0, \\ w|_{\partial\Omega} = 0, \qquad \{w, w_t\}(0) \in X. \end{gathered} \tag{5.25}$$

Assume for simplicity that there are only finitely many solutions $y \in W_0^{1,2}(\Omega)$ of the steady-state problem

$$\Delta y = \bar{\phi}(y). \tag{5.26}$$

As before we may write (5.25) in the form

$$\dot{u} = Au + \bar{f}(u),$$

where $A(\{w, v\}) = \{v, \Delta w\}$, $\bar{f}(\{w, v\}) = \{0, -(1 + \| \nabla w \|^2)v - \bar{\phi}(w)\}$, etc. But $\bar{f}$ is not sequentially weakly continuous, so that the preceding theory does not apply. Nevertheless any weak solution $\{w, w_t\}(t)$ of (5.25) converges strongly in X as $t \to \infty$ to $\{y, 0\}$ for some solution $y \in W_0^{1,2}(\Omega)$ of (5.26). To prove this let X have the limit space structure of weak convergence, and let $\mathscr{X} = \{u \in X^{\mathscr{R}^+} : u = \{w, w_t\}$ is a weak solution of (5.25)$\}$. For $\tau \in \mathscr{R}^+$, $u \in \mathscr{X}$, define $R(\tau)u$ to be the τ-translate u_τ of u. We claim that R satisfies the relaxed continuity property:

(*) if $t_n \to \infty$ and $R(t_n)u \xrightarrow{X^{\mathscr{R}^+}} v$, then $v = \{w, 0\}$ and

$$R(t)\, R(t_n)u \xrightarrow{X^{\mathscr{R}^+}} R(t)v \qquad \text{for any} \quad t \in \mathscr{R}^+.$$

To prove (*) note that the energy equation

$$[\tfrac{1}{2} \|\{w, w_t\}\|_X^2 + \Phi(w)]_s^t + \int_s^t (1 + \| \nabla w \|^2) \| w_t \|^2 \, d\tau = 0$$

holds for weak solutions of (5.25), so that $\| \nabla w \|^2(t)$ is bounded for all $t \in \mathscr{R}^+$ and $\int_0^\infty \| w_t \|^2 \, d\tau < \infty$. Property (*) follows by using these facts, the variation of constants formula, and our previous techniques. We now observe that (*) is sufficient for the arguments of Section 2 to go through. Strong convergence of $\{w, w_t\}(t)$ to $\{y, 0\}$ as $t \to \infty$ can then be proved as for special case 1. Certain other damping terms depending on ∇w may be treated similarly.

Note added in proof. The methods of Section 5 are applied in Ball and Slemrod [58] to some semilinear control problems in Hilbert space.

In a recent paper Webb [59] has shown that one may prove strong precompactness of bounded orbits for autonomous equations of the form (5.4) when $\bar{f}: x \to x$ is compact and $\| T(t)\| \leq Me^{\omega t}$ for all $t \in \mathbb{R}^+$, with $\omega < 0$. Some results for equations (5.18) and (5.19) can be obtained by this method.

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