CONTINUITY PROPERTIES AND GLOBAL ATTRACTORS OF GENERALIZED SEMIFLOWS AND THE NAVIER-STOKES EQUATIONS

J.M. BALL

ABSTRACT. A class of semiflows having possibly nonunique solutions is defined. The measurability and continuity properties of such generalized semiflows are studied. It is shown that a generalized semiflow has a global attractor if and only if it is pointwise dissipative and asymptotically compact. The structure of the global attractor in the presence of a Lyapunov function, and its connectedness and stability properties are studied. In particular, examples are given in which the global attractor is a single point but is not Lyapunov stable.

The existence of a global attractor for the 3D incompressible Navier-Stokes equations is established under the (unproved) hypothesis that all weak solutions are continuous from $(0,\infty)$ to L^2.

Dedicated to the memory of Juan Simo.

1. INTRODUCTION

Generalized semiflows are an abstraction of autonomous dynamical systems for which there may be more than one solution corresponding to given initial data. The need for a theory of such systems arises for various reasons. First, there may be genuine nonuniqueness of solutions. Second, solutions may not be known to be unique (as, for example, for certain semilinear wave equations with high power nonlinearities, or for the incompressible Navier-Stokes equations in three space dimensions, an example studied in some detail in the paper). Third, there may be free parameters or controls that are not specified and lead to various possible solutions. The paper discusses the measurability and continuity properties of generalized semiflows on a metric space X, and their global attractors.

There are various possible ways of abstracting dynamical systems with nonunique solutions. One method (see Sell [38]) is to recover uniqueness of solutions by working in a space of semitrajectories $\varphi : [0,\infty) \to X$ and defining a corresponding semiflow $T(\cdot)$ by $T(t)\varphi = \varphi^t$, for $t \geq 0$, where $\varphi^t(\tau) := \varphi(t+\tau)$. An interesting example of the use of this method is the recent proof by Sell [39] of the existence of a global attractor for the 3D incompressible Navier-Stokes equations. (For further results see Chepyzhov & Vishik [14].) However, a disadvantage is that the direct connection with the evolution of the system in the 'physical' state space is lost. (In fact, the existence of a global attractor for the Navier-Stokes equations in the original phase space remains an open problem; the existence is proved under a continuity hypothesis on solutions in this paper.) A second method is to consider a set-valued trajectory $t \mapsto T(t)z$ in which $T(t)z$ consists of all possible points reached at time t by solutions with initial data z. This approach has been taken, for example by Barbashin [10], Budak [13], Bronstein [12], Minkevic [30], Roxin [36, 35], Szego & Treccani [41], Babin & Vishik [3], Babin [2] and Mel'nik [29].

However, the disadvantage of this method is that it is not phrased directly in terms of solutions; in fact in some situations one has to recover solutions by selecting suitable regular paths from the sets $T(t)z$.

The approach taken in this paper is more closely related to the second method than the first, and takes as the primitive objects the solutions themselves; it is an adaptation of that in [8, 9]. A generalized semiflow is defined in Section 2 to be a family of maps $\varphi : [0,\infty) \to X$ satisfying axioms relating to existence, time translation, concatenation and upper-semicontinuity with respect to initial data. It is shown in Theorem 2.1 that, under a mild technical hypothesis, for generalized semiflows strong measurability of solutions with respect to time implies their continuity on $(0,\infty)$, and hence (Theorem 2.2) that the upper-semicontinuity with respect to initial data is uniform on compact subsets of $(0,\infty)$. These theorems generalize corresponding results for semiflows due to Chernoff & Marsden [16] and the author [6, 7].

In Section 3 global attractors for generalized semiflows are studied. It is shown in Theorem 3.3 that a generalized semiflow has a global attractor if and only if it is point dissipative and asymptotically compact. This result generalizes those for semiflows of Hale [21] and of Ladyzhenskaya [27]. Related results in the the context of set-valued semiflows have recently been announced by Mel'nik [29]. In Section 4 the connectedness of the global attractor is proved (Corollary 4.3) provided X is connected and Kneser's property holds, that is the set $T(t)\{z\}$, consisting of all points $\varphi(t)$ for solutions φ with $\varphi(0) = z$, is connected. In Section 5 the case of an asymptotically compact generalized semiflow with a Lyapunov function is considered. As for semiflows the point dissipativeness can be verified by showing that the set of rest points is bounded (Theorem 5.1).

Section 6 is motivated by results of Sell & You [40] on the Lyapunov stability of attractors. In order to treat the case of a semiflow whose solutions are not necessarily continuous up to $t = 0$, Sell & You were led to change the usual definition of Lyapunov stability. We show by means of two examples, one finite- and the other infinite-dimensional, that without such a change in the definition the global attractor of such a semiflow need not be Lyapunov stable, even if the global attractor consists of a single point. We also give a positive result (Theorem 6.1) giving hypotheses under which the global attractor of a generalized semiflow is in fact stable.

The theory is applied in Section 7 to the case of the 3D incompressible Navier-Stokes equations. The main results are that (Proposition 7.4) weak solutions satisfying an energy inequality form a generalized semiflow, in the usual phase-space H consisting of L^2 vector-fields with zero divergence, if and only if all weak solutions are continuous from $(0,\infty) \to H$, and that (Theorem 7.6) under this hypothesis there is a global attractor in H. Since weak solutions of the Navier-Stokes equations are not known to be continuous in time, these results may turn out to be vacuous. However, it is notable that we assume neither additional regularity nor the uniqueness of weak solutions.

A further application of the theory to damped semilinear wave equations (see Example 2.3) will appear in [5]. This example in fact motivated the paper. This is a more substantial application of the theory (though to a perhaps less interesting example), since the corresponding generalized semiflows are asymptotically compact but not compact, whereas the generalized semiflow for the Navier-Stokes equations (under the assumed continuity of solutions) is compact. For earlier work on attractors for these equations also not assuming uniqueness of solutions see Babin & Vishik [3].

2. Generalized semiflows

Let X be a metric space (not necessarily complete) with metric d. We write $B(a,r)$ for the open ball centre $a \in X$ and radius r. If $C \subset X$ and $b \in X$ we set $\rho(b,C) := \inf_{c\in C} d(b,c)$. If $B \subset X, C \subset X$ we set

$$\mathrm{dist}\,(B,C) := \sup_{b\in B} \rho(b,C),$$

and define the *Hausdorff distance* $d_H(B,C)$ by

$$d_H(B,C) = \max\{\mathrm{dist}\,(B,C), \mathrm{dist}\,(C,B)\}.$$

If $C \subset X$ and $\varepsilon > 0$ we write

$$N_\varepsilon(C) := \{z \in X : \rho(z,C) < \varepsilon\}$$

for the open ε-neighbourhood of C.

Definition 2.1. A *generalized semiflow* G on X is a family of maps $\varphi : [0,\infty) \to X$ (called *solutions*) satisfying the hypotheses:

(H1) (*Existence*) For each $z \in X$ there exists at least one $\varphi \in G$ with $\varphi(0) = z$.

(H2) (*Translates of solutions are solutions*) If $\varphi \in G$ and $\tau \geq 0$, then $\varphi^\tau \in G$, where $\varphi^\tau(t) := \varphi(t+\tau)$, $t \in [0,\infty)$.

(H3) (*Concatenation*) If $\varphi, \psi \in G$, $t \geq 0$, with $\psi(0) = \varphi(t)$ then $\theta \in G$, where

$$\theta(\tau) := \begin{cases} \varphi(\tau) & \text{for } 0 \leq \tau \leq t, \\ \psi(\tau - t) & \text{for } t < \tau. \end{cases}$$

(H4)(*Upper-semicontinuity with respect to initial data*) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

If for each $z \in X$ there is exactly one $\varphi \in G$ with $\varphi(0) = z$ then G is called a *semiflow*. Equivalently, via the correspondence $S(t)z = \varphi(t)$, a semiflow can be defined as a family of continuous maps $S(t) : X \to X$, $t \geq 0$, satisfying the semigroup properties

(a) $S(0) = \text{identity}$,

(b) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$.

The simplest examples of generalized semiflows are those generated on $\mathbf{R}^n$ by autonomous ordinary differential equations of the form $\dot{u} = f(u)$ for appropriate continuous $f : \mathbf{R}^n \to \mathbf{R}^n$ (see Sell [37]).

We consider the following additional measurability and continuity assumptions that may be satisfied by G. Recall that a map $f : (0,\infty) \to X$ is *strongly measurable* if there exists a sequence f_j of measurable countably-valued maps converging almost everywhere to f on $(0,\infty)$.

(C0) Each $\varphi \in G$ is strongly measurable from $(0,\infty)$ to X.

(C1) Each $\varphi \in G$ is continuous from $(0,\infty)$ to X.

(C2) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ uniformly for t in compact subsets of $(0,\infty)$.

(C3) Each $\varphi \in G$ is continuous from $[0,\infty)$ to X.

(C4) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ uniformly for t in compact subsets of $[0, \infty)$.

We illustrate these definitions with some examples:

Example 2.1. Let $X = [-1, 1]$. Define $S(0)\tau = \tau$, and for $t > 0$

$$S(t)\tau = \begin{cases} 1 - e^{-t} & \text{if } -1 \leq \tau \leq 0, \\ 1 - e^{-t}(1-\tau) & 0 < \tau \leq 1. \end{cases}$$

Then it is easily verified that $S(t)$ is a semiflow satisfying (C1) and (C4) but not (C3).

Example 2.2. (The one-dimensional heat equation.)

For $f \in L^\infty(0,1)$ define $S(t)f$ to be the unique solution $u(\cdot, t)$ of the problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \ t > 0, \\ u &= 0 \ \text{ at } x = 0, 1, \\ u(x, 0) &= f(x). \end{aligned}$$

Then $S(t)$ is a semiflow on $L^\infty(0,1)$ satisfying (C1) (since u is smooth for $t > 0$), (C4) (since $\|S(t)f\|_\infty \leq \|f\|_\infty$), but not (C3) (since otherwise we would have $u(\cdot, t) \to f$ in $L^\infty(0,1)$ as $t \to 0+$, implying that f is continuous).

Example 2.3. (A semilinear wave equation.)

Let, $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be bounded and open with boundary $\partial\Omega$. Consider the damped semilinear wave equation

$$u_{tt} + \beta u_t - \Delta u + f(u) = 0 \qquad \text{in } \Omega, \tag{2.1}$$

with boundary condition

$$u|_{\partial\Omega} = 0, \tag{2.2}$$

and initial conditions

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \tag{2.3}$$

where $\beta > 0$ is a constant and $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the growth condition

$$|f(u)| \leq c_0(|u|^{\frac{n}{n-2}} + 1),$$

for some constant $c_0 > 0$, and sign condition

$$\liminf_{|u| \to \infty} \frac{f(u)}{u} \geq -\lambda_1,$$

where λ_1 is the first eigenvalue of $-\Delta$ with the boundary condition (2.2) . Let $X = H_0^1(\Omega) \times L^2(\Omega)$. Then given $\{u_0, u_1\} \in X$ there exists at least one weak solution $\varphi = \{u, u_t\}$ on $[0, \infty)$ to (2.1)-(2.3), and the set G of all such weak solutions is a generalized semiflow satisfying (C3) and (C4). This is proved in [5]. Note that we do not assume any Lipschitz condition on f, so that there is no reason to suppose that solutions are unique.

Our first result is a simple extension to generalized semiflows of that of [7]. We say that G *has unique representatives* if whenever $\varphi, \psi \in G$ with $\varphi(t) = \psi(t)$ for a.e. $t > 0$ we have $\varphi(t) = \psi(t)$ for all $t > 0$; it is easily seen that this property holds for semiflows.

Theorem 2.1. *Let G have unique representatives and satisfy* (C0). *Then G satisfies* (C1).

Proof. Let $\varphi \in G$. Following the proof in [7], which uses an argument of Auerbach [1], let $0 < a < a + \delta < \infty$ and let I, J denote the open intervals $(a, a + \delta)$ and $(a+\delta/3, a+2\delta/3)$ respectively. It suffices to show that φ is continuous in J. Since φ is strongly measurable, by a version of Lusin's theorem (see Oxtoby [31], [7]), there exists in I a closed set F_j of measure greater than $\delta - 1/j^2$ on which the restriction of φ is continuous. The continuity being uniform, there exists $\eta_j \in (0, \delta/3)$ such that $t, t + h \in F_j$ and $|h| < \eta_j$ imply that $d(\varphi(t + h), \varphi(t)) < 1/j$.

Suppose for contradiction that there exist $t_0 \in J$ and a sequence $h_j \to 0$ with $\varphi(t_0 + h_j) \not\to \varphi(t_0)$. Extracting a subsequence, we may assume that

$$d(\varphi(t_0 + h_j), \varphi(t_0)) > \varepsilon \tag{2.4}$$

for some $\varepsilon > 0$ and all j, and that $|h_j| < \eta_j$ for all j. Let

$$\begin{aligned} E_j &= \{t \in J : t, t + h_j \in F_j\} \\ &= F_j \cap (F_j - h_j) \cap J. \end{aligned}$$

Then meas $(J \backslash F_j) \leq 1/j^2$, meas $(J \backslash (F_j - h_j)) =$ meas $((J + h_j) \backslash F_j) \leq 1/j^2$. Thus meas $(J \backslash E_j) \leq 2/j^2$. Hence meas $(J \backslash \liminf_{j\to\infty} E_j) = 0$, and so $\varphi(t + h_j) \to \varphi(t)$ for a.e. $t \in J$. In particular there exists $t_1, t_2 \in J$, $t_1 < t_0 < t_2$, with $\varphi(t_i + h_j) \to \varphi(t_i)$ for $i = 1, 2$. By (H2) $\varphi^{t_1+h_j}$ is a solution, and since $\varphi^{t_1+h_j}(0) \to \varphi(t_1)$, by (H4) there exists a subsequence $\varphi^{t_1+h_\mu}$ and a solution ψ with $\varphi^{t_1+h_\mu}(t) \to \psi(t)$ for all $t \geq 0$. But then $\psi(t) = \varphi(t + t_1)$ for a.e. $t \in (0, a + 2\delta/3 - t_1)$. Now define $\tilde{\psi}$ by

$$\tilde{\psi}(t) = \begin{cases} \varphi(t + t_1) & \text{for } 0 \leq t \leq t_2 - t_1, \\ \psi(t) & \text{for } t > t_2 - t_1. \end{cases}$$

By (H2), (H3) $\tilde{\psi} \in G$, and $\tilde{\psi}(t) = \psi(t)$ for a.e. $t > 0$. Since G has unique representatives it follows that $\tilde{\psi}(t) = \psi(t)$ for all $t > 0$. In particular $\tilde{\psi}(t_0 - t_1) = \psi(t_0 - t_1)$, and hence $\varphi(t_0 + h_\mu) \to \varphi(t_0)$, contradicting (2.4). □

Next, we extend a result for semiflows of Chernoff & Marsden [16] (see also [6]).

Theorem 2.2. *Let G satisfy* (C1). *Let φ_j, φ be solutions with $\varphi_j(t) \to \varphi(t)$ for all $t > 0$. Then $\varphi_j(t) \to \varphi(t)$ uniformly for t in compact subsets of $(0, \infty)$. In particular G satisfies* (C2).

Proof. Let $0 < a < b < \infty$, and for $\varepsilon > 0$, $n = 1, 2, \ldots$ set

$$S_{n,\varepsilon} = \{t \in [a, b] : j \geq n \text{ implies } d(\varphi_j(t), \varphi(t)) \leq \varepsilon\}.$$

$S_{n,\varepsilon}$ is closed by (C1), and by assumption $\cup_{n=1}^{\infty} S_{n,\varepsilon} = [a, b]$. By the Baire Category theorem, some $S_{r,\varepsilon}$ contains an open interval. Since we may apply this argument to any $[a, b] \subset (0, \infty)$ there exists a dense open subset S_ε of $(0, \infty)$ such that if $t_0 \in S_\varepsilon$ there exist an open neighbourhood $N_\varepsilon(t_0)$ of t_0 and $r_\varepsilon(t_0)$ with $d(\varphi_j(t), \varphi(t)) \leq \varepsilon$ whenever $j \geq r_\varepsilon(t_0)$, $t \in N_\varepsilon(t_0)$.

Let $K = \cap_{i=1}^{\infty} S_{1/i}$. Clearly $\varphi_j(t_j) \to \varphi(t)$ whenever $t_j \to t$ and $t \in K$. Again by the Baire Category theorem, K is dense in $(0, \infty)$.

Now let $t > 0$ be arbitrary and $t_j \to t$, and suppose for contradiction that $\varphi_j(t_j) \not\to \varphi(t)$, and without loss of generality that

$$d(\varphi_j(t_j), \varphi(t)) > \delta \tag{2.5}$$

for all j and some $\delta > 0$. Now let $s \in K$, $s < t$ and consider the solutions $\psi_j = \varphi_j^{t_j+s-t}$, which are well defined for j large enough. Since $s \in K$, $\psi_j(0) \to \varphi(s)$, and so by (H4) there exist a subsequence ψ_μ and a solution ψ with $\psi_\mu(\tau) \to \psi(\tau)$ for all $\tau \geq 0$. But if $s+\tau \in K$ then $\psi_j(\tau) = \varphi_j(t_j + s - t + \tau) \to \varphi(s+\tau)$. Since K is dense and φ, ψ continuous on $(0, \infty)$, it follows that $\psi(\tau) = \varphi(s+\tau)$ for all $\tau \geq 0$. Hence $\varphi_\mu(t_\mu) = \psi_\mu(t-s) \to \psi(t-s) = \varphi(t)$, contradicting (2.5). □

If X is locally compact then for semiflows (C3) implies (C4), a result first proved by Dorroh [19]. We next show that the same result holds for generalized semiflows, adapting the simple proof by Chernoff [15] of Dorroh's result.

Theorem 2.3. *Let X be locally compact. Let G satisfy* (C3). *Then G satisfies* (C4)

Proof. Let $\varphi_j \in G$ with $\varphi_j(0) \to z$. By Theorem 2.2 there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ and $\varphi_\mu(t) \to \varphi(t)$ uniformly for t in compact subsets of $(0, \infty)$. It thus suffices to show that if $t_\mu \to 0$ then $\varphi_\mu(t_\mu) \to z$. Suppose not. Then there exists a further subsequence, which we do not relabel, such that $d(\varphi_\mu(t_\mu), z) \geq \varepsilon$ for some $\varepsilon > 0$. We may assume also that $d(\varphi_\mu(0), z) < \varepsilon$. Hence by (C3) there exists $s_\mu \in [0, t_\mu]$ with $d(\varphi_\mu(s_\mu), z) = \varepsilon$. Since X is locally compact we may assume further that $\varphi_\mu(s_\mu) \to y$, where $d(y, z) = \varepsilon$. Thus by (H3),(H4) there exists $\psi \in G$ such that for a further subsequence $\varphi_\mu(s_\mu + t) \to \psi(t)$ for all $t \geq 0$. But for $t > 0$ we have $\varphi_\mu(s_\mu + t) \to \varphi(t)$, and so $\psi(t) = \varphi(t)$ for all $t > 0$. Letting $t \to 0$ we deduce from (C3) that $y = z$, a contradiction. □

Chernoff [15] gives an example of a semiflow on a Hilbert space satisfying (C3) but not (C4); we give another more explicit example in Section 6.2.

3. Existence of global attractors

We first extend to generalized semiflows various standard definitions for semiflows.

Let G be a generalized semiflow and let $E \subset X$. Define for $t \geq 0$

$$T(t)E = \{\varphi(t) : \varphi \in G \text{ with } \varphi(0) \in E\}, \tag{3.1}$$

so that $T(t) : 2^X \to 2^X$, where 2^X is the space of all subsets of X. It follows from (H2), (H3) that $\{T(t)\}_{t\geq 0}$ defines a semigroup on 2^X, i.e. (a), (b) hold for $T(t)$.[1]

Note that (H4) implies that $T(t)\{z\}$ is compact for each $z \in X, t \geq 0$.

The *positive orbit* of $\varphi \in G$ is the set $\gamma^+(\varphi) = \{\varphi(t) : t \geq 0\}$. If $E \subset X$ then the *positive orbit* of E is the set

$$\gamma^+(E) = \bigcup_{t\geq 0} T(t)E = \bigcup\{\gamma^+(\varphi) : \varphi \in G \text{ with } \varphi(0) \in E\}.$$

[1]This semigroup has various interesting properties; for example, it is monotone with respect to the partial order of set inclusion (i.e. $E \subset F$ implies $T(t)E \subset T(t)F$ for all $t \geq 0$) and its rest points are the invariant sets of G. When restricted, for example, to the space $K(X)$ of compact subsets of X endowed with the Hausdorff metric, it inherits from (H4) the upper semicontinuity property that $K_j \to K$ implies that $\text{dist}(T(t)K_j, T(t)K) \to 0$ for all $t \geq 0$. If G is a semiflow then we have the stronger property that $K_j \to K$ implies $T(t)K_j \to T(t)K$ for all $t \geq 0$, so that $T(\cdot)$ is a semiflow on $K(X)$.

If $\tau \geq 0$ we set
$$\gamma^\tau(E) = \bigcup_{t\geq\tau} T(t)E = \gamma^+(T(\tau)E).$$
The $\omega-$*limit set* of $\varphi \in G$ is the set
$$\omega(\varphi) = \{z \in X : \varphi(t_j) \to z \text{ for some sequence } t_j \to \infty\}.$$
A *complete orbit* is a map $\psi : \mathbf{R} \to X$ such that for any $s \in \mathbf{R}$, $\psi^s \in G$. If ψ is a complete orbit then the α-*limit set* of ψ is the set
$$\alpha(\psi) = \{z \in X : \psi(t_j) \to z \text{ for some sequence } t_j \to -\infty\}.$$
If $E \subset X$ the $\omega-$*limit set* of E is the set
$$\omega(E) = \{z \in X : \text{there exist } \varphi_j \in G \text{ with } \varphi_j(0) \in E,\ \varphi_j(0) \text{ bounded, and a sequence } t_j \to \infty \text{ with } \varphi_j(t_j) \to z\}.$$
(When E is unbounded this definition differs from the usual one, in which it is not assumed that the $\varphi_j(0)$ are bounded.)

The subset $A \subset X$ *attracts* a set E if $\text{dist}(T(t)E, A) \to 0$ as $t \to \infty$, and is *locally attracting* if A attracts a neighbourhood of A.

We say that A is *positively invariant* if $T(t)A \subset A$ for all $t \geq 0$, that A is *quasi-invariant* if for each $z \in A$ there exists a complete orbit ψ with $\psi(0) = z$ and $\psi(t) \in A$ for all $t \in \mathbf{R}$, and that A is *invariant* if $T(t)A = A$ for all $t \geq 0$. Note that if A is quasi-invariant then $A \subset T(t)A$ for all $t \geq 0$ (this is taken as the definition of quasi-invariance by Barbashin [10]); from this it follows easily that A is invariant if and only if A is positively invariant and quasi-invariant. Note that even for semiflows a set A may be invariant but there may be solutions $\varphi \in G$ with $\varphi(0) \notin A$ and $\varphi(t) \in A$ for some $t > 0$.

The subset A is a *global attractor* if A is compact, invariant, and attracts all bounded sets.

The generalized semiflow G is *eventually bounded* if given any bounded $B \subset X$ there exists $\tau \geq 0$ with $\gamma^\tau(B)$ bounded.

G is *point dissipative* if there is a bounded set B_0 such that for any $\varphi \in G$ $\varphi(t) \in B_0$ for all sufficiently large t.

G is *asymptotically compact* if for any sequence $\varphi_j \in G$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to \infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.

G is *compact* if for any sequence $\varphi_j \in G$ with $\varphi_j(0)$ bounded there exists a subsequence φ_μ such that $\varphi_\mu(t)$ is convergent for each $t > 0$.

Proposition 3.1. *Let G be asymptotically compact. Then G is eventually bounded.*

Proof. Let $a \in X$, let $B \subset X$ be bounded, and suppose for contradiction that $\gamma^\tau(B)$ is unbounded for all $\tau \geq 0$. Then there exist $\varphi_j \in G$ with $\varphi_j(0) \in B$ and $t_j \to \infty$ with $d(\varphi_j(t_j), a) \to \infty$. But $\varphi_j(t_j)$ has a convergent subsequence by asymptotic compactness. □

Proposition 3.2. *Let G be eventually bounded and compact. Then G is asymptotically compact.*

Proof. Let $\varphi_j \in G$ with $\varphi_j(0)$ bounded, and let $t_j \to \infty$. Since G is eventually bounded $\varphi_j^{t_j-1}(0)$ is bounded. Since G is compact, for some subsequence $\varphi_\mu^{t_\mu-1}(1) = \varphi_\mu(t_\mu)$ is convergent. □

Theorem 3.3. *A generalized semiflow G has a global attractor if and only if G is point dissipative and asymptotically compact. The global attractor A is unique and given by*

$$A = \bigcup\{\omega(B) : B \text{ a bounded subset of } X\} = \omega(X). \tag{3.2}$$

Furthermore A is the maximal compact invariant subset of X.

Theorem 3.3 generalizes a corresponding result for semiflows given in Hale [21], Ladyzhenskaya [27] and having antecedents in the work of Billotti & LaSalle [11] and Hale, LaSalle & Slemrod [26]. Note, however, that we make no assumption that the positive orbits of bounded sets are bounded. A closely related result in the context of a set-valued semiflow $T(t)$ is announced in Mel'nik [29]; the main differences are that in [29] (i) the global attractor A is not asserted to be invariant, but only to satisfy the property $A \subset T(t)A$ for all $t \geq 0$ (this can be traced to the weaker hypothesis made in [29] that $T(t)$ satisfies $T(s+t)\{z\} \subset T(s)T(t)\{z\}$ for all $z \in X$, $s, t \geq 0$, whereas our concatenation hypothesis $(H3)$ implies equality), (ii) the definition of point dissipative is stronger, (iii) the semiflow is assumed to be eventually bounded. For other results on global attractors and applications see Temam [42] and Babin & Vishik [4].

In order to prove Theorem 3.3 we need to suitably modify the corresponding arguments for semiflows. Our treatment is closest to that of Ladyzhenskaya [27].

Lemma 3.4. *Let G be asymptotically compact.*
(i) Let $B \subset X$ be nonempty and bounded. Then $\omega(B)$ is nonempty, compact, quasi-invariant and attracts B. If $T(t_0)\omega(B) \subset B$ for some $t_0 \geq 0$ then $\omega(B)$ is invariant.
(ii) If $\varphi \in G$ then $\omega(\varphi)$ is nonempty, compact, quasi-invariant, and $\lim_{t\to\infty} \rho(\varphi(t), \omega(\varphi)) = 0$.
(iii) If ψ is a bounded complete orbit then $\alpha(\psi)$ is nonempty, compact and quasi-invariant, and $\lim_{t\to-\infty} \rho(\psi(t), \alpha(\psi)) = 0$.

Proof. (i) Let $v \in B$. By (H1) there exists some $\varphi \in G$ with $\varphi(0) = v$. By the asymptotic compactness $\varphi(j)$ has a convergent subsequence, and so $\omega(B)$ is nonempty. Since $\omega(B) \subset \overline{\gamma^\tau(B)}$ for any $\tau \geq 0$, and since G is eventually bounded, $\omega(B)$ is bounded. It is easily seen that $\omega(B)$ is also closed.

Let $z \in \omega(B)$. By definition there exist $\varphi_j \in G$ with $\varphi_j(0) \in B$ and a sequence $t_j \to \infty$ such that $\varphi_j(t_j) \to z$. By (H2), $\varphi_j^{t_j} \in G$. Since $\varphi_j^{t_j}(0) \to z$, by (H4) there exist a subsequence, which we do not relabel, and a solution ψ_0 with $\psi_0(0) = z$, such that $\varphi_j^{t_j}(t) \to \psi_0(t)$ for all $t \geq 0$. Clearly $\psi_0(t) \in \omega(B)$ for all $t \geq 0$. Now consider the sequence $\varphi_j^{t_j-1}$. Since $\varphi_j^{t_j-1}(0) = \varphi_j(t_j - 1)$, by the asymptotic compactness and (H4) we have (after extraction of a further subsequence) that $\varphi_j^{t_j-1}(t) \to \psi_1(t)$ for all $t \geq 0$, where $\psi_1 \in G$. Clearly $\psi_1^1 = \psi_0$. Proceeding inductively, we find for each $r = 1, 2, ...$ a solution ψ_r such that $\psi_r^1 = \psi_{r-1}$ and $\psi_r(t) \in \omega(B)$ for all $t \geq 0$. Given $t \in \mathbf{R}$ define $\psi(t)$ to be the common value of $\psi_r(t+r)$ for $r \geq -t$. Then ψ is a complete orbit with $\psi(0) = z$ and $\psi(t) \in \omega(B)$ for all $t \in \mathbf{R}$. Hence $\omega(B)$ is quasi-invariant.

Now suppose $z_k \in \omega(B)$. By the quasi-invariance $z_k = \psi_k(k)$ where $\psi_k \in G$ and $\psi_k(0) \in \omega(B)$. By the boundedness of $\omega(B)$ and the asymptotic compactness z_k has a convergent subsequence. Hence $\omega(B)$ is compact.

Suppose $\omega(B)$ does not attract B. Then there exist $\varepsilon > 0$, $\varphi_j \in G$ with $\varphi_j(0) \in B$ and a sequence $t_j \to \infty$ with $\varphi_j(t_j) \notin N_\varepsilon(\omega(B))$. But by asymptotic compactness $\varphi_j(t_j)$ has a convergent subsequence, and the limit belongs to $\omega(B)$, a contradiction.

If $T(t_0)\omega(B) \subset B$ for some $t_0 \geq 0$ then by the quasi-invariance of $\omega(B)$ we have $\omega(B) \subset B$. Let $\varphi \in G$ with $\varphi(0) \in \omega(B)$ and let $t \geq 0$. By the quasi-invariance and concatenation we have that, for each $k \geq 0$, $\varphi(t) = \psi_k(k)$ for some $\psi_k \in G$ with $\psi_k(0) \in \omega(B)$. But then $\psi_k(0) \in B$ and so $\varphi(t) \in \omega(B)$. Thus $\omega(B)$ is invariant.

(ii) The proof is similar to (i) but easier.

(iii) If $t_j \to -\infty$ then $\psi(t_j) = \psi^{2t_j}(-t_j)$ and $\psi^{2t_j}(0)$ is bounded, so that by asymptotic compactness $\psi(t_j)$ has a convergent subsequence. The rest of the proof is as for (ii).

□

Lemma 3.5. *Let G be pointwise dissipative and asymptotically compact. Then there exists a bounded set B_1 such that given any compact $K \subset X$ there exist $\varepsilon = \varepsilon(K) > 0, t_1 = t_1(K) > 0$ such that $T(t)N_\varepsilon(K) \subset B_1$ for all $t \geq t_1$.*

Proof. Let $\delta > 0$. Since by Proposition 3.1 G is eventually bounded there exists $\tau \geq 0$ such that $B_1 := \gamma^\tau(N_\delta(B_0))$ is bounded. Suppose for contradiction that there exist a compact set K and sequences $\varepsilon_j \to 0$, $t_j \to \infty$, $\varphi_j \in G$ with $\varphi_j(0) \in N_{\varepsilon_j}(K)$ and $\varphi_j(t_j) \notin B_1$. Since $\varphi_j(t_j) = \varphi_j^t(t_j - t)$ it follows that $\varphi_j^t(0) = \varphi_j(t) \notin N_\delta(B_0)$ for $0 \leq t \leq t_j - \tau$. We may also assume that $\varphi_j(0) \to z \in K$. Hence by (H4) there is a subsequence φ_μ and $\varphi \in G$ with $\varphi_\mu \to \varphi$ pointwise, $\varphi(0) = z$, and $\varphi(t) \notin B_0$ for all $t \geq 0$. This contradicts the point dissipativeness of G. □

Proof of Theorem 3.3. Let A be a global attractor for G and let $B_0 = N_\delta(A)$ for some $\delta > 0$. Given $\varphi \in G$ the set consisting of the single point $\{\varphi(0)\}$ is bounded and thus attracted to A. Hence $\varphi(t) \in B_0$ for t sufficiently large, and hence G is point dissipative. If $\varphi_j \in G$ with $\varphi_j(0)$ bounded, the set $\{\varphi_j(0)\}$ is attracted to A, and thus if $t_j \to \infty$ we have $\rho(\varphi_j(t_j), A) \to 0$. Since A is compact this implies that $\varphi_j(t_j)$ has a convergent subsequence, and thus G is asymptotically compact.

Conversely, let G be point dissipative and asymptotically compact. Let B_1 be as in Lemma 3.5 and let $A = \omega(B_1)$. By Lemma 3.4 A is compact and attracts B_1. We show that A attracts bounded sets. Let B be bounded and let $K = \omega(B)$. By Lemma 3.4 K is compact and attracts B. Let $\varepsilon(K)$, $t_1 = t_1(K)$ be as in Lemma 3.5, and let $0 < \varepsilon < \varepsilon(K)$. Since K attracts B, $T(t_0)B \subset N_\varepsilon(K)$ for some $t_0 > 0$. Hence $T(t_0 + t_1)B = T(t_1)T(t_0)B \subset T(t_1)N_\varepsilon(K) \subset B_1$. Thus $T(t_0 + t_1 + t)B \subset T(t)B_1$ for all $t \geq 0$, and since B_1 is attracted to A so is B. Since by Lemma 3.5 we also have that $T(t_2)\omega(B_1) \subset B_1$ for some $t_2 \geq 0$, it follows from Lemma 3.4 that A is invariant. This proves that A is a global attractor, and that $\omega(B) \subset A$ for any bounded B, so that (3.2) holds.

Suppose A_1 is compact and invariant. Then $\omega(A_1) = A_1$ and so $A_1 \subset A$ by (3.2). Hence A is the maximal compact invariant subset of X. □

4. Connectedness

Proposition 4.1. *Let G be asymptotically compact and satisfy $(C1)$. If $\varphi \in G$ then $\omega(\varphi)$ is connected. If ψ is a complete orbit then $\alpha(\psi)$ is connected.*

Proof. This is standard. By Lemma 3.4 $\omega(\varphi)$ and $\alpha(\psi)$ are compact. If $\omega(\varphi)$, say, were not connected then $\omega(\varphi) = A_1 \cup A_2$ for nonempty disjoint compact sets

A_1, A_2. Let U_1, U_2 be disjoint open sets with $A_1 \subset U_1$, $A_2 \subset U_2$. By (C1) there exist $t_j \to \infty$ with $\varphi(t_j) \notin U_1 \cup U_2$, and by asymptotic compactness this implies that there exists $z \in \omega(\varphi) \backslash (A_1 \cup A_2)$, a contradiction. □

We say that G has *Kneser's property* if $T(t)\{z\}$ is connected for each $z \in X, t \geq 0$. Any semiflow has Kneser's property since $T(t)\{z\}$ is a point.

Theorem 4.2. *Let G satisfy $(C1)$. If G has Kneser's property and if $E \subset X$ is connected then $\omega(E)$ is connected.*

For a related result see Mel'nik [29].

Proof of Theorem 4.2. Suppose E is connected but $\omega(E)$ is not. Then $\omega(E) = A_1 \cup A_2$, for nonempty sets A_1, A_2, where $A_1 \cap \overline{A_2} = A_2 \cap \overline{A_1} = \emptyset$. Since X is a metric space, it is completely normal (see [23, p. 42]), so that there exist disjoint open sets U_1, U_2 with $A_1 \subset U_1$, $A_2 \subset U_2$. (If $\omega(E)$ is compact this conclusion is obvious.) For $i = 1, 2$ let $E_i = \{z \in E : \text{dist}\,(T(t)\{z\}, A_i) \to 0 \text{ as } t \to \infty\}$. We claim that E_1, E_2 are disjoint nonempty relatively open subsets of E with $E_1 \cup E_2 = E$. Since E is connected this is a contradiction.

To show that the E_i are disjoint, note that if $z \in E_1 \cap E_2$ then $T(t)\{z\} \subset U_1$ for t large enough and $T(t)\{z\} \subset U_2$ for t large enough, which is impossible since $U_1 \cap U_2 = \emptyset$.

To show that $E_1 \cup E_2 = E$ suppose $z \in E$. Then $\text{dist}\,(T(t)\{z\}, \omega(E)) \to 0$ as $t \to \infty$, and so there exists $T > 0$ such that $T(t)\{z\} \subset U_1 \cup U_2$ for all $t > T$. By Kneser's property we thus have that for each $t > T$ either $T(t)\{z\} \subset U_1$ or $T(t)\{z\} \subset U_2$. But if $T(r)\{z\} \subset U_1$, $T(s)\{z\} \subset U_2$ for $T < r < s$ then there exists $\varphi \in G$ with $\varphi(0) = z$, $\varphi(r) \in U_1$, $\varphi(s) \in U_2$, and $\varphi(t) \in U_1 \cup U_2$ for $t \in [r, s]$. This is impossible by (C1). Hence $z \in E_1 \cup E_2$.

To show that E_2, say, is nonempty suppose that $E = E_1$. Let $a \in A_2$, so that there exist $\varphi_j \in G$ with $\varphi_j(0) \in E$, $\varphi_j(0)$ bounded and $t_j \to \infty$ with $\varphi_j(t_j) \to a$. Let $B = \{\varphi_j(0)\}$. Since B is bounded, by Lemma 3.4 B is attracted to $\omega(B) \subset \omega(E)$. Hence there exists $T > 0$ with $\varphi_j(t) \in U_1 \cup U_2$ for all $t > T$ and all j. But since $\varphi_j(0) \in E_1$, $\varphi_j(t) \in U_1$ for all j and all $t > T$. This contradicts $a \in A_2$.

Finally, to show that E_2, say, is relatively open, let $z \in E_2$ and suppose for contradiction that there exist $z_j \to z$ with $z_j \in E_1$ for all j. Thus there exist $\varphi_j \in G$ with $\varphi_j(0) = z_j$ and for each j and for all sufficiently large t we have $\varphi_j(t) \in U_1$. Since $\{\varphi_j(0)\}$ is bounded, as argued above we have that there exists $T > 0$ with $\varphi_j(t) \in U_1 \cup U_2$ for all $t > T$ and thus $\varphi_j(t) \in U_1$ for all j and all $t > T$. But by (H4) we may assume that $\varphi_j(t) \to \varphi(t)$ for all $t > 0$, where $\varphi \in G$ with $\varphi(0) = z$. Since $z \in E_2$, there exists $\tau > T$ with $\varphi(\tau) \in U_2$, and hence $\varphi_j(\tau) \in U_2$ for j sufficiently large, a contradiction. □

Corollary 4.3. *Let X be connected, and let G satisfy $(C1)$ and have Kneser's property. If A is a global attractor then A is connected.*

Proof. This follows since $A = \omega(X)$. □

For similar but not identical results for semiflows see Ladyzhenskaya [27] and Sell & You [40][2].

[2] However the proof in [27] that the global attractor is connected if X is connected makes use of the incorrect remark that a bounded subset B of a connected metric space X is contained in a bounded connected subset of X; a counterexample is provided by the metric subspace X of $\mathbf{R}^2$

5. Lyapunov functions

A complete orbit $\psi \in G$ is *stationary* if $\psi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in X$. Each such z is called a *rest point.* (Note that in general, if z is a rest point there may also exist nonconstant $\psi \in G$ with $\psi(0) = z$.) We denote the set of rest points of G by $Z(G)$. It follows easily from (H4) that $Z(G)$ is closed.

We say that $V : X \to \mathbf{R}$ is a *Lyapunov function* for G provided

(*i*) V is continuous,
(*ii*) $V(\varphi(t)) \leq V(\varphi(s))$ whenever $\varphi \in G$ and $t \geq s \geq 0$.
(*iii*) if $V(\psi(t))$ = constant for some complete orbit ψ and all $t \in \mathbf{R}$ then ψ is stationary.

Since a global attractor A is quasi-invariant, given any $a \in A$ there exists a complete orbit ψ with $\psi(0) = a$. In the presence of a Lyapunov function the behaviour of such complete orbits can be characterized.

Theorem 5.1. *Let G be asymptotically compact, let $(C1)$ hold, and suppose there exists a Lyapunov function V for G. Suppose further that $Z(G)$ is bounded. Then G is point dissipative, so that there exists a global attractor A. For each complete orbit ψ lying in A the limit sets $\alpha(\psi), \omega(\psi)$ are connected subsets of $Z(G)$ on which V is constant. If $Z(G)$ is totally disconnected (in particular, if $Z(G)$ is countable) the limits*

$$z_- = \lim_{t \to -\infty} \psi(t), \quad z_+ = \lim_{t \to \infty} \psi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to \infty$ for every $\varphi \in G$.

Proof. Let $\varepsilon > 0$, $B_0 = N_\varepsilon(Z(G))$. If $\varphi \in G$, by properties (i), (ii) of V and Lemma 3.4 we have that $V(z) = \lim_{t\to\infty} V(\varphi(t)) \in \mathbf{R}$ for all $z \in \omega(\varphi)$. Since $\omega(\varphi)$ is quasi-invariant, by property (ii) $\omega(\varphi) \subset Z(G)$ and hence $\varphi(t) \in B_0$ for t sufficiently large. Thus G is point dissipative. If ψ is a complete orbit in A then we have by the above and Proposition 4.1 that $\omega(\psi)$ is a connected subset of $Z(G)$, and the corresponding result for $\alpha(\psi)$ holds similarly. The rest of the theorem is then obvious. □

6. Stability

The subset $A \subset X$ is *Lyapunov stable* if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset X$ with $\operatorname{dist}(E, A) < \delta$ then $\operatorname{dist}(T(t)E, A) < \varepsilon$ for all $t \geq 0$. It is easily seen that a subset A is Lyapunov stable if and only if given $\varphi \in G$ with $\rho(\varphi_j(0), A) \to 0$ and $t_j \geq 0$ we have $\rho(\varphi_j(t_j), A) \to 0$. We say that A is *uniformly asymptotically stable* if A is Lyapunov stable and is locally attracting.

Since a global attractor is compact, invariant and locally attracting, the following theorem gives in particular conditions under which a global attractor is uniformly asymptotically stable.

Theorem 6.1. *Let G satisfy* (C1) *and* (C4), *and let A be a compact invariant set that is locally attracting. Then A is uniformly asymptotically stable.*

---

given by the union of the sets $C_j \backslash Q_j$, $j = 1, 2, \ldots$, where C_j is the circle centre $(0, j)$ of radius j and Q_j is the square $(0, 1/j)^2$. The intersection B of X with the ball $B(0, 2)$ is not connected, but the only connected set containing B is X itself.

Proof. We must show that A is Lyapunov stable. Suppose not. Then there exist $\varphi_j \in G$ and $t_j \geq 0$ such that $\rho(\varphi_j(0), A) \to 0$ but $\rho(\varphi_j(t_j), A) \geq \varepsilon$ for some $\varepsilon > 0$. Since A is compact we may suppose that $\varphi_j(0) \to z \in A$ and thus by (H4) that $\varphi_j(t) \to \varphi(t)$ for all $t \geq 0$ for some $\varphi \in G$ with $\varphi(0) = z$. Since A is invariant $\varphi(t) \in A$ for all $t \geq 0$. We may also suppose that either $t_j \to \infty$ or $t_j \to t \geq 0$.

If $t_j \to \infty$, then since A is locally attracting we have $\rho(\varphi_j(t_j), A) \to 0$, a contradiction. If $t_j \to t > 0$ then since G satisfies (C1), by Theorem 2.2 we have $\varphi_j(t_j) \to \varphi(t) \in A$, a contradiction. So it remains to consider the case $t_j \to 0$. By (C4) we have, after the possible extraction of a further subsequence, that $d(\varphi_j(t_j), \varphi(t_j)) \to 0$. Since A is invariant there exists $\theta \in G$ with $\theta(1+\tau) = \varphi(\tau)$ for all $\tau \geq 0$, and so by (C1) $\varphi(t_j) \to \varphi(0)$. Hence $\varphi_j(t_j) \to \varphi(0) \in A$, a contradiction. □

Corollary 6.2. *Let X be locally compact, let G satisfy* (C3), *and let A be a compact invariant set that is locally attracting. Then A is uniformly asymptotically stable.*

Proof. This follows immediately from Theorems 2.3, 6.1. □

Let G satisfy (C1) and let A be a compact invariant set that is locally attracting. Then the proof of Theorem 6.1 shows that given $\varepsilon > 0$, $\tau > 0$ there exists $\delta > 0$ such that if $E \subset X$ with $\operatorname{dist}(E, A) < \delta$ then $\operatorname{dist}(T(t)E, A) < \varepsilon$ for all $t \geq \tau$. For semiflows Sell & You [40] take this conclusion as the definition of Lyapunov stability, and prove an equivalent result.

We now give two examples of semiflows $\{S(t)\}_{t\geq 0}$ on a metric space X satisfying (C1) for which A is a single point but is not Lyapunov stable.

6.1. **An example with $X = \mathbf{R}^2$.** In the first example $X = \mathbf{R}^2$ is locally compact. Thus by Corollary 6.2 (C3) cannot be satisfied.

Consider the differential equations in $\mathbf{R}^2$

$$\dot{r} = -r^{\frac{1}{2}} h(\theta), \tag{6.1a}$$

$$\dot{\theta} = -\,2r^{-\frac{1}{2}} \sin(\theta/2) h(\theta), \tag{6.1b}$$

where (r, θ) are plane polar coordinates with $0 \leq \theta < 2\pi$ and where

$$h(\theta) := \theta^{-2}(2\pi - \theta)^{-2}. \tag{6.2}$$

Writing $x = r\cos\theta$, $y = r\sin\theta$, we see that (6.1) defines a C^∞ vector field in $U = \mathbf{R}^2 \backslash \bar{L}$, where $L = \{(x,0) : x > 0\}$ denotes the positive semi x-axis. Note that the integral curves of (6.1) are given by

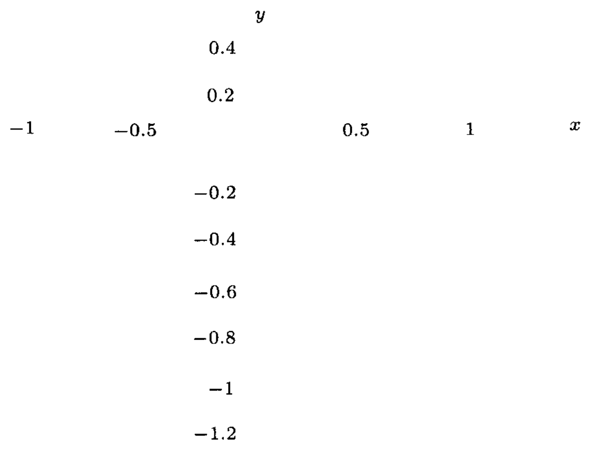
$$r = C\tan(\theta/4) \tag{6.3}$$

where $C > 0$ is a constant. These curves do not intersect L, and approach the origin tangent to it from the first quadrant (see Fig. 1a). Since $h(\theta) \geq \pi^{-4}$, we have that

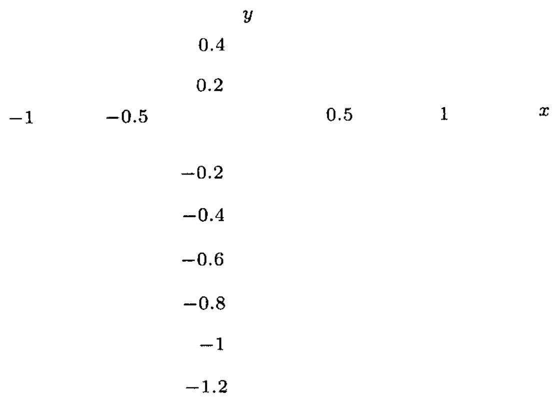
$$r^{\frac{1}{2}}(t) \leq r^{\frac{1}{2}}(0) - \frac{1}{2}\pi^{-4} t. \tag{6.4}$$

Hence for any $z \in U$, the solution $(x(t), y(t))$ of (6.1) with initial data $(x(0), y(0)) = z$ reaches the origin in a finite time $t_c = t_c(z) > 0$. We define for $z \in U$

$$R(t)z = \begin{cases} (x(t), y(t)) & \text{if } 0 \leq t < t_c, \\ (0,0) & \text{if } t \geq t_c. \end{cases} \tag{6.5}$$



(a)



(b)

FIGURE 1. a. Phase portrait for the semiflow $R(t)$. b. Phase portrait for the semiflow $S(t)$.

For $z = (x, 0)$ with $x > 0$ we define

$$R(t)z = \begin{cases} z & \text{if } t = 0, \\ (0,0) & \text{if } t > 0. \end{cases} \tag{6.6}$$

Thus $R(t) : \mathbf{R}^2 \to \mathbf{R}^2$ is defined for all $t \geq 0$, and clearly $R(0) = \text{identity}$, $R(s+t) = R(s)R(t)$ for all $s, t \geq 0$. We show that $R(t)$ is continuous for each $t > 0$. Let $t > 0$ and $z_j \to z$. We must show that $R(t)z_j \to R(t)z$. This is easily proved if $z \in U$ or $z = 0$, using the fact that by (6.4) the origin is stable, so we assume that $z \in L$. By (6.6) we may also assume that $z_j \notin L$ for all j. The result then follows provided we can show that $t_c(z_j) \to 0$, since then $R(t)z_j = R(t)z = 0$ for sufficiently large j.

Let (r_j, θ_j) be the solutions of (6.1) corresponding to the initial data z_j. We first claim that given $\varepsilon > 0$ there exists $J(\varepsilon)$ such that if $j \geq J(\varepsilon)$ and $\tau \in [0, t_c(z_j))$

then either $r_j(\tau) \leq \varepsilon$ or $h(\theta_j(\tau)) \geq \varepsilon^{-1}$. If not there would exist a subsequence j_k and $\tau_k \in [0, t_c(z_{j_k}))$ with $r_{j_k}(\tau_k) > \varepsilon$ and $h(\theta_{j_k}(\tau_k)) < \varepsilon^{-1}$. From (6.3) we have that

$$\frac{r_{j_k}(\tau_k)}{\tan(\theta_{j_k}(\tau_k)/4)} = \frac{r_{j_k}(0)}{\tan(\theta_{j_k}(0)/4)} \tag{6.7}$$

and the left-hand side of (6.7) is bounded away from zero and infinity. But this is not true for the right-hand side, since $z \in L$, establishing the claim. Now for $\varepsilon > 0$ let $t^\varepsilon(z_j) = \inf\{s > 0 : r_j(s) = \varepsilon\}$. From (6.4) applied to $r(t) = r_j(t + t^\varepsilon(z_j))$ we have that

$$t_c(z_j) - t^\varepsilon(z_j) \leq 2\pi^4 \varepsilon^{\frac{1}{2}}. \tag{6.8}$$

But for $j \geq J(\varepsilon)$, $\tau \in [0, t^\varepsilon(z_j))$ we have $h(\theta_j(\tau)) \geq \varepsilon^{-1}$ and thus $\dot{r_j}(\tau) \leq -r_j(\tau)^{\frac{1}{2}} \varepsilon^{-1}$. Hence

$$\varepsilon^{\frac{1}{2}} = r_j(t^\varepsilon(z_j))^{\frac{1}{2}} \leq r_j(0)^{\frac{1}{2}} - \frac{1}{2} t^\varepsilon(z_j)\varepsilon^{-1}. \tag{6.9}$$

Combining (6.8), (6.9) we deduce that for $j \geq J(\varepsilon)$

$$t_c(z_j) \leq 2\pi^4 \varepsilon^{\frac{1}{2}} + 2\varepsilon(r_j(0)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}})$$

and letting $\varepsilon \to 0$ we obtain $t_c(z_j) \to 0$ as required.

We have thus shown that $\{R(t)\}_{t \geq 0}$ is a semiflow on $\mathbf{R}^2$. Also, the map $t \mapsto R(t)z$ is clearly continuous on $(0, \infty)$ for any $z \in \mathbf{R}^2$.

We now modify $R(t)$ using a map $P : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$P(x, y) = (x - f(x, y), y),$$

where $f \in C^1(\mathbf{R}^2 \backslash \{0\})$ satisfies $f(x, y) = 0$ if $(x, y) \notin R := (-1, 0) \times (-1, 1)$, $\lim_{x \to 0-} f(x, 0) = a$, where $0 < a < 1$, and $f_x(x, y) \leq \frac{1}{2}$ for $(x, y) \neq (0, 0)$. Since $\partial_x P(x, y) = (1 - f_x(x, y), 0)$ for $(x, y) \neq (0, 0)$, P is monotone on lines $y =$ const., and hence P restricted to $\mathbf{R}^2 \backslash \{0\}$ is a diffeomorphism with range $\mathbf{R}^2 \backslash I$, where $I := \{(x, 0) : -a \leq x \leq 0\}$. We define $S(t) : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\begin{aligned} S(t)z &= P(R(t)P^{-1}z) && \text{if } z \notin I, t \geq 0, \\ S(0)(x, 0) &= (x, 0) && \text{if } (x, 0) \in I, \\ S(t)(x, 0) &= (0, 0) && \text{if } (x, 0) \in I, t > 0. \end{aligned}$$

(See Fig. 1b, where we have chosen

$$f(x, y) = \begin{cases} \frac{1}{4}(x+1)^2(1-y^2)^2 & \text{if } -1 < x < -y^2, \\ \frac{1}{4}(x+1)^2(1-y^2)^2 \sin^2(\pi x / 2y^2) & \text{if } -y^2 \leq x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

for which $a = \frac{1}{2}$.)

It is easily checked that $S(0) =$ identity, $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$. Each solution $t \mapsto S(t)z$ is continuous on $(0, \infty)$. The only case that is not immediately obvious is when $z \notin I$ and $P^{-1}z \notin L$. But then $R(t)P^{-1}z$ is continuous on $(0, \infty)$ and is zero for $t \geq t_c = t_c(P^{-1}z)$. So we just need to show that $\lim_{t \to t_c-} P(R(t)P^{-1}z) = 0$. But this follows since $R(t)P^{-1}z$ belongs to the first quadrant for $t \in (t_c - \varepsilon, t_c)$ for some $\varepsilon > 0$, and there $P =$ identity.

We now prove that each $S(t)$ is continuous from $\mathbf{R}^2 \to \mathbf{R}^2$. Let $t > 0$, $z_j \to z$. We must show that $S(t)z_j \to S(t)z$. There are various cases.

(*i*) Suppose $z \in I$, so that $S(t)z = 0$. We may assume that either $z_j \in I$ for all j or $z_j \notin I$ for all j, and in the former case $S(t)z_j = 0$ and we are done. If $z_j \notin I$ for all j then we may assume that $P^{-1}z_j \to w$, and clearly $w = 0$. But then by (6.4) $R(t)P^{-1}z_j = 0$ for j large enough, and hence $S(t)z_j = 0$ for j large enough.
(*ii*) Suppose $z \notin I$, so that $P^{-1}z \neq 0$ and $P^{-1}z_j \to P^{-1}z$. Then $R(t)P^{-1}z_j \to R(t)P^{-1}z$, so that if $R(t)P^{-1}z \neq 0$ we have $S(t)z_j \to S(t)z$. If $R(t)P^{-1}z = 0$ there are two subcases. If $P^{-1}z \in L$ then we may assume that either $P^{-1}z_j \in L$ for all j or that $P^{-1}z_j \notin L$ for all j. In the first case we have $S(t)z_j = S(t)z = 0$. In the second we already showed that $t_c(P^{-1}z_j) \to 0$. Hence $R(t)P^{-1}z_j = 0$ and $S(t)z_j = 0$ for j large enough. The second subcase is when $P^{-1}z \notin L$. Then $P^{-1}z_j \notin L$ for j large enough, and since $R(t)P^{-1}z$ approaches 0 from the first quadrant, so $R(t)P^{-1}z_j$ belongs to the closed first quadrant for j large enough. Hence $S(t)z_j = R(t)P^{-1}z_j \to 0 = S(t)z$.

It remains to show that $\{0\}$ is a global attractor for $\{S(t)\}_{t\geq 0}$ that is not Lyapunov stable. To show that $\{0\}$ is a global attractor we just need to prove that it attracts bounded sets. Let $M \geq 2$ and $|z| \leq M$. If $z \notin I$ then, since $P =$ identity outside $B(0,2)$, $|P^{-1}z| \leq M$. Hence by (6.4) $R(t)P^{-1}z = 0$ for $t \geq 2M^{\frac{1}{2}}\pi^4$, and so $S(t)z = 0$ for such t. If $z \in I$ then $S(t)z = 0$ for $t > 0$. Hence $S(t)B(0,M) = \{0\}$ for $t \geq 2M^{\frac{1}{2}}\pi^4$ and thus $S(\cdot)$ attracts bounded sets.

To see that $\{0\}$ is not Lyapunov stable, let $z_j = (j^{-1}, -j^{-1})$. Then by (6.3) there exists $t_j > 0$ such that $R(t_j)z_j = (-\sqrt{2}/(j\cot 7\pi/16), 0)$. Then $|S(t_j)z_j| = |P(R(t_j)z_j)| \geq a$.

6.2. **An example with X a Hilbert space.** In this example we take $X = H$ to be a real Hilbert space with inner product $(\ ,\ )$, norm $\|\cdot\|$ and orthonormal basis consisting of the vectors $e_i, \hat{e_i}$, $i = 1, 2, \ldots$. We construct a semiflow $\{S(t)\}_{t\geq 0}$ on H satisfying (C3) (i.e. $t \mapsto S(t)w$ is continuous from $[0,\infty) \to H$ for each $w \in H$) and such that $\{0\}$ is a global attractor which is not Lyapunov stable. By Corollary 6.2 such an example cannot occur for X locally compact. By Theorem 6.1 the semigroup $S(t)$ cannot satisfy (C4), and so in particular we obtain an explicit example of a semigroup on a Hilbert space satisfying (C3) but not (C4); the example of Chernoff [15] is not explicit and is based on a nontrivial result of infinite-dimensional topology.

Fix β with $0 < \beta < 1$. For each i we let $L_i = \operatorname{span}\{e_i, \hat{e_i}\}$ and

$$K_i := \{u + v : u \in L_i, v \in L_i^{\perp}, \|v\|^2 < \beta\|u\|^2\},$$

where $L_i^{\perp}$ denotes the orthogonal complement of L_i. Let $S = \{w \in H : \|w\| = 1\}$. We have the following elementary result.

Lemma 6.3.

$$\operatorname{dist}(K_i \cap S, K_j \cap S) = c(\beta)$$

for $i \neq j$, where $c(\beta) := \sqrt{2}(1-\sqrt{\beta})/\sqrt{1+\beta} > 0$.

Proof. Let $w_i \in K_i \cap S$, $w_j \in K_j \cap S$. Then we have

$$w_i = u_i + z_j + v,$$
$$w_j = u_j + z_i + \bar{v},$$

with $u_i, z_i \in L_i$, $u_j, z_j \in L_j$, $v, \bar{v} \in (L_i \oplus L_j)^{\perp}$ and

$$\|u_i\|^2 + \|z_j\|^2 + \|v\|^2 = \|u_j\|^2 + \|z_i\|^2 + \|\bar{v}\|^2 = 1, \tag{6.10}$$

$$\|z_j\|^2 + \|v\|^2 < \beta\|u_i\|^2, \quad \|z_i\|^2 + \|\bar{v}\|^2 < \beta\|u_j\|^2. \tag{6.11}$$

It follows from (6.10), (6.11) that

$$\|u_i\|^2 > \frac{1}{1+\beta}, \qquad \|u_j\|^2 > \frac{1}{1+\beta},$$
$$\|z_j\|^2 < \frac{\beta}{1+\beta}, \qquad \|z_i\|^2 < \frac{\beta}{1+\beta}.$$

Hence

$$\begin{aligned}\|w_i - w_j\|^2 &= \|u_i - z_i\|^2 + \|u_j - z_j\|^2 + \|v - \bar{v}\|^2 \\ &\geq (\|u_i\| - \|z_i\|)^2 + (\|u_j\| - \|z_j\|)^2 \\ &> \frac{2(1-\sqrt{\beta})^2}{1+\beta},\end{aligned}$$

and so $\operatorname{dist}(K_i \cap S, K_j \cap S) \geq c(\beta)$.

The opposite inequality follows on choosing $v = \bar{v} = 0$, $u_i = (1/\sqrt{1+\beta})e_i$, $u_j = (1/\sqrt{1+\beta})e_j$, $z_i = (\sqrt{\beta}/\sqrt{1+\beta})e_i$, $z_j = (\sqrt{\beta}/\sqrt{1+\beta})e_j$, and noting that $w_i \in \overline{K_i \cap S}$, $w_j \in \overline{K_j \cap S}$. □

Corollary 6.4. *The sets K_i are disjoint.*

Proof. If $w \in K_i \cap K_j$ for $i \neq j$ then $w/\|w\| \in K_i \cap K_j \cap S$. □

Let $\eta \in C^\infty([0,\infty))$ satisfy $\eta \geq 0$ and

$$\eta(\tau) = \begin{cases} 1 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \geq \frac{1}{3}c(\beta). \end{cases}$$

Define $h : S \to \mathbf{R}$ by

$$h(\theta) = 1 + \sum_{i=1}^{\infty}(i-1)\eta(\operatorname{dist}(\theta, K_i \cap S)).$$

Given $\theta \in S$, all the terms in the sum vanish in a neighbourhood of θ except perhaps one. Thus h is locally Lipschitz. Clearly $h \geq 1$ and $h(\theta) = i$ for $\theta \in K_i \cap S$.

We construct $S(t)$ through an ordinary differential equation

$$\dot{w} = F(w) \tag{6.12}$$

on $H\backslash\{0\}$, where $F : H\backslash\{0\} \to H$. For $w \notin \bigcup_{i=1}^{\infty} K_i$ we define

$$F(w) = -h\left(\frac{w}{\|w\|}\right)w.$$

To define F on each K_i we first define F on $L_i\backslash\{0\}$. Let $\psi_i \in C_0^\infty(\mathbf{R})$ with $0 \leq \psi_i \leq 1$, $\operatorname{supp}\psi_i \subset (1/i, 3/i)$, $\psi_i(2/i) = 1$, and let $\theta_i \in C_0^\infty(\mathbf{R})$ with $0 \leq \theta_i \leq 1$, $\operatorname{supp}\theta_i \subset (1/i, 4)$, $\theta_i(2/i) = 1$, $\theta_i' > -1$. Define $P_i : L_i \to L_i$ by

$$P_i(xe_i + y\hat{e}_i) = xe_i + (y + \psi_i(x)\theta_i(y))\hat{e}_i.$$

Since

$$\frac{\partial}{\partial y}(y + \psi_i(x)\theta_i(y)) = 1 + \psi_i(x)\theta_i'(y) > 0$$

it is easily seen that P_i is a diffeomorphism satisfying $P_i =$ identity if $x \notin (1/i, 3/i)$ or $y \notin (1/i, 4)$. Consider the ordinary differential equation

$$\dot{p} = -ip \tag{6.13}$$

on L_i, whose trajectories are given by

$$p(t) = \exp(-it)p_0, \qquad p_0 \in L_i,$$

and the ordinary differential equation

$$\dot{q} = f_i(q)$$

on L_i whose trajectories are given by

$$q(t) = P_i(p(t)). \tag{6.14}$$

Differentiating (6.14) and using (6.13) we see that

$$f_i(q) = -iDP_i(P_i^{-1}(q))P_i^{-1}(q).$$

Note that f_i has the form

$$f_i(xe_i + y\hat{e}_i) = -ixe_i + g_i(x,y)\hat{e}_i,$$

where g_i is smooth with $g_i(x,y) = -iy$ if $x \notin (1/i, 3/i)$ or $y \notin (1/i, 4)$.

Let $w \in K_i$, so that $w = xe_i + y\hat{e}_i + v$ with $(v, e_i) = (v, \hat{e}_i) = 0$ and $\|v\|^2 < \beta(x^2+y^2)$. Define

$$F(w) = -ixe_i + h_i(w)\hat{e}_i - iv,$$

where

$$h_i(w) := \left(1 - \frac{\|v\|^2}{\beta(x^2+y^2)}\right) g_i(x,y) - \frac{\|v\|^2}{\beta(x^2+y^2)} iy.$$

Note that $F(w) = f_i(w)$ if $w \in L_i \backslash \{0\}$, that $F(w) = -iw$ if $x \notin (1/i, 3/i)$ or $y \notin (1/i, 4)$, and that $F(w) + iw \to 0$ as $w \to \partial K_i \backslash \{0\}$.

We have thus defined $F : H\backslash\{0\} \to H$, and it is easily seen that F is locally Lipschitz on $H\backslash\{0\}$. If $w_0 \in H\backslash\{0\}$ there thus exists a unique continuous solution $w(t)$ of (6.12) satisfying $w(0) = w_0$, defined and remaining in $H\backslash\{0\}$ on a maximal time interval $[0, t_c)$, where $0 < t_c \leq \infty$. If $w_0 \notin \bigcup_{i=1}^{\infty} K_i$ then this solution is given by

$$w(t) = \exp\left(-h\left(\frac{w_0}{\|w_0\|}\right)t\right)w_0 \tag{6.15}$$

and so $t_c = \infty$. Also, since $h \geq 1$,

$$\|w(t)\| \leq \exp(-t)\|w_0\|. \tag{6.16}$$

Suppose $w_0 = x_0e_i + y_0\hat{e}_i + v \in K_i$. We have that

$$w(t) = x(t)e_i + y(t)\hat{e}_i + \exp(-it)v,$$

where $x(t) = \exp(-it)x_0$ and $\dot{y}(t) = h_i(w(t))$. By the backwards uniqueness of the solution (6.15) $w(t)$ cannot belong to $\partial K_i\backslash\{0\}$ for any t. Also $\dot{w}(t) = -iw(t)$ if $\|w(t)\| \leq \sqrt{2}/i$ or if $\|w(t)\| \geq 4\sqrt{2}$, and so $t_c = \infty$. If $x_0 \leq 1/i$ then $x(t) \leq 1/i$ for all $t \geq 0$ and thus $w(t) = \exp(-it)w_0$. So let $x_0 > 1/i$. Then $x(t_0) = 1/i$ where $t_0 = i^{-1}\ln(ix_0)$, and therefore

$$y(t) = \exp(-i(t-t_0))y(t_0) \qquad \text{for all } t \geq t_0.$$

But $h_i(w(t)) = -iy(t)$ if $|y(t)| \geq 4$, and so

$$|y(t_0)| \leq \max\{|y_0|, 4\}.$$

Hence if $t \geq i^{-1}\ln(ix_0)$

$$|y(t)| \leq \exp(-it)ix_0 \max\{|y_0|, 4\}$$

and thus

$$\begin{aligned}\|w(t)\|^2 &\le \exp(-2it)(|x_0|^2 + i^2|x_0|^2\max\{|y_0|^2, 16\} + \|v_0\|^2)\\ &\le i^2\exp(-2it)(17|x_0|^2 + |x_0|^2|y_0|^2 + \|v_0\|^2)\\ &\le i^2\exp(-2it)(17\|w_0\|^2 + \|w_0\|^4)\\ &\le \frac{1}{(te)^2}(17\|w_0\|^2 + \|w_0\|^4). \end{aligned} \tag{6.17}$$

For $t \ge 0$ we define $S(t)w_0 = w(t)$ if $w_0 \ne 0$ and $S(t)0 = 0$. Then the map $t \mapsto S(t)w_0$ is continuous on $[0,\infty)$ for all $w_0 \in H$, and from standard properties of ordinary differential equations we also have that $S(t)w_{0j} \to S(t)w_0$ whenever $w_{0j} \to w_0 \ne 0$. To show that $S(t)$ is continuous we must therefore show that $w_{0j} \to 0$ implies that $S(t)w_{0j} \to 0$. But if $t > 0$ and $w_0 = xe_i + y\hat{e}_i + v \in H$ with $\|w_0\| \le te$, we have that

$$\|w(t)\| \le \exp(-t)\|w_0\|$$

if $w_0 \notin \bigcup_{i=1}^\infty K_i$ (by (6.16)) or if $w_0 \in K_i$ with $x_0 \le 1/i$, and that (6.17) holds if $w_0 \in K_i$ with $x_0 > 1/i$, since $i^{-1}\ln(ix_0) \le i^{-1}\ln(ite) \le t$. Hence $S(t)w_{0j} \to 0$ if $w_{0j} \to 0$.

The same inequalities clearly imply that $\{0\}$ attracts bounded sets, and so it remains to show that the global attractor $A = \{0\}$ is not Lyapunov stable. Let $w_{0i} = 4i^{-1}(e_i + \hat{e}_i)$, $t_i = i^{-1}\ln 2$. Then by (6.14)

$$\begin{aligned}S(t_i)w_{0i} &= P_i\left(2i^{-1}(e_i + \hat{e}_i)\right)\\ &= 2i^{-1}e_i + (2i^{-1} + 1)\hat{e}_i\end{aligned}$$

and thus $\|S(t_i)w_{0i}\| > 1$ for all i.

7. The incompressible Navier-Stokes equations

Let $\Omega \subset \mathbf{R}^3$ be a bounded open set with boundary $\partial\Omega$. Let $f \in L^2(\Omega)^3$ and consider the incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u = \nu\Delta u - \nabla p + f, \tag{7.1a}$$

$$\operatorname{div} u = 0 \tag{7.1b}$$

with boundary condition

$$u|_{\partial\Omega} = 0, \tag{7.2}$$

where $\nu > 0$ is a constant. (Similar results to those below can be established for the more realistic case of the nonzero boundary condition $u|_{\partial\Omega} = U$ provided Ω is of class C^2 and that $U = \operatorname{curl} V$ for a sufficiently smooth V, using the well-known device of Hopf [25].) As is customary, we use the function spaces

$$\begin{aligned}\mathcal{V} &= \{u \in C_0^\infty(\Omega)^3; \operatorname{div} u = 0\},\\ H &= \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3,\\ V &= \{u \in H_0^1(\Omega)^3; \operatorname{div} u = 0\}.\end{aligned}$$

We denote by V' the dual space of V, and by H_w the space H endowed with its weak topology. We denote respectively by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)^3$, and for $u, v \in V$ write

$$(Du, Dv) = \int_\Omega \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}dx, \qquad \|Du\| = (Du, Du)^{\frac{1}{2}}.$$

For $u, v, w \in V$ we let

$$b(u,v,w) = \int_\Omega \sum_{i,j=1}^{3} u_j v_{i,j} w_i dx.$$

We say that $u : [0,\infty) \to H$ is a *weak solution* of (7.1), (7.2) if $u \in C([0,T]; H_w) \cap L^2(0,T;V)$, $du/dt \in L^1(0,T;V')$ for all $T > 0$, if

$$(7.3) \qquad (\frac{du}{dt}, v) + \nu(Du, Dv) + b(u,u,v) = (f,v) \quad \text{for a.e. } t > 0,$$

for all $v \in V$, and if u satisfies the energy inequality

$$(7.4) \qquad V(u)(t) \le V(u)(s) \text{ for all } t \ge s,$$

for a.e. $s \in (0,\infty)$ and for $s = 0$, where

$$(7.5) \qquad V(u)(t) := \frac{1}{2}\|u(t)\|^2 + \nu \int_0^t \|Du(\tau)\|^2 d\tau - \int_0^t (f, u(\tau))\, d\tau.$$

Standard theory [24, 28, 17, 42, 43] shows that given any $u_0 \in H$ there exists at least one weak solution with $u(0) = u_0$, constructed via a Galerkin method. Let G_{NS} denote the set of all weak solutions. G_{NS} satisfies (H1), (H3) but it is not known whether (H4) holds. (H2) would hold if $V(u)(t)$ were nonincreasing, but this does not follow directly from (7.4) which is consistent, for example, with the behaviour $V(u)(t) = 1$ for $t \in [0,1)$, $V(u)(1) = 0$, $V(u)(t) = a$ for $t \in (1,\infty)$, where $0 < a \le 1$. This undesirable behaviour cannot be eliminated simply by redefining the weak solution on a set of times of measure zero, since we have already chosen a representative which is continuous from $[0,\infty) \to H_w$.

In Proposition 7.4 below we show that G_{NS} is a generalized semiflow on H if and only if each weak solution u is continuous from $(0,\infty) \to H$. In preparation for this result we give in Proposition 7.3 a consequence of the energy inequality (7.4) that is well known to hold for any weak solution constructed via the Galerkin method; however, our proof does not assume that the weak solution is constructed in this way. We need two lemmas.

Lemma 7.1. *Let $\rho \in L^1_{loc}(0,\infty)$. Then the following conditions are equivalent:*
(i) ρ has a nonincreasing representative $\bar\rho : (0,\infty) \to \mathbf{R}$,
(ii) $\dot\rho \le 0$ in $\mathcal{D}'(0,\infty)$.

If in addition $\rho : [0,\infty) \to \mathbf{R}$ is lower semicontinuous and continuous at zero then (i) and (ii) are equivalent to
(iii) $\rho(t) \le \rho(s)$ for all $t \ge s$, for a.e. $s \in (0,\infty)$ and for $s = 0$.

Proof. The equivalence of (i) and (ii) is standard. For (i)⇒(ii) one takes $\varphi \in \mathcal{D}(0,\infty)$, $\varphi \ge 0$, and passes to the limit $h \to 0+$ in

$$\int_0^\infty \frac{\rho(t) - \rho(t+h)}{h} \varphi(t)\, dt = \int_0^\infty \rho(t) \frac{\varphi(t) - \varphi(t-h)}{h}\, dt \ge 0,$$

which is valid for $h > 0$ sufficiently small, while (ii)⇒(i) follows from mollifying ρ.

Suppose now that $\rho : [0,\infty) \to \mathbf{R}$ is lower semicontinuous, continuous at zero, and satisfies (i). Then $\rho(\tau) = \bar\rho(\tau)$ for all $\tau \notin N$, where N is a null set. Let $s > 0, s \notin N$, $t > s$ and $t_j \to t$ with $t_j \notin N$. Then

$$(7.6) \qquad \rho(t) \le \liminf_{j\to\infty} \rho(t_j) = \liminf_{j\to\infty} \bar\rho(t_j) \le \bar\rho(s) = \rho(s),$$

while if $s = 0$ we obtain $\rho(t) \leq \rho(0)$ by passing to the limit $s_k \to 0$ in (7.6) with $s_k \notin N$. Hence (iii) holds.

Conversely if (iii) holds then $\bar{\rho}(t) := \sup_{\tau \geq t} \rho(\tau)$ defines a nonincreasing representative of ρ. □

Lemma 7.2. *Let $\theta : [0,\infty) \to \mathbf{R}$ be lower semicontinuous, continuous at zero, $\theta \in L^1(0,T)$ for all $T > 0$, and let θ satisfy, for some constant $c \geq 0$,*

$$\theta(t) + c\int_0^t \theta(\tau)\,d\tau \leq \theta(s) + c\int_0^s \theta(\tau)\,d\tau \tag{7.7}$$

for all $t \geq s$, for a.e. $s > 0$ and for $s = 0$. Then

$$\theta(t)e^{ct} \leq \theta(s)e^{cs} \tag{7.8}$$

for all $t \geq s$, for a.e. $s > 0$ and for $s = 0$.

Proof. Let $\rho(t) = \theta(t) + c\int_0^t \theta(\tau)\,d\tau$. Then $\rho \in L^1_{\mathrm{loc}}(0,\infty)$, is lower semicontinuous and continuous at zero. Hence by Lemma 7.1, $\dot{\rho} \leq 0$ in $\mathcal{D}'(0,\infty)$. Hence $\dot{\theta} + c\theta \leq 0$ in $\mathcal{D}'(0,\infty)$ and so $\frac{d}{dt}(\theta e^{ct}) \leq 0$ in $\mathcal{D}'(0,\infty)$. The result then follows from Lemma 7.1 applied to θe^{ct}. □

Let λ_1 denote the lowest eigenvalue for the Stokes operator on Ω; thus

$$\|Dv\|^2 \geq \lambda_1 \|v\|^2 \quad \text{for all } v \in V. \tag{7.9}$$

Proposition 7.3. *Let u be a weak solution. Then*

$$\|u(t)\|^2 - \frac{1}{(\nu\lambda_1)^2}\|f\|^2 \leq e^{-\nu\lambda_1 t}\left(\|u(0)\|^2 - \frac{1}{(\nu\lambda_1)^2}\|f\|^2\right) \tag{7.10}$$

for all $t \geq 0$.

Proof. By (7.4)

$$\frac{1}{2}\|u(t)\|^2 + \nu\int_s^t \|Du(\tau)\|^2 d\tau \leq \frac{1}{2}\|u(s)\|^2 + \int_s^t \|f\| \cdot \|u(\tau)\|\,d\tau \tag{7.11}$$

for all $t \geq s$, for a.e. $s > 0$ and for $s = 0$. Using (7.9) and the inequality

$$\|f\| \cdot \|u(\tau)\| \leq \frac{1}{2}\left(\nu\lambda_1\|u(\tau)\|^2 + \frac{1}{\nu\lambda_1}\|f\|^2\right)$$

it follows that $\theta(t) := \|u(\tau)\|^2 - \frac{1}{(\nu\lambda_1)^2}\|f\|^2$ satisfies (7.7) with $c = \nu\lambda_1$. Further, since $u : [0,\infty) \to H_w$ is continuous it follows that θ is lower semicontinuous and hence also, from (7.7) with $s = 0$, continuous at zero. The result then follows from Lemma 7.2, taking $s = 0$ in (7.8). □

Proposition 7.4. *The following conditions are equivalent:*
(i) G_{NS} is a generalized semiflow on H.
(ii) Each weak solution u is continuous from $(0,\infty)$ to H.
(iii) Each weak solution u is continuous from $[0,\infty)$ to H.

Proof. (i)⇒(ii). Suppose G_{NS} is a generalized semiflow on H. Clearly G_{NS} has unique representatives and so by Theorem 2.1 we just need to show that each weak solution u is strongly measurable. But u is weakly continuous, hence weakly measurable, and so by a well-known result [22, p 73] u is strongly measurable.

(ii)⇒(iii). Let u be a weak solution that is continuous from $(0,\infty) \to H$. Let $t_j \to 0+$. Since $u \in C([0,T];H_w)$ for all $T > 0$ we have that $\|u(0)\| \le \liminf_{j\to\infty} \|u(t_j)\|$. But from (7.4) with $s = 0$ we have that $\limsup_{j\to\infty} \|u(t_j)\|^2 \le \|u(0)\|^2$, and so $\|u(t_j)\| \to \|u(0)\|$. Hence $u(t_j) \to u(0)$ in H strongly, as required.

(iii)⇒(i). Suppose each weak solution is continuous from $[0,\infty)$ to H. Then $V(u)(t)$ is continuous for $t \ge 0$ and hence $V(u)(t)$ is nonincreasing. In particular, (H2) holds. Also, if we define

$$\tilde{V}(u)(t) := \frac{1}{2}\|u(t)\|^2 - \int_0^t (f, u(\tau))\, d\tau,$$

then $\tilde{V}(u)(t)$ is continuous for $t \ge 0$ and nonincreasing in t.

Let $u^{(j)}$ be a sequence of weak solutions with $u^{(j)}(0) \to u_0$ in H. By Proposition 7.3 $u^{(j)}$ is bounded in $L^\infty(0,\infty;H)$, and thus from the energy inequality (7.4) $u^{(j)}$ is bounded in $L^2(0,T;V)$ for every $T > 0$. A standard estimate then shows that

$$du^{(j)}/dt \quad \text{is bounded in } L^{4/3}(0,T;V') \tag{7.12}$$

for every $T > 0$, and hence using the usual compactness results that for a diagonal subsequence, which we do not relabel, there exists $u : [0,\infty) \to H$ with $u \in C([0,T];H_w) \cap L^2(0,T;V)$ for all $T > 0$ such that

$$u^{(j)}(t) \rightharpoonup u(t) \text{ in } H \text{ for all } t \ge 0, \tag{7.13}$$

$$u^{(j)} \rightharpoonup u \text{ in } L^2(0,T;V) \text{ for all } T > 0, \tag{7.14}$$

$$du^{(j)}/dt \rightharpoonup du/dt \text{ in } L^{4/3}(0,T;V') \text{ for all } T > 0, \tag{7.15}$$

$$u^{(j)} \to u \text{ strongly in } L^2(0,T;H) \text{ for all } T > 0. \tag{7.16}$$

It follows from (7.16) that extracting a further subsequence we have that

$$u^{(j)}(t) \to u(t) \text{ in } H, \text{ a.e. } t > 0. \tag{7.17}$$

From (7.13)-(7.17) we deduce that u satisfies (7.3) and that $u(0) = u_0$. Also $\tilde{V}(u^{(j)})(s) \to \tilde{V}(u)(s)$ for a.e. $s > 0$ and for $s = 0$. If $t \ge s$ then since $V(u^{(j)})(t)$ is nonincreasing, by (7.13) and weak lower semicontinuity, we have that u satisfies the energy inequality (7.4). Hence u is a weak solution. Since, therefore, each $\tilde{V}(u^{(j)})(t)$ and $\tilde{V}(u)(t)$ are nonincreasing and continuous it follows that $\tilde{V}(u^{(j)})(t) \to \tilde{V}(u)(t)$ for *all* $t \ge 0$.

But this implies that $\|u^{(j)}(t)\| \to \|u(t)\|$ and so $u^{(j)}(t) \to u(t)$ in H. Hence (H4) holds and G_{NS} is a generalized semiflow. □

Corollary 7.5. *If G_{NS} is a generalized semiflow then it satisfies* (C4).

Proof. Let $u^{(j)}$ be as in the proof of Proposition 7.4. By Theorem 2.2 we have to show that if $t_j \to 0+$ then $u^{(j)}(t_j) \to u_0$ in H. We first show that $u^{(j)}(t_j) \rightharpoonup u_0$ in H. For $v \in V$ we have that

$$(u^{(j)}(t_j) - u^{(j)}(0), v) = \int_0^{t_j} (\frac{du^{(j)}}{dt}, v)\, d\tau. \tag{7.18}$$

Since $du^{(j)}/dt$ is bounded in $L^{4/3}(0,T;V')$, $(du^{(j)}/dt, v)$ is bounded in $L^{4/3}(0,T)$, and hence by Hölder's inequality the right-hand side of (7.18) tends to zero. Hence $(u^{(j)}(t_j), v) \to (u_0, v)$ for all $v \in V$, and since $u^{(j)}(t_j)$ is bounded in H it follows that $u^{(j)}(t_j) \rightharpoonup u_0$ in H.

To prove strong convergence, note that in fact the convergence of $V(u^{(j)})(t)$ to $V(u)(t)$ is uniform on compact subsets of $[0, \infty)$, and so $V(u^{(j)})(t_j) \to V(u_0)$. From this it follows that $\limsup_{j\to\infty} \|u^{(j)}(t_j)\| \leq \|u_0\|$, and the strong convergence follows. □

As an application of Theorem 3.3 we prove

Theorem 7.6. *Under the hypothesis that G_{NS} is a generalized semiflow there exists a global attractor for G_{NS}.*

Proof. By Proposition 7.4 G_{NS} is a generalized semiflow if and only if all weak solutions are continuous from $[0, \infty) \to H$. It then follows from Proposition 7.3 that G_{NS} is point dissipative. By Proposition 7.3 we also have that G_{NS} is eventually bounded. Using Proposition 3.2, to show that G_{NS} is asymptotically compact we thus need only show that G_{NS} is compact. But this follows from the argument in Proposition 7.4 (note that we do not need $u^{(j)}(0) \to u_0$ to conclude that $u^{(j)}(t) \to u(t)$ for all $t > 0$). □

The connectedness of the attractor for G_{NS}, assuming that G_{NS} is a generalized semiflow, depends on whether Kneser's property holds. This does not seem easy to prove.

It follows from Theorem 6.1, Corollary 7.5 that the attractor for G_{NS} (assuming that G_{NS} is a generalized flow) is asymptotically stable. However, this does not use the full strength of Theorem 6.1 since in this case (C3) holds.

It might still be true that there is a global attractor in H if there exist weak solutions that are not continuous from $(0, \infty) \to H$. However, such a global attractor would not exist if there was a complete orbit that was bounded but not continuous, since such an orbit would have to be contained in the global attractor and would not be relatively compact on account of the continuity of weak solutions into H_w. It might be possible to eliminate any discontinuous solutions from $(0, \infty) \to H$ by means of some unknown admissibility criterion; if this could be done in such a way that the resulting family of solutions $\hat{G}_{NS} \subset G_{NS}$ formed a generalized semiflow, then the above methods would guarantee the existence of a global attractor for $\hat{G}_{NS}$ in H.

The existence and properties of a global attractor have been studied by Constantin, Foias & Temam [18] under the *a priori* assumption that all solutions with initial data in V remain bounded in V for all finite times. The only results on the existence of a global attractor in H which do not make unsubstantiated assumptions on the solutions seem to be those of Raugel & Sell [32],[33],[34] for suitable thin 3D domains.

Foias & Temam [20] have introduced the concept of a *universal attractor* $\tilde{A}$ for the 3D Navier-Stokes equations. In the definition of a weak solution they drop the requirement that the energy equation (7.4) hold with $s = 0$ (as does Sell [39]). With this definition translates of weak solutions are weak solutions. Let us call the corresponding (possibly larger) set of weak solutions $\tilde{G}_{NS}$. For $\tilde{G}_{NS}$ the statements of our theorems need slight modification. Proposition 7.3 holds only for solutions that are continuous at zero, and Proposition 7.4 is replaced by the assertions (i) that $\tilde{G}_{NS}$ is a generalized semiflow implies that each weak solution is continuous from $(0, \infty) \to H$, and (ii) that if each weak solution is continuous from $[0, \infty) \to H$ (i.e. (C3) holds) then $\tilde{G}_{NS}$ is a generalized semiflow. Corollary 7.5 is dropped, and Theorem 7.6 is replaced by the statement that a global attractor exists for $\tilde{G}_{NS}$

provided (C3) holds. Foias & Temam then define $\tilde{A}$ as the set of all $u_0 \in H$ through which there passes a complete orbit $u : \mathbf{R} \to H$. If (C3) holds then it is easily seen that $\tilde{A}$ coincides with the global attractor A in Theorem 7.6. It does not seem to be obvious without making *a priori* assumptions on solutions that $\tilde{A}$ attracts bounded sets in the sense of weak convergence in H, i.e. that if $u^{(j)}$ are weak solutions with $u^{(j)}(0)$ bounded, and if $t_j \to \infty$, then $u^{(j)}(t_j)$ has a subsequence converging to a point of $\tilde{A}$. This is because the estimate (7.10) is not proved, so that it is not even clear that $u^{(j)}(t_j)$ is bounded. This weak attraction property would hold, however, if we knew that $V(u)(t)$ were nonincreasing for all weak solutions.

Acknowledgements.
I am grateful to G.R. Sell and Y. You for providing me with prepublication versions of [40], and to P. Constantin, J.K. Hale and J. Howie for helpful comments. This version of the paper incorporates a modification to the proofs of Proposition 7.4 and Corollary 7.5 (see Erratum, J. Nonlinear Science 8(1998)p.233). I am grateful to G. Francfort for querying this point.

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Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford OX1 3LB, U.K.

E-mail address: `ball@maths.ox.ac.uk`