JOURNAL OF DIFFERENTIAL EQUATIONS 14, 399–418 (1973)

Stability Theory for an Extensible Beam

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Received February 15, 1972; revised March 21, 1973

1. INTRODUCTION

In this paper we use topological methods to study the asymptotic properties of the equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left(\beta + k \int_0^l \left[\frac{\partial u(\xi, t)}{\partial \xi}\right]^2 d\xi\right) \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^5 u}{\partial x^4 \, \partial t}$$
$$-\sigma \int_0^l \frac{\partial u}{\partial \xi} \frac{\partial^2 u}{\partial \xi \, \partial t} \, d\xi \, \frac{\partial^2 u}{\partial x^2} + \delta \frac{\partial u}{\partial t} = 0, \tag{1.1}$$

where the constants α, k, γ, σ are positive, and where the constants β and δ are unrestricted in sign.

When $\gamma = \sigma = \delta = 0$, (1.1) reduces to the equation studied in [1] and introduced by Woinowsky-Krieger as a model for the transverse motion of an extensible beam whose ends are held a fixed distance apart. Here we modify the model by introducing terms to account for the effects of internal (structural) and external damping. Specifically, we assume that the beam is linearly viscoelastic and that the (possibly negative) external damping is proportional to the velocity. Our methods, however, can cater for a variety of damping terms. In particular they may be simply adapted to the case when $\gamma = \sigma = 0$ and $\delta > 0$ in (1.1). This is a situation studied recently by Reiss and Matkowsky [20], who used a formal "two-time" asymptotic expansion method.

The relative simplicity and tractability of (1.1) make it a useful prototype for the study of more complex stability problems. Our aim is to show that as time t tends to infinity, provided δ is not large and negative, any solution of (1.1) converges in a suitable topology to an equilibrium position of the beam. When the beam is constrained to lie along the x-axis, an axial force H

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is set up. (H is taken as positive when tensile.) If $H \geqslant H_E$, the Euler load of the beam, then the only equilibrium position is the trivial one. In this case we give an elementary proof that if $\delta > \delta_0$ any solution converges to the trivial position in the energy norm. The constant δ_0 is negative and depends on the boundary conditions. When $H < H_E$, however, there are $2n + 1$ equilibrium positions for some $n \geqslant 1$ and there is no obvious criterion for determining to which state a particular solution will converge—a glance at Fig. 4 of [20] will confirm this. Following a suggestion given in conversation by Professor R. J. Knops, we use for this case an invariance principle in the theory of dynamical systems on a Banach space.

If an orbit in a dynamical system on a reflexive Banach space B belongs to a compact set then this ensures that the limit set is nonempty and invariant. In practice, as pointed out by Hale [8], it is easier to prove that a given orbit is *bounded* for all time. If a smaller Banach space C is compactly embedded in B then bounded sets in C are precompact in B. Hale exploited this fact to prove a Lyapunov-type theorem for this situation. One disadvantage of Hale's method is that it gives information only on the smoother orbits in B, those that belong to C. An alternate approach used by Slemrod [21] is based on the fact that bounded sets in B belong to weakly compact sets. Thus, if a dynamical system is continuous in the weak topology a bounded orbit will possess a nonempty and invariant weak limit set. Slemrod proved a theorem analogous to Hale's for the case when a weakly continuous Lyapunov function exists. In our case we prove using the methods of Lions [15] that (1.1) defines a weakly continuous dynamical system on a suitable Banach space. However, the natural Lyapunov function, the total energy of the beam, unfortunately fails to be weakly continuous, and so we are unable to use Slemrod's result. As an example of his theory, Slemrod considered a modification of Van der Pol's nonlinear partial differential equation, for which the natural Lyapunov function is also not weakly continuous. (The treatment of this equation in [8] contains algebraic errors.) By putting the equation into a canonical form Slemrod nevertheless found a weakly continuous Lyapunov function. Such a method seems particularly difficult when, as in (1.1), the space part of the equation is nonlinear and when one cannot guess the final state of an orbit. Our method involves studying functions representing energy lost through dissipation. It turns out to be simple to prove that if $\delta > \delta_0$ any orbit converges weakly to an equilibrium position. This is a similar result to Slemrod's for the modified Van der Pol equation. We then show that convergence to an equilibrium position also takes place in the energy norm. We would not have to consider this point if we restricted attention to orbits in a finite-dimensional space, which is the effect of the assumptions of hinged ends and "quiescent" initial data made by Reiss and Matkowsky. For the hinged beam we show that if $\delta < \delta_0$ and $H \geqslant H_E$ then there is at least one periodic solution to (1.1).

It is shown by Dafermos [5] that for a ***uniform*** dynamical system on a Banach space weak lower semicontinuity of a Lyapunov functional may be exploited to determine the structure of the weak limit set. We cannot use his results since our dynamical system is nonuniform.

One result we prove is that for the case $H < H_E$, $\delta > \delta_0$ the two equilibrium states which minimize the potential energy of the beam are (dynamically) stable. See in this connection Dickey [6]. In further work we have for some cases proved that orbits in the regions of attraction of the remaining equilibrium states are Lyapunov unstable.

A number of interesting papers by Hsu [9–12] are relevant to our work. Hsu develops, and applies to shallow arch problems, sufficiency criteria for dynamic stability based on the concept of expanding level surfaces of energy about an equilibrium position. It seems that a result along the lines of Lemma 4 is needed to apply these criteria to shallow arches. Hsu uses his results to derive bounds on the magnitude of impulsive loads necessary to prevent snap through.

It is also of interest to compare our study with the paper by Chafee and Infante [3]. These authors study a nonlinear partial differential equation of parabolic type and use Hale's invariance principle to prove that any solution converges to an equilibrium state as time t tends to infinity. They then study the stability properties of each equilibrium state.

The rest of this paper is divided into eight sections. In Section 2 we establish some notation. In Section 3 we discuss the model and in Section 4 prove existence, uniqueness, and regularity theorems for the case when the ends of the beam are clamped. In Section 5 we discuss weak dynamical systems and in Section 6 prove that (1.1) defines such a system. Section 7 contains our main results. In Section 8 we consider the case when the ends of the beam are hinged and in Section 9 discuss the stability of the equilibrium positions of the beam.

2. Preliminaries

Let Ω be the open interval $]0, l[$ of $\mathbf{R}^1$, where $l > 0$ is the natural length of the beam. Write $Q = \Omega \times]0, T[$, where $T > 0$ is fixed. Let $\mathbf{R}^+$ denote the interval $[0, \infty)$.

We shall need explicitly or implicitly the following spaces; their definitions are well known and are given in Section 2 [1].

Spaces of continuous functions: $C^m(\Omega)$, $C^\infty(\Omega)$.

Spaces of integrable functions: $L^2(\Omega)$, $L^2(Q)$, $L^\infty(0, T)$, $L^p(0, T; X)$ for X a Banach space.

Sobolev spaces: $H^m(\Omega)$, $H^m(Q)$, $H_0{}^m(\Omega)$.

Spaces of distributions: $\mathscr{D}(\Omega)$, $\mathscr{D}'(\Omega)$, $\mathscr{D}(]0, T[)$, $\mathscr{D}'(]0, T[)$, $\mathscr{D}'(0, T; X)$ for X a Banach space.

Define $C^m(\bar{\Omega}) \times C^n([0, T]) \equiv \{f = f(x, t)$: for each $x_0 \in \bar{\Omega}$ $f(x_0, \cdot) \in C^n([0, T])$, and for each $t_0 \in [0, T]$ $f(\cdot, t_0) \in C^m(\bar{\Omega})\}$.

By $L^\infty(\mathbf{R}^+)$ we mean the space of essentially bounded real-valued measurable functions on $\mathbf{R}^+$. Throughout we write

$$|f| = \|f\|_{L^2(\Omega)} = \left(\int_0^l (f(x))^2 \, dx\right)^{1/2}$$

and

$$(f, g) = \int_0^l f(x) g(x) \, dx.$$

Weak convergence in a Banach space is written $\rightharpoonup$, while weak star convergence is written $\overset{*}{\rightharpoonup}$.

From now on derivatives may be denoted by

$$\frac{\partial}{\partial t}(\ ) = (\dot{\ }) \qquad \text{and} \qquad \frac{\partial}{\partial x}(\ ) = (\ )'.$$

Constants are denoted generically by C and C_i $(i = 1, 2, \ldots)$.

3. The Model

Consider transverse motion, at small strains, in the $X - Y$ plane, of a linear viscoelastic beam in a viscous medium whose resistance is proportional to the velocity. We neglect rotational inertia and shear deformation. In the reference, stress-free state the beam occupies the interval $[0, l]$ of the X-axis. The ends are then fixed at $(0, 0)$ and $(l + \Delta, 0)$. Let an arbitrary point P of the neutral axis, whose position is $(x, 0)$ in the reference state, be displaced to $(x + \omega, u)$. For our model the axial force N and the bending moment M are given by

$$N = \frac{EA\Delta}{l} + \frac{EA}{2l} |u'|^2 + \frac{A\eta}{l} (u', \dot{u}'), \tag{3.1}$$

$$M = -EIu'' - \eta I\dot{u}'', \tag{3.2}$$

where E is the Young's modulus, A the cross-sectional area, η the effective viscosity, and I the cross-sectional second moment of area. The equation of motion in the Y-direction is

$$\rho\ddot{u} + EIu'''' + \eta I\dot{u}'''' - \left[\frac{EA\Delta}{l} + \frac{EA}{2l}\,|\,u'\,|^2 + \frac{A\eta}{l}\,(u', \dot{u}')\,u''\right] + \rho\delta\dot{u} = 0, \tag{3.3}$$

which is (1.1) with $\alpha = EI/\rho$, $\beta = EA\Delta/l\rho$, $\gamma = \eta I/\rho$, $k = EA/2l\rho$ and $\sigma = A\eta/l\rho$. In (3.3) ρ is the mass per unit length in the reference configuration and δ the coefficient of external damping.

(3.3) is equivalent to the model used by Huang and Nachbar [13]. A derivation of (3.1)–(3.3) can be made following the treatment of Mettler [17] and by reference to [13]. Detail is given in the author's thesis [2].

One unfortunate feature of the model is that the axial force N is independent of x so that effects are transmitted instantaneously along the beam.

Our aim is to study the initial-boundary value problem consisting of (1.1), the initial conditions

$$u(o) = u_0\,, \qquad \dot{u}(0) = u_1\,, \tag{3.4}$$

and the boundary conditions corresponding either to hinged ends, when

$$u = u'' = 0 \qquad \text{at} \qquad x = 0,\ l, \tag{3.5}$$

or to clamped ends, when

$$u = u' = 0 \qquad \text{at} \qquad x = 0,\ l. \tag{3.6}$$

(3.5) is a sufficient, though not necessary, condition for a smooth enough u to have zero bending moment at $x = 0, l$.

The equilibrium states of the beam have been studied by, for example, Reiss [19] and satisfy $\alpha u'''' = (\beta + k\,|\,u'\,|^2)u''$ subject to either (3.5) or (3.6). Any nonzero equilibrium position v_j is an eigenfunction satisfying $\alpha v_j'''' + \lambda_j v_j'' = 0$ subject to the relevant boundary conditions, where $|\,v_j'\,|^2 = -(\beta + \lambda_j)/k$. The positive sequence $\{\lambda_j\}$ is strictly increasing and has no finite accumulation point. So if $\beta \geqslant -\lambda_1$, there are no nonzero equilibrium positions, while if $\beta < -\lambda_1$, there are $2n$ such corresponding to those $\lambda_j < -\beta$. $\lambda_1 = \alpha\pi^2/l^2$ or $4\alpha\pi^2/l^2$ for hinged or clamped ends, respectively. Define the *load* H by $H = EA\Delta/l$, and the *Euler buckling load* H_E by $H_E = -\lambda_1\rho$. Then the conditions $\beta \geqslant -\lambda_1$ and $\beta < -\lambda_1$ are equivalent to $H \geqslant H_E$ and $H < H_E$, respectively.

The potential energy $V(u)$ of the beam when the deflection equals u is

$$V(u) = (\alpha/2)|\,u''\,|^2 + (\beta/2)|\,u'\,|^2 + (k/4)|\,u'\,|^4. \tag{3.7}$$

Therefore, for $H < H_E$,

$$V(v_j) = -(\beta + \lambda_j)^2/4k, \qquad j = 1, 2, \ldots, n. \tag{3.8}$$

It is easy to prove that the zero position ($H \geqslant H_E$) and v_1, $-v_1$ ($H < H_E$) minimize V. If $H < H_E$ and $j > 1$ then neither v_j nor $-v_j$ minimize V locally, since a little calculation shows that

$$V(v_j + \epsilon v_1) - V(v_j) = \frac{\epsilon^2}{2} \mid v_1' \mid^2 (\lambda_1 - \lambda_j) + \frac{\epsilon^4 k}{4} \mid v_1' \mid^4, \tag{3.9}$$

which is negative if ϵ is small enough. Similarly, if $H < H_E$, the zero position does not minimize V locally.

4. Existence, Uniqueness, and Regularity

For brevity, in the next four sections we restrict ourselves to the case of clamped ends.

In this section we assume that α, k, σ, k are positive but do not restrict δ, thus admitting the possibility of large negative external damping. The proofs follow closely those of [1] and in places implicit use is made of certain lemmas in that paper. We restate one such lemma (due to Wilcox) as we will need it in a different context later.

Lemma 1. *Let X be a Banach space. If $f \in L^2(0, T; X)$ and $\dot{f} \in L^2(0, T; X)$, then f, possibly after redefinition on a set of measure zero, is continuous from $[0, T] \to X$. Indeed, for almost all s, $t \in [0, T]$*

$$f(t) - f(s) = \int_s^t \dot{f}(\sigma)\, d\sigma.$$

If X is a reflexive Banach space, $\tilde{X}$ denotes X endowed with the weak topology. The following lemma is an immediate consequence of Lemma 1 and a result of Lions and Magenes [16, p. 297, Lemma 8.1].

Lemma 2. *Let X, Y be Banach spaces with X continuously embedded in Y. Suppose X is reflexive. If $f \in L^\infty(0, T; X)$ and $\dot{f} \in L^2(0, T; X)$ then f is a continuous map of $[0, T]$ into $\tilde{X}$.*

In our first two theorems we prove the existence of weak solutions to the initial-boundary value problem and show that these solutions have continuity properties with respect to time and the initial data.

THEOREM 1. *Suppose* $u_0 \in H_0^2(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then there exists a function* $u \equiv u(x, t)$ *with*

$$u \in L^\infty(0, T; H_0^2(\Omega)) \quad \text{and} \quad u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T, H_0^2(\Omega)),$$

such that u *satisfies the initial conditions* (3.4) *and Eq.* (1.1) *in the sense that*

$$(\ddot{u}, \phi) + \alpha(u'', \phi'') - (\beta + k \mid u' \mid^2)(u'', \phi) + \gamma(\dot{u}'', \phi'') - \sigma(u', \dot{u}')(u'', \phi) + \delta(\dot{u}, \phi) = 0 \quad \text{for all} \quad \phi \in H_0^2(\Omega). \tag{4.1}$$

Proof. Let $\{\omega_j\}$ be a C^∞ basis of $H_0^2(\Omega)$. Consider the approximating equations

$$(\ddot{u}_m, \omega_j) + \alpha(u''_m, \omega''_j) - (\beta + k \mid u_m' \mid^2)(u''_m, \omega_j) + \gamma(\dot{u}''_m, \omega''_j) - \sigma(u_m', \dot{u}_m')(u''_m, \omega_j) + \delta(\dot{u}_m, \omega_j) = 0, \qquad 1 \leqslant j \leqslant m, \tag{4.2}$$

where

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, \omega_i, \tag{4.3}$$

$$u_m(0) \equiv u_{m0} = \sum_{i=1}^{m} \alpha_{im} \omega_i \to u_0 \text{ in } H_0^2(\Omega), \tag{4.4}$$

$$\dot{u}_m(0) \equiv u_{m1} = \sum_{i=1}^{m} \beta_{im} \omega_i \to u_1 \text{ in } L^2(\Omega). \tag{4.5}$$

These equations have a solution u_m, valid in a subinterval $[0, t_m]$ of $[0, T]$, which satisfies the energy equation

$$\mid \dot{u}_m(t)\mid^2 + \alpha \mid u''_m(t)\mid^2 + \beta \mid u_m'(t)\mid^2 + (k/2) \mid u_m'(t)\mid^4 + 2\gamma \int_0^t \mid \dot{u}''_m(s)\mid^2 ds + 2\sigma \int_0^t (u_m'(s), \dot{u}_m'(s))^2\, ds + 2\delta \int_0^t \mid \dot{u}_m(s)\mid^2 ds = 2E_{0m}, \tag{4.6}$$

where

$$E_{0m} = \tfrac{1}{2} \mid u_{1m} \mid^2 + (\alpha/2) \mid u''_{0m} \mid^2 + (\beta/2) \mid u'_{0m} \mid^2 + (k/4) \mid u'_{0m} \mid^4.$$

Since $\gamma, \sigma > 0$

$$\begin{aligned} \mid \dot{u}_m(t)\mid^2 + \alpha \mid u''_m(t)\mid^2 &\leqslant 2E_{0m} + \int_0^t [\beta(u''_m, \dot{u}_m) - 2\delta \mid \dot{u}_m \mid^2]\, ds \\ &\leqslant 2E_{0m} + C \int_0^t [\mid \dot{u}_m \mid^2 + \alpha \mid u''_m \mid^2]\, ds. \end{aligned}$$

Gronwall's lemma and (4.6) now give the bounds

$$|\dot{u}_m|,\ |u_m''|,\ \int_0^t |\dot{u}_m''|^2\, ds < C \text{ (independent of } m \text{ and } t \in [0, T]). \quad (4.7)$$

(4.7) ensures that $t_m = T$ and that

$$\{u_m\} \text{ is bounded in } L^\infty(0, T; H_0^2(\Omega)),$$

$$\{\dot{u}_m\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^2(\Omega)).$$

We may thus extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ such that

$$u_\mu \xrightarrow{*} u \text{ in } L^\infty(0, T; H_0^2(\Omega)),$$

$$\dot{u}_\mu \xrightarrow{*} \dot{u} \text{ in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^2(\Omega)),$$

$$u_\mu \longrightarrow u \text{ in } L^2(Q) \text{ strongly and a.e.},$$

$$|u_\mu'|^2 u_\mu'' \xrightarrow{*} \aleph_1 \text{ in } L^\infty(0, T; L^2(\Omega)),$$

and

$$(u_\mu', \dot{u}_\mu')\, u_\mu'' \xrightarrow{*} \aleph_2 \text{ in } L^\infty(0, T; L^2(\Omega)).$$

That $\chi_1 = |u'|^2 u''$ follows as in [1]. To prove that $\chi_2 = (u', \dot{u}')u''$ first note that

$$(u_\mu', \dot{u}_\mu') = -(u'', \dot{u}_\mu) - (u_\mu - u, \dot{u}_\mu''). \quad (4.8)$$

Now $(u'', \dot{u}_\mu) \to (u'', \dot{u})$ in $L^\infty(0, T)$, while

$$\int_0^T |(u_\mu - u, \dot{u}_\mu'')|\, dt \leqslant \left(\int_0^T |u_\mu - u|^2\, dt\right)\left(\int_0^T |\dot{u}_\mu''|^2\, dt\right)^{1/2}$$

$$\leqslant C \left(\int_0^T |u_\mu - u|^2\, dt\right)^{1/2} \text{ by (4.7).}$$

Hence, $(u_\mu - u, \dot{u}_\mu'') \to 0$ in $L^1(0, T)$ and so from (4.8)

$$(u_\mu', \dot{u}_\mu') \to (u', \dot{u}') \qquad \text{in} \quad L^1(0, T). \quad (4.9)$$

Now let $\theta \in L^1(0, T; L^2(\Omega))$. Then

$$\int_0^T (u', \dot{u}')(u'', \theta)\, dt = \int_0^T (u_\mu', \dot{u}_\mu')(u_\mu'', \theta)\, dt$$

$$+ \int_0^T [(u', \dot{u}') - (u_\mu', \dot{u}_\mu')](u_\mu'', \theta)\, dt$$

$$+ \int_0^T (u'' - u_\mu'', (u', \dot{u}')\theta)\, dt. \quad (4.10)$$

The second and third integrals on the right side of (4.10) tend to zero as $\mu \to \infty$ by (4.7) and (4.9). The arbitrariness of θ implies that $\chi_2 = (u', \dot{u}')u''$. Thus, passing to the limit in (4.2) we obtain for any j

$$(\ddot{u}, \omega_j) + \alpha(u'', \omega_j'') - (\beta + k \mid u' \mid^2)(u'', \omega_j) + \gamma(\dot{u}'', \omega_j'') - \sigma(u', \dot{u}')(u'', \omega_j) + \delta(\dot{u}, \omega_j) = 0. \quad (4.11)$$

(4.1) follows from the denseness of $\{\omega_j\}$ in $H_0^2(\Omega)$.

As $u_\mu \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T; L^2(\Omega))$ and $\dot{u}_\mu \overset{*}{\rightharpoonup} \dot{u}$ in $L^\infty(0, T; L^2(\Omega))$, it follows from Lemma 1 that

$$(u_{\mu 0}, \phi) \to (u(0), \phi) \qquad \text{for all} \quad \phi \in L^2(\Omega).$$

Our assumptions on $u_{\mu 0}$ imply that $u_{\mu 0} \rightharpoonup u_0$ in $H_0^2(\Omega)$, and so $u(0) = u_0$. From (4.11), $(\ddot{u}_\mu, \omega_j) \rightharpoonup (\ddot{u}, \omega_j)$ in $L^2(0, T)$. So from Lemma 1 with $X = \mathbf{R}$, $(\dot{u}_\mu(0), \omega_j) \to (u(0), \omega_j)$. But by assumption $u_{\mu 1} \rightharpoonup u_1$ in $L^2(\Omega)$, and so $\dot{u}(0) = u_1$. □

THEOREM 2. *Let u be as in Theorem 1, and let v be another such weak solution with initial conditions $v(0) = v_0$, $\dot{v}(0) = v_1$ where $v_0 \in H_0^2(\Omega)$ and $v_1 \in L^2(\Omega)$. Set $y = u - v$. Then*

$$| \dot{y}(t)|^2 + \alpha \mid y''(t)|^2 + 2\gamma \int_0^t \mid \dot{y}'' \mid^2 ds \leqslant [\mid u_1 - v_1 \mid^2 + \alpha \mid u_0'' - v_0'' \mid^2] \exp(Kt), \quad (4.12)$$

where K is a continuous function of T, $| u_0'' |$, $| u_1 |$, $| v_0'' |$ and $| v_1 |$. In particular the solution u in Theorem 1 is unique. Furthermore, the functions u: $[0, T] \to H_0^2(\Omega)$ and $\dot{u}$: $[0, T] \to L^2(\Omega)$ are continuous and satisfy the energy equation

$$E(t) + \gamma \int_0^t \mid \dot{u}'' \mid^2 ds + \sigma \int_0^t (u', \dot{u}')^2 ds + \delta \int_0^t \mid \dot{u} \mid^2 ds = E(0), \quad (4.13)$$

where $E(t) \equiv E(u(t)) = \frac{1}{2} \mid \dot{u}(t)|^2 + (\alpha/2)| u''(t)|^2 + (\beta/2)| u'(t)|^2 + (k/4)| u'(t)|^4$.

Proof. From (4.1), u satisfies

$$\ddot{u} + \alpha u'''' - (\beta + k \mid u' \mid^2)u'' + \gamma \dot{u}'''' - \sigma(u', \dot{u}')u'' + \delta \dot{u} = 0. \quad (4.14)$$

In (4.14), $\alpha u'''' \in L^\infty(0, T; H^{-2}(\Omega))$ and $\gamma \dot{u}'''' \in L^2(0, T; H^{-2}(\Omega))$, and so $\ddot{u} \in L^2(0, T; H^{-2}(\Omega))$. Similarly, v satisfies

$$\ddot{v} + \alpha v'''' - (\beta + k \mid v' \mid^2)v'' + \gamma \dot{v}'''' - \sigma(v', \dot{v}')v'' + \delta \dot{v} = 0 \quad (4.15)$$

with $\ddot{v} \in L^2(0, T; H^{-2}(\Omega))$. Subtract (4.15) from (4.14) and take the inner product with $\dot{y}$, which is possible since $\dot{y} \in L^2(0, T; H_0^2(\Omega))$. Thus,

$$(\ddot{y}, \dot{y}) + \alpha(y'', \dot{y}'') - \beta(y'', \dot{y}) + \gamma \mid \dot{y}'' \mid^2 + \delta \mid \dot{y} \mid^2 \\ - k(\mid u' \mid^2 u'' - \mid v' \mid^2 v'', \dot{y}) - \sigma((u', \dot{u}')u'' - (v', \dot{v}')v'', \dot{y}) = 0. \quad (4.16)$$

But $k(\mid u' \mid^2 u'' - \mid v' \mid^2 v'', \dot{y}) \leqslant C \mid y'' \mid \mid \dot{y} \mid$ and

$$\sigma((u', \dot{u}')u'' - (v', \dot{v}')v'', \dot{y}) = -\sigma(v'', \dot{y})^2 - \sigma(v'', \dot{y})(\dot{u}, y'') - \sigma(\dot{u}, u'')(y'', \dot{y}) \\ \leqslant -\sigma(v'', \dot{y})^2 + C \mid \dot{y} \mid \mid y'' \mid.$$

Thus,

$$d/dt\, [\tfrac{1}{2} \mid \dot{y} \mid^2 + (\alpha/2 \mid y'' \mid^2] + \gamma \mid \dot{y}'' \mid^2 + \sigma(v'', \dot{y})^2 \leqslant C(\mid \dot{y} \mid^2 + \alpha \mid y'' \mid^2).$$

(4.12) now follows from Lemma 1 and Gronwall's lemma. The energy equation (4.13) follows from (4.16) with $v \equiv 0$, and (4.13) implies that K is a continuous function of $\mid u_0'' \mid$, $\mid u_1 \mid$, $\mid v_0'' \mid$, and $\mid v_1 \mid$. Setting $u_0 = v_0$, $u_1 = v_1$ in (4.12) shows that u is unique.

Since $u, \dot{u} \in L^2(0, T; H_0^2(\Omega))$, it follows from Lemma 1 that u is continuous from $[0, T] \to H_0^2(\Omega)$. It remains to show that $\dot{u}$ is continuous from $[0, T] \to L^2(\Omega)$. Let $t_n \to t$. Since $\ddot{u} \in L^2(0, T; H^{-2}(\Omega))$ and $\dot{u} \in L^\infty(0, T; L^2(\Omega))$, it follows from Lemma 2 that $\dot{u}(t_n) \rightharpoonup \dot{u}(t)$ in $L^2(\Omega)$. But from (4.13) and the fact that $u: [0, T] \to H_0^2(\Omega)$ is continuous, it is clear that $\mid \dot{u}(t_n) \mid \to \mid \dot{u}(t) \mid$. Therefore, $\dot{u}(t_n) \to \dot{u}(t)$ in $L^2(\Omega)$. □

Remark. The structural damping terms make the proof of Theorem 2 much simpler than for the case of the undamped beam [1], where a regularization argument was used to establish (4.12) and (4.13). A regularization argument is still necessary, however, for the case $\alpha = \sigma = 0$, $\delta \neq 0$.

We next prove a regularity result. Let $X = H_0^2(\Omega) \cap H^4(\Omega)$.

THEOREM 3. *Suppose $u_0 \in X$ and $u_1 \in H_0^2(\Omega)$. Then there exists a unique function $u \equiv u(x, t)$ with*

$$u \in L^\infty(0, T; X),$$

$$\dot{u} \in L^\infty(0, T; H_0^2(\Omega)) \cap L^2(0, T; X),$$

and

$$\ddot{u} \in L^2(0, T; L^2(\Omega)),$$

such that u satisfies the initial conditions (3.4) *and the equation*

$$\ddot{u} + \alpha u'''' - (\beta + k \mid u' \mid^2)u'' + \gamma \dot{u}'''' - \sigma(u', \dot{u}')u'' + \delta \dot{u} = 0 \quad (4.17)$$

in $L^2(0, T; L^2(\Omega))$.

Proof. Let $\{\omega_j\}$ consist of the eigenfunctions of $\omega'''' = \lambda\omega$ subject to $\omega = \omega' = 0$.

$\{\omega_j\}$ is a basis of X (cf. [1, Theorem 11]). The approximating solutions u_m are of the form (4.3) and satisfy (4.2) and the initial conditions

$$u_m(0) \equiv u_{m0} \to u_0 \quad \text{in } X \text{ and}$$

$$\dot{u}_m(0) \equiv u_{m1} \to u_1 \quad \text{in } H_0^2(\Omega).$$

The bounds (4.7) are satisfied. Multiply (4.2) by $\lambda_j \dot{g}_{jm}(t)$ and sum for $j = 1,..., m$. Then

$$\tfrac{1}{2}\, d/dt\,(|\dot{u}_m''|^2 + \alpha\,|u_m''''|^2) + \gamma\,|\dot{u}_m''''|^2 = (\beta + k\,|u_m'|^2)(u_m'', \dot{u}_m'''') + \sigma(u_m', \dot{u}_m')(u_m'', \dot{u}_m'''') - \delta\,|\dot{u}_m''|^2.$$

Therefore, using (4.7), it follows that

$$|\dot{u}_m''(t)|^2 + \alpha\,|u_m''''(t)|^2 + 2\gamma \int_0^t |\dot{u}_m''''|^2\, ds \leqslant C_1 + C_2 \int_0^t |(u_m'', \dot{u}_m'''')|\, ds$$
$$\leqslant C_1 + C_3 \int_0^t |u''|^2\, ds + \gamma \int_0^t |\dot{u}_m''''|^2\, ds.$$

Hence,

$$|\dot{u}_m''|,\ |u_m''''|,\ \int_0^t |\dot{u}_m''''|^2\, ds < C. \tag{4.18}$$

Proceeding as in Theorem 1, we establish the existence of a unique u satisfying (4.17) and (3.4) such that $u \in L^\infty(0, T; X)$ and $\dot{u} \in L^\infty(0, T; H_0^2(\Omega)) \cap L^2(0, T; X)$. It follows from (4.17) that $\dot{u} \in L^2(0, T; L^2(\Omega))$. □

Remark. For the case $\gamma = \sigma = 0$ it is possible to prove a similar theorem using the methods of [1, Theorem 9]. For that case one can proceed further (following [1]) and establish the existence of a classical solution if the initial data is smooth enough and satisfies the appropriate compatibility conditions. However, the method of choosing a special basis used in [1] appears to break down for γ, $\sigma > 0$.

5. Weak Dynamical Systems

By a *dynamical system on a Banach space B* we mean a function $w : \mathbf{R}^+ \times B \to B$ which satisfies

(i) $w^t : \phi \to w(t, \phi)$ is continuous for fixed $t \in \mathbf{R}^+$,

(ii) $w^\phi : t \to w(t, \phi)$ is continuous for fixed $\phi \in B$,

(iii) $w(0, \phi) = \phi$ for all $\phi \in B$, and

(iv) (semigroup property) $w(t + \tau, \phi) = w(t, w(\tau, \phi))$ for all $t, \tau \in \mathbf{R}^+$, $\phi \in B$.

Here (i) and (ii) replace the requirement of joint continuity in t and ϕ in the definition of a dynamical system used by Hale [8].

Let $\Sigma = H_0^2(\Omega) \times L^2(\Omega)$. Σ is a Hilbert space under the "energy" norm

$$\|\{\psi, \chi\}\|_\Sigma = [| \chi |^2 + \alpha | \psi'' |^2]^{1/2}. \tag{5.1}$$

Theorems 1, 2, and 3 show that $\{u, \dot{u}\}$ generates a dynamical system on Σ and on the space $X \times H_0^2(\Omega)$.

A weak dynamical system on a reflexive Banach space B is a function $w : \mathbf{R}^+ \times B \to B$ which satisfies

(i) $w^t : \phi \to w(t, \phi)$ is (sequentially) weakly continuous for fixed $t \in \mathbf{R}^+$, (i.e., if $\phi_n \rightharpoonup \phi$ then $w(t, \phi_n) \rightharpoonup w(t, \phi)$),

(ii) $w^\phi : t \to w(t, \phi)$ is a continuous map of $[0, T]$ into $\tilde{B}$ for fixed $\phi \in B$,

(iii) $w(0, \phi) = \phi$, and

(iv) $w(t + \tau, \phi) = w(t, w(\tau, \phi))$ for all $t, \tau \in \mathbf{R}^+, \phi \in B$.

Again, we have replaced by (i) and (ii) the joint continuity requirement in the definition of a weak dynamical system used by Slemrod [21].

The *positive orbit* $0^+(\phi)$ through $\phi \in B$ is defined by $0^+(\phi) = \bigcup_{t \geq 0} w(t, \phi)$. A set M in B is an *invariant set* of the weak dynamical system w if for each $\phi \in M$, there exists a function $W \equiv W(s, \phi)$ such that

(a) $W(s, \phi) \in M$ for all $s \in (-\infty, \infty)$,

(b) $W(0, \phi) = \phi$, and

(c) for any $\sigma \in (-\infty, \infty)$, $w(t, W(\sigma, \phi)) = W(t + \sigma, \phi)$ for all $t \in \mathbf{R}^+$.

For any $\phi \in B$, define the *weak limit set* $\tilde{\Omega}(\phi)$ of an orbit through ϕ by $\tilde{\Omega}(\phi) = \{\psi \in B \mid$ there exists an increasing sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $w(t_n, \phi) \rightharpoonup \psi$ in B as $n \to \infty\}$. With these definitions we can state the following basic theorem.

THEOREM 4. *Let B be a separable, reflexive Banach space, and let w be a weak dynamical system on B. Also let $\phi \in B$ be such that $0^+(\phi)$ is bounded in B. Then $\tilde{\Omega}(\phi)$ is a nonempty, weakly compact, invariant, weakly connected set in B.*

Proof. Since B is reflexive $0^+(\phi)$ belongs to a weakly compact set A. Therefore, $\tilde{\Omega}(\phi)$ is nonempty. Since B is separable, A may be regarded as a compact set in a metric space with metric d induced by the weak topology, cf.

Dunford and Schwartz [7]. Apart from the connectedness the theorem is now a consequence of the proof of Proposition 2.2 [5]. A detailed proof of our special case, differing slightly from that of Dafermos, is given in [2].

Suppose $\tilde{\Omega}(\phi)$ is not connected in $\tilde{B}$. Then there exist sets M_1, M_2, A_1, and A_2 with A_1 and A_2 weakly open such that $M_1 \cap M_2$ and $A_1 \cap A_2$ are empty, $\tilde{\Omega}(\phi) = M_1 \cup M_2$, $M_1 \subset A_1$, and $M_2 \subset A_2$. From the continuity properties, there exists an increasing sequence $\{t_n\}$, $t_n \to \infty$, such that $\{w(t_n, \phi)\} \subset A \backslash (A_1 \cup A_2)$. As $A \backslash (A_1 \cup A_2)$ is weakly closed, it follows that $\tilde{\Omega}(\phi) \cap (A \backslash (A_1 \cap A_2))$ is nonempty. This contradiction proves that $\tilde{\Omega}(\phi)$ is weakly connected. □

Remark. Slemrod's proof of invariance relies on the joint continuity of $w(t, \phi)$ in t and ϕ.

6. Proof That (1.1) Generates a Weak Dynamical System

It turns out that the methods of Lions used in Section 4 are easily adapted to show that many initial-boundary value problems for partial differential equations generate weak dynamical systems. Such is the case for (1.1). Again let $\Sigma = H_0^2(\Omega) \times L^2(\Omega)$. If $\phi_0 = \{u_0, u_1\}$ belongs to Σ, define $w(t, \phi_0) \in \Sigma$ by $w(t, \phi_0) = \{u(t), \dot{u}(t)\}$ where u is the unique weak solution of the initial-boundary value problem for the clamped beam, with initial data $\{u(0), \dot{u}(0)\} = \phi_0$, of Theorem 1.

Theorem 5. *w forms a weak dynamical system on Σ.*

Proof. Properties (ii), (iii), and (iv) in the definition of a weak dynamical system are immediate from Theorems 1 and 2. It remains to prove (i). So fix $T > 0$ and let $\phi_j \rightharpoonup \phi_0$ in Σ, where $\phi_j = \{u_{0j}, u_{1j}\}$, and let u_j be the weak solution corresponding to initial data ϕ_j. The sequence $\| \phi_j \|_\Sigma$ is bounded. From the energy equation (4.10) it follows that

$$\| w(t, \phi_j)\|_\Sigma < C, \qquad \int_0^t | \dot{u}_j'' |^2 \, ds < C_1,$$

where C, C_1 are independent of j and of $t \in [0, T]$.

So we may extract a subsequence $\{\phi_\mu\}$ of $\{\phi_j\}$ such that for some χ

$$\{u_\mu, \dot{u}_\mu\} \xrightarrow{*} \{\chi, \dot{\chi}\} \text{ in } L^\infty(0, T; \Sigma)$$

and

$$\dot{u}_\mu \xrightarrow{*} \dot{\chi} \text{ in } L^2(0, T; H_0^2(\Omega)).$$

The u_μ satisfy, for $\psi \in H_0^2(\Omega)$,

$$(\ddot{u}_\mu, \psi) + \alpha(u''_\mu, \psi'') - (\beta + k \mid u_\mu' \mid^2)(u''_\mu, \psi) + \gamma(\dot{u}''_\mu, \psi'') + \delta(\dot{u}_\mu, \psi) = 0. \tag{6.1}$$

It follows, by passing to the limit as in Theorem 1, that

$$(\ddot{\chi}, \psi) + \alpha(\chi'', \psi'') - (\beta + k \mid \chi' \mid^2)(\chi'', \psi) + \gamma(\dot{\chi}'', \psi'') + \delta(\dot{\chi}, \psi) = 0. \tag{6.2}$$

But the argument at the end of the proof of Theorem 1 can be applied to show that $\chi(0) = u_0$ and $\dot{\chi}(0) = u_1$. Thus, χ is a weak solution satisfying the same initial data as u. Hence, $\chi = u$. Thus, $\{u_\mu, u_\mu\} \overset{*}{\rightharpoonup} \{u, \dot{u}\}$ in $L^\infty(0, T; \Sigma)$, and so $\{u_j, \dot{u}_j\} \overset{*}{\rightharpoonup} \{u, \dot{u}\}$ in $L^\infty(0, T; \Sigma)$. Also, from (6.1), $(\ddot{u}_\mu, \psi) \rightharpoonup (\ddot{u}, \psi)$ in $L^2(0, T)$ for any $\psi \in H_0^2(\Omega)$. Thus, using Lemma 1 for the functions u_μ and $(\dot{u}_\mu, \psi)$, it follows that for any t $w(t, \phi_j) \rightharpoonup w(t, \phi_0)$. $\square$

7. Asymptotic Behavior of the Clamped Beam

In this section we restrict δ by

$$\delta > \delta_0 = -\mu_0\gamma/l^4, \tag{7.1}$$

where $\mu = \mu_0$ is the lowest eigenvalue of $y'''' = (\mu/l^4)y$ subject to $y = y' = 0$ at $x = 0, l$. μ_0 has the approximate value 500.56. It is well known that $\mid y'' \mid^2 \geqslant (\mu_0/l^4) \mid y \mid^2$ for all $y \in H_0^2(\Omega)$ (see, for example, Mikhlin [18]). We have shown that the unique weak solution u of

$$\ddot{u} + \alpha u'''' - (\beta + k \mid u' \mid^2)u'''' + \gamma \dot{u}'''' - \sigma(u', \dot{u}')u'' + \delta \dot{u} = 0, \tag{7.2}$$

with $w = \{u, \dot{u}\} \in L^\infty(0, T; \Sigma)$, $\dot{u} \in L^2(0, T; H_0^2(\Omega))$ and $\{u(0), \dot{u}(0)\} = \{u_0, u_1\} = \phi_0 \in \Sigma$, satisfies the energy equation (4.10). Thus, (7.1) implies that $\dot{E}(t) \leqslant 0$, so that the energy $E(t)$ of the beam is nonincreasing. It follows that

$$\mid \dot{u}(t) \mid, \mid u''(t) \mid, \int_0^t \mid \dot{u}'' \mid^2 ds, \int_0^t \mid \dot{u} \mid^2 ds \leqslant C \quad (\text{independent of } t \in \mathbf{R}^+). \tag{7.3}$$

Recall that for the clamped beam $H_E = -4\alpha\pi^2\rho/l^2$. We first show that in the case $H \geqslant H_E$ the beam approaches the undeflected equilibrium position.

Theorem 6. *If $H \geqslant H_E$ then $w(t, \phi_0) \to \{0, 0\}$ strongly in Σ as $t \to \infty$.*

Proof. Take the inner product of (7.2) with u to obtain

$$(\ddot{u}, u) + \alpha \mid u'' \mid^2 + \beta \mid u' \mid^2 + k \mid u' \mid^4 + \gamma(\dot{u}'', u'') + \sigma(u', \dot{u}) \mid u' \mid^2 + \delta(\dot{u}, u) = 0. \tag{7.4}$$

Integrate (7.4) with the aid of Lemma 1. Thus,

$$[(\dot{u}, u) + (\gamma/2) \mid u'' \mid^2 + (\delta/2) \mid u \mid^2 + (\sigma/4) \mid u' \mid^4](t) + \int_0^t [2E(s) + (k/2) \mid u'(s)|^4] \, ds - 2 \int_0^t \mid \dot{u}(s)|^2 \, ds = \text{constant}. \tag{7.5}$$

It follows from (7.3) and (7.5) that

$$\int_0^t [2E(s) + (k/2) \mid u'(s)|^4] \, ds < C \quad \text{(independent of } t\text{)}. \tag{7.6}$$

But the integrand in (7.6) is nonnegative (since $H \geqslant H_E$) and $E(s)$ is nonincreasing. Thus $E(t)$ and $\mid u'(t)|$ both tend to zero as $t \to \infty$. The theorem follows. □

For the case of arbitrary H we need the following lemma, which is a consequence of Lemma 1 and a well known result on uniformly continuous functions integrable on $\mathbf{R}^+$.

LEMMA 3. *If a nonnegative measurable function f satisfies*

(i) $f, \dot{f} \in L^\infty(\mathbf{R}^+)$ and

(ii) $\int_0^t f(s) \, ds \leqslant C$ (independent of $t \in \mathbf{R}^+$),

then $f(t) \to 0$ as $t \to \infty$.

THEOREM 7. *For any H, $w(t, \phi_0) \to \{v, 0\}$ strongly in Σ, where v is an equilibrium position.*

Proof. The proof consists of two stages.

(a) We first show that for some equilibrium position v, $\{u(t), \dot{u}(t)\} \rightharpoonup \{v, 0\}$ in Σ. For $\psi \in X$, consider the function $f = (\dot{u}, \psi)^2$. By the Schwarz inequality $f \in L^\infty(\mathbf{R}^+)$. But $\dot{f} = 2(\dot{u}, \psi)(\ddot{u}, \psi)$, and from (7.2)

$$(\ddot{u}, \psi) = -\alpha(u'', \psi'') + (\beta + k \mid u' \mid^2)(u'', \psi) - \gamma(\dot{u}, \psi'''') - \sigma(u'', \dot{u})(u'', \psi) - \delta(\dot{u}, \psi),$$

and, thus, from (7.3) $\dot{f} \in L^\infty(\mathbf{R}^+)$. Lemma 3 now shows that $f(t) \to 0$ as $t \to \infty$. Since $\mid \dot{u}(t)|$ is bounded it follows that $\dot{u}(t) \rightharpoonup 0$ in $L^2(\Omega)$. But from (7.3) $O^+(\phi_0)$ is bounded, and thus the weak limit set $\tilde{\Omega}(\phi_0)$ in Σ is nonempty and invariant from Theorem 4. Hence, $\tilde{\Omega}(\phi_0)$ consists only of equilibrium positions $\{v_j, 0\}$. But $\tilde{\Omega}(\phi_0)$ is weakly connected. Therefore, $\tilde{\Omega}(\phi_0) = \{v, 0\}$ for some equilibrium position v. It follows that $\{u(t), \dot{u}(t)\} \rightharpoonup \{v, 0\}$ in Σ.

(b) We now show that $w(t, \phi_0) \to \{v, 0\}$ strongly in Σ. By (a) $u(t) \rightharpoonup v$ in $H_0^2(\Omega)$ as $t \to \infty$. Hence, $|u'(t)|^4 \to |v'|^4$ as $t \to \infty$. But $E(t)$ is non-increasing and so tends to its greatest lower bound E_∞ as $t \to \infty$. By (7.3) and (7.5)

$$\int_0^t [2E(s) + (k/2)\,|u'(s)|^4]\,ds \text{ is bounded on } [0, \infty).$$

Therefore, $E_\infty = -(k/4)\,|v'|^4$. Hence, from (a) $\|w(t, \phi_0)\|_\Sigma \to \|\{v, 0\}\|_\Sigma$ and so again from (a) $w(t, \phi_0) \to \{v, 0\}$ strongly in Σ. □

We now study the stability of equilibrium positions of the beam. If $v \in H_0^2(\Omega)$ is an equilibrium position we also denote by v the element $\{v, 0\}$ of Σ. We say that an equilibrium position $v \in \Sigma$ is *stable* if there is a neighborhood N of v in Σ such that $O^+(\phi) \to v$ in Σ for any $\phi \in N$. An equilibrium position is said to be *unstable* if it is not stable.

It is clear from Theorem 6 that when $H \geqslant H_E$ the zero position is stable. When $H < H_E$, the discussion at the end of Section 3 and the fact that $\dot{E}(t) \leqslant 0$ for any orbit show that the zero position and the equilibrium positions $\pm v_j$, $1 < j \leqslant n$, are unstable. To prove that $+v_1$ and $-v_1$ are stable when $H < H_E$ we need the following lemma.

Lemma 4. *Let $H < H_E$, and suppose that Γ is a continuous arc in $H_0^2(\Omega)$ joining v_1 and $-v_1$. Then there is a point ψ on Γ with $V(\psi) \geqslant V(v^*)$ where $v^* = v_2$ if $n \geqslant 2$ and $v^* = 0$ if $n = 1$.*

Proof. $\Gamma \equiv \Gamma(t)$ is a continuous map from $[0, 1]$ into $H_0^2(\Omega)$ with $\Gamma(0) = v_1$, $\Gamma(1) = -v_1$. Define $g\colon [0, 1] \to \mathbf{R}$ by $g(\tau) = (\Gamma'(\tau), v_1')$. g is continuous and $g(0) = |v_1'|^2$, $g(1) = -|v_1'|^2$. Therefore, there exists $\tau_0 \in [0, 1]$ with $g(\tau_0) = 0$. Let $\psi = \Gamma(\tau_0)$. Then

$$V(\psi) = (\alpha/2)\,|\psi''|^2 + (\beta/2)\,|\psi'|^2 + (k/4)\,|\psi'|^4.$$

But v_2 minimizes $\alpha\,|y''|^2/|y'|^2$ subject to $y \neq 0$, $(y', v_1') = 0$, cf. Courant and Hilbert [4] and Mikhlin [18]. Therefore,

$$V(\psi) \geqslant |\psi'|^2[\beta + \lambda_2 + (k/2)\,|\psi'|^2] \geqslant V(v^*). \quad \square$$

Theorem 8. *If $H < H_E$ then v_1 and $-v_1$ are stable.*

Proof. Let $V(v^*) = M$. By the continuity of E in Σ, there are spherical neighborhoods $N(v_1), N(-v_1)$ of $v_1, -v_1$, respectively, such that $\phi \in N(v_1) \cup N(-v_1)$ implies $E(\phi) < M$. If $E(\phi) < M$ then since $\dot{E}(w(t, \phi)) \leqslant 0$, by Theorem 7 $w(t, \phi) \to v_1$ or $-v_1$. But suppose $\phi_1 \in N(v_1)$ and $w(t, \phi_1) \to -v_1$. Then we may construct an arc $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ with Γ_0 an arc in $N(v_1)$ joining v_1

and ϕ_1, Γ_1 a part of $O^+(\phi_1)$ joining ϕ_1 and some $\phi_2 \in N(-v_1)$, and Γ_2 an arc in $N(-v_1)$ joining ϕ_2 and $-v_1$. By Lemma 4 there is a point ψ on Γ_1 with $E(\psi) \geqslant M$, contradicting $E(\phi_1) < M$. Therefore, $w(t, \phi_1) \to v_1$. Similarly, $\psi_1 \in N(-v_1)$ implies $w(t, \psi_1) \to -v_1$. Therefore, v_1 and $-v_1$ are stable. □

For $H < H_E$, we may partition Σ into the $2n + 1$ sets

$$S(0) = \{\psi \mid w(t, \psi) \to 0 \text{ as } t \to \infty\},$$
$$S(\pm v_j) = \{\psi \mid w(t, \psi) \to \pm v_j \text{ as } t \to \infty\}, \qquad j = 1, 2, \ldots, n.$$

It is clear from Theorems 2 and 8 that $S(+v_1)$ and $S(-v_1)$ are open and arcwise connected. We conjecture that

$$\Sigma \backslash (S(+v_1) \cup S(-v_1)) = \partial S(+v_1) = \partial S(-v_1),$$

where ∂ denotes "the boundary of."

8. Hinged-End Boundary Conditions

In this section we consider the initial-boundary value problem for (1.1) subject to the initial conditions $u(0) = u_0$, $\dot{u}(0) = u_1$ and the boundary conditions for hinged ends $u = u'' = 0$ at $x = 0, l$.

Let m be a positive integer. Define $G_m = \{u \in H^{2m}(\Omega) : u^{(2r)} \in H_0^1(\Omega)$ for $0 \leqslant r \leqslant m - 1\}$, so that for example

$$G_1 = H_0^1(\Omega) \cap H^2(\Omega) \qquad \text{and} \qquad G_2 = \{u \in H^4(\Omega) : u, u'' \in H_0^1(\Omega)\}.$$

G_1, G_2, G_3 correspond to the Hilbert spaces S_2, S_1, S_0 in [1], respectively. We first state three theorems on existence, uniqueness and regularity. The proofs are omitted since they are straightforward adaptations of those in [1]. In these theorems we do not restrict δ but require that $\alpha, k, \gamma, \sigma > 0$.

THEOREM 9. *Suppose $u_0 \in G_1$, $u_1 \in L^2(\Omega)$. Then there exists u with*

$$u \in L^\infty(0, T; G_1),$$
$$\dot{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; G_1),$$

satisfying the initial conditions $u(0) = u_0$, $\dot{u}(0) = u_1$, and

$$(\ddot{u}, \phi) + \alpha(u'', \phi'') - (\beta + k \mid u' \mid^2)(u'', \phi) + \gamma(\dot{u}'', \phi'') - \sigma(u', \dot{u}')(u'', \phi) + \delta(\dot{u}, \phi) = 0 \tag{8.1}$$

for all $\phi \in G_1$.

THEOREM 10. *Let u be as in Theorem 9 and let v be another such weak solution with initial conditions $v(0) = v_0$, $\dot{v}(0) = v_1$, where $v_0 \in G_1$ and $v_1 \in L^2(\Omega)$. Set $y = u - v$. Then*

$$\begin{aligned} &| \dot{y}(t)|^2 + \alpha \,| y''(t)|^2 + 2\gamma \int_0^t | \dot{y}'' |^2 \, ds \\ &\qquad \leqslant [| u_1 - v_1 |^2 + \alpha \,| u_0'' - v_0'' |^2] \exp(K_1 t), \end{aligned} \tag{8.2}$$

where K_1 is a continuous function of T, $| u_0'' |$, $| u_1 |$, $| v_0'' |$, and $| v_1 |$.

In particular the solution u in Theorem 9 is unique. Furthermore, the functions u: $[0, T] \to G_1$ and $\dot{u}$: $[0, T] \to L^2(\Omega)$ are continuous and satisfy the energy equation

$$E(t) + \gamma \int_0^t | \dot{u}'' |^2 \, ds + \sigma \int_0^t (u', \dot{u}')^2 \, ds + \delta \int_0^t | \dot{u} |^2 \, ds = E(0), \tag{8.3}$$

where $E(t) = (1/2) \,| \dot{u}(t)|^2 + (\alpha/2) \,| u''(t)|^2 + (\beta/2) \,| u'(t)|^2 + (k/4) \,| u'(t)|^4$.

THEOREM 11. *Suppose $r \geqslant 2$ and that $u_0 \in G_r$, $u_1 \in G_{r-1}$. Then*

$$u \in L^\infty(0, T; G_r),$$

$$\dot{u} \in L^\infty(0, T; G_{r-1}) \cap L^2(0, T; G_r)$$

if $r \geqslant 4$

$$\partial^j u/\partial t^j \in L^\infty(0, T; G_{r-j-1}) \cap L^2(0, T; G_{r-j}) \quad \text{for} \quad 2 \leqslant j \leqslant r - 2,$$

if $r \geqslant 3$

$$\partial^{r-1} u/\partial t^{r-1} \in L^\infty(0, T; L^2) \cap L^2(0, T; G_1),$$

and

$$\partial^r u/\partial t^r \in L^2(Q).$$

Furthermore,

$$u \in C^{r-2}(\bar{Q}) \cap [C^{2r-1}(\bar{\Omega}) \times C^{r-1}([0, T])]. \tag{8.4}$$

Let $\Sigma_1 = G_1 \times L^2(\Omega)$. The preceding theorems imply that $\{u, \dot{u}\}$ forms a dynamical system on Σ_1 and on $G_r \times G_{r-1}$. Next, using the same proof as for Theorem 5, we have the following theorem.

THEOREM 12. *$\{u, \dot{u}\}$ forms a weak dynamical system on Σ_1.*

We now restrict δ by

$$\delta > \delta_0 = -\pi^4 \gamma / l^4, \tag{8.5}$$

so that $| y'' |^2 \geqslant (\pi^4/l^4) | y |^2$ for all $y \in S_2$, and thus $\dot{E}(t) \leqslant 0$. For the case of hinged ends $H_E = -\alpha\pi^2\rho/l^2$ and when $H \geqslant H_E$ we can prove simply, as in Theorem 6, that $\{u(t), \dot{u}(t)\} \to \{0, 0\}$ strongly in Σ_1. For arbitrary H we have the following theorem.

THEOREM 13. *$\{u(t), \dot{u}(t)\} \to \{v, 0\}$ strongly in Σ_1, where v is an equilibrium position.*

The proof of Theorem 13 follows that of Theorem 7. The discussion in Section 7 on the stability of equilibrium positions carries over in the obvious way.

We now investigate what can happen if $\delta < \delta_0$, so that (8.5) does not hold. If $u(x, t) = T_n(t) \sin(n\pi x/l)$ is to be a solution of (8.4), then T_n must satisfy the generalized Liénard equation

$$\ddot{T}_n + A_n T_n + B_n T_n^3 + (C_n + D_n T_n^2)\, \dot{T}_n = 0, \tag{8.6}$$

where $A_n = \alpha(n\pi/l)^4 + \beta(n\pi/l)^2$, $B_n = kn^4\pi^4/2l^3$, $C_n = \delta + \gamma(n\pi/l)^4$ and $D_n = \sigma n^4\pi^4/2l^3$. Clearly, B_n and D_n are positive. Suppose $A_n \geqslant 0$ and $C_n < 0$. Then the conditions are satisfied of a theorem of Lefschetz [14, p. 268] which guarantees that there exists a unique nonzero periodic solution to (8.6). This result is to be expected since it follows from (8.6) that when T_n is small the beam gains more energy from the negative external damping than it loses from the internal damping, while the opposite occurs for T_n large.

Hence, if $\delta < \delta_0$ then in particular for any $H \geqslant H_E$ there is at least one periodic solution to (8.4) satisfying the hinged-end boundary conditions.

ACKNOWLEDGMENTS

The author would like to thank Professor D. E. Edmunds under whose invaluable guidance this paper was written and who critically read earlier versions. He would also like to thank Professor R. J. Knops for his comments and the referee for a number of suggestions which led to substantial improvements in the paper.

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