Calc. Var. 14, 1–27 (2002) **Calculus of Variations**

DOI (Digital Object Identifier) 10.1007/s005260100085

J.M. Ball · A. Taheri · M. Winter

Local minimizers in micromagnetics and related problems

Received: 20 November 2000 / Accepted: 4 December 2000 /
Published online: 4 May 2001 – © Springer-Verlag 2001

Abstract. Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain and consider the energy functional

$$\mathcal{J}_\varepsilon(m;\Omega) := \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + \psi(m) + \frac{1}{2}|h-m|^2 \right) dx + \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2\, dx.$$

Here $\varepsilon > 0$ is a small parameter and the admissible function m lies in the Sobolev space of vector-valued functions $W^{1,2}(\Omega;\mathbf{R}^3)$ and satisfies the pointwise constraint $|m(x)| = 1$ for a.e. $x \in \Omega$. The induced magnetic field $h_m \in L^2(\mathbf{R}^3;\mathbf{R}^3)$ is related to m via Maxwell's equations and the function $\psi : \mathbf{S}^2 \to \mathbf{R}$ is assumed to be a sufficiently smooth, non-negative energy density with a multi-well structure. Finally $h \in \mathbf{R}^3$ is a constant vector. The energy functional $\mathcal{J}_\varepsilon$ arises from the continuum model for ferromagnetic materials known as *micromagnetics* developed by W.F. Brown [9].

In this paper we aim to construct local energy minimizers for this functional. Our approach is based on studying the corresponding Euler-Lagrange equation and proving a *local existence* result for this equation around a fixed constant solution. Our main device for doing so is a suitable version of the implicit function theorem. We then show that these solutions are local minimizers of $\mathcal{J}_\varepsilon$ in appropriate topologies by use of certain sufficiency theorems for local minimizers.

Our analysis is applicable to a much broader class of functionals than the ones introduced above and on the way to proving our main results we reflect on some related problems.

1 Introduction

The micromagnetic theory of ferromagnetic materials as developed by Brown [9] consists of studying the minimizers of the energy functional

$$\mathcal{J}_\varepsilon(m;\Omega) = \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + \psi(m) + \frac{1}{2}|h-m|^2 \right) dx + \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2\, dx.$$

Here $\Omega \subset \mathbf{R}^3$ is a sufficiently regular open set representing the region occupied by the body and the unknown function $m : \Omega \to \mathbf{S}^2$ denotes an arbitrary magnetization state for the body.

The various terms appearing in this energy functional are respectively

J.M. Ball, A. Taheri*: Mathematical Institute, University of Oxford, Oxford, UK

M. Winter: Mathematisches Institut A, Universität Stuttgart, Stuttgart, Germany

* *Present address:* Department of Mathematics, Heriot-Watt University, Edinburg, UK

(i) **The exchange energy:** This term penalizes spatial changes in the direction of the magnetization m and hence reflects the tendency of the body to maintain a spatially uniform magnetization state.

(ii) **The anisotropy energy:** This term describes the existence of preferred directions of magnetization or the so-called *easy axes* for the magnetic state of the material. To be more specific the anisotropy energy density $\psi : \mathbf{S}^2 \to \mathbf{R}$ is such that $\psi(m) \geq 0$ for all $m \in \mathbf{S}^2$ and $\psi(m) = 0$ if and only if $m \in K$ where K is a finite set of unit vectors representing the preferred directions for magnetization.

(iii) **The external field energy:** If the body lies in a region of space where an external magnetic field $h : \Omega \to \mathbf{R}^3$ is present, the magnetization m tends to align itself with the direction of this field. The external field energy thus penalizes any deviation from this field inside the body. In this paper we assume that the applied field h is spatially uniform.

(iv) **The field energy:** The magnetization state m in the body generates a magnetic field $h_m : \mathbf{R}^3 \to \mathbf{R}^3$ that satisfies Maxwell's equations:

$$\begin{cases} \textbf{curl}\, h_m = 0, \\ \textbf{div}\, (h_m + m\chi_\Omega) = 0. \end{cases}$$

The above equations show that the field h_m is nothing but the gradient part of the Helmholtz decomposition of $-m\chi_\Omega$.

We recall that James and Müller [20], following some earlier work by Lorentz [23] (cf. also Toupin [28]), have obtained this field energy by studying the corresponding energy for a lattice of magnetic dipoles and passing to the continuum limit by letting a typical lattice parameter go to zero.

In this paper we are interested in studying the limiting behaviour of the family of functionals $\mathcal{J}_\varepsilon$ as the parameter $\varepsilon \to 0$. This is usually referred to in the literature as the small particle limit, and can be justified by observing that the functional $\mathcal{J}_1$ satisfies the simple rescaling property $\mathcal{J}_1(m, \varepsilon^{\frac{1}{2}}\, \Omega) = \varepsilon^{\frac{3}{2}} \mathcal{J}_\varepsilon(m_\varepsilon, \Omega)$ for any $\varepsilon > 0$, where $m_\varepsilon(x) = m(\varepsilon^{\frac{1}{2}} x)$. It is clear that this property enables one to keep the domain Ω fixed and instead study the rescaled functionals $\mathcal{J}_\varepsilon$ as $\varepsilon \to 0$.

Our primary aim is to construct local minimizers for $\mathcal{J}_\varepsilon$. We note that prior work on this problem due to De Simone [11] employs ideas of De Giorgi, or more precisely the notion of Γ-convergence, which itself has been developed for the study of local minimizers by Kohn and Sternberg [21]. Our method is more direct. To be more specific we *construct* stationary points for the energy functional J_ε using an appropriate version of the implicit function theorem and then apply certain sufficiency theorems to establish the desired minimality property for these stationary points. It turns out that our results are stronger than the known ones in the sense that the stationary points constructed are local minimizers of $\mathcal{J}_\varepsilon$ in weaker norms. Our analysis can be regarded as a very modest first step towards the rigorous study of the pattern formation problem for magnetic domains as ε increases. A major difficulty in carrying out such a study is that of understanding bifurcations of solutions that are not known explicitly.

At this stage we should like to remark that the idea of applying versions of the implicit function theorem to achieve *local existence* for various equilibrium equations of continuum mechanics has been employed before in different contexts (cf. Stoppelli, Valent [30], Zhang [32], Ball *et. al.* [5] for examples within elasticity theory). The idea in this paper however is to combine such local existence theorems together with certain sufficiency theorems to ensure the existence of a continuous branch of local energy minimizers.

Throughout the paper we assume that $\Omega \subset \mathbf{R}^n$ is a bounded domain (open connected set) with a smooth boundary $\partial\Omega$. We denote the unit outward normal to the boundary at a point x by $\nu(x)$, and as usual $\mathcal{L}^n(\cdot)$ stands for n-dimensional Lebesgue measure. As regards the energy functional $\mathcal{J}_\varepsilon$ the dimension $n = 3$. However we do not restrict our analysis to this case only and allow n to be any positive integer.

For the admissible class of functions we use the Sobolev spaces of vector-valued functions $W^{m,p}(\Omega;\mathbf{R}^N)$ where m is a positive integer and the exponent $1 \leq p \leq \infty$. Our terminology for these spaces is in accordance with [1], [15] and [33] and we refer the interested reader to these books for relevant properties of these functions.

Assume now that $\mathcal{A} \subset W^{m,p}(\Omega;\mathbf{R}^N)$ is a given set of admissible functions and $\mathcal{J} : \mathcal{A} \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, \infty\}$ a given functional. For later reference we state the following

Definition 1.1. *Let $1 \leq r \leq \infty$. The function $m_0 \in \mathcal{A}$ is an L^r local minimizer of $\mathcal{J}$ if and only if there exists $\delta > 0$ such that*

$$\mathcal{J}(m_0) \leq \mathcal{J}(m)$$

for all $m \in \mathcal{A}$ satisfying

$$||m - m_0||_{L^r(\Omega;\mathbf{R}^N)} < \delta.$$

To gain a clear understanding of the energy minimization problem described above we proceed by considering two related but slightly simplified problems each having some ingredients of the original micromagnetic energy functional.

In the first problem we consider the family of functionals

$$\mathcal{I}_\varepsilon(u) := \int_\Omega \left(\frac{1}{2\varepsilon} |\nabla u|^2 + F(x,u) \right) dx,$$

with $\varepsilon > 0$, and $F \in C^2(\overline{\Omega} \times \mathbf{R})$. Here the function u is assumed to belong to the class

$$\mathcal{A}_1 := \{u \in W^{1,2}(\Omega) : \mathcal{I}_\varepsilon \text{ is well defined}\}.$$

By well defined we mean that the function $F(\cdot, u(\cdot))$ has a well-defined integral, i.e. that at least one of the functions $F^+ := \max\{F(\cdot, u(\cdot)), 0\}$ or $F^- := \min\{F(\cdot, u(\cdot)), 0\}$ has a finite integral. It is therefore to be understood that $\mathcal{I}_\varepsilon : \mathcal{A}_1 \to \overline{\mathbf{R}}$.

Note that we have dropped the pointwise constraint $|u(x)| = 1$ for the admissible functions that occurs in the micromagnetic problem. In addition we have restricted attention to scalar valued functions, that is to $N = 1$. However this latter assumption is not a technical obstacle and almost all the statements and results in this case extend to the case $N > 1$ without any difficulty.

In our analysis special attention is paid to a limiting problem corresponding to the case $\varepsilon = 0$. We start by imposing conditions on the integrand F and a given point $\tilde{u} \in \mathcal{A}_1$ that turn out to be sufficient for $\tilde{u}$ to be a local minimizer of an appropriate functional corresponding to the $\varepsilon = 0$ problem. We then apply the implicit function theorem to prove the existence of a branch of stationary points u^ε for $\mathcal{I}_\varepsilon$ when $\varepsilon > 0$ is sufficiently small. Note that the Euler-Lagrange equation corresponding to $\mathcal{I}_\varepsilon$ takes the simple form

$$\begin{cases} \Delta u = \varepsilon F_u(x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Having established the existence of such stationary points we then proceed to study the second variation of the functional $\mathcal{I}_\varepsilon$ at these points. Our starting assumptions on F and $\tilde{u}$ imply that the second variation at each u^ε is indeed positive and thus according to the sufficiency theorem in Section 2 (Theorem 2.2) these points are L^r local minimizers of the corresponding $\mathcal{I}_\varepsilon$, where the exponent r depends on the growth of F at infinity.

By imposing further assumptions on the integrand F we are able to show that for a sufficiently small range of the parameter ε the stationary points of $\mathcal{I}_\varepsilon$ obtained by the application of the implicit function theorem are the *only* stationary points of $\mathcal{I}_\varepsilon$. This in particular means that if the limiting functional has only a finite number of *nondegenerate* stationary points the same holds true for $\mathcal{I}_\varepsilon$ when ε is small.

Having a clear understanding of the first problem we then proceed to the second family that consists of functionals of the form

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(\frac{1}{2\varepsilon}|Du|^2 + V(x,u) \right) dx,$$

where $V \in C^2(\overline{\Omega} \times \mathbf{S}^{N-1})$. Here we aim to deal with the pointwise constraint $|u(x)| = 1$ and leave out the only remaining task, i.e. handling the *non-local* term in the original micromagnetics problem, to the final stage. Thus we introduce the class of admissible functions

$$\mathcal{A}_2 := \{u \in W^{1,2}(\Omega;\mathbf{R}^N) : |u(x)| = 1 \quad a.e.\}.$$

It follows immediately from the constraint on u and the continuity assumption on V that $\mathcal{F}_\varepsilon$ is well defined and in fact finite over $\mathcal{A}_2$. In this setting it is also possible to assume without loss of generality that $V \in C^2(\overline{\Omega} \times \mathbf{R}^N)$ and vanishes for large $|u|$.

As in the first problem our analysis is linked to studying a limiting functional corresponding to the $\varepsilon = 0$ case. We impose conditions on the integrand V and a given $\tilde{u} \in \mathbf{S}^{N-1}$ that in turn imply $\tilde{u}$ to be a *constrained* local minimizer of this latter functional.

It can be shown that here the Euler-Lagrange equation corresponding to $\mathcal{F}_\varepsilon$ takes the form

$$\begin{cases} \Delta u + |Du|^2 u - \varepsilon(I - u \otimes u)V_u(x,u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

We again apply the implicit function theorem to prove the existence of a continuous branch of solutions to the Euler-Lagrange equation corresponding to $\mathcal{F}_\varepsilon$.

To deal with the pointwise constraint $|u(x)| = 1$ in applying Theorem 2.2, we extend the functional $\mathcal{F}_\varepsilon$ to $\tilde{\mathcal{F}}_\varepsilon : W^{1,2}(\Omega; \mathbf{R}^N) \to \mathbf{R}$ in such a way that

(i) $\tilde{\mathcal{F}}_\varepsilon(u) = \mathcal{F}_\varepsilon(u)$ for every $u \in \mathcal{A}_2$,
(ii) If u^ε is a stationary point of $\mathcal{F}_\varepsilon$ it is also a stationary point of $\tilde{\mathcal{F}}_\varepsilon$, and
(iii) $\delta^2 \tilde{\mathcal{F}}_\varepsilon(u^\varepsilon) > 0$ for ε sufficiently small provided a similar condition hold for the solution to the $\varepsilon = 0$ problem.

It then follows from Theorem 2.2 that u^ε is an L^1 local minimizer of $\tilde{\mathcal{F}}_\varepsilon$ and so (i) implies the same to be true for $\mathcal{F}_\varepsilon$ as $u \in \mathcal{A}_2$.

To end this introduction we give a brief description of the plan of the paper. In Section 2 we gather some known results and key tools that will be frequently referred to throughout the article. This in particular includes the statements of both an appropriate version of the implicit function theorem and a sufficiency theorem for L^r local minimizers of certain functionals. In Section 3 we study the first problem, namely the family of functionals $\mathcal{I}_\varepsilon$. Section 4 continues with the first problem and includes a detailed analysis of the second variation of $\mathcal{I}_\varepsilon$ along the branch of stationary points constructed in Section 3. In addition we study the number of such solutions for fixed values of ε when this parameter is sufficiently small. In Section 5 we move on to the constrained problem, that is the study of the functionals $\mathcal{F}_\varepsilon$. Finally in section 6 we return to the micromagnetics problem and apply the same ideas to construct L^1 local minimizers for the functional $\mathcal{J}_\varepsilon$.

2 Preliminaries

In this section we gather some well-known results needed for our later analysis.

As pointed out in Section 1, our main tool for constructing solutions to the Euler-Lagrange equations is the implicit function theorem. For the following version we refer the interested reader to the monographs by Ambrosetti and Prodi [2] or Zeidler [31] for the proofs and further discussions.

Theorem 2.1. *Let X, Y, and Z be Banach spaces, U an open subset of $X \times Y$, and $T = T(\varepsilon, u)$ a C^1 map from U into Z. Let $(\varepsilon_0, u_0) \in U$ be such that $T(\varepsilon_0, u_0) = 0$ and $D_u T(\varepsilon_0, u_0)$ is a bijection of Y onto Z. Then there exist an open neighbourhood U_0 of (ε_0, u_0) in $X \times Y$, an open neighbourhood V_0 of ε_0 in X, and a C^1 function $\omega : V_0 \to Y$ such that*

$$\{(\varepsilon, u) \in U_0 : T(\varepsilon, u) = 0\} = \{(\varepsilon, u) : \varepsilon \in V_0, u = \omega(\varepsilon)\}.$$

Furthermore, U_0 can be chosen so that $D_u T(\varepsilon, u)$ is a bijection of Y onto Z for all $(\varepsilon, u) \in U_0$. In this case, if $\varepsilon \in V_0$ then

$$D\omega(\varepsilon) = -(D_u T(\varepsilon, \omega(\varepsilon)))^{-1} D_\varepsilon T(\varepsilon, \omega(\varepsilon)), \tag{2.1}$$

while if T is analytic at $(\varepsilon, \omega(\varepsilon))$ then ω is analytic at ε.

While the implicit function theorem can be applied to the Euler-Lagrange equation to establish the existence of a branch of solutions starting from a given function, we need certain sufficiency theorems to guarantee that such stationary points are under suitable conditions local minimizers for the corresponding functional.

We now state a sufficiency theorem for L^r local minimizers of functionals of the type appearing in this article. For this let $F : \overline{\Omega} \times \mathbf{R}^N \to \mathbf{R}$ be given and consider the functional

$$\mathcal{I}(u) := \int_\Omega \left(\frac{1}{2}|Du|^2 + F(x,u) \right) dx,$$

over the class of admissible functions

$$\tilde{\mathcal{A}} := \{u \in W^{1,2}(\Omega;\mathbf{R}^N) : \mathcal{I} \text{ is well-defined}\}.$$

We can now state the following result from [26].

Theorem 2.2. *Let $F \in C^2(\overline{\Omega} \times \mathbf{R}^N)$ and assume that there are constants $C > 0$ and $p \geq 1$ such that*

$$F(x,u) \geq -C(1+|u|^p) \tag{2.2}$$

for all $x \in \Omega$ and all $u \in \mathbf{R}^N$. Furthermore let $u_0 \in \tilde{\mathcal{A}}$ be of class $L^\infty(\Omega;\mathbf{R}^N)$ and satisfy

$$(i) \quad \frac{d}{dt}\mathcal{I}(u_0 + t\varphi)|_{t=0} = 0, \qquad (ii) \quad \frac{d^2}{dt^2}\mathcal{I}(u_0 + t\varphi)|_{t=0} \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^n)},$$

for all $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$ and some $\gamma > 0$. Finally let $r = r(n,p,2) = \max(1, \frac{n}{2}(p-2))$. Then there exist $\sigma, \rho > 0$ such that

$$\mathcal{I}(u) - \mathcal{I}(u_0) \geq \sigma ||u - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $u \in \tilde{\mathcal{A}}$ satisfying $||u-u_0||_{L^r(\Omega;\mathbf{R}^N)} < \rho$.

Remark 2.1. Following exactly the same argument as in the proof of Theorem 2.1 in [26], one can show that the conclusion above holds if we replace F by $F(x,u) + a(x)\cdot u + b(x)(|u|^2 - 1)$ where $a \in L^2(\Omega;\mathbf{R}^N)$ and $b \in L^\infty(\Omega)$.

Remark 2.2. The lower bound on $\mathcal{I}(u) - \mathcal{I}(u_0)$ in the theorem shows that $\mathcal{I}(u) > \mathcal{I}(u_0)$ whenever $u \neq u_0$ and $||u-u_0||_{L^r(\Omega)}$ is sufficiently small, i.e. u_0 is a *strict* local minimizer of $\mathcal{I}$. But it says more than this. Suppose, for example, that F is bounded from below, so that $r = 1$. Then there is a *potential well* at u_0 in the sense that for all sufficiently small $\varepsilon > 0$,

$$\mathcal{I}(u_0) < \inf_{\{u \in \tilde{\mathcal{A}} : ||u-u_0||_{L^1(\Omega)} = \varepsilon\}} \mathcal{I}(u).$$

(The same holds if we use the $W^{1,2}$ norm in place of the L^1 norm.) We refer the interested reader to [3] and [26] for more discussion on this and its connection to dynamic stability of u_0.

As pointed out earlier, the magnetization m and the field h_m are related to one another by the following system of differential equations

$$\begin{cases} \mathbf{curl}\, h_m = 0, \\ \mathbf{div}\,(h_m + m\chi_\Omega) = 0. \end{cases} \tag{2.3}$$

In the next theorem we gather some of the important properties of the solution operator of this system.

Theorem 2.3. *There exists a continuous linear operator* $\mathcal{H} : L^2(\mathbf{R}^3;\mathbf{R}^3) \to L^2(\mathbf{R}^3;\mathbf{R}^3)$ *such that*

(i) *Given* $m \in L^2(\Omega;\mathbf{R}^3)$, (2.3) *holds in the sense of distributions on* $\mathbf{R}^3$*, for* $h_m := \mathcal{H}(m\chi_\Omega)$.

(ii) *For every* m_1 *and* $m_2 \in L^2(\Omega;\mathbf{R}^3)$,

$$\int_{\mathbf{R}^3} h_{m_1} \cdot h_{m_2}\, dx = -\int_\Omega m_1 \cdot h_{m_2}\, dx = -\int_\Omega h_{m_1} \cdot m_2\, dx,$$

and so in particular $||h_{m_1}||^2_{L^2(\mathbf{R}^3;\mathbf{R}^3)} = -\int_\Omega m_1 \cdot h_{m_1}$.

(iii) *There exists a positive definite, symmetric matrix* D_e *such that for every constant function* m

$$\int_\Omega h_m\, dx = -D_e m.$$

For a proof of (i) we refer the reader to [18]. Part (ii) follows from (2.3) and a simple integration by parts. The proof of (iii) is a consequence of the linearity of $\mathcal{H}$ and (ii). See [11] for more details.

3 The unconstrained problem

We begin this section by formally deriving the Euler-Lagrange equation corresponding to the functional $\mathcal{I}_\varepsilon$. In its weak form this is the condition

$$\frac{d}{dt}\mathcal{I}_\varepsilon(u + t\varphi)|_{t=0} = 0, \tag{3.1}$$

where the variation $\varphi \in C^\infty(\overline{\Omega})$. First, since equation (3.1) holds for all $\varphi \in C_0^\infty(\Omega)$ we deduce that

$$\Delta u = \varepsilon F_u(x, u). \tag{3.2}$$

Second, since (3.1) holds for all $\varphi \in C^\infty(\overline{\Omega})$ we get the natural boundary condition

$$\frac{\partial u}{\partial \nu} = 0. \tag{3.3}$$

Now we introduce the setting for the application of the implicit function theorem (cf. Theorem 2.1). A key point in the application of this theorem is the choice of the spaces X, Y and Z in order to ensure that the linearization of T at (ε_0, u_0) is a bijection. To discuss this, as a first attempt in applying the implicit function theorem to (3.2) and (3.3) let us consider the map $T_1 : \mathbf{R} \times W^{2,s}(\Omega) \to L^s(\Omega) \times W^{1-\frac{1}{s},s}(\partial\Omega)$ given by

$$T_1(\varepsilon, u) = \begin{pmatrix} \Delta u(x) - \varepsilon F_u(x, u(x)) \\ \frac{\partial u}{\partial \nu}(x) \end{pmatrix},$$

for some $s > \frac{n}{2}$. Clearly if $u^\varepsilon \in W^{2,s}(\Omega)$ is such that $T_1(\varepsilon, u^\varepsilon) = 0$, then u^ε would be the required branch of stationary points of I_ε, that is a continuous family of solutions to (3.2) and (3.3) in $W^{2,s}(\Omega)$. However it is a trivial matter to see that for the above choice of spaces the linearization of T_1 at any point $(0, u) \in \mathbf{R} \times W^{2,s}(\Omega)$ is not a bijection. To overcome this difficulty and also to motivate the proof of Theorem 3.1 let us formally seek a solution to (3.2) and (3.3) in the form

$$u(\varepsilon) = u + \varepsilon v + \varepsilon^2 w + ...$$

Substituting this into the equation it immediately follows that $u = \tilde{u}$ is constant. Moreover other powers of ε lead to further equations, namely

$$\begin{cases} \Delta v = F_u(x, \tilde{u}) \\ \frac{\partial v}{\partial \nu}(x) = 0, \end{cases} \tag{3.4}$$

for the coefficients of ε, and similarly

$$\begin{cases} \Delta w = F_{uu}(x, \tilde{u})\, v \\ \frac{\partial w}{\partial \nu}(x) = 0, \end{cases} \tag{3.5}$$

for the coefficients of ε^2. It follows that a necessary condition for solvability of (3.4) is that

$$\int_\Omega F_u(x, \tilde{u})\, dx = 0.$$

Moreover the solution obtained in this way is unique up to an additive constant. Substituting this solution v into (3.5) and using the necessary condition for solvability of (3.5), that is

$$\int_\Omega F_{uu}(x, \tilde{u}) v(x)\, dx = 0,$$

it follows that this constant is uniquely determined provided

$$\int_\Omega F_{uu}(x, \tilde{u})\, dx \neq 0.$$

Following this informal discussion we proceed with the detailed analysis by introducing the map

$$T : \mathbf{R} \times W^{2,s}(\Omega) \to E^s(\Omega) \times \mathbf{R}$$

defined by

$$T(\varepsilon, u) = \begin{pmatrix} \Delta u(x) - \varepsilon \left(F_u(x, u(x)) - ⨍_\Omega F_u(x, u(x))\, dx \right) \\ \frac{\partial u}{\partial \nu}(x) \\ ⨍_\Omega F_u(x, u(x)) dx \end{pmatrix},$$

where $\fint_\Omega$ denotes averaging over Ω,

$$E^s(\Omega) = \left\{ (f,g) \in L^s(\Omega) \times W^{1-1/s,s}(\partial\Omega) : \int_\Omega f\,dx = \int_{\partial\Omega} g\,d\mathcal{H}^{n-1} \right\}, \tag{3.6}$$

and we set

$$s > \frac{n}{2}. \tag{3.7}$$

It is clear that for $\varepsilon \neq 0$ a function $u \in W^{2,s}(\Omega)$ is a solution of the Euler-Lagrange equation (3.2) with boundary condition (3.3) if and only if it satisfies

$$T(\varepsilon, u) = 0.$$

We now claim that for the choice of s given by (3.7), $T \in C^1(\mathbf{R} \times W^{2,s}(\Omega); E^s(\Omega) \times \mathbf{R})$. To show this we look at the partial Gateaux derivative $D_u T$ at an arbitrary point $(\varepsilon, u) \in \mathbf{R} \times W^{2,s}(\Omega)$. Indeed we have

$$D_u T(\varepsilon, u)(U) = \begin{pmatrix} \Delta U - \varepsilon \left(F_{uu}(x,u)U - \fint_\Omega F_{uu}(x,u)U\,dx \right) \\ \frac{\partial U}{\partial \nu} \\ \fint_\Omega F_{uu}(x,u)U(x)\,dx \end{pmatrix},$$

for each $U \in W^{2,s}(\Omega)$. Now $D_u T$ is continuous if and only if for all sequences $u^{(k)} \to u$ in $W^{2,s}(\Omega)$, and $\varepsilon^{(k)} \to \varepsilon$ in $\mathbf{R}$, it follows that

$$\sup \left\{ ||(D_u T(\varepsilon^{(k)}, u^{(k)}) - D_u T(\varepsilon, u))(U)||_{E^s(\Omega)\times\mathbf{R}} : ||U||_{W^{2,s}(\Omega)} \leq 1 \right\} \to 0.$$

But this is an immediate consequence of $u^{(k)} \to u$ in $L^\infty(\Omega)$ as a result of (3.7) and the continuity of the embedding $W^{1,s}(\Omega) \to W^{1-\frac{1}{s},s}(\partial\Omega)$. A similar argument can be applied to $D_\varepsilon T$ and so the claim is justified.

To check that the remaining assumptions of Theorem 2.1 are true we begin by solving the equation $T(0,u) = 0$, i.e.

$$\begin{cases} \Delta u = 0, \\ \frac{\partial u}{\partial \nu} = 0, \\ \fint_\Omega F_u(x, u(x))\,dx = 0. \end{cases}$$

It follows from the first two equations that u is a constant. Call this constant $\tilde{u}$. We are therefore left with the third equation,

$$\fint_\Omega F_u(x, \tilde{u})\,dx = 0. \tag{3.8}$$

Assume there exists $\tilde{u}$ such that (3.8) holds. To check the second assumption of Theorem 2.1 we need to show that the linear operator $D_u T(0, \tilde{u}) : W^{2,s} \to E^s(\Omega) \times \mathbf{R}$ is bijective. This amounts to proving that the system

$$\begin{cases} \Delta U = f \\ \frac{\partial U}{\partial \nu} = g \\ \fint_\Omega F_{uu}(x, \tilde{u})\,U(x)\,dx = t \end{cases}$$

has a unique solution $U \in W^{2,s}(\Omega)$ for all $(f, g, t) \in E^s(\Omega) \times R$. It is well known (see e.g. [8] or [30]) that given $(f, g) \in E^s(\Omega)$, the system

$$\begin{cases} \Delta U = f \\ \frac{\partial U}{\partial \nu} = g \end{cases}$$

has a solution $U \in W^{2,s}(\Omega)$, which is unique up to an additive constant. If

$$⨍_\Omega F_{uu}(x, \tilde{u})\, dx \neq 0$$

this constant can be determined in a unique way by solving the third equation,

$$⨍_\Omega F_{uu}(x, \tilde{u})\, U(x)\, dx = t.$$

Thus we have proved

Theorem 3.1. *Suppose there exists a constant $\tilde{u}$ such that*

$$\int_\Omega F_u(x, \tilde{u}) dx = 0 \tag{3.9}$$

and that

$$\int_\Omega F_{uu}(x, \tilde{u}) dx \neq 0. \tag{3.10}$$

Then for ε small enough the Euler-Lagrange equation (3.2) subject to the boundary condition (3.3) has a solution u^ε which is contained in the Sobolev space $W^{2,s}(\Omega)$ and is close to $\tilde{u}$ in the corresponding norm. Furthermore if the neighbourhood of $\tilde{u}$ in $W^{2,s}(\Omega)$ is taken small enough, u^ε is the only solution to (3.2), (3.3) lying in this neighbourhood.

Having proved the existence of a continuous branch of stationary points for $\mathcal{I}_\varepsilon$, we proceed to address the question of under what conditions on the integrand F and $\tilde{u}$ the solution u^ε is a local minimizer for $\mathcal{I}_\varepsilon$. We pursue this in the following section.

4 Local minimizers and the positivity of the second variation

We first consider the question of positivity of the quadratic functional

$$\mathcal{Q}(\varphi) = \int_\Omega \left(|\nabla \varphi|^2 + a(x)\varphi^2 \right) \, dx, \tag{4.1}$$

over $W^{1,2}(\Omega)$ for given $a \in L^\infty(\Omega)$. Setting φ to be constant it follows immediately that the condition

$$\int_\Omega a\, dx > 0 \tag{4.2}$$

is necessary. We can however prove

Proposition 4.1. *Let $\mathcal{Q}$ be as in* (4.1) *and let a satisfy* (4.2)*. Then there exists $\gamma > 0$ such that*

$$\mathcal{Q}(\varphi) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega)} \tag{4.3}$$

provided $||a||_{L^\infty(\Omega)}$ is sufficiently small.

Proof. Given $\varphi \in W^{1,2}(\Omega)$ we can write $\varphi = \tilde{\varphi} + \fint_\Omega \varphi\, dx$, where $\fint_\Omega \tilde{\varphi}\, dx = 0$. Thus setting $c = \fint_\Omega \varphi\, dx$ we can write

$$\begin{aligned} \mathcal{Q}(\varphi) &= \int_\Omega \left(|\nabla\tilde{\varphi}|^2 + a\,(\tilde{\varphi} + c)^2 \right)\, dx \\ &= \int_\Omega \left(|\nabla\tilde{\varphi}|^2 + a\tilde{\varphi}^2 + 2ac\tilde{\varphi} + ac^2 \right)\, dx \\ &\geq \int_\Omega \frac{1}{2}|\nabla\tilde{\varphi}|^2\, dx + \int_\Omega \left(\frac{1}{2}|\nabla\tilde{\varphi}|^2 + a\tilde{\varphi}^2 \right)\, dx - \tau \int_\Omega \tilde{\varphi}^2\, dx \\ &\quad + c^2 \int_\Omega \left(a(1 - \frac{a}{\tau}) \right)\, dx, \end{aligned} \tag{4.4}$$

that holds for every $\tau > 0$. If now $||a||_{L^\infty(\Omega)} < \frac{1}{2}\lambda_2$ where $\lambda_2 > 0$ denotes the second eigenvalue of the Laplacian subject to Neumann boundary conditions on $\partial\Omega$ and τ is sufficiently small the sum of the second and third terms in (4.4) will be positive. Choosing $||a||_{L^\infty(\Omega)}$ smaller if necessary, it follows from the Poincaré inequality that there exists $\gamma > 0$ such that

$$\mathcal{Q}(\varphi) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega)}.$$

The proof is thus complete. □

As a consequence of the above proposition and Theorem 2.2 we can state the following

Theorem 4.1. *Assume that the hypotheses of Theorem 3.1 hold and that*

$$\int_\Omega F_{uu}(x, \tilde{u})\, dx > 0. \tag{4.5}$$

Then the solution u^ε given by Theorem 3.1 is an L^∞ local minimizer of $\mathcal{I}_\varepsilon$. Furthermore if the growth of F from below is restricted by

$$F(x, u) \geq -C(1 + |u|^p)$$

for some $C > 0$ and $p \geq 1$, then u^ε is an L^r local minimizer with $r(n, p) = \max(1, \frac{n}{2}(p-2))$. In particular if F is bounded from below then u^ε is an L^1 local minimizer of $\mathcal{I}_\varepsilon$.

Proof. We start by calculating the second variation of $\mathcal{I}_\varepsilon$ at the stationary point u^ε. Indeed for $\varphi \in C^\infty(\overline{\Omega})$

$$\begin{aligned} \delta^2\mathcal{I}_\varepsilon(u^\varepsilon,\varphi) &= \frac{d^2}{dt^2}\mathcal{I}_\varepsilon(u^\varepsilon+t\varphi)\,|_{t=0} \\ &= \frac{1}{\varepsilon}\int_\Omega \left(|\nabla\varphi|^2+\varepsilon F_{uu}(x,u^\varepsilon)\varphi^2\right)\,dx. \end{aligned} \tag{4.6}$$

Note that

$$\int_\Omega F_{uu}(x,u^\varepsilon)\,dx \geq \int_\Omega F_{uu}(x,\tilde{u})\,dx - \int_\Omega |F_{uu}(x,u^\varepsilon)-F_{uu}(x,\tilde{u})|\,dx > 0$$

provided ε is sufficiently small. Thus it follows from Proposition 4.1 that for ε small enough

$$\delta^2\mathcal{I}_\varepsilon(u^\varepsilon,\varphi) \geq \gamma||\varphi||^2_{W^{1,2}(\Omega)}$$

for some $\gamma=\gamma(\varepsilon)>0$ and all $\varphi\in W^{1,2}(\Omega)$. The result is now a consequence of Theorem 2.2. □

Remark 4.1. Consider the function $I_0:\mathbf{R}\to\mathbf{R}$ given by

$$I_0(u) := \int_\Omega F(x,u)\,dx,$$

and the corresponding functional $\mathcal{I}_0(u)=I_0(u)$ if $u\in\mathcal{A}_1$ is constant and $\mathcal{I}_0(u)=+\infty$ elsewhere. It is clear that conditions (3.9) and (4.5) are sufficient for $\tilde{u}$ to be a local minimizer of I_0. In Theorem 4.1 we have shown that under these conditions one can construct a continuous branch of local minimizers for $\mathcal{I}_\varepsilon$ that starts off from a local minimizer of $\mathcal{I}_0$.

We now wish to make a simple observation regarding the global minimizers of $\mathcal{I}_\varepsilon$ and their possible connection to those of $\mathcal{I}_0$.

Proposition 4.2. *Let $F(x,u)\geq C_1+C_2|u|$ for some $C_2>0$ and let u^ε be a sequence such that $\mathcal{I}_\varepsilon(u^\varepsilon)<M$ for some constant M. Then by passing to a subsequence if necessary $u^\varepsilon\to\tilde{u}$ in $W^{1,2}(\Omega)$ where $\tilde{u}$ is a constant.*

Proof. It follows from the coercivity condition above that

$$\begin{cases} u^\varepsilon & \text{is bounded in } L^1(\Omega), \\ \nabla u^\varepsilon & \text{is bounded in } L^2(\Omega;\mathbf{R}^n). \end{cases}$$

Hence u^ε is bounded in $W^{1,2}(\Omega)$ and therefore by passing to a subsequence

$$u^\varepsilon \rightharpoonup \tilde{u} \quad \text{in } W^{1,2}(\Omega), \qquad u^\varepsilon\to\tilde{u} \quad \text{a.e.} \tag{4.7}$$

for some $\tilde{u}\in W^{1,2}(\Omega)$. Also it follows that

$$\frac{1}{2\varepsilon}\int_\Omega |\nabla u^\varepsilon|^2\,dx \leq M-C_1$$

and so $\nabla u^\varepsilon\to 0$ in $L^2(\Omega;\mathbf{R}^n)$. Hence $\nabla\tilde{u}=0$ which means $\tilde{u}$ is constant and consequently the weak convergence in (4.7) is strong. □

Remark 4.2. It can be easily checked that under the assumptions of Proposition 4.2 from every sequence of global minimizers of $\mathcal{I}_\varepsilon$ we can extract a subsequence that converges strongly in $W^{1,2}(\Omega)$ to a global minimizer of $\mathcal{I}_0$. Indeed let u^ε be such a sequence; then

$$\mathcal{I}_\varepsilon(u^\varepsilon) \le \mathcal{I}_\varepsilon(u) = \mathcal{I}_0(u) \tag{4.8}$$

where u is an arbitrary constant. It now follows from the above proposition that, by passing to a subsequence, $u^\varepsilon \to \tilde{u}$ in $W^{1,2}(\Omega)$ for some constant $\tilde{u}$. According to Fatou's Lemma

$$\int_\Omega F(x,\tilde{u})\,dx \le \liminf_{\varepsilon\to 0} \int_\Omega F(x,u^\varepsilon)\,dx$$

and therefore $\mathcal{I}_0(\tilde{u}) \le \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u^\varepsilon)$, which together with (4.8) gives the result.

Proposition 4.3. *Let the partial derivative of F with respect to u satisfy*

$$(\mathbf{G}) \qquad \begin{cases} F_u(x,u) \to +\infty \text{ as } u \to +\infty \text{ uniformly in } x, \\ F_u(x,u) \to -\infty \text{ as } u \to -\infty \text{ uniformly in } x. \end{cases}$$

If $n \ge 3$ assume further that for some $1 \le q \le 2^*$

$$|F_u(x,u)| \le C(1+|u|^q), \tag{4.9}$$

for all $x \in \overline{\Omega}$ and all $u \in \mathbf{R}$ where $C > 0$. Then if u^ε is a sequence of stationary points of I_ε in $W^{1,2}(\Omega)$, by passing to a subsequence if necessary we have $u^\varepsilon \to \tilde{u}$ in $W^{1,2}(\Omega)$ where $\tilde{u}$ is a constant.

Proof. It follows from $(\mathbf{G})$ that there exists a constant $C_0 > 0$ such that $F_u(x,u)\,u \ge -C_0$ for all $x \in \overline{\Omega}$ and all $u \in \mathbf{R}$. As u^ε is a stationary point of $\mathcal{I}_\varepsilon$, it satisfies (3.2) and (3.3). This in particular implies that

$$\int_\Omega |\nabla u^\varepsilon|^2\,dx = -\varepsilon \int_\Omega F_u(x,u^\varepsilon)\,u^\varepsilon\,dx \le \varepsilon C_0\,\mathcal{L}^n(\Omega). \tag{4.10}$$

Hence $u^\varepsilon = v^\varepsilon + c^\varepsilon$ with $c^\varepsilon = \fint_\Omega u^\varepsilon\,dx$ and $v^\varepsilon \to 0$ in $W^{1,2}(\Omega)$. We now claim that c^ε is bounded and therefore by passing to a subsequence if necessary $c^\varepsilon \to \tilde{u}$. Indeed if c^ε is unbounded without loss of generality we can extract a subsequence such that $c^\varepsilon \to +\infty$. Now let $K > 0$ be such that $F_u(x,u) \ge 1$ when $u \ge K$. We can write

$$\int_\Omega F_u(x,u^\varepsilon)\,dx = \int_{\{u^\varepsilon \ge K\}} F_u(x,u^\varepsilon)\,dx + \int_{\{u^\varepsilon < K\}} F_u(x,u^\varepsilon)\,dx,$$

where the first integral

$$\int_{\{u^\varepsilon \ge K\}} F_u(x,u^\varepsilon)\,dx \ge \mathcal{L}^n(\{u^\varepsilon \ge K\}) \to \mathcal{L}^n(\Omega),$$

and using the fact that $u^\varepsilon(x) < K$ implies $|u^\varepsilon(x)| \le \max(K, |v^\varepsilon(x)|)$, the second integral converges to zero as if $n \ge 3$

$$\int_{\{u^\varepsilon < K\}} |F_u(x, u^\varepsilon)|\, dx \le C \int_{\{u^\varepsilon < K\}} (1 + \max(K^q, |v^\varepsilon|^q))\ dx \to 0.$$

The contradiction now follows by recalling that

$$\int_\Omega F_u(x, u^\varepsilon)\, dx = 0 \tag{4.11}$$

for all $\varepsilon > 0$ as u^ε is a stationary point of $\mathcal{I}_\varepsilon$. □

We now look at the set of stationary points of $\mathcal{I}_\varepsilon$ when $\varepsilon > 0$. Let us assume that $\mathcal{I}_0$ has at most finitely many critical points all of which satisfy (3.10). In other words there is a finite set $\mathbf{P^0} \subset \mathbf{R}$ containing all the points $\tilde{u}$ satisfying (3.9) and such that (3.10) holds for every $\tilde{u} \in \mathbf{P^0}$. According to Theorem 3.1, for any such $\tilde{u}$ there is a continuous branch of solutions starting from $\tilde{u}$. Moreover as $\mathbf{P^0}$ is finite there exists an $\varepsilon_0 > 0$ such that for any $\tilde{u} \in \mathbf{P^0}$ the solution u^ε obtained by the application of the implicit function theorem exists for all $\varepsilon \le \varepsilon_0$. We denote the set of all such solutions for each fixed $0 \le \varepsilon \le \varepsilon_0$ by $\mathbf{P}^\varepsilon$.

The following result shows that under certain growth condition on F_u, the above class contains all possible solutions when ε is sufficiently small.

Proposition 4.4. *Let F satisfy condition $(\mathbf{G})$ in Proposition 4.3 and suppose that if $n \ge 3$*

$$|F_u(x, u)| \le C(1 + |u|^q) \tag{4.12}$$

for some $1 \le q < \frac{n+2}{n-2}$. Then there exists $\varepsilon_1 > 0$ such that the complete set of stationary points of $\mathcal{I}_\varepsilon$ for $0 \le \varepsilon \le \varepsilon_1$ is given by $\mathbf{P}^\varepsilon$.

Proof. We argue by contradiction. Assume the conclusion of the proposition does not hold. Then there exist a sequence $\varepsilon_k \to 0$ and corresponding stationary points u^{ε_k} of $\mathcal{I}_{\varepsilon_k}$ which do not lie in $\mathbf{P}^{\varepsilon_k}$. According to Theorem 3.1 this sequence is bounded away from $\mathbf{P}^{\varepsilon_k}$, i.e. there exists $\rho > 0$ independent of k such that

$$||u^{\varepsilon_k} - v||_{W^{2,s}(\Omega)} \ge \rho \tag{4.13}$$

for all $v \in \mathbf{S}^{\varepsilon_k}$, where s is as (3.7). It follows from Proposition 4.3, that for a further subsequence, $u^{\varepsilon_k} \to \tilde{u}$ in $W^{1,2}(\Omega)$ for some constant $\tilde{u}$.

It is clear that u^{ε_k} satisfies

$$\begin{cases} \Delta u^{\varepsilon_k} = f^{\varepsilon_k} & \text{in } \Omega \\ \frac{\partial u^{\varepsilon_k}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.14}$$

for each k with $f^{\varepsilon_k} = \varepsilon_k F_u(x, u^{\varepsilon_k})$. As u^{ε_k} is bounded in $W^{1,2}(\Omega)$, using (4.12) for $n \ge 3$ we can bootstrap this to u^{ε_k} being bounded in $W^{2,p}(\Omega)$ for every $p < \infty$ and hence in $L^\infty(\Omega)$. Thus $f^{\varepsilon_k} \to 0$ in $L^\infty(\Omega)$ and so $u^{\varepsilon_k} \to \tilde{u}$ in $W^{2,s}(\Omega)$. This contradicts (4.13) provided we can show that $\tilde{u}$ is a stationary point of I_0. But by integrating (i) in (4.14) and using the boundary condition (ii), we deduce that

$$\int_\Omega F_u(x, u^{\varepsilon_k})\, dx = 0$$

for all k. Thus by passing to the limit the same holds for $\tilde{u}$, which is thus a stationary point of I_0. □

5 The constrained problem

This section is devoted to the study of the second problem introduced earlier in Section 1. Here the energy functional is defined over the space of vector-valued functions $u : \Omega \to \mathbf{R}^N$ whose values are restricted to lie on the unit sphere $\mathbf{S}^{N-1}$. Let us recall that the energy functional in this case is given by

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \left(\frac{1}{2\varepsilon}|Du|^2 + V(x,u) \right)\, dx, \tag{5.1}$$

where the admissible function u belongs to the class

$$\mathcal{A}_2 = \{u \in W^{1,2}(\Omega; \mathbf{R}^N) : |u(x)| = 1 \quad a.e.\}.$$

The integrand V is initially assumed to belong to the class $C^2(\overline{\Omega} \times \mathbf{S}^{N-1})$. However we may extend V to a function in $C^2(\overline{\Omega} \times \mathbf{R}^N)$. We proceed by showing one such extension that is convenient for later purposes. Given $u \in \mathbf{S}^{N-1}$, we denote by $u^\perp$ the tangent space to $\mathbf{S}^{N-1}$ at u, i.e. the orthogonal complement in $\mathbf{R}^N$ of the subspace $\mathbf{R}u$. The projections of $\mathbf{R}^N$ onto $\mathbf{R}u$ and $u^\perp$ are given by $\mathbf{P}^u = u \otimes u$ and $\mathbf{P}^{u^\perp} = I - u \otimes u$ respectively. We now claim that for any $K > 0$, V has an extension $V^K \in C^2(\overline{\Omega} \times \mathbf{R}^N)$ such that $V^K(x,u) = 0$ for $|u|$ sufficiently large, and in addition for all $x \in \overline{\Omega}$, $u \in \mathbf{S}^{N-1}$, $V^K_u(x,u) \cdot u = 0$ and

$$V^K_{uu}(x,u)v \cdot v \geq V^K_{uu}(x,u)\mathbf{P}^{u^\perp} v \cdot \mathbf{P}^{u^\perp} v - c|\mathbf{P}^u v||\mathbf{P}^{u^\perp} v| + K|\mathbf{P}^u v|^2$$

for all $v \in \mathbf{R}^N$, where $c > 0$ is a constant depending only on V.

To this end, for $x \in \overline{\Omega}$ and $u \neq 0$, we first define $\overline{V}(x,u) = V(x, \frac{u}{|u|})$. Then clearly $\overline{V} \in C^2(\overline{\Omega} \times (\mathbf{R}^N \backslash \{0\}))$. A simple calculation now shows that for $x \in \overline{\Omega}$, $u \in \mathbf{S}^{N-1}$, $v \in \mathbf{R}^N$

$$\begin{aligned} \overline{V}_u(x,u) \cdot v &= \frac{d}{dt}\overline{V}(x, u+tv)|_{t=0} \\ &= V_u(x,u) \cdot \mathbf{P}^{u^\perp} v, \end{aligned} \tag{5.2}$$

so that in particular

$$\overline{V}_u(x,u) \cdot u = 0. \tag{5.3}$$

Similarly

$$\begin{aligned}\overline{V}_{uu}(x,u)v\cdot v &= \frac{d^2}{dt^2}\overline{V}(x,u+tv)|_{t=0}\\ &= V_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v\\ &\quad -V_u(x,u)\cdot\left(2\mathbf{P}^{u^\perp}v(u\cdot v)+u|\mathbf{P}^{u^\perp}v|^2\right). \qquad (5.4)\end{aligned}$$

It is clear that the right-hand sides of (5.2) and (5.4) are independent of the particular extension of V to $\overline{\Omega}\times\mathbf{R}^N$. Note that we can replace V by $\overline{V}$ in the right-hand side of (5.4) so that by (5.3)

$$\overline{V}_{uu}(x,u)v\cdot v = \overline{V}_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v - 2\overline{V}_u(x,u)\cdot\mathbf{P}^{u^\perp}v(u\cdot v).$$

We now let $\rho\in C_0^\infty(0,\infty)$ with $\rho(s)=1$ for s in a neighbourhood of 1, and $\rho(s)=0$ for $s\geq 2$. For $K>0$ we set

$$V^K(x,u)=\rho(|u|)\left(\overline{V}(x,u)+\frac{K}{2}\left(|u|^2-1\right)^2\right). \qquad (5.5)$$

Clearly $V^K\in C^2(\overline{\Omega}\times\mathbf{R}^N)$, and $V^K_u(x,u)\cdot u=0$ for $x\in\overline{\Omega}$, $|u|=1$, and $V^K(x,u)=0$ for $|u|\geq 2$. Therefore for $x\in\overline{\Omega}$ and $|u|=1$,

$$V^K_{uu}(x,u)v\cdot v = \overline{V}_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v - 2\overline{V}_u(x,u)\cdot\mathbf{P}^{u^\perp}v(u\cdot v)+K(u\cdot v)^2.$$

Hence $V^K_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v=\overline{V}_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v$ and so

$$\begin{aligned}V^K_{uu}(x,u)v\cdot v &= V^K_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v - 2\overline{V}_u(x,u)\cdot\mathbf{P}^{u^\perp}v(u\cdot v)+K(u\cdot v)^2\\ &\geq V^K_{uu}(x,u)\mathbf{P}^{u^\perp}v\cdot\mathbf{P}^{u^\perp}v - c|\mathbf{P}^u v||\mathbf{P}^{u^\perp}v|+K|\mathbf{P}^u v|^2.\end{aligned}$$

This justifies the claim. In what follows we always assume that $V\in C^2(\overline{\Omega}\times\mathbf{R}^N)$.

We proceed now by formally deriving the Euler-Lagrange equation associated to $\mathcal{F}_\varepsilon$. For this we consider variations $\varphi\in C^\infty(\overline{\Omega};\mathbf{R}^N)$ and deduce from the condition

$$\frac{d}{dt}\mathcal{F}_\varepsilon\Big(\frac{u+t\varphi}{|u+t\varphi|}\Big)\Big|_{t=0}=0$$

that

$$\int_\Omega\left(Du\cdot D(\mathbf{P}^{u^\perp}\varphi)+\varepsilon V_u(x,u)\mathbf{P}^{u^\perp}\varphi\right)\,dx=0,$$

from which we obtain the equation

$$\mathbf{P}^{u^\perp}(\Delta u-\varepsilon V_u(x,u))=0$$

in Ω, and the natural boundary condition

$$\mathbf{P}^{u^\perp}\frac{\partial u}{\partial\nu}=0,$$

on $\partial\Omega$. Noting that $\nabla(|u|^2) = 0$, we can rewrite the above as

$$\begin{cases} \Delta u + |Du|^2 u - \varepsilon(I - u \otimes u)\, V_u(x,u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

When $\varepsilon = 0$ the Euler-Lagrange equation reduces to

$$\begin{cases} \Delta u + |Du|^2 u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.7}$$

which is the well-known equation of harmonic maps into the unit sphere. It is clear that in this case any function $u = \tilde{u}$ with $\tilde{u} \in \mathbf{S}^{N-1}$ is a solution to this system in $\mathcal{A}_2$. However such functions are far from being the only solutions to this system. For example when $\Omega = B_1$ is the unit ball in $\mathbf{R}^n$ with $n = N \geq 3$ the function $u(x) = x/|x|$ is a solution to (5.7) that lies in $\mathcal{A}_2$. In fact this function is the unique global minimizer of the Dirichlet integral over $\mathcal{A}_2$ subject to the linear boundary condition $u = x$ on $\partial\Omega$ (cf. [7], [22]).

In a similar way to Section 3 we proceed by formally seeking a solution to the system (5.6) in the form

$$u(\varepsilon) = u + \varepsilon v + \varepsilon^2 w + \cdots, \tag{5.8}$$

where $u = \tilde{u}$ for some $\tilde{u} \in \mathbf{S}^{N-1}$. Notice that unlike the problem studied in Section 3, the fact that $u = \tilde{u}$ is constant does not follow by substituting the above ansatz in the equation and solving it for u. Indeed as explained in the previous paragraph the system (5.7) in general has non-constant solutions.

Substituting (5.8) into (5.6) we get

$$\begin{cases} \Delta v = (I - \tilde{u} \otimes \tilde{u}) V_u(x,\tilde{u}) & \text{in } \Omega \\ v \cdot \tilde{u} = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.9}$$

for the coefficients of ε. A necessary condition for the solvability of the system (5.9) is that

$$\int_\Omega (I - \tilde{u} \otimes \tilde{u}) V_u(x,\tilde{u})\, dx = 0.$$

Moreover in this case the solution is unique up to an additive constant vector. Note that the second equation in (5.9) implies that this constant vector is normal to $\tilde{u}$. The coefficient of ε^2 gives

$$\begin{cases} \Delta w + |Dv|^2 \tilde{u} = (I - \tilde{u} \otimes \tilde{u}) V_{uu}(x,\tilde{u}) v \\ \qquad\qquad - (\tilde{u} \otimes v + v \otimes \tilde{u}) V_u(x,\tilde{u}) & \text{in } \Omega \\ |v|^2 + 2w \cdot \tilde{u} = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

Again a necessary condition for the solvability of (5.10) is that

$$\int_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x,\tilde{u}) v - (\tilde{u} \otimes v + v \otimes \tilde{u}) V_u(x,\tilde{u}) - |Dv|^2\, \tilde{u} \right)\, dx = 0. \tag{5.11}$$

Multiplying the first equation in (5.9) by v and integrating over Ω we get

$$\int_{\Omega} \left(|Dv|^2 + V_u(x,\tilde{u}) \cdot v\right) \, dx = 0,$$

and therefore (5.11) can be written as

$$\int_{\Omega} \left((I - \tilde{u} \otimes \tilde{u})V_{uu}(x,\tilde{u}) - \tilde{u} \cdot V_u(x,\tilde{u})\, I\right) \, v \, dx = 0.$$

Note that the linear transformation

$$V_0 := \fint_{\Omega} \left((I - \tilde{u} \otimes \tilde{u})V_{uu}(x,\tilde{u}) - \tilde{u} \cdot V_u(x,\tilde{u})\, I\right) \, dx$$

maps $\tilde{u}^{\perp}$ to $\tilde{u}^{\perp}$, and thus the constant vector mentioned above (normal to $\tilde{u}$) can be uniquely determined provided $V_0 : \tilde{u}^{\perp} \to \tilde{u}^{\perp}$ is invertible.

Following this informal discussion and to establish rigorously the existence of a continuous branch of solutions to (5.6) we proceed as follows. Assume that the coordinate system is such that

$$\tilde{u} = e_N = (0, 0, \cdots, 1).$$

Let $u(x) = (u_1(x), u_2(x), \cdots, u_N(x))$ where

$$u_N(x) = \sqrt{1 - \Sigma_{i=1}^{N-1} u_i^2(x)} \tag{5.12}$$

and $||u_i||_{L^\infty(\Omega)}$ are small for all $1 \leq i \leq N-1$. Then clearly

$$u(x) \in \mathbf{S}^{N-1}$$

for a.e. $x \in \Omega$. Furthermore we claim that if $(u_1, ..., u_N)$ satisfy the first $N-1$ equations in (5.6), then the last one is automatically satisfied. Indeed proceeding formally, it follows from the constraint $\Sigma_{i=1}^{N} u_i^2 = 1$ that $\Sigma_{i=1}^{N} u_i \nabla u_i = 0$ and so

$$\sum_{i=1}^{N} (|\nabla u_i|^2 + u_i \Delta u_i) = 0.$$

As $|Du|^2 = \Sigma_{i=1}^{N} |\nabla u_i|^2$, we have that $u_N \Delta u_N = -\Sigma_{i=1}^{N-1} u_i \Delta u_i - |Du|^2$. The result now follows by multiplying the i-th equation by u_i, summing over $i = 1$ to $N-1$ and recalling that $((I - u \otimes u)\, V_u(x,u)) \cdot u = 0$. Similarly, from $\partial u_i / \partial \nu = 0$ for $1 \leq i \leq N-1$, we deduce that $\partial u_N / \partial \nu = 0$. These computations are rigorous for $u_i \in W^{2,s}(\Omega)$, $s > \frac{n}{2}$ and $||u_i||_{L^\infty(\Omega)}$ sufficiently small, $1 \leq i \leq N-1$, as is the case for the solution constructed via the implicit function theorem below.
Let us now set $U = (u - e_N)/\varepsilon$ and solve the first $N-1$ equations of the system

$$\begin{cases} \Delta U + \varepsilon |DU|^2 (e_N + \varepsilon U) & \\ \qquad - (I - (e_N + \varepsilon U) \otimes (e_N + \varepsilon U))\, V_u(x, e_N + \varepsilon U) = 0 & \text{in } \Omega \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

For this we introduce the map

$$T : \mathbf{R} \times (W^{2,s}(\Omega))^{N-1} \to (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$$

for $s > \frac{n}{2}$, defined by

$$T(\varepsilon, U') = \begin{pmatrix} \Delta U_1 - \varepsilon(h_1(\varepsilon,x,U') - \fint_\Omega h_1(\varepsilon,x,U')\,dx) \\ \frac{\partial U_1}{\partial\nu}(x) \\ \dots \\ \Delta U_{N-1} - \varepsilon(h_{N-1}(\varepsilon,x,U') - \fint_\Omega h_{N-1}(\varepsilon,x,U')\,dx) \\ \frac{\partial U_{N-1}}{\partial\nu}(x) \\ \fint_\Omega h_1(\varepsilon,x,U')\,dx \\ \dots \\ \fint_\Omega h_{N-1}(\varepsilon,x,U')\,dx \end{pmatrix} \tag{5.14}$$

first for $\varepsilon \neq 0$, where $U' = (U_1, ..., U_{N-1})$ and

$$h(\varepsilon,x,U') := -\varepsilon|DU|^2 U + \frac{1}{\varepsilon}\left((I - (e_N + \varepsilon U) \otimes (e_N + \varepsilon U))\; V_u(x, e_N + \varepsilon U)\right),$$

where $U_N = \varepsilon^{-1}(u_N - 1)$ is calculated using (5.12). As a simple Taylor expansion shows,

$$V_u(x, e_N + \varepsilon U) = V_u(x, e_N) + \varepsilon V_{uu}(x, e_N)U + O(\varepsilon^2).$$

We can therefore write

$$\begin{aligned} &\fint_\Omega h(\varepsilon,x,U')\,dx = \\ &\fint_\Omega (-\varepsilon|DU|^2 U + (I - e_N \otimes e_N)V_{uu}(x,e_N)U \\ &\quad -(e_N \otimes U + U \otimes e_N)V_u(x,e_N) + O(\varepsilon))\,dx, \end{aligned} \tag{5.15}$$

where we have assumed the following to hold

$$\int_\Omega (I - e_N \otimes e_N)\; V_u(x, e_N)\,dx = 0.$$

This suggests that we can extend the map T to $\varepsilon = 0$ by substituting (5.15) into the last $N-1$ columns in (5.14). Extending T in this way we find that $T \in C^1(\mathbf{U}; (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1})$, where $\mathbf{U} = \{(\varepsilon, U') \in \mathbf{R} \times W^{2,s}(\Omega; \mathbf{R}^{N-1}) : |e_N + \varepsilon U| < 1\}$. (Here we use, for example, that $U \in W^{2,s}(\Omega; \mathbf{R}^{N-1})$ implies

$|DU|^2 U \in L^s(\Omega;\mathbf{R}^N)$ since $s > \frac{n}{2}$.)

$$T(0,U') = \begin{pmatrix} \Delta U_1 - V_{u_1}(x,e_N) \\ \frac{\partial U_1}{\partial \nu}(x) \\ \dots \\ \Delta U_{N-1} - V_{u_{N-1}}(x,e_N) \\ \frac{\partial U_{N-1}}{\partial \nu}(x) \\ \fint_\Omega \left(V_{u_1 u_j}(x,e_N) U_j(x) - V_{u_N}(x,e_N) U_1(x) \right) \, dx \\ \dots \\ \fint_\Omega \left(V_{u_{N-1}\, u_j}(x,e_N) U_j(x) - V_{u_N}(x,e_N) U_{N-1}(x) \right) \, dx \end{pmatrix},$$

where the repeated suffix j is summed from 1 to N. Assume now that the map $V_0 : \tilde{u}^\perp \to \tilde{u}^\perp$ is invertible. With our choices of coordinates this reduces to the requirement that the matrix

$$\fint_\Omega \left(V_{u_i u_j}(x,e_N) - V_{u_N} \delta_{ij} \right) \, dx \tag{5.16}$$

with $1 \leq i,j \leq N-1$ is nonsingular. We proceed by considering the equation $T(0,U') = 0$. It is well known that for all $(f,g) \in (E^s(\Omega))^{N-1}$ the system

$$\begin{cases} \Delta U' = f \\ \frac{\partial U'}{\partial \nu} = g \end{cases}$$

has a solution $U' \in (W^{2,s}(\Omega))^{N-1}$, which is unique up to an additive constant $c \in \mathbf{R}^{N-1}$. This constant is determined uniquely by solving the last $N-1$ equations in $T(0,U') = 0$ because of (5.16). Corresponding to the formal expansion (5.8) we denote the unique solution of $T(0,U') = 0$ by v'. So as to apply the implicit function theorem we need to show that the linear map

$$D_{U'} T(0,v') : (W^{2,s}(\Omega))^{N-1} \to (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$$

is a bijection. This amounts to showing that the system

$$\begin{cases} \Delta z_1 = f_1 \\ \dfrac{\partial z_1}{\partial \nu} = g_1 \\ \dots \\ \Delta z_{N-1} = f_{N-1} \\ \dfrac{\partial z_{N-1}}{\partial \nu} = g_{N-1} \\ \displaystyle\fint_\Omega \left(V_{u_1 u_j}(x,e_N) z_j(x) - V_{u_N}(x,e_N) z_1 \right) \, dx = t_1 \\ \dots \\ \displaystyle\fint_\Omega \left(V_{u_{N-1}\, u_j}(x,e_N) z_j(x) - V_{u_N}(x,e_N) z_{N-1} \right) \, dx = t_{N-1} \end{cases}$$

has a unique solution $z \in (W^{2,s}(\Omega))^{N-1}$ for all $(f, g, t) \in (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$. But this follows easily from our assumptions.

By the implicit function theorem we thus have a solution $U' = (U^\varepsilon)'$ of $T(\varepsilon, U') = 0$ which is a C^1 map from some small interval $(-\varepsilon_0, \varepsilon_0) \to W^{2,s}(\Omega, \mathbf{R}^{N-1})$ with $(U^0)' = v'$. Clearly $(u^\varepsilon)' = \varepsilon (U^\varepsilon)'$ is C^1 in ε for $|\varepsilon| < \varepsilon_0$. Since the map $t \to \sqrt{1 - |t|^2}$ is smooth for $t \in \mathbf{R}^{N-1}$, $|t|$ sufficiently small, and since $s > n/2$, it follows also that $u^\varepsilon_N = \sqrt{1 - |(u^\varepsilon)'|^2}$ is C^1 from $(-\varepsilon_1, \varepsilon_1)$ to $W^{2,s}(\Omega)$ for some $0 < \varepsilon_1 < \varepsilon_0$, with $u^0_N = 1$. Therefore we have proved the following

Theorem 5.1. *Let $\tilde{u} \in \mathbf{S}^{N-1}$ satisfy*

$$\int_\Omega (I - \tilde{u} \otimes \tilde{u}) \, V_u(x, \tilde{u}) \, dx = 0 \tag{5.17}$$

and let the linear map $V_0 : \tilde{u}^\perp \to \tilde{u}^\perp$ corresponding to the matrix

$$V_0 := \fint_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) - \tilde{u} \cdot V_u(x, \tilde{u}) I \right) \, dx \tag{5.18}$$

be invertible. Then the system (5.6) has a solution u^ε that is a C^1 map from some small interval $(-\varepsilon_1, \varepsilon_1)$ to $W^{2,s}(\Omega; \mathbf{R}^N)$ that lies in $\mathcal{A}_2$ with $u^0 = \tilde{u}$.

Remark 5.1. Note that it follows from the proof of the theorem that $U^\varepsilon = (u^\varepsilon - \tilde{u})/\varepsilon \to v$ in $W^{1,\infty}(\Omega; \mathbf{R}^N)$ as $\varepsilon \to 0$, where v is the unique solution of (5.9). In fact we already showed in the proof that $(U^\varepsilon)' \to v'$ as $\varepsilon \to 0$ in $W^{2,s}(\Omega; \mathbf{R}^{N-1})$. In the coordinates of the proof we have that

$$\begin{aligned}
(U^\varepsilon)_N &= \frac{1}{\varepsilon} \left(\sqrt{1 - \varepsilon^2 |(U^\varepsilon)'|^2} - 1 \right) \\
&= -\frac{\varepsilon}{2} |(U^\varepsilon)'|^2 \int_0^1 \frac{ds}{\sqrt{1 - \varepsilon^2 |(U^\varepsilon)'|^2 s}} \\
&= -\frac{\varepsilon}{2} g(\varepsilon, |(U^\varepsilon)'|^2)
\end{aligned}$$

where g is smooth on $(-\varepsilon_M, \varepsilon_M) \times [0, M]$, $M = 2||\,|v'|\,||_{L^\infty(\Omega)}$, $\varepsilon_M > 0$ is sufficiently small. Hence $(U^\varepsilon)_N \to 0$ in $W^{2,s}(\Omega)$, and thus $U^\varepsilon \to v$ in $W^{2,s}(\Omega; \mathbf{R}^N)$. So $DU^\varepsilon \to Dv$ in $L^{s^*}(\Omega; \mathbf{R}^{N \times n})$, where as usual, $s^* = \frac{ns}{n-s}$ is the Sobolev conjugate of s. Using this together with a simple bootstrap argument on equation (5.13) and recalling that $s > \frac{n}{2}$ implies the claim.

Remark 5.2. It does not seem obvious whether or not the solution u^ε obtained in the above theorem is unique for sufficiently small ε and in a sufficiently small neighbourhood of u^ε.

Having proved the existence of a continuous branch of stationary points for the functional $\mathcal{F}_\varepsilon$, we now study conditions under which u^ε is a local minimizer for $\mathcal{F}_\varepsilon$.

Theorem 5.2. *Assume that the hypotheses of Theorem* 5.1 *hold and that* $V_0 : \tilde{u}^\perp \to \tilde{u}^\perp$ *is positive definite* (*equivalently there exists* $\mu > 0$ *such that the matrix*

$$V = \fint_\Omega \left(V_{uu}(x,\tilde{u}) - \tilde{u} \cdot V_u(x,\tilde{u})\, I\right)\, dx \tag{5.19}$$

satisfies $Vv \cdot v \geq \mu |v|^2$ *for all* $v \in \tilde{u}^\perp$). *Then the solution* u^ε *given by Theorem* 5.1 *is an* L^1 *local minimizer of* $\mathcal{F}_\varepsilon$.

Proof. Consider the unconstrained functional

$$\begin{aligned}\tilde{\mathcal{F}}_\varepsilon(u) = \int_\Omega \Bigg(&\frac{1}{2\varepsilon}|Du|^2 + V(x,u) \\ &+ \frac{1}{2}(|u|^2 - 1)\Big(- \frac{1}{\varepsilon}|Du^\varepsilon|^2 - u^\varepsilon \cdot V_u(x,u_\varepsilon)\Big)\Bigg)\, dx.\end{aligned}$$

As the integrand V has compact support, $\tilde{\mathcal{F}}_\varepsilon$ is well defined and finite over the class of admissible functions $W^{1,2}(\Omega, \mathbf{R}^N)$. Moreover it is clear that $\tilde{\mathcal{F}}_\varepsilon(u) = \mathcal{F}_\varepsilon(u)$ for every $u \in \mathcal{A}_2$. The Euler-Lagrange equation associated with this functional can be easily checked to be

$$\begin{cases} \Delta u + |Du^\varepsilon|^2 u - \varepsilon(V_u(x,u) - u^\varepsilon \cdot V_u(x,u^\varepsilon)u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Thus u^ε is a stationary point of $\tilde{\mathcal{F}}_\varepsilon$ as a consequence of being a solution to the system (5.6).

Let us now consider the second variation of $\tilde{\mathcal{F}}_\varepsilon$ at u^ε. Indeed for $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ we can write

$$\begin{aligned}&\delta^2 \tilde{\mathcal{F}}_\varepsilon(u^\varepsilon, \varphi) = \frac{d^2}{dt^2}\tilde{\mathcal{F}}_\varepsilon(u^\varepsilon + t\varphi)|_{t=0} \\ &= \int_\Omega \left(\frac{1}{\varepsilon}|D\varphi|^2 + V_{u_i u_j}(x, u^\varepsilon(x))\varphi_i\varphi_j + |\varphi|^2(-\frac{1}{\varepsilon}|Du^\varepsilon|^2 - u^\varepsilon \cdot V_u(x,u^\varepsilon)\right) dx \\ &= \frac{1}{\varepsilon}\int_\Omega \Big(|D\varphi|^2 + \varepsilon((V_{u_i u_j}(x,u^\varepsilon(x)) - u^\varepsilon \cdot V_u(x,u^\varepsilon)\delta_{ij}) \\ &\qquad -\frac{1}{\varepsilon}|Du^\varepsilon|^2\delta_{ij})\varphi_i\varphi_j\Big)\, dx,\end{aligned}$$

Proceeding in a similar way as in Proposition 4.1 we can show this to be uniformly positive if and only if the matrix

$$V^\varepsilon = \fint_\Omega \left((V_{uu}(x,u^\varepsilon(x)) - u^\varepsilon \cdot V_u(x,u^\varepsilon)I) - \frac{1}{\varepsilon}|Du^\varepsilon|^2 I\right)\, dx$$

is positive definite. But for the extension of V^K of V constructed at the beginning of this section, with K sufficiently large, we have

$$\begin{aligned} V^K v\cdot v &= \fint_\Omega V^K_{uu}(x,\tilde u)\,dx\, v\cdot v \\ &\geq \fint_\Omega V^K_{uu}(x,\tilde u)\,dx\,\mathbf{P}^{\tilde u^\perp} v\cdot \mathbf{P}^{\tilde u^\perp} v - c|\mathbf{P}^{\tilde u} v||\mathbf{P}^{\tilde u^\perp} v| + K|\mathbf{P}^{\tilde u} v|^2 \\ &\geq \mu|\mathbf{P}^{\tilde u} v|^2 - c|\mathbf{P}^{\tilde u} v||\mathbf{P}^{\tilde u^\perp} v| + K|\mathbf{P}^{\tilde u} v|^2 \\ &\geq \frac{\mu}{2}|v|^2. \end{aligned}$$

We now note that

$$(V^K)^\varepsilon - V^K = \fint_\Omega \left(V^K_{uu}(x,u^\varepsilon) - V^K_{uu}(x,\tilde u)\right)\,dx - \frac{1}{\varepsilon}\left(\fint_\Omega |Du^\varepsilon|^2\,dx\right) I$$

satisfies $\lim_{\varepsilon\to 0}|(V^K)^\varepsilon - V^K| = 0$ using Remark 5.1, and the positive definiteness of $(V^K)^\varepsilon$ follows. The proof is completed by applying Theorem 2.2 (see Remark 2.1). □

6 The energy functional of micromagnetics

In this section we focus on the energy functional of micromagnetics in the case of a spatially uniform applied field

$$\mathcal{J}_\varepsilon(m) = \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + W(m)\right)\,dx + \frac{1}{2}\int_{\mathbf{R}^3}|h_m|^2\,dx. \qquad (6.1)$$

Here we have set $W(m) := \psi(m) + \frac{1}{2}|h-m|^2$ for the functional to agree with the original form introduced in Section 1. It is initially assumed that the anisotropy energy density $\psi \in C^2(\mathbf{S}^2)$. However as explained in Section 5 we can extend ψ to any neighbourhood of $\mathbf{S}^2$, and in particular to $\mathbf{R}^3$. Since a substantial part of the analysis in this section is similar to that of Section 5, we shall abbreviate the arguments and refer the reader to the earlier discussions. We also mention that our main device in dealing with the non-local term is Theorem 2.3.

Recall that here $\Omega\subset\mathbf{R}^3$ is a bounded domain with smooth boundary. Regarding the admissible functions we set

$$\mathcal{A}_3 := \{m\in W^{1,2}(\Omega;\mathbf{R}^3) : |m(x)| = 1 \quad a.e.\}.$$

It is clear that $\mathcal{J}_\varepsilon$ is well-defined and finite over this class. The Euler-Lagrange equation associated to $\mathcal{J}_\varepsilon$ is easily seen to be

$$\begin{cases} \Delta m + |Dm|^2 m - \varepsilon(I - m\otimes m)(W_m(m) - h_m) = 0 & \text{in } \Omega \\ \frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (6.2)$$

In a similar way to Section 5 we proceed by formally seeking a solution to (6.2) in the form

$$m(\varepsilon) = \tilde m + \varepsilon v + \varepsilon^2 w + \cdots, \qquad (6.3)$$

where $\tilde{m} \in \mathbf{S}^2$. Substituting this into (6.2) we get

$$\begin{cases} \Delta v = (I - \tilde{m} \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}}) & \text{in } \Omega \\ v \cdot \tilde{m} = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{6.4}$$

for the coefficients of ε. It follows that a necessary condition for solvability of (6.4) is that

$$\int_\Omega (I - \tilde{m} \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}})\, dx = 0.$$

Moreover in this case the solution is unique up to an additive constant. Note that the second equation in (6.4) implies that this constant is normal to $\tilde{m}$. The coefficient of ε^2 gives

$$\begin{cases} \Delta w + |Dv|^2 \tilde{m} = (I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m})v - h_v) \\ \qquad\qquad -(\tilde{m} \otimes v + v \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}}) & \text{in } \Omega \\ |v|^2 + 2w \cdot \tilde{m} = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Again a necessary condition for the solvability of (6.5) is that

$$\int_\Omega ((I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m})v - h_v) - (\tilde{m} \otimes v + v \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}}) \\ - |Dv|^2 \tilde{m})\, dx = 0. \tag{6.6}$$

Multiplying the first equation in (6.4) by v and integrating over Ω we get

$$\int_\Omega \left(|Dv|^2 + (W_m(\tilde{m}) - h_{\tilde{m}}) \cdot v\right)\, dx = 0,$$

and thus (6.6) can be written as

$$\int_\Omega \left((I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m})v - h_v) - \tilde{m} \cdot (W_m(\tilde{m}) - h_{\tilde{m}})v\right)\, dx = 0.$$

Thus the constant mentioned above (normal to $\tilde{m}$) can be uniquely determined provided the linear map $W_0 : \tilde{m}^\perp \to \tilde{m}^\perp$ corresponding to the matrix

$$W_0 := \fint_\Omega \left((I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m}) + D_e) - \tilde{m} \cdot (W_m(\tilde{m}) - h_{\tilde{m}})\, I\right)\, dx$$

is invertible. This informal discussion leads us in exactly the same way as in Section 5 to the following

Theorem 6.1. *Let $\tilde{m} \in \mathbf{S}^2$ satisfy*

$$\int_\Omega (I - \tilde{m} \otimes \tilde{m})\, (W_m(\tilde{m}) - h_{\tilde{m}})\, dx = 0 \tag{6.7}$$

and let the linear map $W_0 : \tilde{m}^\perp \to \tilde{m}^\perp$ corresponding to the matrix W_0 be invertible. Then the system (6.2) has a solution m^ε that is a C^1 map from some small interval $(-\varepsilon_1, \varepsilon_1)$ to $W^{2,s}(\Omega; \mathbf{R}^3)$ that lies in $\mathcal{A}_3$ with $m^0 = \tilde{m}$.

Remark 6.1. Similarly to Remark 5.1 it follows from the proof of this theorem that $\frac{1}{\varepsilon}|Dm^\varepsilon|^2 \to 0$ in $L^\infty(\Omega)$. We recall that equation (2.3) together with standard elliptic theory and the fact that $m^\varepsilon \to \tilde{m}$ in $W^{2,s}(\Omega;\mathbf{R}^3)$ imply that $h_{m_\varepsilon} \to h_{\tilde{m}}$ in $L^\infty(\Omega;\mathbf{R}^3)$. We use this fact later in the proof of Theorem 6.2.

Remark 6.2. Similarly to Remark 5.2, it is not obvious whether or not the solution m^ε obtained in the above theorem is unique for sufficiently small ε and in a sufficiently small neighbourhood of m^ε.

Having proved the existence of a continuous branch of stationary points for the functional $\mathcal{J}_\varepsilon$ we now study conditions under which m^ε is a local minimizer of $\mathcal{J}_\varepsilon$.

Theorem 6.2. *Assume that the hypotheses of Theorem* 6.1 *hold and that the linear map $W : \tilde{m}^\perp \to \tilde{m}^\perp$ corresponding to the matrix*

$$W := \int_\Omega \big(W_{m_i m_j}(\tilde{m}) - \tilde{m}\cdot(W_m(\tilde{m}) - h_{\tilde{m}})I\big)\, dx \tag{6.8}$$

is positive definite. Then the solution m_ε given by Theorem 6.1 *is an L^1 local minimizer of $\mathcal{J}_\varepsilon$.*

Proof. Consider the unconstrained functional

$$\tilde{\mathcal{J}}_\varepsilon(m) := \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2\, dx + \int_\Omega \Bigg(\frac{1}{2\varepsilon}|Dm|^2 + W(m) \\ -\frac{1}{2}(|m|^2-1)\Big(\frac{1}{\varepsilon}|Dm^\varepsilon|^2 + m^\varepsilon\cdot(W_m(m^\varepsilon) - h_{m^\varepsilon})\Big)\Bigg)\, dx. \tag{6.9}$$

As we may assume that the integrand W has compact support, $\tilde{\mathcal{J}}_\varepsilon$ is well-defined and finite over the class of admissible functions $W^{1,2}(\Omega;\mathbf{R}^3)$. Moreover $\tilde{\mathcal{J}}_\varepsilon(m) = \mathcal{J}_\varepsilon(m)$ for every $m \in \mathcal{A}_3$. We now use Theorem 2.3 to write

$$\tilde{\mathcal{J}}_\varepsilon(m_\varepsilon+\varphi) - \tilde{\mathcal{J}}_\varepsilon(m_\varepsilon) = \overline{\mathcal{J}}_\varepsilon(m_\varepsilon+\varphi) - \overline{\mathcal{J}}_\varepsilon(m_\varepsilon) + \frac{1}{2}\int_{\mathbf{R}^3}|h_\varphi|^2\, dx,$$

where φ lies in $W^{1,2}(\Omega;\mathbf{R}^3)$ and

$$\overline{\mathcal{J}}_\varepsilon(m) := -\int_\Omega h_{m_\varepsilon}\cdot m + \int_\Omega \Bigg(\frac{1}{2\varepsilon}|Dm|^2 + W(m) \\ -\frac{1}{2}(|m|^2-1)\Big(\frac{1}{\varepsilon}|Dm^\varepsilon|^2 + m^\varepsilon\cdot(W_m(m^\varepsilon) - h_{m^\varepsilon})\Big)\Bigg)\, dx. \tag{6.10}$$

Thus the conclusion of the theorem follows if we show that m_ε is an L^1 local minimizer of $\overline{\mathcal{J}}_\varepsilon$. It is easy to verify that the Euler-Lagrange equation associated with this functional is

$$\begin{cases} \Delta m + |Dm^\varepsilon|^2 m - \varepsilon(W_m(m) - h_{m_\varepsilon} \\ -(m^\varepsilon\cdot(W_m(m^\varepsilon) - h_{m^\varepsilon}))\, m) = 0 & \text{in } \Omega \\ \frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus m^ε is a stationary point of $\overline{\mathcal{J}}_\varepsilon$ as a consequence of being a solution to the system (6.2).

We now look at the second variation of $\overline{\mathcal{J}}_\varepsilon$ at m^ε. For this let $\varphi \in C^\infty(\overline{\Omega};\mathbf{R}^3)$ and consider

$$\begin{aligned}\delta^2\overline{\mathcal{J}}_\varepsilon(m^\varepsilon,\varphi) &= \frac{d^2}{dt^2}\overline{\mathcal{J}}_\varepsilon(m^\varepsilon + t\varphi)|_{t=0}\\ &= \int_\Omega \Bigg(\frac{1}{\varepsilon}|D\varphi|^2 + W_{m_im_j}(m^\varepsilon)\varphi_i\varphi_j\\ &\quad -\Big(\frac{1}{\varepsilon}|Dm^\varepsilon|^2 + m^\varepsilon\cdot(W_m(m^\varepsilon) - h_{m^\varepsilon})\Big)|\varphi|^2\Bigg)\,dx\\ &= \frac{1}{\varepsilon}\int_\Omega \Bigg(|D\varphi|^2 + \varepsilon\Big((W_{m_im_j}(m^\varepsilon) - m^\varepsilon\cdot(W_m(m^\varepsilon)\\ &\quad - h_{m^\varepsilon})\delta_{ij}) - \frac{1}{\varepsilon}|Dm^\varepsilon|^2\delta_{ij}\Big)\varphi_i\varphi_j\Bigg)\,dx.\end{aligned}$$

Proceeding in a similar way to Proposition 4.1, in particular choosing an appropriate extension of W, we can show this to be uniformly positive if and only if the matrix

$$W_{ij}^\varepsilon = \int_\Omega \left((W_{m_im_j}(m^\varepsilon) - m^\varepsilon\cdot(W_m(m^\varepsilon) - h_{m^\varepsilon})\delta_{ij}) - \frac{1}{\varepsilon}|Dm^\varepsilon|^2\delta_{ij}\right)\,dx$$

is positive definite. But this follows in a similar way to Theorem 5.2 recalling Remark 6.1. □

Acknowledgements. An earlier version of this work appeared in the second author's Ph.D thesis at Heriot-Watt University. This research initiated in Spring 1996 when MW held an EC and postdoctoral fellowship at Heriot-Watt University. It was partly supported by HCM contract ERBCHBI CT 93-0744, by the EC TMR contract ERBFMRX CT 98-229 on 'Phase transitions in crystalline solids' and by EPSRC grant GR/L90262.

References

1. R. Adams, Sobolev Spaces, Academic Press, 1975.
2. A. Ambrosetti and G. Prodi, A Primer of Nonlinear Analysis, First paperback edition, Cambridge studies in advanced mathematics, **34**, Cambridge University Press, Cambridge, 1995.
3. J.M. Ball, J.E. Marsden, Quasiconvexity at the boundary, positivity of the second variation and elastic stability, Arch. Rational Mech. Anal., Vol. **86**, 1984, pp. 251-277.
4. J.M. Ball, R.D. James, Fine phase mixtures as minimizers of energy, Arch. Rational Mech. Anal., Vol. **100**, 1987, pp. 15-52.
5. J.M. Ball, P.K. Jimack, and T. Qi, Elastostatics in the presence of a temperature distribution or inhomogeneity, Z. Angew. Math. Phys., Vol. **43**, 1992, pp. 943-973.
6. D. Brandon, R.C. Rogers, The coercivity paradox and nonlocal ferromagnetism, Continuum Mech. Thermodyn., Vol. **4**, 1992, pp. 1-21.
7. H. Brezis, J.M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys., Vol. **107**, 1986, pp. 649-705.

8. H. Brezis, Analyse Fonctionnelle, Masson, 1987.
9. W.F. Brown, Micromagnetics, John Wiley and Sons, 1963.
10. A. De Simone, Energy minimizers for large ferromagnetic bodies, Arch. Rational Mech. Anal., Vol. **125**, 1993, pp. 99-143.
11. A. De Simone, Hysteresis and imperfection sensitivity in small ferromagnetic particles, Meccanica, Vol. **30**, 1995, pp. 591-603.
12. G. Dal Maso, An Introduction to Γ Convergence, Birkhäuser, 1993.
13. E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., Vol. **68**, 1975, pp. 842-850.
14. L.C. Evans, Weak Convergence Methods for Non-linear Partial Differential Equations, Expository lectures from the CBMS regional conference, Vol. **74**, 1990
15. L.C. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
16. D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.
17. E. Hewitt, K. Stromberg, Real and Abstract Analysis, Springer-Verlag, 1965.
18. R.D. James and D. Kinderlehrer, Frustration in ferromagnetic materials, Cont. Mech. Thermodyn. , Vol. **2**, 1990, pp. 215-239.
19. R.D. James and D. Kinderlehrer, Theory of magnetostriction with applications to $Tb_xD_{1-x}Fe_2$, Phil. Mag., Vol. **68**, 1993, pp. 237-274.
20. R.D. James and S. Müller, Internal variables and fine-scale oscillations in micromagnetics, Continuum Mech. Thermodyn., Vol. **6**, 1994, pp. 291-336.
21. R.V. Kohn, P. Sternberg, Local minimisers and singular perturbations, Proc. Roy. Soc. Edin. A, Vol. **111**, 1989, pp. 69-84.
22. F.H. Lin, Une remarque sur l'application $x/|x|$, C. R. Acad. Sci. Paris, Vol. **305**, 1987, pp. 529-531.
23. H.A. Lorentz, Theory of Electrons, B.G. Teubner, 1909.
24. W. Rudin, Real and Complex Analysis, Mc Graw Hill, 1987.
25. A. Taheri, Local minimizers in the calculus of variations, Ph.D Thesis, Heriot-Watt University, 1997.
26. A. Taheri, Strong versus weak local minimizers of the perturbed Dirichlet functional, Comm. Appl. Anal., to appear.
27. A. Taheri, Sufficiency theorems for local minimizers of the multiple integrals of the calculus of variations, Proc. Roy. Soc. Edin A, Vol. 131, 2001.
28. R. Toupin, The elastic dielectric, J. Rational Mech. Anal. Vol. **5**, 1956, pp. 849-915.
29. M.M. Vainberg, Variational Methods for the Study of Nonlinear Operators, Holden Day, 1964.
30. T. Valent, Boundary Value Problems of Finite Elasticity, Springer-Verlag, 1987.
31. E. Zeidler, Nonlinear Functional Analysis and its Applications I, Fixed-Point Theorems, Springer, New York, Berlin, Heidelberg, Tokyo, 1986.
32. K. Zhang, Energy minimizers in nonlinear elastostatics and the implicit function theorem, Arch. Rational Mech. Anal., Vol. **114**, 1991, pp. 95-117.
33. W.P. Ziemer, Weakly Differentiable Functions, Graduate Texts in Mathematics, Springer-Verlag, 1989.