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SETS OF GRADIENTS WITH NO RANK-ONE CONNECTIONS

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ABSTRACT. — Examples are given for $m \geqq n \geqq 3$ and $n=2$, $m \geqq 4$ of rank-one convex integrands $W: M^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ which are not quasiconvex in the sense of Morrey. The method is based on the construction of smooth mappings $u: \bar{\Omega} \to \mathbb{R}^m$ with linear boundary values, where $\Omega \subset \mathbb{R}^n$ is bounded and open, such that the set of gradients $Du(\bar{\Omega})$ has no rank-one connections. The integrands W take the value $+\infty$ in an essential way.

Using similar ideas examples are exhibited of bounded sequences of mappings $u^{(j)}$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $Du^{(j)}$ has essential support in a closed set $K \subset M^{m\times n}$ having no rank-one connections, but such that the Young measure corresponding to $Du^{(j)}$ is not a Dirac mass.

Finally it is shown that the same construction does not work for $m=n=2$, at least for sufficiently smooth mappings u.

RÉSUMÉ. — On donne ici, pour $m \geqq n \geqq 3$ et $n=2$, $m \geqq 4$, des exemples d'intégrandes $W: M^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ qui sont rang-un convexes mais ne sont pas quasiconvexes au sens de Morrey. La méthode repose sur la construction d'applications régulières $u: \bar{\Omega} \to \mathbb{R}^m$, où $\Omega \subset \mathbb{R}^n$ est un ouvert borné, avec valeurs au bord linéaires, telles que l'ensemble de gradients $Du(\bar{\Omega})$ n'a aucune relation de rang un; il est d'autre part essentiel que les intégrandes W prennent la valeur $+\infty$.

En utilisant des idées semblables, on présente des exemples de suites d'applications $u^{(j)}$ bornées dans $W^{1,\infty}(\Omega; \mathbb{R}^m)$ telles que les supports des $Du^{(j)}$ sont contenus dans un ensemble fermé $K \subset M^{m\times n}$ sans relations de rang un, mais telles que la mesure de Young qui correspond à $Du^{(j)}$ n'est pas une masse de Dirac.

On démontre enfin que la même construction ne conduit pas à un contre-exemple pour $m=n=2$, au moins pour les applications u suffisamment régulières.

1. Introduction

This paper is motivated by the problem of minimizing integrals of the form

$$I(u) = \int_{\Omega} W(Du(x))\,dx \tag{1.1}$$

among a suitable class of mappings $u: \Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is bounded and open, $Du(x)$ denotes the gradient matrix of u, and $W: M^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is a given integrand. (Here $M^{m\times n}$ denotes the set of real $m \times n$ matrices.) The integrand W is said to be *rank-one convex* if it is convex along all line segments in $M^{m\times n}$ whose endpoints differ by a matrix of rank one (*see* Definition 2.1), while it is *quasiconvex* if linear

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mappings u minimize I among Lipschitz mappings having the same boundary values (*see* Definition 2.2).

A significant problem of the calculus of variations, posed by Morrey ([16], [17]), is to decide whether or not every rank-one convex W is quasiconvex, and the main purpose of this paper is to provide a counterexample in the cases $m \geqq n \geqq 3$ and $n=2$, $m \geqq 4$. Unfortunately, the integrand $\bar{W}$ in the counterexample takes the value $+\infty$ except on a small set K of matrices, and it does not seem possible to modify it to give a counterexample which is everywhere finite. In particular, the mapping u with linear boundary values that violates the quasiconvexity inequality for $\bar{W}$ does not do so for any everywhere finite rank-one convex function (Proposition 3.4). Thus the counterexample does not satisfactorily resolve Morrey's question, and in particular is consistent with the possibility that all sufficiently regular rank-one convex functions are quasiconvex.

The idea of the counterexample is to construct a smooth mapping u with linear boundary values $A x$, such that the set $K_\Omega(u)=\{Du(x): x \in \bar{\Omega}\}$ has no *rank-one connections*, that is, there is no pair of matrices $A_1, A_2 \in K_\Omega(u)$ with $\operatorname{rank}(A_1-A_2)=1$. If, in addition, $A \notin K_\Omega(u)$, then a suitable $\bar{W}$ may be defined by $\bar{W}(B)=0$ if $B \in K_\Omega(u)$, $=+\infty$ otherwise. For $m=n \geqq 3$ it turns out (Proposition 3.2) that a *radial* mapping u can be constructed with the above properties (with $A=\lambda \mathbf{1}$, λ a real constant). On the other hand, we prove that for $m=n=2$ there is no such smooth u, even among non-radial mappings (Theorem 5.1).

The counterexample is easily adapted (Theorem 4.2) to provide an example in the cases $\min(m, n) \geqq 3$ and $n=2$, $m \geqq 4$ of a bounded sequence $u^{(j)}$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $Du^{(j)}$ has essential support in a set K of matrices having no rank-one connections, but such that the Young measure (ν_x) corresponding to $Du^{(j)}$ is not a Dirac mass. The contrary was conjectured by Tartar [24] in the more general context of the theory of compensated compactness, on the basis that eliminating one-dimensional oscillations might eliminate all oscillations. In the example, the diameter of $\operatorname{supp} \nu_x$ may be made arbitrarily small without changing K. Some cases when Tartar's conjecture is known to be true are listed in Section 4; these examples partly motivated this paper. The negative observations in Theorem 5.1 might perhaps suggest that similar examples cannot be constructed for $m=n=2$, but the author has not succeeded in adapting the arguments to give a proof.

The reader is referred to [2] for a survey of the rank-one convexity/quasiconvexity problem. We now briefly discuss two more recent developments. First, in the case $m=n=2$ Aubert [1] and Gurvich [12] have constructed explicit examples of quartic integrands $\hat{W}$ that are rank-one convex but not polyconvex. An algebraically simpler version of Aubert's example with the same properties, found by Dacorogna & Marcellini [9], is given by $\hat{W}(A)=|A|^4-(4/\sqrt{3})|A|^2(\det A)$, where $|A|=[\operatorname{tr}(A^T A)]^{1/2}$. It is not known whether these integrands are quasiconvex. Note that if it were known that any finite quasiconvex function W satisfied the lower growth condition

$$W(A) \geqq -\text{Const.}(|A|^{\min(n, m)}+1), \tag{1.2}$$

then it would follow immediately that $\hat{W}$ is not quasiconvex.

Secondly, Sivaloganathan [22] has proved an interesting result, in the spirit of the field theory of the calculus of variations, implying that if $W \in C^2$ is rank-one convex, and if $u(\alpha, \cdot)$ is a one-parameter family of smooth solutions to the Euler-Lagrange equations for I, then $u(0, \cdot)$ minimizes I among all mappings u_φ of the form $u_\varphi(x) = u(\varphi(x), x)$, where $\varphi \in C^1$ with $\varphi|_{\partial\Omega} = 0$. He has further suggested how such ideas might form part of a proof that rank-one convexity implies quasiconvexity for smooth W. Another suggestion for a possible proof, motivated by a result of Knops & Stuart [15] can be found in [2]; however implementation of this idea would seem to require some new way of using rank-one convexity to obtain some regularity or compactness properties for solutions to the Euler-Lagrange equations, and at present the only regularity assertions that come close to that required are those based on Evans [11], which have quasiconvexity as a principal hypothesis.

2. Definitions and preliminaries

Let $m \geqq 1$, $n \geqq 1$. If $E \subset \mathbb{R}^n$ is open we write $\|v\|_\infty = \operatorname*{ess\,sup}_{x \in E} |v(x)|$ and denote by $W^{1,\infty}(E; \mathbb{R}^m)$ the Sobolev space consisting of measurable mappings $u: E \to \mathbb{R}^m$ with finite norm $\|u\|_{1,\infty} \stackrel{\text{def}}{=} \|u\|_\infty + \|Du\|_\infty$. We denote by $W_0^{1,\infty}(E; \mathbb{R}^m)$ the closure of $C_0^\infty(E; \mathbb{R}^m)$ in the weak* topology of $W^{1,\infty}(E; \mathbb{R}^m)$, *i.e.* in the subspace topology induced by regarding $W^{1,\infty}(E; \mathbb{R}^m)$ as a closed subspace of a finite product of $L^\infty(E)$ spaces each endowed with the weak* topology.

Let $W: M^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable and bounded below.

DEFINITION 2.1. — W is *rank-one convex* if

$$W(tA + (1-t)B) \leqq tW(A) + (1-t)W(B) \tag{2.1}$$

whenever $t \in (0, 1)$ and $A, B \in M^{m \times n}$ with rank $(A - B) = 1$.

DEFINITION 2.2. — W is *quasiconvex* if

$$\int_E W(A + D\varphi(x))\, dx \geqq \int_E W(A)\, dx = (\text{meas } E)\, W(A) \tag{2.2}$$

for every bounded open set $E \subset \mathbb{R}^n$ with meas $\partial E = 0$, each $A \in M^{m \times n}$, and all $\varphi \in W_0^{1,\infty}(E; \mathbb{R}^m)$.

DEFINITION 2.3. — Let $K \subset M^{m \times n}$. K has *no rank-one connections* if there is no pair of matrices $A, B \in K$ with rank $(A - B) = 1$.

Remarks 2.4. — 1. If (2.2) holds for one nonempty bounded open subset $E \subset \Omega$, and for all $A \in M^{m \times n}$, all $\varphi \in W_0^{1,\infty}(E; \mathbb{R}^m)$, then (*cf.* Ball & Murat [8], Prop. 2.3) W is quasiconvex.

2. Both quasiconvexity and rank-one convexity of W are necessary conditions for $I_E(u)=\int_E W(Du(x))\,dx$ to be sequentially weak* lower semicontinuous on $W^{1,\infty}(E;\mathbb{R}^m)$ (see Morrey [16], Ball & Murat [8]).

3. If W is quasiconvex and *continuous* (with respect to the usual topology on $\mathbb{R}\cup\{+\infty\}$) then W is rank-one convex. This is false for general Borel measurable W (Ball & Murat [8], Example 3.5).

Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and a point $x\in\mathbb{R}^n$, we write $x'=\sum_{i=1}^{n-1}(x\cdot e_i)\,e_i$, $x^n=x\cdot e_n$. To avoid possible confusion we recall for the convenience of the reader two standard definitions.

Definition 2.5. — Let $\Omega\subset\mathbb{R}^n$ be open. We say that Ω has a boundary of class C^1 if given any $x_0\in\partial\Omega$ there exist a corresponding orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, a neighbourhood U of x_0 and a C^1 function $f:\mathbb{R}^{n-1}\to\mathbb{R}$, such that

$$\partial\Omega\cap U=\{x\in U : x^n=f(x')\}$$

and

$$\Omega\cap U=\{x\in U : x^n>f(x')\}.$$

Definition 2.6. — Let $\Omega\subset\mathbb{R}^n$ be open and let $r\geqq 1$ be an integer. Given $u\in C^r(\Omega;\mathbb{R}^m)$ we say that $u\in C^r(\bar\Omega;\mathbb{R}^m)$ if there exists $v\in C^r(\mathbb{R}^n;\mathbb{R}^m)$ such that $v(x)=u(x)$ for all $x\in\Omega$.

Remark 2.7. — As is well known this is stronger than requiring that the derivatives up to order r of u in Ω extend to continuous functions on $\bar\Omega$ (*cf.* Whitney [26]).

It follows from Definition 2.6 that if $u\in C^r(\bar\Omega;\mathbb{R}^m)$ then the boundary values $D^\alpha u|_{\partial\Omega}$, $|\alpha|\leqq r$, are well defined.

Lemma 2.8. — *Let Ω be open with $\Omega=\operatorname{int}\bar\Omega$. Then a necessary and sufficient condition for Ω to have C^1 boundary is that for some $m\geqq 1$ there exists a function $u\in C^1(\bar\Omega;\mathbb{R}^m)$ such that $u(x)=0$ and $Du(x)\neq 0$ for every $x\in\partial\Omega$.*

Proof. — *Necessity.* — This is easily proved (taking $m=1$) using the functions in Definition 2.5 and a partition of unity. We omit the details since the necessity is not used in this paper.

Sufficiency. — Since $u\in C^1(\bar\Omega;\mathbb{R}^m)$ there exists $v\in C^1(\mathbb{R}^n;\mathbb{R}^m)$ extending u. Let $x_0\in\partial\Omega$. For some $i=1, \ldots, m$, $Dv^i(x_0)\neq 0$. Let $\{e_j\}$ be an orthonormal basis of $\mathbb{R}^n$ with e_n parallel to $Dv^i(x_0)$, and write $v^i(x)=v^i(x', x^n)$. Thus $(\partial v^i/\partial x^n)(x'_0, x^n_0)\neq 0$, and so by the implicit function theorem there exist a C^1 function $f:\mathbb{R}^{n-1}\to\mathbb{R}$ and a cylindrical open neighbourhood $U=\{x\in\mathbb{R}^n : |x'-x'_0|<\delta, |x^n-x^n_0|<\varepsilon\}$ of x_0 such that (i) $|f(x')-x^n_0|<\varepsilon/2$ whenever $|x'-x'_0|<\delta$, and (ii) if $x\in U$ then $v^i(x)=0$ if and only if

$x^n = f(x')$. Since $\Omega = \operatorname{int} \bar{\Omega}$ there are points from each of the open sets Ω, $\mathbb{R}^n \setminus \Omega$ in U, and $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$. Let $U^+ = \{x \in U : x^n > f(x')\}$, $U^- = \{x \in U : x^n < f(x')\}$. Since any two points in U^+ (resp. U^-) can be joined by a continuous arc in U^+ (resp. U^-), and since $v^i(x) = 0$ if $x \in \partial\Omega$, it follows that $U^+ = \Omega \cap U$, $U^- = U \setminus \bar{\Omega}$ or vice versa. We may suppose the first alternative holds by if necessary changing the orientation of e_n. Clearly $\partial\Omega \cap U = \{x \in U : x^n = f(x')\}$. Hence Ω has a boundary of class C^1. □

Remark 2.9. — Hörmander [13], p. 59, gives a definition of an open set with C^1 boundary which is close in spirit to the above lemma.

If $\Omega \subset \mathbb{R}^n$ is open and $u \in C^1(\bar{\Omega}; \mathbb{R}^m)$ we write

$$K_\Omega(u) = \operatorname{range} Du(\cdot) = \{A \in M^{m \times n} : Du(x) = A \text{ for some } x \in \bar{\Omega}\}.$$

3. Rank-one convexity and quasiconvexity

The analysis is based on consideration of radial mappings $u : \bar{B} \to \mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, *i.e.* mappings of the form

$$u(x) = \frac{r(R)}{R} x, \tag{3.1}$$

where $R = |x|$ and $r : [0, 1] \to \mathbb{R}$. If $r \in C^1([0, 1])$ with $r(0) = 0$, then defining $u(0) = 0$ we have $u \in C^1(\bar{B}; \mathbb{R}^n)$ and

$$\begin{aligned} Du(x) &= \frac{r(R)}{R} \mathbf{1} + \left(r'(R) - \frac{r(R)}{R} \right) \vartheta \otimes \vartheta, \quad R > 0, \\ Du(0) &= r'(0) \mathbf{1}, \end{aligned} \tag{3.2}$$

where $\vartheta = x/R$. Given such a radial mapping we investigate whether $\operatorname{rank}(Du(x) - Du(y)) = 1$ for some pair of points $x, y \in \bar{B}$. The following elementary lemma is useful.

LEMMA 3.1. — *Let $n \geq 2$, $q, s, t \in \mathbb{R}$, $\vartheta, \vartheta_1 \in \mathbb{R}^n$ with $|\vartheta| = |\vartheta_1| = 1$. The following table gives necessary and sufficient conditions that the $n \times n$ matrix*

$$A = q\mathbf{1} + s\vartheta \otimes \vartheta + t\vartheta_1 \otimes \vartheta_1 \tag{3.3}$$

have rank 1.

	$n=2$	$n=3$	$n \geq 4$
$\vartheta = \pm \vartheta_1$	$q(q+s+t)=0$ $s+t \neq 0$	$q=0$ $s+t \neq 0$	$q=0$ $s+t \neq 0$
$\vartheta \neq \pm \vartheta_1$	$(q+s)(q+t)$ $= (\vartheta \cdot \vartheta_1)^2 st$ $2q+s+t \neq 0$	$q=s=0,\ t \neq 0$ or $q=t=0,\ s \neq 0$ or $q+s=q+t=0,\ \vartheta \cdot \vartheta_1 = 0,\ q \neq 0$	$q=s=0,\ t \neq 0$ or $q=t=0,\ s \neq 0$

Necessary and sufficient conditions that rank $A \leqq 1$ *are given by omitting the inequality conditions in the table.*

Proof. — If $\vartheta = \pm \vartheta_1$ then $\vartheta \otimes \vartheta = \vartheta_1 \otimes \vartheta_1$ and so in an orthonormal basis of $\mathbb{R}^n$ extending ϑ the matrix of A is $\operatorname{diag}(q+s+t, q, \ldots, q)$. The conditions given are thus obvious.

If $\vartheta \neq \pm \vartheta_1$ then $\vartheta_1 = \alpha \vartheta + \beta w$, where w is a unit vector perpendicular to ϑ, $\alpha^2 + \beta^2 = 1$, $\alpha = \vartheta \cdot \vartheta_1 \neq \pm 1$. Hence in an orthonormal basis extending $\{\vartheta, w\}$, the matrix of A is

$$\begin{bmatrix} q+s+\alpha^2 t & t\alpha\beta & & & 0 \\ t\alpha\beta & q+\beta^2 t & & & \\ & & q & & \\ & 0 & & \ddots & \\ & & & & q \end{bmatrix}.$$

The determinant of the 2×2 matrix in the top left hand corner is $(q+s)(q+t) - \alpha^2 st$. Since a symmetric 2×2 matrix is of rank one if and only if it has zero determinant and nonzero trace the result for $n=2$ follows immediately.

If $n=3$ then rank $A \leqq 1$ if and only if either

$$q=0 \quad \text{and} \quad \beta^2 st = 0$$

or

$$q+s+\alpha^2 t = q + \beta^2 t = t\alpha\beta = 0.$$

If $n \geqq 4$ then rank $A \leqq 1$ if and only if $q=0$ and $\beta^2 st = 0$. Since $\beta \neq 0$ this gives the conditions stated in the table. □

PROPOSITION 3.2. — *Let* $n \geqq 3$, *let* $r \in C^1([0, 1])$ *and let* u *be defined as above. Then* $K_B(u)$ *has no rank-one connections if and only if* $r(R)/R$ *is either nonincreasing or nondecreasing on* [0, 1].

Proof. — Given $x, y \in \bar{B}$ we write $x = R\vartheta$, $y = R_1 \vartheta_1$, where $|\vartheta| = |\vartheta_1| = 1$, R, $R_1 \geqq 0$. We can suppose that $R_1 \geqq R$. Then by (3.2)

$$(3.4) \quad Du(x) - Du(y) = (R^{-1} r(R) - R_1^{-1} r(R_1))\mathbf{1} + (r'(R) - R^{-1} r(R))\vartheta \otimes \vartheta - (r'(R_1) - R_1^{-1} r(R_1))\vartheta_1 \otimes \vartheta_1$$

with the appropriate interpretation if R or R_1 is zero.

Let $K_B(u)$ have no rank-one connections, and suppose for contradiction that $f(R) = r(R)/R$ is neither nonincreasing nor nondecreasing on [0, 1]. Then there exist R, $R_1 \in (0, 1)$ with $f(R) = f(R_1)$ but $Rf'(R) \neq R_1 f'(R_1)$. Choosing $\vartheta = \vartheta_1$ we obtain from (3.4) that

$$Du(x) - Du(y) = (Rf'(R) - R_1 f'(R_1))\vartheta \otimes \vartheta,$$

which is of rank one, a contradiction.

Let $f(\mathrm{R})$ be nonincreasing or nondecreasing. We apply Lemma 3.1 with

$$\mathrm{A}=\mathrm{D}u(x)-\mathrm{D}u(y), \qquad q=\mathrm{R}^{-1}r(\mathrm{R})-\mathrm{R}_1^{-1}r(\mathrm{R}_1),$$
$$s=r'(\mathrm{R})-\mathrm{R}^{-1}r(\mathrm{R}), \qquad t=-(r'(\mathrm{R}_1)-\mathrm{R}_1^{-1}r(\mathrm{R}_1)).$$

If $n \geqq 4$ then $q=0$ implies that f is constant on the line segment $[\mathrm{R}, \mathrm{R}_1]$. If $\mathrm{R}=\mathrm{R}_1$ then $s+t=0$, while if $\mathrm{R}<\mathrm{R}_1$ then $s=t=0$. Hence by the lemma rank $\mathrm{A} \neq 1$. If $n=3$ we have to check the additional possibility $q+s=q+t=0$, $\vartheta\cdot\vartheta_1=0$, $q \neq 0$. But this gives

$$(3.5) \qquad r'(\mathrm{R})=r(\mathrm{R}_1)/\mathrm{R}_1, \qquad r'(\mathrm{R}_1)=r(\mathrm{R})/\mathrm{R}, \qquad r(\mathrm{R}_1)/\mathrm{R}_1 \neq r(\mathrm{R})/\mathrm{R}.$$

Suppose that f is nondecreasing. Then $r'(\mathrm{R}) \geqq r(\mathrm{R})/\mathrm{R}$, $r'(\mathrm{R}_1) \geqq r(\mathrm{R}_1)/\mathrm{R}_1$, contradicting (3.5). A similar contradiction obtains for f nonincreasing. This completes the proof. □

Fix $\lambda \in \mathbb{R}$, and choose $r \in \mathrm{C}^1([0, 1])$ such that $r(\mathrm{R})/\mathrm{R}$ is nondecreasing or nonincreasing, $r(1)=\lambda$, $r'(1) \neq \lambda$. [For example, $r(\mathrm{R})=\lambda\mathrm{R}+(\mathrm{R}-\mathrm{R}^2)$.] Let u be the corresponding radial mapping given by (3.1). Let $g: \mathrm{M}^{n\times n} \to \mathbb{R}$ be any finite and continuous function (we can take $g \equiv 0$). Define

$$(3.6) \qquad \bar{\mathrm{W}}(\mathrm{A})=\begin{cases} g(\mathrm{A}), & \mathrm{A} \in \mathrm{K}_{\mathrm{B}}(u), \\ +\infty, & A \notin \mathrm{K}_{\mathrm{B}}(u). \end{cases}$$

THEOREM 3.3. – *Let $n \geqq 3$. Then $\bar{\mathrm{W}}: \mathrm{M}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, bounded below and rank-one convex, but is not quasiconvex.*

Proof. – Since $\mathrm{K}_{\mathrm{B}}(u)$ is compact, $\bar{\mathrm{W}}$ is lower semicontinuous and bounded below. Given A, B with rank $(\mathrm{A}-\mathrm{B})=1$, by Proposition 3.1 at least one of A, B does not belong to $\mathrm{K}_{\mathrm{B}}(u)$. Hence if $t \in (0, 1)$ the right-hand side of (2.1) is $+\infty$. Hence $\bar{\mathrm{W}}$ is rank-one convex.

To show that $\bar{\mathrm{W}}$ is not quasiconvex we note first that since $r'(1) \neq \lambda$ and $r(\mathrm{R})/\mathrm{R}$ is monotone, $r(\mathrm{R})/\mathrm{R}=\lambda$ only for $\mathrm{R}=1$. Hence, by (3.2), $\lambda\mathbf{1} \notin \mathrm{K}$. Define $\varphi(x)=u(x)-\lambda x$. Then $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}; \mathbb{R}^n)$ and

$$\int_{\mathrm{B}} \bar{\mathrm{W}}(\lambda\mathbf{1}+\mathrm{D}\varphi(x))\,dx=\int_{\mathrm{B}} \bar{\mathrm{W}}(\mathrm{D}u(x))\,dx=\int_{\mathrm{B}} g(\mathrm{D}u(x))\,dx<\infty,$$
$$\int_{\mathrm{B}} \bar{\mathrm{W}}(\lambda\mathbf{1})\,dx=+\infty,$$

contradicting (2.2). □

The next result concerns *everywhere finite* rank-one convex functions.

PROPOSITION 3.4. — *Let $n \geqq 1$, let $W: M^{n\times n} \to \mathbb{R}$ be rank-one convex, and let u be given by* (3.1) *with* $r \in C^1([0, 1])$, $r(0)=0$, $r(1)=\lambda$. *Then*

$$\int_B W(Du(x))\,dx \geqq \int_B W(\lambda\,\mathbf{1})\,dx = (\text{meas } B)\,W(\lambda\,\mathbf{1}). \tag{3.7}$$

Proof (*cf.* Ball [2], Sivaloganathan [22]). — First suppose that W is smooth and rank-one convex. By (3.2) and rank-one convexity

$$W(Du(x)) \geqq W(R^{-1} r(R)\,\mathbf{1}) + (r'(R) - R^{-1} r(R))\,DW(R^{-1} r(R)\,\mathbf{1})\,\vartheta \otimes \vartheta.$$

Integrating over B and using the fact that $\int_{S^{n-1}} (\vartheta \otimes \vartheta - n^{-1}\,\mathbf{1})\,d\vartheta = 0$,

$$\begin{aligned}\int_B W(Du(x))\,dx &\geqq \int_{S^{n-1}} \int_0^1 R^{n-1} \\ &\quad \times (W(R^{-1} r(R)\,\mathbf{1}) + n^{-1}(r'(R) - R^{-1} r(R)) \cdot DW(R^{-1} r(R)\,\mathbf{1})\,\mathbf{1})\,dR\,d\vartheta \\ &\geqq n^{-1} \int_{S^{n-1}} \int_0^1 \frac{d}{dR}(R^n W(R^{-1} r(R)\,\mathbf{1}))\,dR\,d\vartheta \\ &= n^{-1} \int_{S^{n-1}} W(\lambda\,\mathbf{1})\,d\vartheta = \int_B W(\lambda\,\mathbf{1})\,dx.\end{aligned}$$

Now suppose that W is everywhere finite and rank-one convex. Then (*cf.* Morrey [17], p. 112) W is continuous. Let $\rho \in C_0^\infty(M^{n\times n})$, $\rho \geqq 0$, $\int_{M^{n\times n}} \rho\,dA = 1$, $\varepsilon > 0$, $\rho_\varepsilon(A) = \varepsilon^{-n^2} \rho(\varepsilon^{-1} A)$. Then (*cf.* Morrey [17]) $W_\varepsilon = \rho_\varepsilon * W$ is smooth and rank-one convex, and $W_\varepsilon \to W$ uniformly on compact subsets of $M^{n\times n}$ as $\varepsilon \to 0$, so that passing to the limit $\varepsilon \to 0$ we obtain (3.7) for W. □

PROPOSITION 3.5. — *Let $n \geqq 3$ and $\mathcal{S} = \{W: M^{n\times n} \to \mathbb{R}: W$ rank-one convex, $W \leqq \bar{W}\}$. Then*

$$\sup_{\mathcal{S}} W \neq \bar{W}.$$

Proof: It suffices to show that $\sup_{W \in \mathcal{S}} W(\lambda\,\mathbf{1}) < \bar{W}(\lambda\,\mathbf{1})$. Let *u* be the radial mapping used in the construction of $\bar{W}$. Then $\bar{W}(\lambda\mathbf{1}) = +\infty$, but by Proposition 3.4, for any $W \in \mathcal{S}$,

$$W(\lambda\,\mathbf{1}) \leqq (\text{meas } B)^{-1} \int_B W(Du(x))\,dx \leqq (\text{meas } B)^{-1} \int_B g(Du(x))\,dx < \infty. \quad \square$$

Remark 3.6. — The question of whether a lower semicontinuous rank-one convex function $W: M^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ can be written as the supremum of finite rank-one convex functions, answered negatively in Proposition 3.5, was raised several years ago in discussions between R. V. Kohn and the author. In response, Tartar [25] produced an interesting model example of a lower semicontinuous separately convex function $f: \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ which is not the supremum of everywhere finite separately convex functions, and for which this supremum can be calculated explicitly.

We now show how the above construction can be generalized to the other dimensions mentioned in the introduction. To handle the cases $n=2$, $m \geqq 4$ we use the following lemma.

LEMMA 3.7. — *Let* $n=2$. *Define* $u: \bar{B} \to \mathbb{R}^4$ *by*

$$u(x)=\begin{pmatrix} R x \\ R^2 x \end{pmatrix}, \qquad R=|x|. \tag{3.8}$$

Then $K_B(u)$ has no rank-one connections, and

$$F=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \notin K_B(u).$$

Proof. — Let $v(x)=R x$, $w(x)=R^2 x$, $x \in \bar{B}$. Now let x, $y \in \bar{B}$ and write $x=R\vartheta$, $y=R_1 \vartheta_1$. By (3.4)

$$D v(x)-D v(y)=(R-R_1) \mathbf{1}+R \vartheta \otimes \vartheta-R_1 \vartheta_1 \otimes \vartheta_1,$$

$$D w(x)-D w(y)=(R^2-R_1^2) \mathbf{1}+2 R^2 \vartheta \otimes \vartheta-2 R_1^2 \vartheta_1 \otimes \vartheta_1.$$

Suppose for contradiction that $\operatorname{rank}(Du(x)-Du(y))=1$. Then $\operatorname{rank}(Dv(x)-Dv(y)) \leqq 1$, $\operatorname{rank}(Dw(x)-Dw(y)) \leqq 1$, and so by Lemma 3.1 ($n=2$),

$$2(R-R_1)^2=(1-(\vartheta \cdot \vartheta_1)^2) R R_1, \tag{3.9}$$

$$3(R^2-R_1^2)^2=4(1-(\vartheta \cdot \vartheta_1)^2) R^2 R_1^2. \tag{3.10}$$

Eliminating $1-(\vartheta \cdot \vartheta_1)^2$ it follows easily that $R=R_1$. Hence by (3.9), (3.10) either $R=R_1=0$ or $R=R_1$ and $\vartheta= \pm \vartheta_1$, and in both cases $Du(x)=Du(y)$, a contradiction.

That $F \notin K_B(u)$ is an easy calculation. □

Suppose that for some m, n there exists $u \in C^1(\bar{B}; \mathbb{R}^m)$ with $u|_B=F x$ for some $F \in M^{m \times n}$, and such that $K_B(u)$ has no rank-one connections and $F \notin K_B(u)$. Then if $g \in C(M^{m \times n})$ and $\bar{W}$ is defined by (3.6), the proof of Theorem 3.3 remains valid, showing that $\bar{W}$ is lower semicontinuous and rank-one convex, but not quasiconvex. If $M > m$ we may define $\hat{u} \in C^1(\bar{B}; \mathbb{R}^M)$ by $\hat{u}(x)=(u(x), 0)$. Then $\hat{u}|_B=\hat{F} x$, where $\hat{F}=\begin{pmatrix} F \\ 0 \end{pmatrix}$ and 0 is the zero $(M-m) \times n$ matrix. Clearly $K_B(\hat{u})$ has no rank-one connections and $\hat{F} \notin K_B(\hat{u})$, so we may repeat the same construction. On account of Theorem 3.2, Lemma 3.6 we have thus proved

THEOREM 3.8. — *Let $m \geqq n \geqq 3$ or $n=2$, $m \geqq 4$. Then there exists* $\bar{W}$: $M^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ *which is lower semicontinuous, bounded below and rank-one convex, but is not quasiconvex.*

It seems probable that similar counterexamples could be constructed for all $m, n \geqq 2$, except perhaps in the case $m=n=2$ (*see* Theorem 5.1), but we do not attempt this here.

4. Rank-one connections, compensated compactness and Young measures

Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \mathbb{R}^s$ be closed. Given any bounded sequence $\vartheta^{(j)}$ in $L^\infty(\Omega; \mathbb{R}^s)$ with $\vartheta^{(j)}(x) \in K$ a.e., there exists (*cf.* Tartar [24]) a subsequence $\vartheta^{(\mu)}$ and a family of probability measures (ν_x) on $\mathbb{R}^s$, depending measurably on $x \in \Omega$, with $\operatorname{supp} \nu_x \subset K$ a.e., such that for any continuous $f : \mathbb{R}^s \to \mathbb{R}$,

$$f(\vartheta^{(\mu)}) \overset{*}{\rightharpoonup} \langle \nu_x, f \rangle = \int_{\mathbb{R}^s} f(A)\, d\nu_x(A) \tag{4.1}$$

in $L^\infty(\mathbb{R}^n)$. (ν_x) is called the *Young measure* associated with $\vartheta^{(\mu)}$. If for a.e. $x \in \mathbb{R}^n$ ν_x is a Dirac mass, that is $\nu_x = \delta_{\vartheta(x)}$ for some $\vartheta(x) \in \mathbb{R}^s$, then by (4.1)

$$f(\vartheta^{(\mu)}) \overset{*}{\rightharpoonup} f(\vartheta(\cdot)) \quad \text{in } L^\infty(\Omega),$$

which implies (taking $f(\sigma) = \sigma$, $f(\sigma) = |\sigma|^2$) that

$$\vartheta^{(\mu)} \to \vartheta \quad \text{in } L^p_{\text{loc}}(\Omega; \mathbb{R}^s) \text{ strongly},$$

for all p, $1 \leqq p < \infty$.

Suppose now that $\vartheta^{(j)}$ is a bounded sequence in $L^\infty(\Omega; \mathbb{R}^s)$ and that a suitable subsequence, which we now again call $\vartheta^{(j)}$ for convenience, is extracted as above so that

$$f(\vartheta^{(j)}) \overset{*}{\rightharpoonup} \langle \nu_x, f \rangle \quad \text{in } L^\infty(\Omega), \tag{4.2}$$

for every continuous $f : \mathbb{R}^s \to \mathbb{R}$. Suppose further that

$$\sum_{k, l} a^{ikl} (\partial/\partial x^l)\, \vartheta_k^{(j)} = 0, \qquad i = 1, \ldots, q, \tag{4.3}$$

where the a^{ikl} are real constants. (More generally we may assume, for example, that the left-hand sides of (4.3) belong to a compact subset of $H^{-1}(\Omega; \mathbb{R}^s)$.) Following Murat [18] and Tartar [23] define the characteristic cone

$$\Lambda = \{\lambda \in \mathbb{R}^s : \exists\, \xi \in \mathbb{R}^n \setminus \{0\} \text{ such that } \sum_{k, l} a^{ikl} \lambda_k \xi_l = 0,\ i = 1, \ldots, q\}. \tag{4.4}$$

If K has a Λ-*connection*, that is $A - B \in \Lambda$ for some pair $A, B \in K$, then it is easy to construct a bounded sequence $\vartheta^{(j)}$ in $L^\infty(\mathbb{R}^n; \mathbb{R}^s)$ satisfying (4.2), (4.3), and with

$\vartheta^{(j)}(x) \in K$ a. e., such that ν_x is not a Dirac mass. For example, if $A - B = \lambda$ we may take $\vartheta^{(j)}(x) = B + \vartheta(jx \cdot \xi)\lambda$, where ξ is the corresponding vector in (4.4) and ϑ is the 1-periodic extension to $\mathbb{R}$ of the characteristic function of $[0, 1/2)$; in this case $\nu_x = (1/2)(\delta_A + \delta_B)$.

Motivated by the idea that preventing such one-dimensional oscillations might eliminate all oscillations, Tartar [24] made the conjecture that if K has no Λ-connections and $\vartheta^{(j)}$ satisfies (4.2), (4.3) with supp $\nu_x \subset K$ a. e., then (ν_x) is a Dirac mass a. e.. In applications of compensated compactness to systems of nonlinear differential equations, the linear partial differential equations (4.3) are supplemented by nonlinear relations (e. g. constitutive equations) that are incorporated in the set K. Thus the conclusion of the conjecture is that one can pass to the limit in the resulting nonlinear equations using weak convergence.

For the case of gradients considered in this paper, we identify $M^{m \times n}$ with $\mathbb{R}^{mn} = \mathbb{R}^s$ and $\vartheta^{(j)}$ with $Dz^{(j)}$, where $z^{(j)}$ is a bounded sequence in $W^{1, \infty}(\Omega; \mathbb{R}^m)$. The differential equations (4.3) are taken to be the compatibility conditions

$$\frac{\partial \vartheta_{ik}^{(j)}}{\partial x^l} - \frac{\partial \vartheta_{il}^{(j)}}{\partial x^k} = 0, \quad \text{for all } i = 1, \ldots, m, \quad k, l = 1, \ldots, n. \tag{4.5}$$

The corresponding cone Λ is the set of all rank-one $m \times n$ matrices. Thus for this case Tartar's conjecture implies the following:

Conjecture 4.1. — Let $K \subset M^{m \times n}$ be closed and have no rank-one connections. Let $z^{(j)}$ be a bounded sequence in $W^{1, \infty}(\mathbb{R}^n, \mathbb{R}^m)$ satisfying $Dz^{(j)}(x) \in K$ a. e. and such that $f(Dz^{(j)})$ is weak* convergent in $L^\infty(\mathbb{R}^n)$ for every continuous $f: M^{m \times n} \to \mathbb{R}$. Then the Young measure (ν_x) associated with $Dz^{(j)}$ is a Dirac mass.

(Note that Tartar's hypothesis supp $\nu_x \subset K$ has been strengthened to $Dz^{(j)}(x) \in K$ a. e.).

To construct a counterexample, suppose that for some m, n there exists $u \in C^1(\bar{B}; \mathbb{R}^m)$ such that $u|_{\partial B} = Fx$ for some $F \in M^{m \times n}$, $K_B(u)$ has no rank-one connections, and $u \not\equiv Fx$. We construct $z^{(j)}$ using the method of Ball & Murat [8]. Let $Q = (0, 1)^n$. By Vitali's covering theorem (Saks [21]) there exist $\varepsilon_i > 0$, $a_i \in \mathbb{R}^n$ and a subset $N \subset \mathbb{R}^n$ of measure zero such that

$$Q = \bigcup_{i=1}^{\infty} (a_i + \varepsilon_i B) \cup N,$$

and such that the sets $a_i + \varepsilon_i B$ are disjoint. Define $z: Q \to \mathbb{R}^m$ by

$$z(x) = F a_i + \varepsilon_i u(\varepsilon_i^{-1}(x - a_i)), \qquad x \in a_i + \varepsilon_i B. \tag{4.6}$$

It is easily shown (Ball & Murat [8]) that $z(x) = Fx + \varphi(x)$, a. e. $x \in Q$, with $\varphi \in W_0^{1, \infty}(Q; \mathbb{R}^m)$. Extend z to $\mathbb{R}^n$ so that Dz is periodic with respect to Q, and define $z^{(j)}: \mathbb{R}^n \to \mathbb{R}^m$ by

$$z^{(j)}(x) = j^{-1} z(jx). \tag{4.7}$$

Then

$$Dz^{(j)}(x) = Dz(jx) = Du(\varepsilon_i^{-1}(x-a_i))$$

for some $i = i(x, j)$. Hence $Dz^{(j)}(x) \in K_B(u)$ a. e. $x \in \mathbb{R}^n$. If $f: M^{m \times n} \to \mathbb{R}$ is continuous,

$$f(Dz^{(j)}) \overset{*}{\rightharpoonup} \int_Q f(Dz)\, dx \quad \text{in } L^\infty(\mathbb{R}^n).$$

But

$$\int_Q f(Dz)\, dx = \sum_{i=1}^{\infty} \int_{a_i + \varepsilon_i B} f(Du(\varepsilon_i^{-1}(x-a_i)))\, dx$$

$$= \sum_{i=1}^{\infty} \varepsilon_i^n \int_B f(Du(x))\, dx = \fint_B f(Du(x))\, dx,$$

where $\fint_B (\cdot)\, dx \overset{\text{def}}{=} (\text{meas } B)^{-1} \int_B (\cdot)\, dx$. Hence the Young measure (ν_x) corresponding to $Dz^{(j)}$ is independent of x a. e. and given by

$$\langle \nu_x, f \rangle = \fint_B f(Du(y))\, dy. \tag{4.8}$$

It follows easily from (4.8) that supp $\nu_x = K_B(u)$ a. e. and in particular that ν_x is not a Dirac mass. Since by Proposition 3.2, Lemma 3.7, a suitable u can be constructed in the cases $m = n \geqq 3$ and $n = 2$, $m = 4$, we have shown that Conjecture 4.1 is false for these dimensions.

To handle the cases of larger n, note that if $z^{(j)}$ is constructed as above for m, n and if $N \geqq n$ then

$$\hat{z}^{(j)}(x) = z^{(j)}(x'), \qquad x \in \mathbb{R}^N, \quad x' = (x^1, \ldots, x^n)$$

defines a sequence $\hat{z}^{(j)}$ satisfying

$$\hat{z}^{(j)} \overset{*}{\rightharpoonup} F x' \quad \text{in } W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^m).$$

Let $\hat{K} = \{(A, 0) : A \in K_B(u)\}$, where 0 denotes the zero $m \times (N-n)$ matrix, with the obvious convention if $N = n$. Then $D\hat{z}^{(j)}(x) \in \hat{K}$ a. e. and $\hat{K}$ has no rank-one connections. Furthermore the Young measure (ν_x) corresponding to $D\hat{z}^{(j)}$ is given by

$$\langle \nu_x, f \rangle = \fint_B f(Du(x'), 0)\, dx'$$

for $f \in C(M^{m \times N}; \mathbb{R})$ and is not a Dirac mass.

Let $g \in C(M^{m \times N}; \mathbb{R})$ and define $\bar{W}$ by

$$\bar{W}(A) = \begin{cases} g(A), & A \in \hat{K} \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $\bar{W}$ is lower semicontinuous and rank-one convex. But if $\Omega \subset \mathbb{R}^N$ is bounded and open

$$\underline{\lim}_{j \to \infty} \int_\Omega \bar{W}(D\hat{z}^{(j)}(x))\, dx = \underline{\lim}_{j \to \infty} \int_\Omega g(D\hat{z}^{(j)}(x))\, dx = (\text{meas}\,\Omega) \fint_B g(Du(x'), 0)\, dx' < \infty,$$

while provided $F \notin K_B(u)$,

$$\int_\Omega \bar{W}(F, 0)\, dx = +\infty.$$

Hence

$$I_\Omega(z) = \int_\Omega \bar{W}(Dz(x))\, dx$$

is not sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

We have thus proved

THEOREM 4.2. — *Let* $\min(m, n) \geqq 3$ *or* $n=2$, $m \geqq 4$. *Then Conjecture* 4.1 *is false, and there exists a lower semicontinuous rank-one convex function* $\bar{W}: M^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ *such that* $I_\Omega(\cdot)$ *is not sequentially weak* lower semicontinuous on* $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Remark 4.3. — Notice that the mapping $z: \mathbb{R}^n \to \mathbb{R}^m$ constructed above is Lipschitz but is not C^1, since Dz takes all the values of Du in the neighbourhood of an accumulation point of the a_i. Nevertheless $K_{\mathbb{R}^n}(z)$ has no rank-one connections.

There are a number of cases when Tartar's conjecture is known to be true for gradients under supplementary hypotheses on the set K.

(i) $K = \{A_1, A_2\}$ with rank $(A_1 - A_2) > 1$ (Ball and James [5]).

(ii) $m = n > 1$, $K = SO(n)$ (Kinderlehrer [14]). In fact, more generally, let $n > 1$ and

$$K_1 = \{tR;\ t \geqq 0,\ R \in SO(n)\}.$$

Then we have

THEOREM 4.4. — (*a*) K_1 *has no rank-one connections.*

(*b*) *Let* $\Omega \subset \mathbb{R}^n$, $n > 1$, *be bounded, open and connected. Let* $z^{(j)} \in W^{1,n}(\Omega; \mathbb{R}^n)$ *be such that*

$$\sup_j \int_\Omega \psi(|Dz^{(j)}|^n)\, dx < \infty \tag{4.9}$$

for some continuous $\psi: [0, \infty) \to \mathbb{R}$ *satisfying* $\lim_{t \to \infty} \psi(t)/t = \infty$. *Let* (ν_x) *be the Young measure corresponding to* $Dz^{(j)}$ *(it being understood that an appropriate subsequence of* $z^{(j)}$ *has already been extracted so that* (ν_x) *is well defined), and suppose that* $\operatorname{supp} \nu_x \subset K_1$ *a. e.* $x \in \Omega$. *Then*

$$\nu_x = \delta_{Dz(x)} \text{ a. e.,} \tag{4.10}$$

where $z: \Omega \to \mathbb{R}^n$ *is a smooth conformal mapping, and*

$$Dz^{(j)} \to Dz \quad \textit{strongly in } L^n(\Omega; M^{n \times n}).$$

Remark 4.5. — In the statement and proof of the theorem we make use of the following more general construction of the Young measure (*cf.* Ball [3], Ball & Knowles [7]). Given a sequence $\vartheta^{(j)}: \Omega \to \mathbb{R}^s$ of measurable mappings satisfying a mild boundedness condition (satisfied, for example, if $\vartheta^{(j)}$ is bounded in $L^1(\Omega; \mathbb{R}^s)$) there exists a subsequence $\vartheta^{(\mu)}$ and a family of probability measures (ν_x), depending measurably on x, such that

$$f(\vartheta^{(\mu)}) \rightharpoonup \langle \nu_x, f \rangle \quad \text{in } L^1(\Omega)$$

for every continuous f such that $f(\vartheta^{(\mu)})$ is sequentially weakly relatively compact in $L^1(\Omega)$.

Proof of Theorem 4.4. — (*a*) Let $A, B \in K_1$ with $A - B = a \otimes b$, $a, b \in \mathbb{R}^n$. Then $A = sR$, $B = t\bar{R}$ for $s, t \geqq 0$, $R, \bar{R} \in SO(n)$. Thus

$$sQ = t\mathbf{1} + a \otimes c,$$

where $Q = RR^T$, $c = \bar{R}b$, and so

$$s^2 \mathbf{1} = (t\mathbf{1} + a \otimes c)(t\mathbf{1} + c \otimes a) = (t\mathbf{1} + c \otimes a)(t\mathbf{1} + a \otimes c).$$

It is easily deduced from this, using $\det Q = +1$, that $A = B$. Hence K_1 has no rank-one connections.

(*b*) We combine ideas of Kinderlehrer [14] and Reshetnyak [20]. Consider the function

$$f(A) = |A|^n - n^{n/2} \det A. \tag{4.11}$$

Then $f \geqq 0$, and if $\det A \geqq 0$ then $A \in K_1$ if and only if $f(A) = 0$. By (4.9) and the de la Vallée Poussin criterion, $f(Dz^{(j)})$ is sequentially weakly relatively compact in $L^1(\Omega)$, and hence

$$f(Dz^{(j)}) \rightharpoonup \langle \nu_x, f \rangle \quad \text{in } L^1(\Omega).$$

Also

$$Dz^{(j)} \rightharpoonup \langle \nu_x, A \rangle = Dz \quad \text{in } L^n(\Omega),$$

for some $z \in W^{1,n}(\Omega; \mathbb{R}^n)$.

By a result of Reshetnyak [21] (*see* also Ball, Currie and Olver [4])

$$\det \mathrm{D}z^{(j)} \rightharpoonup \det \mathrm{D}z, \quad \text{in } \mathrm{L}^1(\Omega),$$

so that

$$\langle \nu_x, \det \mathrm{A} \rangle = \det \langle \nu_x, \mathrm{A} \rangle \text{ a.e..}$$

Since supp $\nu_x \subset \mathrm{K}_1$, we thus have that

$$0 = \langle \nu_x, f \rangle = \langle \nu_x, |\mathrm{A}|^n \rangle - n^{n/2} \langle \nu_x, \det \mathrm{A} \rangle \geqq |\langle \nu_x, \mathrm{A} \rangle|^n - n^{n/2} \det \langle \nu_x, \mathrm{A} \rangle \geqq 0,$$

where we have used Jensen's inequality. Hence

$$(4.12) \qquad \langle \nu_x, |\mathrm{A}|^n \rangle = |\langle \nu_x, \mathrm{A} \rangle|^n \quad \text{a.e. } x \in \Omega,$$

which implies, since $|\cdot|^n$ is strictly convex, that

$$(4.13) \qquad \nu_x = \delta_{\mathrm{D}z(x)} \quad \text{a.e. } x \in \Omega.$$

Clearly $\mathrm{D}z(x) \in \mathrm{K}_1$ a.e., so that z (by definition) is conformal, and hence (Reshetnyak [20]) smooth. Finally, by (4.12), (4.13) $\|\mathrm{D}z^{(j)}\|_{\mathrm{L}^n} \to \|\mathrm{D}z\|_{\mathrm{L}^n}$, so that $\mathrm{D}z^{(j)} \to \mathrm{D}z$ strongly in $\mathrm{L}^n(\Omega; \mathrm{M}^{n \times n})$. □

(iii) (In this example we consider the general framework of compensated compactness described at the beginning of this section.) K is a finite-dimensional C^1 manifold imbedded in $\mathbb{R}^s$, for every $\mathrm{A} \in \mathrm{K}$ the tangent space $\mathrm{T_A K}$ contains no elements of Λ, and supp ν_x is sufficiently small (DiPerna [10], Tartar [24]). In connection with this result it is interesting to note that our counterexamples can be easily modified so that supp ν_x is arbitrarily small without changing K. To do this we just need to replace $\mathrm{B} = \{x \in \mathbb{R}^n;\ |x| < 1\}$ by $\mathrm{B}_\varepsilon = \{x \in \mathbb{R}^n;\ |x| < \varepsilon\}$ in the above constructions while retaining the same fixed radial mappings u. The set $\mathrm{K} = \mathrm{K_B}(u)$ is not contained in a C^1 manifold having the required properties.

5. Negative results for $m = n = 2$

In this section we prove that the method used to construct the counterexamples does not work in the case $m = n = 2$, at least within the class of smooth functions u.

THEOREM 5.1. — *Let $\Omega \subset \mathbb{R}^2$ be bounded and open, and let* $\mathrm{A} \in \mathrm{M}^{2 \times 2}$.

(i) *There does not exist a mapping $u \in \mathrm{C}^1(\bar{\Omega}; \mathbb{R}^2)$ with $u|_{\partial\Omega} = \mathrm{A}x$, such that $\mathrm{K}_\Omega(u)$ has no rank-one connections and $\mathrm{A} \notin \mathrm{K}_\Omega(u)$.*

(ii) *If $u \in \mathrm{C}^2(\bar{\Omega}; \mathbb{R}^2)$ with $u|_{\partial\Omega} = \mathrm{A}x$ is such that $\mathrm{K}_\Omega(u)$ has no rank-one connections then* $u(x) \equiv \mathrm{A}x$.

Proof. — We assume without loss of generality that $\mathrm{A} = 0$. The general case then follows by replacing $u(x)$ by $u(x) - \mathrm{A}x$.

(i) Suppose for contradiction that u has the given properties. We first show that without loss of generality we may assume that Ω is connected and has a C^1 boundary. In fact let $\Omega_1 = \operatorname{int} \bar{\Omega}$. Then $\bar{\Omega}_1 = \bar{\Omega}$, $\partial\Omega_1 \subset \partial\Omega$, $u|_{\partial\Omega_1} = 0$ and $u \in C^1(\bar{\Omega}_1; \mathbb{R}^2)$. Hence u has the same properties with respect to Ω_1. But $Du|_{\partial\Omega_1} \neq 0$, so that by Lemma 2.7, Ω_1 has a C^1 boundary. Now let Ω_2 be a connected component of Ω_1. Clearly Ω_2 has a C^1 boundary and u has the same properties with respect to Ω_2. At each point $y \in \partial\Omega$ there is a well defined unit outward normal $n(y)$, and $n(y)$ depends continuously on y. Furthermore, since Ω is bounded, by touching $\bar{\Omega}$ from the outside by straight lines it is easily shown that $\{n(y) : y \in \partial\Omega\} = S^1 \overset{\text{def}}{=} \{p \in \mathbb{R}^2 : |p| = 1\}$. Since $u|_{\partial\Omega} = 0$, $Du^i(y)$ is parallel to $n(y)$ for each $y \in \partial\Omega$, $i = 1, 2$. Also $0 \notin K_\Omega(u)$ implies $Du(y) \neq 0$. Thus

$$Du(y) = a(y) \otimes n(y), \qquad y \in \partial\Omega, \tag{5.1}$$

where $a(y) \in \mathbb{R}^2$ is nonzero. Note that if $y, z \in \partial\Omega$ are such that $n(y) \neq \pm n(z)$ then $Du(y) \neq Du(z)$ and hence, by the hypothesis that K has no rank-one connections, $\det(Du(y) - Du(z)) \neq 0$. On the other hand, if $y, z \in \partial\Omega$ with $n(y) = \pm n(z)$ then by (5.1) $\operatorname{rank}(Du(y) - Du(z)) \leq 1$, and since $K_\Omega(u)$ has no rank-one connections it follows that $Du(y) = Du(z)$.

Given $n \in S^1$ denote by $\Delta(n)$ the common value of $Du(y)$ for those $y \in \partial\Omega$ with $n(y) = n$. Note that $\Delta(n) = \Delta(-n)$. Since $\partial\Omega$ is compact and $n(\cdot)$, $Du(\cdot)$ are continuous, $\Delta(\cdot)$ is continuous. We claim that either

$$\det(\Delta(n) - \Delta(m)) > 0 \quad \text{for all } n, m \in S^1, \quad n \neq \pm m, \tag{5.2}$$

or that

$$\det(\Delta(n) - \Delta(m)) < 0 \quad \text{for all } n, m \in S^1, \quad n \neq \pm m. \tag{5.3}$$

We have already shown that $\det(\Delta(n) - \Delta(m)) \neq 0$ for $n \neq \pm m$. If the claim were false there would therefore exist pairs $(n^{(1)}, m^{(1)})$, $(n^{(2)}, m^{(2)})$ with $n^{(i)} \neq \pm m^{(i)}$, $i = 1, 2$ and

$$\det(\Delta(n^{(1)}) - \Delta(m^{(1)})) < 0 < \det(\Delta(n^{(2)}) - \Delta(m^{(2)})). \tag{5.4}$$

However, it is easy to find a continuous path $t \mapsto (n(t), m(t))$, $t \in [0, 1]$, such that $(n(0), m(0)) = (n^{(1)}, m^{(1)})$, $(n(1), m(1)) = (n^{(2)}, m^{(2)})$ or $(m^{(2)}, n^{(2)})$, and

$$(n(t), m(t)) \in S^1 \times S^1, \qquad n(t) \neq \pm m(t) \quad \text{for all } t \in [0, 1]. \tag{5.5}$$

(For example, move $m^{(1)}$ towards $n^{(1)}$ along the shorter of the two arcs joining them until they are a small positive distance part. Then move the two points as a pair, preserving the arclength between them, until they are both on the shorter of the two arcs joining $n^{(2)}$ and $m^{(2)}$. Finally move them apart to meet $n^{(2)}$ and $m^{(2)}$.) Since $\det(\Delta(n(t)) - \Delta(m(t))) \neq 0$ for all $t \in [0, 1]$ and Δ is continuous, this contradicts (5.4) and proves the claim.

We suppose (5.2) to hold. If (5.3) holds the same argument will apply with the inequality signs reversed. Given $n \in S^1$ define

$$S(n) = \{x \in \Omega : \det(Du(x) - \Delta(n)) < 0\}. \tag{5.6}$$

Since $\det(\cdot - \Delta(n))$ is a null Lagrangian and $u|_{\partial\Omega} = 0$, and since $\det \Delta(n) = 0$,

$$\int_\Omega \det(Du(x) - \Delta(n))\, dx = \int_\Omega \det(0 - \Delta(n))\, dx = 0, \tag{5.7}$$

(Since Ω has a C^1 boundary this calculation may be justified directly using integration by parts.) But $\det(Du(x) - \Delta(n))$ is not identically zero in Ω by (5.2) and the continuity of $Du(\cdot)$. Hence $S(n)$ is nonempty and open, and $\Omega \setminus S(n)$ is non-empty. We will prove that $S(n)$ and $S(m)$ are disjoint for $m \neq n$. This gives the desired contradiction because an arc in S^1 of length less than π contains uncountably many vectors n no two of which are parallel, and Ω cannot contain uncountably many disjoint open sets. Suppose that $x \in \partial S(m) \cap \Omega$. Then $\det(Du(x) - \Delta(m)) = 0$, and since K has no rank-one connections it follows that $Du(x) = \Delta(m)$. Thus by (5.2) $x \notin \overline{S(n)}$. Hence $\overline{S(n)} \cap \partial S(m) \cap \Omega$ is empty, and similarly $\overline{S(m)} \cap \partial S(n) \cap \Omega$ is empty. So

$$\overline{S(m)} \cap \overline{S(n)} \cap \Omega = S(m) \cap S(n) \cap \Omega,$$

which implies that $S(m) \cap S(n)$ is both open and closed with respect to the connected open set Ω. Since $\Omega \setminus S(n)$ is non-empty, $S(m) \cap S(n) \cap \Omega \neq \Omega$, so that $S(m)$ and $S(n)$ are disjoint.

(ii) Let $u \in C^2(\bar{\Omega}; \mathbb{R}^2)$ be such that $u|_{\partial\Omega} = 0$ and $K_\Omega(u)$ has no rank-one connections. By part (i) $0 \in K_\Omega(u)$. We use the following lemma.

Lemma 5.2. — *Let* $E \subset \mathbb{R}^2$ *be bounded and open, and let* $u \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ *satisfy* $u^1 u^2_{,2} = u^1 u^2_{,1} = 0$ *on* ∂E. *Then*

$$\int_E \det Du(x)\, dx = 0.$$

Proof. — A standard construction, involving mollification and the fact that the function $\operatorname{dist}(x, \partial E)$ is Lipschitz, shows that there exists a sequence $\{\varphi^{(j)}\} \subset C_0^\infty(E)$ such that for $x \in E$, $\varphi^{(j)}(x) = 1$ if $\operatorname{dist}(x, \partial E) > 1/j$, $0 \leqq \varphi^{(j)}(x) \leqq 1$ and $|D\varphi^{(j)}(x)| \leqq \text{Const.}\, j$. Then

$$\int_E \varphi^{(j)} \det Du\, dx = \int_E (u^1 u^2_{,1} \varphi^{(j)}_{,2} - u^1 u^2_{,2} \varphi^{(j)}_{,1})\, dx. \tag{5.8}$$

Since $u^1 u^2_{,1}$, $u^1 u^2_{,2} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ and are zero on ∂E, it follows that

$$|u^1(x) u^2_{,1}(x)| + |u^1(x) u^2_{,2}(x)| \leqq \text{Const.} \operatorname{dist}(x, \partial E), \qquad x \in E,$$

and hence from (5.8) that

$$\left|\int_{E} \varphi^{(j)} \det Du\, dx\right| \leqq \text{Const. meas}\,\{x\in E : \text{dist}\,(x, \partial E) < 1/j\}.$$

Letting $j\to\infty$ establishes the lemma. □

Remark 5.3. — It is not obvious to the author that Lemma 5.2 remains valid for $u\in C^1(\mathbb{R}^2; \mathbb{R}^2)$ (*cf.* the example of Whitney [27]).

Continuation of proof of Theorem 5.1. — By Lemma 5.2,

$$\int_{\Omega} \det Du(x)\, dx = 0. \tag{5.9}$$

(In fact this conclusion holds even if $u\in C^1(\mathbb{R}^2; \mathbb{R}^2)$, as a simple modification of the proof of Lemma 5.2 shows.) Suppose for contradiction that Du is not identically zero in Ω. Since $K_{\Omega}(u)$ has no rank-one connections and $0\in K_{\Omega}(u)$, $\det Du$ is not identically zero. Hence by (5.9) $\det Du$ takes both positive and negative values in Ω. Let E be a connected component of the open set $\{x\in\Omega : \det Du(x) > 0\}$. If $x\in\partial E\cap\partial\Omega$ then $u(x)=0$. If $x\in\partial E\cap\Omega$ then $\det Du(x)=0$ and since $K_{\Omega}(u)$ has no rank-one connections $Du(x)=0$. Applying Lemma 5.2 to E we deduce that

$$\int_{E} \det Du(x)\, dx = 0.$$

But $\det Du(x) > 0$ on E, a contradiction. □

Theorem 5.1 leaves open the possibility that there could exist a non-affine mapping $u\in W^{1,\infty}(\Omega; \mathbb{R}^2)$ with $u|_{\partial\Omega}=Ax$ and a subset K of $M^{2\times 2}$ without rank-one connections such that $Du(x)\in K$ a.e.. In fact, on the basis of an example in [5], p. 48 (*see* also Ball & James [6]), R. D. James and the author have speculated that such a u may exist whose gradient takes only a *finite* number of values.

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