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23.—Weak Continuity Properties of Mappings and Semigroups.* By

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Synopsis

The relationship between weak and sequential weak continuity for mappings between Banach spaces and semigroups on Banach space is studied.

1. Introduction

In this paper we discuss weak and sequential weak continuity for mappings between Banach spaces, and for semigroups on Banach space. To facilitate discussion we make the following definitions. Let X, Y be Banach spaces and f: $X \to Y$. We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence of *sequences* respectively.

Definition 1.1

(*a*) f is (*strongly*) *continuous* iff $x_n \overset{X}{\to} x$ implies $f(x_n) \overset{Y}{\to} f(x)$.

(*b*) f is *weakly continuous* iff f is continuous with respect to the weak topologies on X and Y.

(*c*) f is *sequentially weakly continuous* iff $x_n \overset{X}{\rightharpoonup} x$ implies $f(x_n) \overset{Y}{\rightharpoonup} f(x)$.

It is clear that (*b*) implies (*c*). If f is linear then (*a*), (*b*) and (*c*) are equivalent, as can be seen from the proof of the equivalence of (*a*) and (*b*) given in Dunford and Schwartz [**8**, p. 422]. In section 2 we present examples to show that (i) for arbitrary non-zero Y there exists functions f satisfying (*a*) and (*c*) but not (*b*), if and only if X is infinite-dimensional; (ii) if H denotes infinite-dimensional separable real Hilbert space and $X = Y = H$ then there exist functions f satisfying the remaining possibilities {(*a*), not (*b*), not (*c*)}, {not (*a*), (*b*), (*c*)} and {not (*a*), not (*b*), (*c*)}. Although the distinction between properties (*a*), (*b*), (*c*) underlies the hypotheses of many theorems in non-linear functional analysis, there seem to be no references in the literature to the relevant counter-examples—this is particularly unfortunate in case (i) above, since it is common practice to define 'weak continuity' to be our 'sequential weak continuity'. We cannot expect analogous results to (ii) for arbitrary Banach spaces X, Y since in the space l_1 strong and weak convergence of sequences coincide (Dunford and Schwartz [**8**, p. 296]).

In section 3 we consider semigroups. Let $\mathbb{R}^+$ denote the non-negative reals. By a *semigroup* on a topological space S we mean a family of maps $T(t)$: $S \to S$, $t \in \mathbb{R}^+$ satisfying (i) $T(0) =$ identity and (ii) $T(s+t) = T(s)T(t)$, s, $t \in \mathbb{R}^+$. Among abstract formulations of semigroup theory various choices of continuity axioms are current.

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From the point of view of mathematical elegance it is natural to require that for each $t \in R^+$, $T(t): S \to S$ is continuous. This approach is taken by numerous writers (e.g. Bhatia and Szëgo [5]). Alternatively (see Dafermos [6]) physical considerations can lead one to postulate the lesser requirement that for each $t \in R^+$, $T(t): S \to S$ is sequentially continuous (i.e. $s_n \xrightarrow{S} s$ implies $T(t)s_n \xrightarrow{S} T(t)s$). Many initial-boundary value problems for partial differential equations can be shown to generate semigroups on a Banach space X which are sequentially weakly continuous. If X is reflexive, 'weak' invariance principles have been successfully used to determine asymptotic properties of such equations (Slemrod [10], Ball [2]). It is not clear whether such semigroups of physical origin are such that for $t \in R^+$, $T(t): X \to X$ is weakly continuous. We are, however, able to use an example from section 2 to construct a (necessarily non-linear) semigroup of maps which for $t > 0$ are strongly continuous and sequentially weakly continuous, but not weakly continuous. Furthermore, when X is reflexive we show that 'topological' continuity may be recovered if we give X the bounded weak (BW) topology.

Throughout the paper all Banach spaces are supposed real.

2. Non-linear Mappings between Banach Spaces

Throughout the rest of this paper X, Y are Banach spaces with dual spaces X^*, Y^*, respectively. We denote the norms in both X and Y by $\|\cdot\|$.

Definition 2.1

A function $g: X \to R^+$ belongs to $S(X)$ iff

(i) $g(0) = 0$,

(ii) $x_n \xrightarrow{X} x$ implies $g(x_n) \to g(x)$,

and there exists a subset E of $X \setminus \{0\}$ such that

(iii) for any finite subset G of X^* the set $E \cap \{x \in X: x^*(x) = 0 \text{ for all } x^* \in G\}$ is non-empty, and

(iv) for any $x \in X$, $e \in E$, $\lim_{\lambda \to \infty} g(x + \lambda e) = \infty$.

Examples of functions $g \in S(X)$ arise naturally as follows (the proofs are straightforward).

Examples

2.1. Let X, X_1 be Sobolev spaces with X compactly embedded in X_1 and let $E = X \setminus \{0\}$. Let $\alpha > 0$ and define $g: X \to R^+$ by $g(x) = \|x\|^{\alpha}_{X_1}$.

2.2. Let H be separable Hilbert space with orthonormal basis $\{e_n\}$. Then the map $h: H \to H$ defined by $h\left(\sum_{r=1}^{\infty} a_r e_r\right) = \sum_{r=1}^{\infty} \frac{a_r}{r} e_r$ maps the closed unit ball of H into the compact Hilbert cube. Let $X = E \cup \{0\} = H$, $\alpha > 0$, and define $g: H \to R^+$ by $g(x) = \|h(x)\|^{\alpha}$, $x \in H$.

For an arbitrary infinite-dimensional Banach space X we have:

Lemma 2.1. *Let $x_i^* \in X^*$, $\|x_i^*\| \leq 1$ $(i = 1, 2, \ldots)$ be linearly independent. Define $g: X \to R^+$ by*

$$g(x) \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i^*(x)|, \quad x \in X. \tag{2.1}$$

Then $g \in S(X)$.

Proof. Clearly g is well defined and satisfies properties (i) and (iv) of Definition 2.1 with $E \equiv \{x \in X: x_i^*(x) \neq 0 \text{ some } i\}$. If property (iii) were false then, for any $x \in X$, $x^*(x) = 0$ for each $x^* \in G$ would imply that $x_i^*(x) = 0$ each i. By Dunford and Schwartz [8, Lemma 10, p. 421] this would imply that, for each i, x_i^* is a finite linear combination of elements of G, which contradicts the linear independence of $\{x_i^*\}$.

To prove g satisfies (ii) let $x_n \xrightarrow{X} x$. Then $\|x_n\| \leq M$ for some M and all n. Hence, given $\varepsilon > 0$, there exists N such that for all n

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} [|x_i^*(x_n)| + |x_i^*(x)|] < \varepsilon.$$

Therefore

$$|g(x_n) - g(x)| \leq \sum_{i=1}^{N} \frac{1}{2^i} \big| |x_i^*(x_n)| - |x_i^*(x)| \big| + \varepsilon,$$

and hence $g(x_n) \to g(x)$.

Definition 2.2

A function $f: X \to Y$ is *bounded* iff there exists a constant M such that $\|f(x)\| \leq M$ for all $x \in X$.

Theorem 2.2. *Let X be infinite-dimensional and Y non-zero. Let $L: X \to Y$ be linear, continuous and non-zero. Let $g \in S(X)$. Let $f_0: X \to Y$ be bounded with $f_0(0) = 0$ and, satisfy (c). Define $f: X \to Y$ by*

$$f(x) \equiv f_0(x) + g(x)Lx, \quad x \in X. \tag{2.2}$$

Then f satisfies (c) and not (b). f satisfies (a) iff f_0 does.

Proof. Let f_0 satisfy (*a*). To show f strongly continuous let $x_n \xrightarrow{X} x$. Then

$$\|f(x_n) - f(x)\| \leq \|f_0(x_n) - f_0(x)\| + \|g(x_n)Lx_n - g(x)Lx_n\| + \|g(x)(Lx_n - Lx)\|, \tag{2.3}$$

which tends to zero as $n \to \infty$ by our assumptions. Similarly if f satisfies (*a*) so does f_0.

Now let $x_n \rightharpoonup x$. Then

$$f(x_n) - f(x) = f_0(x_n) - f_0(x) + (g(x_n) - g(x))Lx_n + g(x)(Lx_n - Lx). \tag{2.4}$$

Since L is linear and continuous it is sequentially weakly continuous and bounded. Hence the right-hand side of (2.4) tends weakly to zero. Thus f is sequentially weakly continuous.

Suppose for contradiction that f is weakly continuous at zero. Note that $f(0) = 0$. Since $L \neq 0$ there exists $y^* \in Y^*$ with $y^*L \neq 0$. Let $N = \{y \in Y: |y^*(y)| < 1\}$, which is a weakly open neighbourhood of zero in Y. By assumption there exists

$$U = \{x \in X: |x_i^*(x)| < \delta \quad i = 1, \ldots, m\},$$

a basic weakly open neighbourhood of zero in X, such that $f(U) \subseteq N$. Let $\chi \in X$ be such that $y^*(L\chi) = 1$. By assumption there exists $e \in X$ such that (iv) of Definition

2.1 holds and $x_1^*(e) = \dots = x_m^*(e) = (y^*L)(e) = 0$. Let $x = a\chi + \lambda e$, where a and λ are real constants to be chosen. Choose $a > 0$ small enough so that for each $i = 1, \dots, m$

$$|x_i^*(x)| = |ax_i^*(\chi)| < \delta.$$

Then $x \in U$. But

$$|y^*(f(x))| = |y^*(f_0(x)) + ag(a\chi+\lambda e)y^*(L\chi)| \geqq ag(a\chi+\lambda e) - \|y^*\| \|f_0(x)\|,$$

which tends to ∞ as $\lambda \to \infty$. Hence for large enough λ, $f(x) \notin N$. This contradiction completes the proof.

Corollary 2.3. *Let Y be non-zero. There exist functions $f: X \to Y$ satisfying (a), (c) and not (b) iff X is infinite-dimensional.*

Proof. Suppose X infinite-dimensional. Let $\theta \in X^*$, $\theta \neq 0$ and $y_0 \in Y$, $y_0 \neq 0$. Define $Lx = \theta(x)y_0$, which is linear and continuous. Let $f_0 \equiv 0$ and g be as in Lemma 2.1. The result now follows from the theorem.

Corollary 2.4. *If X^* is separable and infinite-dimensional, and if $X = Y$, then the function f in Corollary 2.3 can be chosen to be $1-1$ and onto.*

Proof. Let $\{x_i^*\}$ span X^*, L = identity, $f_0 \equiv 0$, g as in Lemma 2.1. The details are easy to verify.

Examples

2.3. Let Ω be the open interval (0, 1). Let $X = Y = H_0^1(\Omega)$ (for the definition of this space see Lions and Magènes [9]). Let $g(u) = |u|^\gamma$, $\gamma > 0$, where

$$|u|^2 \equiv \int_\Omega u^2 dx.$$

Let L = identity, $f_0 = 0$. Then (see example 2.1) since the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact the hypotheses of Theorem 2.2 are satisfied. Hence f: $H_0^1(\Omega) \to H_0^1(\Omega)$ defined by $f(u) = |u|^\gamma u$ satisfies (a), (c), and not (b). Note that f is $1-1$ and onto.

2.4. Let $\Omega = (0, 1)$, $\Sigma = H_0^2(\Omega) \times L^2(\Omega)$, $f_0 = 0$, $X = H_0^2(\Omega)$, $Y = L^2(\Omega)$,

$$Lu \equiv u_{xx}, \quad g(u) = |u_x|^2.$$

Then by Theorem 2.2 the function f: $H_0^2(\Omega) \to L^2(\Omega)$ defined by $f(u) = |u_x^2| u_{xx}$ satisfies (a), (c) and not (b). Thus $F: \Sigma \to \Sigma$ defined by $F(\{\phi_1, \phi_2\}) = \{0, f(\phi_1)\}$ satisfies (a), (c) and not (b).

The proof of the following lemma is left to the reader.

Lemma 2.5. *Let H be separable Hilbert space with orthonormal basis $\{e_i\}$. Define ψ: $R^+ \to H$ by $\psi(0) = 0$, $\psi\left(\frac{1}{n}\right) = e_n$, $n = 1, 2, \dots$, ψ linear in each interval $\left[\frac{1}{n-1}, \frac{1}{n}\right]$, $\psi(t) = e_1$ for $t \in [1, \infty]$. Define f_0: $H \to H$ by $f_0(x) = \psi(|(x, e_1)|)$, $x \in H$. Then f_0 satisfies {not (a), (b), (c)} with $X = Y = H$.*

Corollary 2.6. *There exist functions f: $H \to H$ satisfying {(a), not (b), not (c)}.*

Proof. The reader will easily find an f satisfying {(a), not (b), not (c)}. For the second example let f_0 be as in Lemma 2.5 and note that f_0 is bounded and does not satisfy (a). Let g be as in Lemma 2.1 and L = identity. The result follows from Theorem 2.2.

3. Non-linear Semigroups on Banach Space

It is easy to use Example 2.3 to construct a semigroup $\{T(t)\}t \in R^+$ on $H_0^1(\Omega)$ such that for each $t > 0$ the map $u \to T(t)u$ satisfies {(a), not (b), (c)} and is $1-1$ onto. We seek such a semigroup of the form $T(t)u = |u|^{\gamma(t)}u$. The semigroup property requires that $\gamma(t)$ satisfies the equation

$$\gamma(s+t)+1 = (\gamma(s)+1)(\gamma(t)+1) \quad s, t \in R^+. \tag{3.1}$$

Hence the semigroup (which is in fact a group) has the form

$$T(t)u = |u|^{(e^{kt}-1)}u, \tag{3.2}$$

for some $k > 0$. $\{T(t)\}t \in R^+$ is generated by the equation

$$\dot{v} = v \log\left(\int_\Omega v^2 dx\right)^{\frac{k}{2}}. \tag{3.3}$$

Note that (3.2) defines a semigroup on $L^2(\Omega)$ such that for each $t > 0$ $T(t)$ satisfies (a) but not (c).

Is it true in general that semigroups $\{T(t)\}t \in R^+$ of physical origin are such that the map $u \to T(t)u$ is weakly continuous for each $t \in R^+$? We conjecture that this is not so. In [1–3] a non-linear beam equation was considered whose only non-linear term was that given in Example 2.4. In order to prove that the equation generated a semigroup of sequentially weakly continuous maps it was necessary to use a continuity property of F similar to, but stronger than, sequential weak continuity. Since F is not weakly continuous it is plausible to suppose that the same holds for the generated semigroup, although we have been unable to verify this.

In order to recover continuity for a semigroup of sequentially weakly continuous maps we proceed as follows.

Definition 3.1 (see Dieudonné [7])

The *bounded weak topology* (BW) for X is the finest topology coinciding with the weak topology of X on every closed ball $B_r = \{x \in X: \|x\| \leqq r\}$.

Lemma 3.1. Let X be reflexive. Then f: $X \to Y$ is sequentially weakly continuous iff f is continuous when X, Y are given their BW topologies.

Proof. If f is continuous with respect to the BW topologies, then f is sequentially weakly continuous, since convergence of sequences in the weak and bounded weak topologies coincide. Conversely, let E be a BW closed set in Y. Let $r > 0$ and let x belong to the weak closure of $f^{-1}(E) \cap B_r$. By, for example, Wilansky [11, p. 295] there exists a sequence $\{x_n\} \subset f^{-1}(E) \cap B_r$ such that $x_n \xrightarrow{X} x$. Also $x \in B_r$. Therefore $f(x_n) \xrightarrow{Y} f(x)$. Thus $f(x) \in E$ and so $x \in f^{-1}(E) \cap B_r$. Hence $f^{-1}(E) \cap B_r$ is weakly closed. Thus $f^{-1}(E)$ is BW closed and f is continuous with respect to the BW topologies.

Remark. The lemma may be false if X is not reflexive (e.g. let f: $l_1 \to R$ be defined by $f(x) = \|x\|$, $x \in l_1$).

By the above lemma, if $T(t)$: $X \to X$ satisfies (c) and if X is reflexive, then $T(t)$ is continuous with respect to the BW topology on X, as required. For further information concerning weak continuity properties of non-linear semigroups see [4]. Finally we remark that the result in [11] referred to above enables one to (i) drop the separability assumption in the 'weak' invariance principles of Slemrod [10] and Ball [2] and (ii) prove that any sequentially weakly continuous map f from a reflexive Banach space X to a Banach space Y which satisfies $\| f(x)\| \to 0$ as $\| x \| \to \infty$ is identically zero. To prove (ii) note that for any $x \in X$ and $R > 0$ there exists a sequence $\{x_n\} \subseteq X$ with $\| x_n - x \| = R$ each n such that $x_n \rightharpoonup x$. The result follows by letting $R \to \infty$ in the inequality $\| f(x)\| \leqq \liminf_{n\to\infty} \| f(x_n)\|$.

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