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Global invertibility of Sobolev functions and the interpenetration of matter

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Synopsis

A global inverse function theorem is established for mappings $u: \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ bounded and open, belonging to the Sobolev space $W^{1,p}(\Omega)$, $p > n$. The theorem is applied to the pure displacement boundary value problem of nonlinear elastostatics, the conclusion being that there is no interpenetration of matter for the energy-minimizing displacement field.

1. Introduction

Consider a material body whose particles are labelled by the positions they occupy in a reference configuration $\Omega \subset \mathbb{R}^3$, Ω bounded and open. A basic requirement of continuum mechanics is that interpenetration of matter does not occur, i.e. that in any deformed configuration the mapping u giving the position $u(x)$ of a particle in terms of its position x in the reference configuration be invertible. A complete analysis of invertibility would necessitate a study of the mechanics of self-contact, and we do not attempt this here. Rather, we examine situations in which interpenetration of matter in the *interior* of a body can be ruled out using information about the deformation of its *boundary*. We concentrate attention on nonlinear elasticity, although our methods have applications to other theories of mechanics.

In nonlinear elastostatics, experience with one-dimensional problems (cf. Antman [**2**], Antman and Brezis [**3**], Ball [**6**]) suggests that to ensure the invertibility of equilibrium solutions in three dimensions one should assume that the stored-energy function $W(x, \nabla u(x))$ satisfies the condition

$$W(x, F) \to \infty \quad \text{as} \quad \det F \to 0+. \tag{1.1}$$

This condition expresses the fact that an infinite amount of energy is required to compress a finite volume of the material into zero volume. It follows from (1.1) that any equilibrium solution u of finite energy satisfies

$$\det \nabla u(x) > 0 \tag{1.2}$$

almost everywhere in Ω. If u were C^1 then (1.2) would imply local invertibility of u at x, by the inverse function theorem. However, it is not known under what conditions u is C^1, and in fact for nonhomogeneous materials u may not be this smooth. The existence theorems of Ball [**4, 5**] and Ball *et al.* [**8**] assert only that

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for certain materials and boundary-value problems a function u minimizing the total energy exists in the Sobolev space $W^{1,p}(\Omega)$. In Theorems 3 and 4 of this paper it is shown under certain hypotheses that for a pure displacement boundary-value problem any such minimizer is a homeomorphism of Ω. The hypotheses do not exclude cases in which self-contact of the boundary occurs. These results are an immediate consequence of a new inverse function theorem in $\mathbb{R}^n$ that is the main result of this paper.

Our inverse function theorem, Theorem 2, asserts roughly the following: if $u: \bar{\Omega} \to \mathbb{R}^n$ is a function in $W^{1,p}(\Omega)$, $p>n$, coinciding on $\partial\Omega$ with a homeomorphism u_0 of Ω, if (1.2) holds almost everywhere, and if for some $q>n$

$$\int_\Omega |\nabla u^{-1}(x)|^q \det \nabla u(x)\, dx < \infty, \tag{1.3}$$

then u is a homeomorphism of Ω onto $u_0(\Omega)$. (In (1.3) $\nabla u^{-1}(x)$ denotes the inverse matrix of $\nabla u(x)$.) Examples are given showing that if (1.3) is omitted u need not be a homeomorphism; however, in this case some information on invertibility is given in Theorem 1. Without the hypothesis that u coincide on $\partial\Omega$ with a homeomorphism, u need not even be *locally* invertible. An instructive example is the mapping u of the unit disc $D=\{|x|<1\}$ in $\mathbb{R}^2$ given in polar coordinates (r, θ) by

$$u: (r, \theta) \mapsto \left(\frac{1}{\sqrt{2}} r, 2\theta\right).$$

It is easily checked that $u \in W^{1,\infty}(D)$, that $\det \nabla u(x) = 1$ if $x \neq 0$, and that (1.3) holds. But u is not locally invertible at the origin. This example shows that Theorem 2 is different in character from an interesting result of Meisters and Olech [**15**], who proved in particular that if $u: \bar{\Omega} \to \mathbb{R}^n$ is continuous and locally one-to-one in $\bar{\Omega} \backslash Z$, where $Z \subset \Omega$ is a finite set, if u is one-to-one on $\partial\bar{\Omega}$, and if $\partial\bar{\Omega}$ is connected, then u is a homeomorphism.

For other related literature, though not of immediate relevance here, see Berger [**9**] (local versus global invertibility), Browder [**11**] (orientation-preserving mappings) and Clarke [12] (a local inverse function theorem for Lipschitz functions).

2. Global invertibility

Let $E \subset \mathbb{R}^n$ be a nonempty bounded open set with boundary ∂E. We assume familiarity with the Sobolev spaces $W^{1,\alpha}(E)$, $1 < \alpha \leqq \infty$ (cf. Adams [**1**]). Elements of $W^{1,\alpha}(E)$ may be (equivalence classes of) functions or vectors, depending on the context. We say that E is *strongly Lipschitz* if for each $x \in \partial E$ there exists a neighbourhood U_x of x and a Cartesian coordinate system $\xi = (\xi_1, \ldots, \xi_n)$ in U_x such that $E \cap U_x = \{\xi \in U_x : \xi_n > f(\xi_1, \ldots, \xi_{n-1})\}$ for some Lipschitz continuous function $f: \mathbb{R}^{n-1} \to \mathbb{R}$, and that E satisfies the *cone condition* if there exists a finite cone C such that each point $x \in E$ is the vertex of a finite cone C_x contained in E and congruent to C. If $\alpha > n$ and E is strongly Lipschitz then $W^{1,\alpha}(E)$ is continuously imbedded in the space $C^{0,\mu}(\bar{E})$ of Hölder continuous functions with exponent $\mu = 1 - n/\alpha$ (Morrey [**16**, p. 83]); if $u \in W^{1,\alpha}(E)$ we shall always assume that the Hölder continuous representative of u has been chosen. All derivatives in this paper are distributional derivatives. $B_r(x)$ denotes the open ball in $\mathbb{R}^n$ with centre x and radius r.

If $u: \bar{E} \to \mathbb{R}^n$ is continuous then the Brouwer degree $d(u, E, p)$ of u with respect to E at a point $p \in \mathbb{R}^n \setminus u(\partial E)$ is a well-defined integer depending only on the boundary values of u. (For a discussion of degree theory see Schwartz [**19**].) If V is a connected component of $\mathbb{R}^n \setminus u(\partial E)$ then $d(u, E, p)$ is independent of $p \in V$, the common value being denoted by $d(u, E, V)$. We will use the formula

$$d(u, E, V) = \int_E \rho(u(x)) \det \nabla u(x) \, dx, \tag{2.1}$$

where ρ is any nonnegative real-valued continuous function with compact support in V and satisfying $\int_{\mathbb{R}^n} \rho(v) \, dv = 1$. Formula (2.1) is valid (Nirenberg [**17**, Th. 1.5.5]) if u is continuously differentiable in E. If E is strongly Lipschitz and $u \in W^{1,\alpha}(E)$, $\alpha > n$, then there exists a sequence of smooth functions u_r converging to u in $W^{1,\alpha}(E)$, and hence in $C^{0,\mu}(\bar{E})$. The right-hand side of (2.1) is a continuous functional on $W^{1,\alpha}(E)$. By passing to the limit in (2.1) using the continuity properties of the degree we deduce that (2.1) holds for u. We remark that for functions in $W^{1,\alpha}(E)$ the fact that the degree depends only on $u|_{\partial E}$ follows from (2.1) and the observation that for smooth ρ the Euler-Lagrange equations for $\int_E \rho(u(x)) \det \nabla u(x) \, dx$ are identically satisfied.

THEOREM 1. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded connected strongly Lipschitz open set. Let $u_0: \bar{\Omega} \to \mathbb{R}^n$ be continuous in $\bar{\Omega}$ and one-to-one in Ω. Let $p > n$ and let $u \in W^{1,p}(\Omega)$ take values in $\mathbb{R}^n$ and satisfy $u|_{\partial\Omega} = u_0|_{\partial\Omega}$, $\det \nabla u(x) > 0$ almost everywhere in Ω. Then*

(i) *$u(\bar{\Omega}) = u_0(\bar{\Omega})$,*

(ii) *u maps measurable sets in $\bar{\Omega}$ to measurable sets in $u_0(\bar{\Omega})$, and the change of variables formula*

$$\int_A f(u(x)) \det \nabla u(x) \, dx = \int_{u(A)} f(v) \, dv \tag{2.2}$$

holds for any measurable $A \subset \bar{\Omega}$ and any measurable function $f: \mathbb{R}^n \to \mathbb{R}$, provided only that one of the integrals in (2.2) exists.

(iii) *u is one-to-one almost everywhere; i.e. the set*

$$S = \{v \in u_0(\bar{\Omega}) : u^{-1}(v) \text{ contains more than one element}\}$$

has measure zero,

(iv) *if $v \in u_0(\Omega)$ then $u^{-1}(v)$ is a continuum contained in Ω, while if $v \in \partial u_0(\Omega)$ then each connected component of $u^{-1}(v)$ intersects $\partial\Omega$.*

The following examples show that nontrivial behaviour of the type described in (iv) can occur.

EXAMPLE 1. Let $n \geqq 2$, and consider the cylinder

$$\Gamma = \{x \in \mathbb{R}^n : 0 \leqq R < 1, |x^n| < 2\},$$

where $x = (x^1, \ldots, x^n)$, $R = ((x^1)^2 + \cdots + (x^{n-1})^2)^{\frac{1}{2}}$. Let $u = (u^1, \ldots, u^n)$ be defined by

$$u^i(x) = R^{-\alpha}x^i, \qquad i = 1, \dots, n-1,$$
$$u^n(x) = R^{\beta}x^n \quad \text{for} \quad |x^n| \leqq 1,$$
$$= [2(|x^n| - 1) + (2 - |x^n|)R^{\beta}] \operatorname{sgn} x^n \quad \text{for} \quad 1 \leqq |x^n| \leqq 2,$$

where $0 \leqq \alpha < 1$ and $\beta > 0$.

It is easily verified that $u(\bar{\Gamma}) = \bar{\Gamma}$, $u|_{\partial\Gamma} = \text{identity}$, that for $x \in \Gamma$

$$|\nabla u^i(x)| \leqq \text{const. } R^{-\alpha}, \qquad i = 1, \dots, n-1,$$
$$|\nabla u^n(x)| \leqq \text{const. } R^{\beta - 1},$$

and that

$$\det \nabla u(x) = (1-\alpha) R^{\beta - \alpha(n-1)} \quad \text{for} \quad |x^n| < 1,$$
$$= (1-\alpha)(2 - R^{\beta}) R^{-\alpha(n-1)} \text{ for } 1 < |x^n| < 2.$$

Choosing first $\alpha = 0$, $\beta \geqq 1$, we deduce that $u \in W^{1,\infty}(\Gamma)$, $\det \nabla u(x) > 0$ almost everywhere, but that $u^{-1}(0) = \{\lambda e_n : |\lambda| \leqq 1\}$, where $e_n \stackrel{\text{def}}{=} (0, \dots, 0, 1)$, so that u is not a homeomorphism. In this example $\det \nabla u(x)$ is not essentially bounded away from zero. Secondly, let $n > 2$ and let $n < p < n(n-1)$; then choosing $\beta = \alpha(n-1)$ and

$$\frac{1}{n-1} - \frac{1}{p} < \alpha < \frac{n-1}{p}$$

we obtain that $u \in W^{1,p}(\Gamma)$, $\det \nabla u(x) > 1 - \alpha$ almost everywhere, but that u is not a homeomorphism. Note that if $p > n(n-1)$, $v \in W^{1,p}(\Gamma)$ and $\det \nabla v(x) > k > 0$ almost everywhere in Γ, then

$$\int_{\Gamma} |\nabla v^{-1}(x)|^q \det \nabla v(x)\, dx < \infty$$

for some $q > n$. In the two cases described above

$$\int_{\Gamma} |\nabla u^{-1}(x)|^n \det \nabla u(x)\, dx = \infty.$$

Example 2. Let $n \geqq 2$, let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, let K be an arbitrary closed subset of S^{n-1}, and let f be a C^∞ real-valued function on S^{n-1} satisfying $1 \leqq f < \frac{3}{2}$, $f^{-1}(1) = K$ (for the construction of f one can use, for example, the argument in Golubitsky and Guillemin [**13**, p. 17]). Let $\Omega = \{x \in \mathbb{R}^n : f(x/|x|) < |x| < 2\}$, and define $w : \bar{\Omega} \to \mathbb{R}^n$ by $w(x) = (1 - |x|^{-1})x$. Then Ω has C^∞ boundary, $w \in C^\infty(\bar{\Omega})$, $\det \nabla w > 0$ in $\bar{\Omega}$, w is one-to-one in Ω, and $w^{-1}(0) = K$.

Example 3. Let $n \geqq 2$. We combine Examples 1 and 2. Let Γ, Ω, K, w, f, be as in these examples, and let u be as constructed in Example 1 with $\alpha = 0$, $\beta \geqq 1$. Let $\Gamma^+ = \Gamma \cap \{x^n > 0\}$. Suppose further that $f(a_r) = 1$ for $r = 1, 2, \dots$, where the $a_r \in K$ are distinct. Let U_r be a family of disjoint open subsets of Ω, $\phi_r : \bar{\Gamma}^+ \to \bar{U}_r$ a corresponding family of diffeomorphisms mapping the base $\Gamma \cap \{x^n = 0\}$ of Γ^+ into the inner surface of $\partial\Omega$ and such that $\phi_r(0) = a_r$, $|\nabla \phi_r(x)| \leqq \text{const. } c_r$, $|\nabla \phi_r^{-1}(x)| \leqq \text{const. } c_r^{-1}$, where $c_r > 0$ are suitable constants. Such U_r, ϕ_r are easily constructed.

Define $v(x)=(\phi_r\circ u\circ\phi_r^{-1})(x)$ if $x\in U_r$, $v(x)=x$ if $x\in\bar\Omega\setminus\bigcup_{r=1}^{\infty}U_r$, $z=w\circ v$. Then $z\in W^{1,\infty}(\Omega)$, $\det\nabla z(x)>0$ almost everywhere in Ω, $z|_{\partial\Omega}=w|_{\partial\Omega}$, and $z^{-1}(0)$ is the union of $w^{-1}(0)$ and the countable set of disjoint continua $\phi_r(u^{-1}(0))$.

Proof of Theorem 1. We first prove (i). The invariance of domain theorem implies that u_0 is a homeomorphism of Ω onto the open set $u_0(\Omega)$. Therefore $u_0(\bar\Omega)=\overline{u_0(\Omega)}$, $\partial u_0(\Omega)=u_0(\partial\Omega)$. Furthermore

$$\left.\begin{aligned}&d(u_0,\Omega,u_0(\Omega))=\pm1,\\&d(u_0,\Omega,p)=0\quad\text{if}\quad p\in\mathbb{R}^n\setminus u_0(\bar\Omega).\end{aligned}\right\}$$

(This is a consequence of the multiplicative property of the degree; a detailed discussion is given in Rado and Reichelderfer [**18**, Section IV 4.6].) Since $u|_{\partial\Omega}=u_0|_{\partial\Omega}$, $\det\nabla u(x)>0$ almost everywhere, it follows from (2.1) that

$$\left.\begin{aligned}&d(u,\Omega,u_0(\Omega))=1,\\&d(u,\Omega,p)=0\quad\text{if}\quad p\in\mathbb{R}^n\setminus u_0(\bar\Omega).\end{aligned}\right\}\tag{2.3}$$

Therefore if $p\in u_0(\Omega)$, $u^{-1}(p)$ is nonempty. Hence $u(\bar\Omega)\supset u_0(\bar\Omega)$.

Let $p\notin u_0(\bar\Omega)$ and suppose for contradiction that $u(x)=p$ for some $x\in\Omega$. Apply (2.1) with V the component of $\mathbb{R}^n\setminus u_0(\bar\Omega)$ containing p and with ρ strictly positive in a neighbourhood U of p. The continuity of u implies that a small ball around x is mapped into U. Since $\det\nabla u(x)>0$ almost everywhere the right-hand side of (2.1) is positive. This contradiction completes the proof of (i).

Since Ω is strongly Lipschitz, we may extend u to a function $\bar u\in W^{1,p}(D)$, where D is a bounded open set in $\mathbb{R}^n$ containing $\bar\Omega$ ([**16**, Th. 3.4.3]). For $A\subset\mathbb{R}^n$, $w\colon A\to\mathbb{R}^n$, $v\in\mathbb{R}^n$, we write $N(w\mid A,v)=\text{cardinality}\,\{w^{-1}(v)\cap A\}$. Let $f\colon\mathbb{R}^n\to\mathbb{R}$ be measurable, and let A be a measurable subset of D. Since $p>n$, $\bar u$ maps sets of measure zero to sets of measure zero (Bony [**10**], Marcus and Mizel [**14**]), and hence maps measurable sets to measurable sets. Furthermore, by a result of Marcus and Mizel [**14**] (see also [**18**], Vodop'yanov and Goldshtein [**20**], Vodop'yanov *et al.* [**21**])

$$\int_{\bar u(A)}f(v)N(\bar u\mid A,v)\,dv=\int_A f(\bar u(x))\det\nabla\bar u(x)\,dx\tag{2.4}$$

whenever one of the two integrals exists. In particular, taking $f=1$, and noting that $\bar u(\partial\Omega)=\partial u_0(\Omega)$, we deduce that

$$\int_{u_0(\Omega)}N(u\mid\bar\Omega,v)\,dv=\int_{u^{-1}(u_0(\Omega))}\det\nabla u(x)\,dx.\tag{2.5}$$

By (2.1), (2.3) we have that

$$1=\int_\Omega\rho(u(x))\det\nabla u(x)\,dx\tag{2.6}$$

for any continuous function $\rho\geqq0$ with $\operatorname{supp}\rho\subset\subset u_0(\Omega)$ and $\int_{\mathbb{R}^n}\rho(v)\,dv=1$. Let $\theta_r\in C(\mathbb{R}^n)$ satisfy $\theta_r(p)=1$ if $p\in u_0(\Omega)$ and $\operatorname{dist}(p,\partial u_0(\Omega))\geqq1/r$, $\operatorname{supp}\theta_r\subset\subset u_0(\Omega)$ and $0\leqq\theta_r(p)\leqq1$ for all $p\in\mathbb{R}^n$. Applying (2.6) to $\rho_r\overset{\text{def}}{=}\theta_r/\int\theta_r\,dv$, and passing to the limit $r \to \infty$ using dominated convergence, we obtain

$$m(u_0(\Omega)) = \int_{u^{-1}(u_0(\Omega))} \det \nabla u(x)\, dx. \tag{2.7}$$

Combining (2.5), (2.7) and using the fact that $N(u \mid \bar{\Omega}, v) \geqq 1$ for $v \in u_0(\Omega)$, we deduce that $N(u \mid \bar{\Omega}, v) = 1$ almost everywhere in $u_0(\Omega)$. Since $m(\partial u_0(\Omega)) = m(\bar{u}(\partial\Omega)) = m(\partial\Omega) = 0$, this proves (iii), while (ii) follows from (2.4).

It remains to prove (iv). Let $v \in u_0(\Omega)$ and suppose that the closed set $u^{-1}(v)$ is not connected. Then there exist nonempty subsets M_1, M_2, E_1, E_2 of Ω with E_1, E_2 open, such that $M_1 \cap M_2$ and $E_1 \cap E_2$ are empty, $u^{-1}(v) = M_1 \cup M_2$, $M_1 \subset E_1$ and $M_2 \subset E_2$. By covering M_1, M_2 by a suitable finite collection of cubes we may suppose that E_1, E_2 are strongly Lipschitz. Since $v \notin u(\partial E_1) \cup u(\partial E_2)$, and since $\det \nabla u(x) > 0$ almost everywhere, the degrees $d(u, E_i, p)$, $i = 1, 2$, given by (2.1) are defined and positive for p in a neighbourhood U of v. Hence $U \subset u(E_1) \cap u(E_2)$, contradicting (iii). A similar argument shows that if $v \in \partial u_0(\Omega)$ then any connected component of $u^{-1}(v)$ intersects $\partial\Omega$. ■

The main result gives conditions under which a function u satisfying the hypotheses of Theorem 1 is a homeomorphism.

THEOREM 2. *Let the hypotheses of Theorem 1 hold, let $u_0(\Omega)$ satisfy the cone condition, and suppose that for some $q > n$,*

$$\int_\Omega |\nabla u^{-1}(x)|^q \det \nabla u(x)\, dx < \infty. \tag{2.8}$$

Then u is a homeomorphism of Ω onto $u_0(\Omega)$, and the inverse function $x(u)$ belongs to $W^{1,q}(u_0(\Omega))$. The matrix of weak derivatives of $x(\cdot)$ is given by

$$\nabla x(v) = \nabla u^{-1}(x(v)) \text{ almost everywhere in } u_0(\Omega).$$

If, further, $u_0(\Omega)$ is strongly Lipschitz, then u is a homeomorphism of $\bar{\Omega}$ onto $u_0(\bar{\Omega})$.

Proof. The idea of the proof is to construct a function $x(\cdot) \in W^{1,q}(u_0(\Omega))$ as the limit of a sequence of mollified functions. Roughly speaking, in the first instance $x(v)$ represents a weighted average of the set $u^{-1}(v)$. Theorem 1 and the continuity properties of $u(\cdot)$ and $x(\cdot)$ then imply that $x(\cdot)$ is the inverse of u, so that $u^{-1}(v)$ is a singleton for each $v \in u_0(\Omega)$.

To motivate the construction of $x(\cdot)$, suppose that $x(\cdot)$ is indeed the inverse of u. Let $\varepsilon > 0$ be given and let $\rho_\varepsilon \geqq 0$ be a smooth function with supp $\rho_\varepsilon \subset\subset \bar{B}_\varepsilon(0)$, $\int_{\mathbb{R}^n} \rho_\varepsilon(v)\, dv = 1$. Let

$$x_\varepsilon(v) = \int_{u_0(\Omega)} \rho_\varepsilon(v - u)x(u)\, du.$$

Changing variables we have

$$x_\varepsilon(v) = \int_\Omega \rho_\varepsilon(v - u(y))y \det \nabla u(y)\, dy. \tag{2.9}$$

Proceeding rigorously, we now *define* $x_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by (2.9). Our assumptions ensure that x_ε is smooth. We have that

$$\frac{\partial x_\varepsilon^\alpha(v)}{\partial v^i} = \int_\Omega \rho_{\varepsilon,i}(v-u(y))y^\alpha \det \nabla u(y)\, dy$$
$$= -\int_\Omega \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v-u(y))\nabla u^{-1}(y)_i^\beta y^\alpha \det \nabla u(y)\, dy$$
$$= -\int_\Omega \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v-u(y))y^\alpha \operatorname{adj} \nabla u(y)_i^\beta\, dy,$$

where adj ∇u denotes the transpose of the matrix of cofactors of ∇u, and where

$$\rho_{\varepsilon,i}(v) \overset{\text{def}}{=} \frac{\partial \rho_\varepsilon}{\partial v^i}(v).$$

Let D be an open subset of $\mathbb{R}^n$ with piecewise smooth boundary and satisfying $\bar{D} \subset u_0(\Omega)$. Let u_r be a sequence of smooth functions converging to u in $W^{1,p}(\Omega)$ and thus uniformly in $\bar{\Omega}$. If ε is small enough there exists $\delta = \delta(\varepsilon) > 0$, $r_0 = r_0(\varepsilon)$, such that if $r \geqq r_0$ and dist $(y, \partial\Omega) < \delta$ then

$$\rho_\varepsilon(v-u_r(y)) = 0 \quad \text{for all} \quad v \in D.$$

Since

$$\frac{\partial}{\partial y^\beta}(\operatorname{adj} \nabla u(y)_i^\beta) = 0,$$

it follows that

$$-\frac{\partial \rho_\varepsilon}{\partial y^\beta}(v-u_r(y))y^\alpha \operatorname{adj} \nabla u_r(y)_i^\beta = \rho_\varepsilon(v-u_r(y)) \operatorname{adj} \nabla u_r(y)_i^\alpha$$
$$-\frac{\partial}{\partial y^\beta}[\rho_\varepsilon(v-u_r(y))^\alpha \operatorname{adj} \nabla u_r(y)_i^\beta],$$

and so

$$-\int_\Omega \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v-u_r(y))y^\alpha \operatorname{adj} \nabla u(y)_i^\beta\, dy = \int_\Omega \rho_\varepsilon(v-u_r(y)) \operatorname{adj} \nabla u_r(y)_i^\alpha\, dy$$

for all $v \in D$ and $r \geqq r_0$. Passing to the limit we obtain

$$\frac{\partial x_\varepsilon^\alpha(v)}{\partial v^i} = \int_\Omega \rho_\varepsilon(v-u(y)) \operatorname{adj} \nabla u(y)_i^\alpha\, dy, \quad v \in D. \tag{2.10}$$

Let $K = \sup_{y\in\Omega} |y|$. Then

$$|x_\varepsilon(v)| \leqq K \int_\Omega \rho_\varepsilon(v-u(y)) \det \nabla u(y)\, dy$$
$$= Kd(u, \Omega, v) = K \tag{2.11}$$

for all $v \in D$, provided ε is sufficiently small. Also, from (2.9) and Hölder's inequality

$$\left|\frac{\partial x_\varepsilon^\alpha(v)}{\partial v^i}\right| \leqq \left(\int_\Omega \rho_\varepsilon(v-u(y)) \det \nabla u(y)\, dy\right)^{1/q'}$$
$$\times \left(\int_\Omega \rho_\varepsilon(v-u(y)) \,|\mathrm{adj}\, \nabla u(y)_i^\alpha|^q (\det \nabla u(y))^{1-q}\, dy\right)^{1/q}$$
$$= \left(\int_\Omega \rho_\varepsilon(v-u(y)) \,|\mathrm{adj}\, \nabla u(y)_i^\alpha|^q (\det \nabla u(y))^{1-q}\, dy\right)^{1/q},$$

for all $v \in D$ and ε sufficiently small. Thus

$$\int_D \left|\frac{\partial x_\varepsilon^\alpha(v)}{\partial v^i}\right|^q dv \leqq \int_\Omega \left(\int_D \rho_\varepsilon(\bar{u}-u(y))\, d\bar{u}\right) |\mathrm{adj}\, \nabla u(y)_i^\alpha|^q (\det \nabla u(y))^{1-q}\, dy$$
$$\leqq \int_\Omega |\mathrm{adj}\, \nabla u(y)_i^\alpha|^q (\det \nabla u(y))^{1-q}\, dy. \tag{2.12}$$

Thus $\{x_\varepsilon\}$ is bounded in $W^{1,q}(D)$ for any D if ε is sufficiently small, and therefore there exists a diagonal subsequence, again denoted $\{x_\varepsilon\}$, converging weakly in every $W^{1,q}(D)$ to a function $x(\cdot)$. On account of the imbedding $W^{1,q}(D) \subset C(\bar{D})$ the convergence is uniform on compact subsets of $u_0(\Omega)$. Since the bounds (2.11), (2.12) are independent of D, it follows that $x(\cdot) \in W^{1,q}(u_0(\Omega))$.

We next prove that $x(\cdot)$ is a right inverse of u, that is

$$u(x(v)) = v \quad \text{for all} \quad v \in u_0(\Omega). \tag{2.13}$$

First, let $v \in u_0(\Omega)\backslash S$, where S is defined in Theorem 1. By Theorem 1 there exists $x \in \Omega$ with $u(x) = v$, and x is unique. From (2.6), (2.9)

$$x_\varepsilon(v) - x = \int_\Omega \rho_\varepsilon(u(x)-u(y))(y-x) \det \nabla u(y)\, dy, \tag{2.14}$$

for $\varepsilon \leqq \varepsilon_1$, say. Given any $\eta > 0$, the uniqueness of x and the continuity of u imply the existence of $\delta > 0$ such that $|y-x| < \eta$ whenever $|u(x)-u(y)| \leqq \delta$. So if $\varepsilon \leqq \min(\delta, \varepsilon_1)$,

$$|x_\varepsilon(v) - x| \leqq \eta\, d(u, \Omega, v) = \eta.$$

Hence $x(v) = x$ and $u(x(v)) = v$. Since $x(\cdot)$ is continuous in $u_0(\Omega)$ and $m(S) = 0$, (2.13) holds.

We now prove that $u\colon \Omega \to u_0(\Omega)$. Let $u_0(\Omega)$ satisfy the cone condition with respect to the finite cone $C = \{x = (x', x^n) \in \mathbb{R}^n : 0 < |x'| < \mu x^n, |x| < \sigma\}$, where μ, σ are positive constants. Let K be an integer greater than $m(B_\sigma(0))/m(C)$. Given $\tau > 0$ there exists $\delta = \delta(\tau) > 0$ such that of any K cones congruent to τC and with vertices in a ball of radius δ, two must intersect. Suppose for contradiction that $x \in \Omega$ with $u(x) = p \in \partial u_0(\Omega)$. By Theorem 1 the connected component of $u^{-1}(p)$ containing x intersects $\partial\Omega$, and in particular $u^{-1}(p)$ contains K distinct points $y_1, \ldots, y_K$. Let $\varepsilon = \min_{i \neq j} |y_i - y_j| > 0$. By the estimate of Morrey [**16**, p. 83], there exists a constant $k > 0$ such that for any finite cone $\tilde{C}$ contained in $u_0(\Omega)$ and similar to C,

$$|x(v)-x(w)| \leqq k\,|v-w|^{1-(n/q)}\,\|x(\cdot)\|_{W^{1,q}(u_0(\Omega))} \quad \text{for all} \quad v, w \in \bar{C}. \tag{2.15}$$

Choose $\tau > 0$ small enough so that

$$\varepsilon > 4k(\tau\sigma)^{1-(n/q)}\,\|x(\cdot)\|_{W^{1,q}(u_0(\Omega))}. \tag{2.16}$$

Since u is continuous and $m(u^{-1}(S))=0$, there exist points $z_i \in \Omega \backslash u^{-1}(S)$ such that $|z_i - y_i| < \varepsilon/4$, $|u(z_i)-p| < \delta(\tau)$, $i=1,\ldots,K$. Since $u_0(\Omega)$ satisfies the cone condition there exist cones $C_i \subset u_0(\Omega)$ with vertices $u(z_i)$ and congruent to C. By the above, two of the cones τC_i must intersect, so that there exists $v \in \tau C_i \cap \tau C_j$, say, $i \neq j$. Since $x(\cdot)$ is continuous at $u(z_i)$, $u(z_j)$ we deduce from (2.15), (2.16), that

$$\begin{aligned} |z_1 - z_j| &\leqq |z_i - x(v)| + |z_j - x(v)| \\ &\leqq k[|u(z_i)-v|^{1-(n/q)} + |u(z_j)-v|^{1-(n/q)}]\,\|x(\cdot)\|_{W^{1,q}(u_0(\Omega))}, \\ &\leqq 2k(\tau\sigma)^{1-(n/q)}\,\|x(\cdot)\|_{W^{1,q}(u_0(\Omega))} < \frac{\varepsilon}{2}. \end{aligned}$$

But

$$\begin{aligned} |z_i - z_j| &\geqq |y_i - y_j| - |y_i - z_i| - |y_j - z_j| \\ &\geqq \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

which is the desired contradiction. Hence $u : \Omega \to u_0(\Omega)$. Let $y \in \Omega$ and $y_r \to y$ with $u(y_r) \notin S$. By (2.13),

$$x(u(y_r)) = y_r.$$

Passing to the limit using the continuity of $x(\cdot)$ and $u(\cdot)$ we obtain

$$x(u(y)) = y \quad \text{for all} \quad y \in \Omega. \tag{2.17}$$

Thus u is a homeomorphism of Ω onto $u_0(\Omega)$.

If $u_0(\Omega)$ is strongly Lipschitz then $x(\cdot)$ belongs to $C^{0,1-(n/q)}(u_0(\bar{\Omega}))$ and it follows easily from (2.13), (2.17) that u is a homeomorphism of $\bar{\Omega}$ onto $u_0(\bar{\Omega})$.

It remains to identify the generalized derivatives of $x(\cdot)$. Let G be open, $\bar{G} \subset u_0(\Omega)$, and $m(\partial G)=0$. Integrating (2.10) over G and passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_G \frac{\partial x^\alpha(v)}{\partial v^i}\,dv = \int_{u^{-1}(G)} \nabla u^{-1}(x)_i^\alpha \det \nabla u(x)\,dx,$$

which thus holds for all compact $G \subset u_0(\Omega)$ by approximation. By (2.2) we deduce that

$$\int_{u^{-1}(G)} \frac{\partial x^\alpha}{\partial v^i}(u(x)) \det \nabla u(x)\,dx = \int_{u^{-1}(G)} \nabla u^{-1}(x)_i^\alpha \det \nabla u(x)\,dx$$

for all compact G, which implies that $\nabla x(v) = \nabla u^{-1}(x(v))$ almost everywhere in $u_0(\Omega)$.

Remarks

1. Example 1 shows that in the absence of (2.8) u need not be a homeomorphism.

2. If the assumption that $u_0(\Omega)$ satisfy the cone condition is omitted, the proof still establishes the existence of a continuous right inverse $x\colon u_0(\Omega)\to\Omega$ of u, that $x(\cdot)\in W^{1,q}(u_0(\Omega))$, and that $x(u_0(\Omega))$ is an open subset of Ω of full measure. The author does not know whether in this general case u is a homeomorphism. The point at issue is whether $u(x)$ can belong to $\partial u_0(\Omega)$ for some $x\in\Omega$. Any such x must be the limit of a sequence x_r such that $u(x_r)\in\partial u_0(\Omega)$ and $u(x_r)\neq u(x_s)$ if $r\neq s$. If not there would exist a ball $B_r(x)\subset\Omega$ such that $u(B_r(x))\cap(\partial u_0(\Omega)\backslash u(x))$ is empty. Let $y_i\in B_r(x)$, $i=1,2,3$, be distinct points such that $u(y_i)=u(x)$, $i=1,2,3$, and choose $z_i\notin u^{-1}(S)$ sufficiently close to y_i. Then for each i the largest open ball $B_i=B_{r_i}(u(z_i))$ contained in $u_0(\Omega)$ has $u(x)$ on its boundary, since otherwise (2.15) would imply that some point in $B_r(x)$ is mapped to $p\in\partial u_0(\Omega)$, $p\neq u(x)$. Two of the B_i must intersect, so that applying again (2.15), as in the proof of Theorem 2, we obtain a contradiction. It is hard to believe that such a complicated counterexample could exist for $n=2$.

3. It would be interesting to decide whether Theorems 1 and 2 are valid when $p=q=n>1$, in the sense that a representative of u exists satisfying the conclusions of the theorems. For information that may be relevant here, see **[20]** and [**16**, Th. 4.3.4]. If $p<n$, $q<n$ then unless continuity of u is assumed the theorem can fail drastically in that $u(\Omega)\backslash u_0(\Omega)$ may contain a ball, even if $\det\nabla u(x)=1$ almost everywhere in Ω; for examples from nonlinear elasticity see **[7]**. If $n=1$, then the theorem holds with $p=q=1$. In this case the existence of the inverse $x(u)$ is obvious.

4. The reader may be surprised that in the proof of Theorem 2 we did not smooth u, rather than its putative inverse, in such a way that the smoothed functions u_ε satisfy $\det\nabla u_\varepsilon(x)>0$ and are thus locally invertible. There are actually serious obstacles to such a procedure. Firstly, the set of $n\times n$ matrices F such that $\det F\geqq m$, $m\in\mathbb{R}$, is not convex. If ρ_ε is a mollifier then

$$\nabla(\rho_\varepsilon * u)(x)=\int_{\mathbb{R}^n}\rho_\varepsilon(x-y)\nabla u(y)\,dy$$

is a convex combination of values of ∇u, so that even if $\det\nabla u(x)\geqq m>0$ everywhere $\det\nabla(\rho_\varepsilon * u)$ may take negative values. Secondly, consider the example

$$u\colon(r,\theta)\mapsto\left(\frac{1}{\sqrt{2}}r,2\theta\right)$$

discussed in the introduction. We claim that even though $\det\nabla u(x)=1$ almost everywhere there is no sequence $\{u_r\}\subset C^1(D)$ such that $\det\nabla u_r>0$ and $u_r\to u$ uniformly on $\bar{D}$. Suppose such a sequence existed. Fix r large enough so that $u_r(B_{\frac14}(0))\subset B_{\frac12}(0)$, $u_r^{-1}(0)\subset B_{\frac14}(0)$, $u_r^{-1}(\overline{B_{\frac12}(0)})\subset D$ and $(tu_r+(1-t)u)(\partial D)\cap B_{\frac14}(0)=\emptyset$ for all $t\in[0,1]$. Let $p\in B_{\frac14}(0)$. Then $d(u,D,p)=2$, so that by homotopy invariance $d(u_r,D,p)=2$. Since $\det\nabla u_r>0$, by the definition of degree $u_r^{-1}(p)$

consists of exactly two points. In particular $u_r^{-1}(0)=\{y_0, y_1\}$ for $y_0, y_1 \in B_{\frac{1}{2}}(0)$, $y_0 \neq y_1$. By the implicit function theorem there exists a unique C^1 solution $x_0(\cdot)$ of $u_r(x_0(v))=v$, $x_0(0)=y_0$, defined for v in a neighbourhood of 0. Since $u^{-1}(\overline{B_{\frac{1}{2}}(0)})\subset D$, x_0 may be extended to the whole of $B_{\frac{1}{2}}(0)$. Similarly there exists a unique C^1 solution $x_1: B_{\frac{1}{2}}(0)\to D$ of $u_r(x_1(v))=v$, $x_1(0)=y_1$. The open sets $x_0(B_{\frac{1}{2}}(0))$, $x_1(B_{\frac{1}{2}}(0))$ are disjoint, since if $p\in x_0(B_{\frac{1}{2}}(0))\cap x_1(B_{\frac{1}{2}}(0))$ then $u_r(x(v))=v$, $x(u_r(p))=p$, has a unique C^1 solution in a neighbourhood of $u_r(p)$. Thus x_0 and x_1 coincide in this neighbourhood, and hence in the whole of $B_{\frac{1}{2}}(0)$, contradicting $y_0\neq y_1$. Therefore on the line segment joining y_0, y_1 there exists a point $y\notin x_0(B_{\frac{1}{2}}(0))\cup x_1(B_{\frac{1}{2}}(0))$. Hence $p=u_r(y)$ has at least three inverse images, a contradiction.

3. The displacement boundary-value problem of nonlinear elastostatics

Consider an elastic body which in a reference configuration occupies the bounded open set $\Omega\subset\mathbb{R}^3$. We suppose that Ω is non-empty, connected, and strongly Lipschitz. In a typical deformed configuration the particle P with position vector $x\in\Omega$ moves to the point P' having position vector $u(x)$ with respect to fixed Cartesian axes. The *deformation gradient* F is defined by

$$F=\nabla u; \quad F^i_\alpha = u^i_{,\alpha}.$$

The mechanical properties of the material are characterized by a stored-energy function $W(x, F)$ in terms of which the total stored-energy is

$$E(u)=\int_\Omega W(x, \nabla u(x))\, dx. \tag{3.1}$$

We consider a pure displacement boundary-value problem in which u is prescribed on $\partial\Omega$, so that

$$u\,|_{\partial\Omega} = u_0\,|_{\partial\Omega}, \tag{3.2}$$

where u_0 is a given function. If the body forces are conservative with potential $\psi(x, u)$ then the equilibrium equations are the Euler-Lagrange equations for the functional

$$I(u)=E(u)+\int_\Omega \psi(x, u(x))\, dx. \tag{3.3}$$

Notation: $M^{3\times3}$ *denotes the set of real* 3×3 *matrices,*

$$M^{3\times3}_+=\{F\in M^{3\times3}: \det F>0\}, \quad K=M^{3\times3}\times M^{3\times3}\times(0,\infty).$$

We make the following hypotheses on W, ψ and u_0:

(H1) $W: \bar\Omega\times M^{3\times3}_+\to\mathbb{R}$ is polyconvex; i.e. there exists a function $g: \bar\Omega\times K\to\mathbb{R}$ such that $g(x,\cdot)$ is convex for almost all $x\in\Omega$ and

$$W(x, F)=g(x, F, \operatorname{adj} F, \det F) \tag{3.4}$$

for all $F\in M^{3\times3}_+$ and almost all $x\in\Omega$. We suppose that g is a Carathéodory function, i.e. $g(x, \cdot)$ is continuous for almost all $x \in \Omega$ and $g(\cdot, a)$ is measurable for every $a \in K$.

(H2) There exists a function $k \in L^1(\Omega)$ and constants $C>0$, $p>3$, $q>3$, $s>2q/q-3$ such that

$$W(x, F) \geqq k(x) + C(|F|^p + |\text{adj } F|^q + (\det F)^{-s}) \tag{3.5}$$

for all $F \in M_+^{3\times 3}$ and almost all $x \in \Omega$.

(H4) $\psi: \bar{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function which is bounded below on $\Omega \times G$ for any bounded set $G \subset \mathbb{R}^3$.

(H4) $u_0 \in W^{1,p}(\Omega)$ is one-to-one in Ω, $\det \nabla u_0(x) > 0$ almost everywhere in Ω, $u_0(\Omega)$ satisfies the cone condition, and $I(u_0) < \infty$.

The reader is referred to [**4, 5**] for an extensive discussion of the physical implications of (H1) and (H2). Note that (H2) implies that (1.1) holds almost everywhere.

We now define a set $\mathcal{A}$ of admissible functions by $\mathcal{A} = \{w \in W^{1,1}(\Omega)$: $\det \nabla w(x) > 0$ almost everywhere in Ω, $I(w) < \infty$, and $w|_{\partial\Omega} = u_0|_{\partial\Omega}\}$.

Theorem 3. *Under the above hypotheses there exists $u \in \mathcal{A}$ which minimizes I on $\mathcal{A}$, u is a homeomorphism of Ω onto $u_0(\Omega)$ and the inverse function $x(u)$ belongs to $W^{1,\sigma}(u_0(\Omega))$, where $\sigma = q(1+s)/q+s$. If, further, $u_0(\Omega)$ is strongly Lipschitz, then u is a homeomorphism of $\bar{\Omega}$ onto $u_0(\bar{\Omega})$.*

Proof. Since $u_0 \in \mathcal{A}$, $\mathcal{A}$ is nonempty. Let $w \in \mathcal{A}$; then by (H2) $w \in W^{1,p}(\Omega)$ and

$$\int_\Omega [|\text{adj } \nabla w(x)|^q + (\det \nabla w(x))^{-s}]\, dx < \infty.$$

Using Hölder's inequality we deduce that

$$\int_\Omega |\nabla w^{-1}(x)|^\sigma \det \nabla w(x)\, dx = \int_\Omega |\text{adj } \nabla w(x)|^\sigma (\det \nabla w(x))^{1-\sigma}\, dx < \infty.$$

Since $\sigma > 3$, the hypotheses of Theorems 1 and 2 are satisfied by w. In particular, $w(\bar{\Omega}) = u_0(\bar{\Omega})$, and so by (H3) I is bounded below on $\mathcal{A}$. The existence of a minimizer u for I now follows as in [**8**, Th. 6.2] (see also [**4**, Th. 7.6, 7.7] and [**5**, Th. 4.1], where a slightly stronger version of (H2) is assumed). Since $u \in \mathcal{A}$ the proof is complete. ∎

Since we have made no smoothness assumptions on W and ψ, u will not in general even be C^1. Note that in order to ensure invertibility of u we imposed stronger conditions on p, q in (H2) than those in [**4, 5, 8**], where it was assumed only that $p \geqq 2$, $q \geqq p/p-1$. Provided $p>3$, however, Theorem 1 still gives some information concerning invertibility.

We remark that by Theorems 1 and 2,

$$I(u) = \int_{u_0(\Omega)} \hat{W}(x(v), \nabla x(v))\, dv + \int_{u_0(\Omega)} \psi(x(v), v) \det \nabla x(v)\, dv,$$

where

$$\hat{W}(x, G) \stackrel{\text{def}}{=} \det G\, W(x, G^{-1}).$$

See [**5**] for more information on $\hat{W}$, including a proof that $\hat{W}(x, \cdot)$ is polyconvex for almost every x. For a one-dimensional example see [**6**, Th. 4].

We now give an example of a function w satisfying (H1) and (H2). For $\alpha \geqq 1$, $\beta \geqq 1$, let

$$\rho(\alpha) = v_1^\alpha + v_2^\alpha + v_3^\alpha - 3,\ \chi(\beta) = (v_2 v_3)^\beta + (v_3 v_1)^\beta + (v_1 v_2)^\beta - 3,$$

where the v_i are the eigenvalues of $\sqrt{F^T F}$. Let

$$W(x, F) = \sum_{i=1}^{M} a_i(x)\rho(\alpha_i) + \sum_{j=1}^{N} b_i(x)\chi(\beta_j) + h(\det F),$$

where $\alpha_1 \geqq \cdots \geqq \alpha_M \geqq 1$, $\beta_1 \geqq \cdots \geqq \beta_N \geqq 1$, and where a_i, b_j are continuous functions on $\bar{\Omega}$ satisfying,

$$a_i(x) \geqq 0,\ b_j(x) \geqq 0, \quad \text{for} \quad 1 \leqq i \leqq M, 1 \leqq j \leqq N, x \in \bar{\Omega},$$
$$a_1(x) > 0,\ b_1(x) > 0, \quad \text{for} \quad x \in \bar{\Omega}.$$

Suppose further that $h: (0, \infty) \to \mathbb{R}$ is a convex function satisfying

$$h(\delta) \geqq \text{const.} + \gamma\delta^{-s},$$

with $\gamma > 0$, and that $\alpha_1 > 3$, $\beta_1 > 3$, $s > 2\beta_1/(\beta_1 - 3)$. Then W is isotropic and satisfies (H1) and (H2); for details see [**4, 5**].

Finally, we indicate the modifications to Theorem 2 that are necessary for incompressible materials. In this case we seek a minimum for I in the set

$$\mathcal{A}_1 = \{w \in W^{1,1}(\Omega): \det \nabla w(x) = 1 \text{ almost everywhere in } \Omega,$$
$$I(w) < \infty, w|_{\partial\Omega} = u_0|_{\partial\Omega}\}.$$

We replace (H1)–(H4) by (H1)′–(H4)′ below.

Let $V = \{F \in M^{3\times 3}: \det F = 1\}$.

(H1)′ $W: \bar{\Omega} \times V \to \mathbb{R}$, and there exists a Carathéodory function $g: \bar{\Omega} \times (M^{3\times 3} \times M^{3\times 3}) \to \mathbb{R}$ such that $g(x, \cdot)$ is convex for almost all $x \in \Omega$ and

$$W(x, F) = g(x, F, \operatorname{adj} F)$$

for all $F \in V$ and almost all $x \in \Omega$.

(H2)′ There exists a function $k \in L^1(\Omega)$ and constants $C > 0$, $p > 3$, $q > 3$ such that

$$W(x, F) \geqq k(x) + C(|F|^p + |\operatorname{adj} F|^q)$$

for all $F \in V$ and almost all $x \in \Omega$.

(H3)′ = (H3).

(H4)′ $u_0 \in \mathcal{A}_1$ is one-to-one in Ω, and $u_0(\Omega)$ satisfies the cone condition.

We then have the following theorem.

THEOREM 4. *Let* (H1)′–(H4)′ *hold. Then there exists* $u \in \mathcal{A}_1$ *which minimizes* I *on* $\mathcal{A}_1$, u *is a homeomorphism of* Ω *onto* $u_0(\Omega)$ *and the inverse function* $x(u)$ *belongs to* $W^{1,q}(u_0(\Omega))$.

If, further, $u_0(\Omega)$ *is strongly Lipschitz, then* u *is a homeomorphism of* $\bar{\Omega}$ *onto* $u_0(\bar{\Omega})$.

Proof. This is the same as for Theorem 2, except that we use the incompressible existence theory from [**4**, **5**], modified as in [**8**] to accommodate the weakened form of (H3)′. ■

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References

1 R. A. Adams. *Sobolev spaces* (New York: Academic, 1975).
2 S. S. Antman. Ordinary differential equations of nonlinear elasticity II: Existence and regularity theory for conservative boundary value problems. *Arch. Rational Mech. Anal.* **61** (1976), 353–393.
3 S. S. Antman and H. Brezis. The existence of orientation-preserving deformations in nonlinear elasticity. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium* vol. II, ed. R. J. Knops (London: Pitman, 1978).
4 J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (1977), 337–403.
5 J. M. Ball. Constitutive inequalities and existence theorems in nonlinear elastostatics. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium* vol. I, ed. R. J. Knops (London: Pitman, 1977).
6 J. M. Ball. Remarques sur l'existence et la régularité des solutions d'élastostatique nonlinéaire (London: Pitman, to appear).
7 J. M. Ball. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, to appear.
8 J. M. Ball, J. C. Currie and P. J. Olver. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Functional Analysis*, to appear.
9 M. S. Berger. *Nonlinearity and Functional Analysis* (New York: Academic, 1977).
10 J. M. Bony. Principe du maximum dans les espaces de Sobolev. *C.R. Acad. Sci. Paris Sér. A* **265** (1967), 333–336.
11 F. E. Browder. Nonlinear Operators and Nonlinear Equations of Evolution in Banach spaces. *Proc. Symposia in Pure Mathematics* **18,** Pt 2 (Providence, R.I.: Amer. Math. Soc., 1976).
12 F. H. Clarke. On the inverse function theorem. *Pacific J. Math.* **64** (1976), 97–102.
13 M. Golubitsky and V. Guillemin. *Stable mappings and their singularities* (Berlin: Springer, 1973).
14 M. Marcus and V. J. Mizel. Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. *Bull. Amer. Math. Soc.* **79** (1973), 790–795.
15 G. H. Meisters and C. Olech. Locally one-to-one mappings and a classical theorem on Schlicht functions. *Duke Math. J.* **30** (1963), 63–80.
16 C. B. Morrey. *Multiple integrals in the calculus of variations* (Berlin: Springer, 1966).
17 L. Nirenberg. *Topics in nonlinear functional analysis* (New York: Courant Institute Lecture Notes, 1974).
18 T. Rado and P. V. Reichelderfer. *Continuous transformations in analysis* (Berlin: Springer, 1955).
19 J. T. Schwartz. *Nonlinear functional analysis* (New York: Gordon and Breach, 1969).
20 S. K. Vodop'yanov and V. M. Gol'dshtein. Quasiconformal mappings and spaces of functions with generalized first derivatives. *Sibirsk. Mat. Z.* **17** (1976), 515–531.
21 S. K. Vodop'yanov, V. M. Gol'dshtein and Yu. G. Reshetnyak. On geometric properties of functions with generalized first derivatives. *Russian Math. Surveys* **34** (1979), 19–74.

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