Math. Proc. Camb. Phil. Soc. (1980), **87**, 501
Printed in Great Britain

Strict convexity, strong ellipticity, and regularity in the calculus of variations

BY J. M. BALL*
University of California, Berkeley

(*Received* 17 *September* 1979)

1. *Introduction*. In this paper we investigate the connection between strong ellipticity and the regularity of weak solutions to the equations of nonlinear elastostatics and other nonlinear systems arising from the calculus of variations. The main mathematical tool is a new characterization of continuously differentiable strictly convex functions. We first describe this characterization, and then explain how it can be applied to the calculus of variations and to elastostatics.

Let $U \subset \mathbb{R}^n$ be open and convex. A function $\phi : U \to \mathbb{R}$ is said to be *strictly convex* if $\phi(tx+(1-t)y) < t\phi(x)+(1-t)\phi(y)$ whenever $x, y \in U$, $x \neq y$, and $t \in (0,1)$. (For general information on convex functions see Rockafellar (**22**).) We shall prove (Theorem 1 below) that if ϕ is C^1, then ϕ is strictly convex if and only if (i) $\nabla\phi$ is locally $1-1$, and (ii) ϕ is convex at (at least) one point of U. The necessity of these conditions is obvious, and it is their sufficiency that is interesting. Geometrically, (i) says that neighbouring but distinct points of the graph of ϕ have distinct tangent spaces. The role of (ii) is less obvious. The trivial example $U = (0,1) \subset \mathbb{R}^1$, $\phi(x) = -x^2$, shows that condition (i) alone does not imply strict convexity; however, one might conjecture that if $U = \mathbb{R}^n$, if (i) holds, and if ϕ is bounded below, then ϕ is strictly convex. This conjecture is false if $n > 1$. An example with $n = 2$ is the function

$$\phi(x,y) = e^{y-x^2}$$

which is convex at no point of $\mathbb{R}^2$.

The main idea in the proof of the sufficiency of (i) and (ii) is to study the asymptotic behaviour of solutions to various gradient systems of ordinary differential equations defined on U, and thus, in the spirit of Morse theory (cf. Palais & Smale (**19**)), to establish the existence of a non-trivial critical point of a suitable function. To achieve this we use an idea of Olech (**17**) and Hartman and Olech (**12**). Since we assume only that ϕ is C^1, the gradient systems we consider may possess nonunique solutions for given initial data, and this complicates somewhat the technical details. A much simpler proof of the strict convexity of ϕ under the stronger hypotheses that $\nabla\phi$ is $1-1$ in U and that ϕ satisfies a growth condition, is given in Theorem 2, which applies also to functions that are convex but not strictly convex.

* On leave from Heriot-Watt University, Edinburgh EH14 4AS. Research partially supported by U.S. Army Contract no. DAAG29-79-C-0086.

0305-0041/80/0000-7360 $03.50

We turn now to the applications of Theorem 1. Let Ω be an open subset of $\mathbb{R}^n$, and consider the functional

$$I(u) = \int_\Omega W(\nabla u(x))\,dx, \tag{1·1}$$

where $u:\Omega\to\mathbb{R}^m$. The Euler–Lagrange equations corresponding to (1·1) are

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W}{\partial u^i_{,\alpha}}\right) = 0 \quad (i = 1, \ldots, m). \tag{1·2}$$

The equilibrium equations of nonlinear elasticity for a homogeneous body under zero body forces have the form (1·2) with $m = n = 3$, and in this case W is the stored-energy function of the material. W is said to be *strongly elliptic*, and (1·2) to be a *strongly elliptic system*, if

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha \partial F^j_\beta}\lambda^i\lambda^j\mu_\alpha\mu_\beta > 0 \tag{1·3}$$

for all F and all nonzero vectors $\lambda\in\mathbb{R}^m$, $\mu\in\mathbb{R}^n$. If equality is allowed then (1·3) is known as the *Legendre-Hadamard condition*. The existence of minimizers for $I(u)$ for various boundary problems of nonlinear elasticity under hypotheses implying the Legendre-Hadamard condition has been established in (**2**, **3**), and corresponding results for arbitrary m, n given in (**4**). However, even if W is smooth it is not known under what conditions weak solutions of (1·2) are C^1 functions. Examples of discontinuous equilibrium solutions in nonlinear elasticity with W strongly elliptic will be given in (**6**). In these examples the discontinuity takes the form of a hole appearing at the centre of a solid body under tension. (Other examples of discontinuous weak solutions to strongly elliptic systems with similar singularities have been given by Giusti & Miranda (**9**) and Necas (**16**), but they do not apply to nonlinear elasticity.)

Although strong ellipticity does not prevent the type of singularities mentioned above, under a mild positivity condition on W it is essentially necessary and sufficient for there to be no continuous weak solutions u of (1·2) in which the only singularity is a jump in ∇u across a smooth $(n-1)$-dimensional surface (taken for simplicity in this paper to be a hyperplane). This result (Theorem 3 below) is stated precisely and proved in Section 3, essential use of Theorem 1 being made in the proof. Actually, in the statement of Theorem 3, (1·3) is replaced by the condition that W be *strictly rank* 1 *convex*. Strict rank 1 convexity bears exactly the same relationship to strong ellipticity as does strict convexity of a function $f(t)$ of a single variable to the condition $f'' > 0$. *A fortiori*, Theorem 3 implies that strict rank 1 convexity of W is a necessary condition for all weak solutions of (1·2) to be C^1. Despite this, in non-linear elasticity one should not discard stored-energy functions that are not strictly rank 1 convex, since such functions may correspond to materials that can undergo phase transitions (Ericksen (**7**, **8**)). For more information on non-elliptic problems in elasticity see Knowles & Sternberg (**13–15**).

In Section 4 we use Theorem 1 in a different way to deduce information concerning the nonuniqueness and bifurcation of homogeneous equilibrium states of an elastic cube subjected to given uniform normal surface tractions.

2. *Necessary and sufficient conditions for strict convexity.* Notation: $\langle\,,\rangle$ and $|\cdot|$ denote the standard inner product and norm in $\mathbb{R}^n$ respectively. If $A \subset \mathbb{R}^n$ then ∂A denotes the boundary of A. $B(x, \epsilon)$ (resp. $\bar{B}(x, \epsilon)$) is the open (resp. closed) ball in $\mathbb{R}^n$ with centre x and radius ϵ.

Definition. Let $U \subset \mathbb{R}^n$ be open. A function $u: U \to \mathbb{R}^n$ is *locally* $1-1$ if every $x \in U$ possesses a neighbourhood in which u is $1-1$.

The main result of this section is the following:

THEOREM 1. *Let $U \subset \mathbb{R}^n$ be open and convex, and let $\phi \in C^1(U)$. Necessary and sufficient conditions for ϕ to be strictly convex are that*

(i) *$\nabla\phi$ is locally $1-1$, and*

(ii) *there exists a locally supporting hyperplane for ϕ at some point of U; i.e. there exist $x_0 \in U$, $\epsilon > 0$, such that*

$$\phi(x) \geqslant \phi(x_0) + \langle \nabla\phi(x_0), x - x_0\rangle \tag{2·1}$$

if $x \in B(x_0, \epsilon) \cap U$.

The necessity of conditions (i), (ii) is well known; in fact, the following standard result shows that if ϕ is strictly convex then $\nabla\phi$ is $1-1$ in U, and that (2·1) holds with strict inequality whenever $x, x_0 \in U$, $x \neq x_0$.

LEMMA 1. *Let $U \subset \mathbb{R}^n$ be open and convex, and let $\phi \in C^1(U)$. Then the following are equivalent*:

(*a*) *ϕ is strictly convex.*

(*b*) *$\phi(x) > \phi(y) + \langle \nabla\phi(y), x - y\rangle$ whenever $x, y \in U$, $x \neq y$.*

(*c*) *$\nabla\phi$ is strictly monotone; i.e.*

$$\langle \nabla\phi(x) - \nabla\phi(y), x - y\rangle > 0 \quad \text{whenever} \quad x, y \in U, x \neq y.$$

To prove the sufficiency of conditions (i) and (ii), we will need some auxiliary results. Let $U \subset \mathbb{R}^n$ be open and convex, and let $\phi \in C^1(U)$.

Definitions. Let $x \in U$. We say that *ϕ is convex at x* if

$$\phi(y) \geqslant \phi(x) + \langle \nabla\phi(x), y - x\rangle$$

for all y in a neighbourhood of x, and that ϕ is *strictly convex at x* if

$$\phi(y) > \phi(x) + \langle \nabla\phi(x), y - x\rangle$$

for all $y \neq x$ in a neighbourhood of x.

The following result is elementary.

LEMMA 2. *ϕ is strictly convex if and only if ϕ is strictly convex at x for every $x \in U$.*

Proof. The necessity follows immediately from Lemma 1. Conversely, suppose that ϕ is not strictly convex. Then there exist $x, y \in U$, $x \neq y$, $t_0 \in (0, 1)$ such that

$$\phi(t_0 x + (1 - t_0) y) \geqslant t_0 \phi(x) + (1 - t_0)\phi(y).$$

Hence the maximum of the function

$$\theta(t) = \phi(tx + (1-t) y) - t\phi(x) - (1-t)\phi(y)$$

for $t \in [0, 1]$ is attained at some interior point τ, and in particular

$$\theta'(\tau) = \langle \nabla\phi(\tau x + (1-\tau) y), x - y\rangle - \phi(x) + \phi(y) = 0.$$

Thus

$$\phi(tx+(1-t)\,y)-\phi(\tau x+(1-\tau)\,y) \leqslant \langle \nabla\phi(\tau x+(1-\tau)\,y), (t-\tau)\,(x-y)\rangle$$

for all $t\in[0,1]$, and so ϕ is not strictly convex at $\tau x+(1-\tau)\,y$. ∎

From now on we suppose that $\nabla\phi$ is locally $1-1$.

LEMMA 3. *Let $x\in U$. The following conditions are equivalent:*

(a) ϕ is convex at x,

(b) ϕ is strictly convex at x,

(c) a bounded open set E exists containing x such that $\bar{E}\subset U$, $\nabla\phi$ is $1-1$ in E, and

$$\min_{y\in\partial E}\ \phi(y)-\phi(x)-\langle\nabla\phi(x), y-x\rangle \geqslant 0.$$

Proof. The implications $(b)\Rightarrow(a)\Rightarrow(c)$ are obvious. Let (c) hold, and suppose that (b) does not. Then there exists $\bar{y}\in E$, $\bar{y}\neq x$, such that

$$\phi(\bar{y})-\phi(x)-\langle\nabla\phi(x),\bar{y}-x\rangle \leqslant 0.$$

Hence

$$\min_{y\in\bar{E}}\ \phi(y)-\phi(x)-\langle\nabla\phi(x), y-x\rangle$$

is attained at some interior point $z\in E$ with $z\neq x$. Differentiating, we obtain

$$\nabla\phi(z)=\nabla\phi(x).$$

Since $\nabla\phi$ is $1-1$ in E, $z=x$. This is a contradiction. ∎

Define

$$S=\{x\in U : \phi \text{ strictly convex at } x\}.$$

LEMMA 4. *S is open.*

Proof. Let $x\in S$, and suppose there exists a sequence $\{x_j\}\subset U\backslash S$ such that $x_j\to x$. Let E be a bounded open set containing x such that $\bar{E}\subset U$, $\nabla\phi$ is $1-1$ in E, and

$$\phi(y)-\phi(x)-\langle\nabla\phi(x), y-x\rangle > 0$$

if $y\in\partial E$. By Lemma 3, for each sufficiently large j there exists $z_j\in\partial E$ with

$$\phi(z_j)-\phi(x_j)-\langle\nabla\phi(x_j), z_j-x_j\rangle < 0.$$

Passing to the limit using a convergent subsequence of $\{z_j\}$ we arrive at a contradiction. ∎

LEMMA 5. Let V be a bounded open subset of $\mathbb{R}^n$, and let $\psi\in C^1(\bar{V})$. Let $c\in V$ be a strict local minimizer for ψ and the only critical point of ψ in $\bar{V}$. Consider the differential equation

$$\dot{x}=-\nabla\psi(x). \tag{2·2}$$

Let A denote the *region of attraction of* c, that is

$A=\{y\in V$: if $x(t)$ is any solution of (2·2) satisfying $x(0)=y$, then $x(t)\in V$ for all $t\geqslant 0$ and $x(t)\to c$ as $t\to\infty$.$\}$

Then A is open, and if $w_0\in\partial A\cap V$ there exists a solution $w(t)$ of (2·2) with $w(0)=w_0$, and such that for some $t_{\max}\in(0,\infty)$,

$$w(t)\in\partial A\cap V \quad\text{for}\quad t\in[0,t_{\max}), \quad\text{and}\quad w(t)\to w_1\in\partial V \quad\text{as}\quad t\to t_{\max}.$$

Proof. By hypothesis there exists $\delta > 0$ such that $\bar{B}(c, \delta) \subset V$ and such that

$$\psi(x) > \psi(c) \quad \text{if} \quad 0 < |x-c| \leqslant \delta.$$

Let

$$a = \min_{|x-c|=\delta} \psi(x) > \psi(c).$$

Let $y_0 \in A$ and suppose $\{y_{r0}\} \subset V \backslash A$ satisfies $y_{r0} \to y_0$. By Hartman ((11) Theorem 3·2, p. 14) there exist solutions y_r of (2·2), $y_r(0) = y_{r0}$, maximally defined on $[0, \tau_r)$ and not tending to c as $t \to \infty$, such that for a subsequence $y_\mu, y_\mu \to y$ uniformly on compact intervals of $[0, \tau)$, where y is a solution of (2·2), $y(0) = y_0$, and y is maximally defined on $[0, \tau)$. Furthermore

$$\varliminf_{\mu \to \infty} \tau_\mu \geqslant \tau.$$

But $\tau = \infty$ since $y_0 \in A$. Let $\delta_1 > 0$ be such that $\psi(x) < a$ whenever $|x-c| < \delta_1$. Then there exists $T > 0$ such that $|y(t) - c| < \delta_1$ for $t \geqslant T$. Thus $|y_\mu(T) - c| < \delta_1$ for large enough μ, and since ψ is nonincreasing for solutions of (2·2) it follows that

$$|y_\mu(t) - c| < \delta$$

for all $t \geqslant T$ and that $\tau_\mu = \infty$, provided μ is large enough. For such μ it follows from (2·2) that

$$\int_0^\infty |\nabla \psi(y_\mu(t))|^2 dt < \infty,$$

and hence that $\nabla \psi(y_\mu(t_k)) \to 0$ for some sequence $t_k \to \infty$. Since c is the only critical point of ψ in V, $y_\mu(t_k) \to c$ as $k \to \infty$. Since ψ is nondecreasing, $\psi(y_\mu(t)) \to \psi(c)$ as $t \to \infty$, and thus $y_\mu(t) \to c$ as $t \to \infty$. This contradiction proves that A is open.

Let $w_0 \in \partial A \cap V$. We construct the required solution $w(t)$ on a small interval $[0, \tau]$. The number $\tau > 0$ is chosen sufficiently small so that every solution of (2·2) with initial data w_0 exists and remains in V for $t \in [0, \tau]$. Since A is open, $w_0 \notin A$, and thus at least one such solution $\hat{w}(t)$ exists such that $\hat{w}(t) \in V \backslash A$ for all $t \in [0, \tau]$. Let $z_{r0} \to w_0$, $z_{r0} \in A$. Then, by the result in Hartman (11) quoted above, there exist a subsequence $z_{\mu 0}$ and solutions z_μ, z of (2·2) defined on $[0, \tau]$ such that $z_\mu(0) = z_{\mu 0}$, $z(0) = w_0$, $z_\mu(t) \in A$ for all $t \in [0, \tau]$, and $z_\mu \to z$ uniformly on $[0, \tau]$. Thus $z(t) \in \bar{A}$ for all $t \in [0, \tau]$. By Kneser's theorem (cf. Hartman ((11) Theorem 4·1, p. 15)) the set of points $\{x(\tau) : x(\cdot)$ a solution of (2·2) with $x(0) = w_0\}$ is closed and connected. It contains one point in $\bar{A}$ and one point not in A. Hence it contains a point in ∂A. Thus there is a solution $z^{(1)}(t)$ such that $z^{(1)}(0) = w_0$ and $z^{(1)}(\tau) \in \partial A \cap V$. Similarly we construct $z^{(2)}$ in two steps so that

$$z^{(2)}(0) = w_0, \quad z^{(2)}(\tau/2) \in \partial A \cap V, \quad z^{(2)}(\tau) \in \partial A \cap V,$$

and more generally $z^{(k)}$ so that $z^{(k)}(0) = w_0$, $z^{(k)}(r\tau/2^k) \in \partial A \cap V$ for $r = 1, \ldots, 2^k$. The solutions $z^{(k)}$ are uniformly bounded and equicontinuous. Hence they converge uniformly on $[0, \tau]$ to a solution w satisfying $w(0) = w_0$ and $w(r\tau/2^k) \in \partial A \cap V$ for all k and $r = 1, \ldots, 2^k$. Thus $w(t) \in \partial A \cap V$ for all $t \in [0, \tau]$. Let w be a maximally extended solution on $[0, t_{\max})$ with $w(t) \in \partial A \cap V$ for all $t \in [0, t_{\max})$. If $t_{\max}$ were $+\infty$ then a subsequence $w(t_k)$ would tend to a critical point $\bar{c} \neq c$ in $\bar{V}$ as $t_k \to \infty$, by the same argument used above in the proof that A is open. Hence $t_{\max} < \infty$. But if $t_k \to t_{\max}-$ then $w(t_k)$ is a Cauchy sequence, since

$$|w(t_j) - w(t_k)| = \left| \int_{t_j}^{t_k} \nabla\psi(s)\, ds \right| \leqslant K\, |t_k - t_j|.$$

Thus $w(t) \to w_1 \in \partial V$ as $t \to t_{\max}$. ∎

Lemma 6. *S is closed in U.*

Proof. Suppose not. Then there exist $x_0 \in U\backslash S$ and a sequence $\{x_j\} \subset S$ with $x_j \to x_0$. We claim that there exist a subsequence $\{x_\mu\}$ and a sequence $\{y_\mu\} \subset U$ such that $y_\mu \to x_0$ and

$$\phi(y_\mu) < \phi(x_\mu) + \langle \nabla\phi(x_\mu), y_\mu - x_\mu \rangle. \tag{2·3}$$

If this were not the case, then there would exist an $\epsilon > 0$ such that

$$\phi(y) \geqslant \phi(x_j) + \langle \nabla\phi(x_j), y - x_j \rangle \quad \text{if} \quad |y - x_0| \leqslant \epsilon.$$

Letting $j \to \infty$ we would then have

$$\phi(y) \geqslant \phi(x_0) + \langle \nabla\phi(x_0), y - x_0 \rangle \quad \text{if} \quad |y - x_0| \leqslant \epsilon,$$

contradicting $x_0 \notin S$.

Let $V = B(x_0, \delta)$, and choose $\delta > 0$ small enough so that $\bar{V} \subset U$ and $\nabla\phi$ is $1-1$ in $\bar{V}$. Choose μ large enough so that $x_\mu, y_\mu \in V$. Define

$$\psi_\mu(x) = \phi(x) - \langle \nabla\phi(x_\mu), x \rangle.$$

Then $\psi_\mu \in C^1(\bar{V})$, and $\nabla\psi_\mu(x) = \nabla\phi(x) - \nabla\phi(x_\mu)$. Hence x_μ is the only critical point of ψ_μ in V, and, since $x_\mu \in S$, x_μ is a local minimizer. Let A_μ be the region of attraction of x_μ with respect to the equation

$$\dot{x} = -\nabla\psi_\mu(x). \tag{2·4}$$

Since ψ_μ is non-increasing for solutions of (2·4), and since (2·3) implies that

$$\psi_\mu(y_\mu) < \psi_\mu(x_\mu),$$

it follows that $y_\mu \notin A_\mu$. Therefore there exists $w_{\mu 0} \in \partial A_\mu$ with $|w_{\mu 0} - x_\mu| \leqslant |y_\mu - x_\mu|$. Let w_μ be the solution constructed in Lemma 5. Thus $w_\mu(0) = w_{\mu 0}$, $w_\mu(t) \in \partial A_\mu$ for $t \in [0, t^{(\mu)}_{\max})$ and $|w_\mu(t) - x_0| \to \delta$ as $t \to t^{(\mu)}_{\max}$.

Fix ϵ with $0 < \epsilon < \delta$. For sufficiently large μ, $|w_{\mu 0} - x_0| < \epsilon$, and so there exists a largest time s_μ such that $|w_\mu(s_\mu) - x_0| = \epsilon$. From (2·4),

$$\delta - \epsilon \leqslant |\, w_\mu(t^{(\mu)}_{\max}) - w_\mu(s_\mu)\,| = \left| \int_{s_\mu}^{t^{(\mu)}_{\max}} \nabla\psi_\mu(w_\mu(t))\, dt \right| \leqslant C(t^{(\mu)}_{\max} - s_\mu), \tag{2·5}$$

where C is a constant. But since $w_\mu(t) \in \partial A$ for all $t \in [0, t^{(\mu)}_{\max}]$ we have that

$$\psi_\mu(w_{\mu 0}) \geqslant \psi_\mu(w_\mu(s_\mu)) \geqslant \psi_\mu(w_\mu(t^{(\mu)}_{\max})) \geqslant \psi_\mu(x_\mu).$$

Hence

$$\int_{s_\mu}^{t_{\max}^{(\mu)}} |\nabla\psi_\mu(w_\mu(t))|^2 dt = \psi_\mu(w_\mu(s_\mu)) - \psi_\mu(w_\mu(t_{\max}^{(\mu)})) \leqslant \psi_\mu(w_{\mu 0}) - \psi_\mu(x_\mu).$$

Thus there exists $z_\mu \in \bar{V}$ with $|z_\mu - x_0| \geqslant \epsilon$ and

$$|\nabla\phi(z_\mu) - \nabla\phi(x_\mu)|^2 \leqslant \frac{C}{\delta - \epsilon}[\phi(w_{\mu 0}) - \phi(x_\mu) - \langle\nabla\phi(x_\mu), w_{\mu 0} - x_\mu\rangle].$$

Let $\mu \to \infty$, and let z be a limit point of z_μ. Then $\epsilon \leqslant |z - x_0| \leqslant \delta$ and

$$|\nabla\phi(z) - \nabla\phi(x_0)|^2 \leqslant 0.$$

Hence $\nabla\phi(z) = \nabla\phi(x_0)$, and so $z = x_0$. This is a contradiction. ∎

Proof of Theorem 1 (*Conclusion*). Since S is open and closed in U, since U is convex, and since S is nonempty by hypothesis (ii) and Lemma 3, it follows that $S = U$. Hence by Lemma 2, ϕ is strictly convex. ∎

Remarks. The idea of considering points lying on the boundary of the region of attraction of a stable critical point is taken from Olech (**17**) (see also Hartman & Olech (**12**), Hartman ((**11**), pp. 548–554). A possible alternative to the use in the proof of the ordinary differential equations (2·4), which may in general have non-unique solutions, might be to use the pseudogradient flows of Palais (**18**). Of course, once ϕ is known to be convex uniqueness for (2·4) follows.

The following consequence of Theorem 1 will be used in Section 3.

COROLLARY 1. *Let $U \subset \mathbb{R}^n$ be open and convex, let $\phi \in C^1(U)$, and suppose that $\nabla\phi$ is $1-1$ in U but that ϕ is not strictly convex. If E is any bounded open set with $\bar{E} \subset U$, and if $x \in E$, then*

$$\min_{y\in\bar{E}} \ \phi(y) - \phi(x) - \langle\nabla\phi(x), y - x\rangle$$

is negative and is attained on ∂E.

Proof. Let E be bounded and open, $\bar{E} \subset U$, and $x \in E$. By Theorem 1, ϕ is not strictly convex at x. The result now follows by the argument used in Lemma 3. ∎

It is not clear whether there is a natural generalization of Theorem 1 to convex functions that are not strictly convex. However, we now give a simple global result that does apply to such functions.

THEOREM 2. *Let $U \subset \mathbb{R}^n$ be open and convex, and let $\phi \in C^1(U)$. Suppose that*

(1) $\dfrac{\phi(x_r)}{1 + |x_r|} \to \infty$ *if* $x_r \to x \in \partial U$ *or* $|x_r| \to \infty$,

(2) $\nabla\phi^{-1}(x)$ *is a convex set for every* $x \in U$.

Then ϕ is convex.

Proof. Let $x \in U$. Consider the problem

$$\underset{y\in U}{\text{minimize}} \ \ \phi(y) - \langle\nabla\phi(x), y\rangle.$$

On account of (1), the minimum is attained at some point $z \in U$. Differentiating, we obtain $\nabla\phi(z) = \nabla\phi(x)$, and hence

$$\phi(y) \geqslant \phi(z) + \langle \nabla\phi(x), y-z \rangle$$

for all $y \in U$. Thus, since

$$\phi(z) - \phi(x) = \int_0^1 \langle \nabla\phi(x+t(z-x)), z-x \rangle \, dt$$
$$= \langle \nabla\phi(x), z-x \rangle,$$

where we have used (2), it follows that

$$\phi(y) \geqslant \phi(x) + \langle \nabla\phi(x), y-x \rangle$$

for all $y \in U$. Thus ϕ is convex. ▌

Note that the same proof establishes the sufficiency part of Theorem 1 under the stronger assumptions that $\nabla\phi$ is $1-1$ in U and that (1) holds.

3. *Strong ellipticity and the regularity of weak solutions*. Let Ω be an open subset of $\mathbb{R}^n$. Let $M^{m\times n}$ denote the set of real $m\times n$ matrices with the induced topology of $\mathbb{R}^{mn}$, and let E be an open subset of $M^{m\times n}$. Let $W \in C^1(E)$. Consider the functional

$$I(u) = \int_\Omega W(\nabla u(x))\, dx, \tag{3·1}$$

where $u: \Omega \to \mathbb{R}^m$. The Euler–Lagrange equations corresponding to (3·1) are

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W}{\partial u^i_{,\alpha}}\right) = 0 \quad (i = 1, \ldots, m), \tag{3·2}$$

where the repeated suffix α indicates summation over $\alpha = 1, \ldots, n$. A function u which, together with its first partial derivatives (in the sense of distributions), is locally integrable over Ω, is said to be a *weak solution of* (3·2) if (3·2) holds in the sense of distributions, i.e.

$$\int_\Omega \frac{\partial W}{\partial u^i_{,\alpha}}(\nabla u(x))\, \phi^i_{,\alpha}(x)\, dx = 0 \tag{3·3}$$

for all $\phi \in (C_0^\infty(\Omega))^m$, where the integral in (3·3) exists and in particular $\nabla u(x) \in E$ almost everywhere in Ω. Here $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω.

The equilibrium equations of nonlinear elasticity are of the form (3·2) with $m = n = 3$, it being assumed that the material is homogeneous and that there are no external forces. In this case $W(F)$ is the stored-energy function and $u(x)$ denotes the position of the particle that occupied the point x in the reference configuration Ω. The equations (3·3) can then be interpreted as a statement of the principle of virtual work; Antman & Osborn (1) have shown that under certain conditions they are equivalent to the requirement that the resultant force on an arbitrary sub-body be zero.

We now consider a basic construction due originally to Hadamard (**10**). Let $\mu \in \mathbb{R}^n$, $\mu \neq 0$, $k \in \mathbb{R}$, and consider the hyperplane π of $\mathbb{R}^n$ with equation $\langle x, \mu \rangle = k$. Let $F, G \in E$. We seek a continuous function $u : \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$\left.\begin{aligned} u(x) &= Fx + a \quad \text{if} \quad \langle x, \mu \rangle > k, \\ u(x) &= Gx + b \quad \text{if} \quad \langle x, \mu \rangle < k, \end{aligned}\right\} \tag{3·4}$$

where $a, b \in \mathbb{R}^m$. It is easily verified that such a function exists if and only if

$$F - G = \lambda \otimes \mu, \quad (\lambda \otimes \mu)^i_\alpha \stackrel{\text{def}}{=} \lambda^i \mu_\alpha,$$

for some $\lambda \in \mathbb{R}^m$, and that in this case $k\lambda = b - a$. A simple calculation then shows that u is a weak solution of (3·2) (for any Ω intersected by π) if and only if the jump condition

$$\left[\frac{\partial W}{\partial F^i_\alpha} \mu_\alpha\right] = 0$$

holds, i.e. if and only if

$$\left(\frac{\partial W}{\partial F^i_\alpha}(G + \lambda \otimes \mu) - \frac{\partial W}{\partial F^i_\alpha}(G)\right)\mu_\alpha = 0. \tag{3·5}$$

In nonlinear elasticity this jump condition exactly expresses the fact that in equilibrium the traction is continuous across π.

We now suppose that E is *rank 1 convex*, i.e. that

$$tF + (1-t)\,G \in E$$

whenever $F, G \in E$, $F - G = a \otimes b$ is a matrix of rank 1, and $t \in [0, 1]$. Examples of open, rank 1 convex, sets are $E = M^{m \times n}$ and, in the case $m = n$,

$$E = \{F \in M^{n \times n} : \alpha < \det F < \beta\},$$

where $\alpha, \beta \in \mathbb{R} \cup \{+\infty, -\infty\}$.

Definition. $W : E \to \mathbb{R}$ is said to be *strictly rank 1 convex* if the inequality

$$W(tF + (1-t)\,G) < tW(F) + (1-t)\,W(G)$$

holds whenever $F, G \in E$, $F - G = a \otimes b \neq 0$, and $t \in (0, 1)$.

(Note that this definition makes sense without any regularity assumptions on W. We repeat, however, that we always assume that $W \in C^1(E)$.) If W is C^2, and if W satisfies the strong ellipticity condition

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha\, \partial F^j_\beta} a^i a^j b_\alpha b_\beta > 0 \quad \text{for all nonzero} \quad a \in \mathbb{R}^m, b \in \mathbb{R}^n,$$

then W is strictly rank 1 convex, but strict rank 1 convexity does not imply strong ellipticity. (See (**2**), section 3, for information and references concerning the relationship between (non-strict) rank 1 convexity and the Legendre–Hadamard condition.)

We can now state the main result of this section.

Theorem 3. *Necessary and sufficient conditions for W to be strictly rank 1 convex are that*

(i) *all weak solutions of* (3·2) *of the form* (3·4) *are C^1, and*

(ii) *there exist $G_0 \in E$, $\mu_0 \neq 0$, $\epsilon > 0$ such that*

$$W(G_0+\lambda\otimes\mu_0) \geqslant W(G_0)+\frac{\partial W}{\partial F^i_\alpha}(G_0)\,\lambda^i\mu_{0\alpha} \quad \text{if} \quad |\lambda| \leqslant \epsilon.$$

Proof. First note that W is strictly rank 1 convex if and only if the function

$$\lambda \mapsto W(G+\lambda\otimes\mu)$$

is strictly convex for every $G\in E$ and $\mu \neq 0$. (Since E is rank 1 convex the domain of each such function is an open convex subset of $\mathbb{R}^m$.)

Let W be strictly rank 1 convex. Since

$$\frac{\partial}{\partial\lambda^i}W(G_0+\lambda\otimes\mu_0)\bigg|_{\lambda=0} = \frac{\partial W}{\partial F^i_\alpha}(G_0)\,\mu_{0\alpha},$$

it follows from the above remark and Lemma 1 that (ii) holds (for every $G_0\in E$, μ_0). Let u be a weak solution of (3·2) of the form (3·4). Multiplying (3·5) by λ^i we obtain

$$\langle \nabla_\lambda W(G+\lambda\otimes\mu)-\nabla_\lambda W(G),\, \lambda-0\rangle = 0.$$

By Lemma 1, $\lambda\otimes\mu = 0$ and hence $F = G$. Thus u is C^1.

Conversely, suppose that (i) and (ii) hold. We claim that the function

$$\lambda \mapsto \frac{\partial W}{\partial\lambda}(G+\lambda\otimes\mu)$$

is 1–1 for every $G\in E$, $\mu \neq 0$. If not there would exist $\lambda \neq \bar\lambda$ such that

$$\left[\frac{\partial W}{\partial F^i_\alpha}(G+\bar\lambda\otimes\mu+(\lambda-\bar\lambda)\otimes\mu)-\frac{\partial W}{\partial F^i_\alpha}(G+\bar\lambda\otimes\mu)\right]\mu_\alpha = 0.$$

By (i) and (3·5) this happens only if $\lambda = \bar\lambda$.

Let $K = E\times(\mathbb{R}^n\backslash\{0\})$, and let $B = \{(G,\mu)\in K: \lambda\mapsto W(G+\lambda\otimes\mu)$ is strictly convex$\}$. By (ii) and Theorem 1, $(G_0,\mu_0)\in B$. Hence B is nonempty. Let $(G_r,\mu_r)\to(G,\mu)\in K$ with $(G_r,\mu_r)\in B$ for each r. Choose $\epsilon > 0$ small enough so that $G+\lambda\otimes\mu\in E$ if $|\lambda| \leqslant \epsilon$. Then for sufficiently large r, $G_r+\lambda\otimes\mu_r\in E$ if $|\lambda|\leqslant\epsilon$, and so

$$W(G_r+\lambda\otimes\mu_r) \geqslant W(G_r)+\frac{\partial W}{\partial F^i_\alpha}(G_r)\,\lambda^i\mu_{r\alpha} \quad (|\lambda|\leqslant\epsilon).$$

Passing to the limit as $r\to\infty$ we deduce that $\lambda\mapsto W(G+\lambda\otimes\mu)$ is convex at $\lambda = 0$, and hence, by Theorem 1, $(G,\mu)\in B$. Thus B is closed in K.

Let $(G,\mu)\in B$ and suppose that $(G_r,\mu_r)\to(G,\mu)$ with $(G_r,\mu_r)\notin B$ for each r. Choose $\epsilon > 0$ small enough so that $G+\lambda\otimes\mu\in E$ if $|\lambda|\leqslant\epsilon$. Since $\lambda\mapsto W(G_r+\lambda\otimes\mu_r)$ is *not* strictly convex, Corollary 1 implies that for sufficiently large r there exists λ_r with $|\lambda_r| = \epsilon$ and

$$W(G_r+\lambda_r\otimes\mu_r) < W(G_r)+\frac{\partial W}{\partial F^i_\alpha}(G_r)\,\lambda^i_r\mu_{r\alpha}. \tag{3.6}$$

Choosing a convergent subsequence λ_{r_k} and passing to the limit in (3·6) we find that

$$W(G+\bar{\lambda}\otimes\mu) \leqslant W(G)+\frac{\partial W}{\partial F^i_\alpha}(G)\,\bar{\lambda}^i\mu_\alpha$$

for some $\bar{\lambda}$ with $|\bar{\lambda}| = \epsilon$. Thus $\lambda \mapsto W(G+\lambda\otimes\mu)$ is not strictly convex, contradicting $(G,\mu)\in B$. Hence B is open.

Since B is a non-empty, open and closed subset of the connected set K it follows that $B = K$. Hence W is strictly rank 1 convex. ∎

Remarks. (1) That strong ellipticity of $W\in C^2(E)$ implies (i) was shown by Knowles and Sternberg (**15**) (see also (**5**)).

(2) It follows in particular from Theorem 3 that if (ii) holds then strict rank 1 convexity of W is a necessary condition for all weak solutions of (3·2) to be C^1. In nonlinear elasticity it is usually assumed that there exists a *natural state*; i.e. that for some $G_0\in E$,

$$W(F) \geqslant W(G_0) \quad \text{for all} \quad F\in E.$$

In this case (ii) holds trivially.

(3) Let S be a smooth $(n-1)$-dimensional surface with normal μ at the point $x\in S$. Suppose that, in a neighbourhood of x, u is continuous across S and C^1 on either side of S, and let F, G denote the limits at x of ∇u from either side of S. Then the jump condition (3·5) still holds, and hence $F = G$ if W is strictly rank 1 convex.

(4) An examination of the proof of Theorem 3 shows that condition (i) may be replaced by the weaker condition (i)′ for every $H\in E$ there is a neighbourhood N of H in $M^{m\times n}$ such that any weak solution of (3·2) of the form (3·4) with $F, G\in N$ is C^1.

4. *Equilibrium configurations of an elastic cube.* In this section we apply Theorem 1 to the problem of the equilibrium of an elastic cube subjected to given uniform normal surface tractions.

Consider an elastic body occupying in a reference configuration the unit cube $\Omega = (0,1)^3$ of $\mathbb{R}^3$. We suppose that the stored-energy function W of the body is homogeneous and isotropic. Thus (cf. Truesdell & Noll (**24**))

$$W = W(F) = \Phi(\lambda_1,\lambda_2,\lambda_3),$$

where F is the deformation gradient, the λ_i are the eigenvalues of $\sqrt{F^TF}$, and where Φ is symmetric in its arguments. We consider only homogeneous deformations of the cube given by

$$u(x) = (\lambda_1 x^1, \lambda_2 x^2, \lambda_3 x^3), \quad x = (x^1,x^2,x^3)\in\Omega,$$

where the λ_i are positive constants. In this case

$$F \overset{\text{def}}{=} \nabla u(x) = \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3),$$

and so the equilibrium equations (3·2) are trivially satisfied. To maintain equilibrium, equal and opposite normal forces of magnitude $T_i (i = 1,2,3)$ must be applied to the two faces of the cube normal to the x^i axis. These forces are given in terms of

$$\lambda = (\lambda_1,\lambda_2,\lambda_3)$$

by the equations (see Truesdell & Noll ((24), p. 317))

$$T_1 = \frac{\partial \Phi(\lambda)}{\partial \lambda_1}, \quad T_2 = \frac{\partial \Phi(\lambda)}{\partial \lambda_2}, \quad T_3 = \frac{\partial \Phi(\lambda)}{\partial \lambda_3}.$$

or, more concisely, by

$$\nabla \Phi(\lambda) = T, \tag{4·1}$$

where $T = (T_1, T_2, T_3)$. If we regard T as given, then (4·1) must be solved for λ. We note that (4·1) is the Euler–Lagrange equation corresponding to the function

$$I(\lambda) = \Phi(\lambda) - \langle T, \lambda \rangle.$$

Let $E = \{\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_i > 0,\ i = 1, 2, 3\}$, and suppose that $\Phi \in C^1(E)$. (If Φ is defined only on a subset of E then the arguments below still apply with appropriate modifications.) Suppose further that the reference configuration is a natural state, so that

$$\Phi(\mu) \geqslant \Phi(1,1,1) \quad \text{for all} \quad \mu \in E.$$

It is known that for natural rubbers Φ is not a strictly convex function. (For a discussion and references see (2, 3).) Supposing, then, that Φ is not strictly convex, we deduce immediately from Theorem 1 that $\nabla \phi$ is not locally $1-1$. That is, there exist $\lambda^* \in E$ and sequences $\lambda^{(r)} \to \lambda^*$, $\bar{\lambda}^{(r)} \to \lambda^*$, with $\lambda^{(r)} \neq \bar{\lambda}^{(r)}$ for each r, such that

$$\nabla \Phi(\lambda^{(r)}) = \nabla \Phi(\bar{\lambda}^{(r)}).$$

This means that $(\lambda^*, \nabla \Phi(\lambda^*))$ is a bifurcation point for (4·1). The same argument shows that there is a bifurcation point in any convex subset of E containing both a point where Φ is convex and a point where Φ is not strictly convex. In particular, if any neighbourhood of $\lambda^* \in E$ contains points of convexity and points where Φ is not strictly convex, then $(\lambda^*, \nabla \Phi(\lambda^*))$ is a bifurcation point.

We can apply our argument to study bifurcation from the solution $\lambda = (\alpha, \alpha, \alpha)$ in which all the principal stretches are equal. Suppose, as is not unreasonable, that Φ in convex in a neighbourhood of $\lambda = (1, 1, 1)$ and that $\Phi_{,1}(\alpha, \alpha, \alpha)$ is a strictly increasing function of α. Let

$$\alpha^* = \inf\{\alpha > 1 : \Phi \quad \text{not strictly convex at} \quad (\alpha, \alpha, \alpha)\}.$$

Clearly $\alpha^* \geqslant 1$. Our argument show that if $\alpha^* < \infty$ then $(\lambda^*, \nabla \phi(\lambda^*))$ is a bifurcation point for $\lambda^* = (\alpha^*, \alpha^*, \alpha^*)$. Since $\nabla \Phi(\alpha, \alpha, \alpha) \neq \nabla \Phi(\beta, \beta, \beta)$ if $\alpha \neq \beta$, it follows that there exist bifurcating solutions in which the principal stretches are not all equal. A similar argument applies in compression. Of course, more detailed information is special cases can be obtained using standard techniques of bifurcation theory, particularly under additional smoothness hypotheses on Φ; on the other hand using Theorem 1 does bring out rather clearly the role of strict convexity. In general, bifurcations into nonhomogeneous deformations will also occur. Finally, we remark that for the case of an incompressible neo-Hookean material, an interesting and detailed study of the set of homogeneous equilibrium solutions has been given by Rivlin (20, 21). (See also Sawyers & Rivlin (23).)

I would like to thank John Guckenheimer for some stimulating discussions.

(1) Antm
Arc
(2) Ball,
Rat
(3) Ball,
Nor
(Lo
(4) Ball,
Ord
Mat
(5) Ball,
line
(6) Ball,
(7) Erick
(8) Erick
New
(9) Giust
min
Ital
(10) Hada
(11) Hart
(12) Hart
enti
(13) Know
stat
(14) Know
elas
(15) Know
disc
329
(16) Neca
coe
Ak
(17) Olec
Dif
(18) Pala
An
(19) Pala
165
(20) Rivli
the
(19
(21) Rivli
loa
(22) Rock
(23) Sawy
thr
(24) True
Ph
196

REFERENCES

(1) Antman, S. S. and Osborn, J. E. The principle of virtual work and integral laws of motion, *Arch. Rational Mech. Anal.* **69** (1979), 231–262.

(2) Ball, J. M. Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.

(3) Ball, J. M. Constitutive inequalities and existence theorems in nonlinear elastostatics. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium*, vol. 1, ed. R. J. Knops, (London, Pitman, 1977).

(4) Ball, J. M. On the calculus of variations and sequentially weakly continuous maps. In *Ordinary and partial differential equations* Dundee, 1976, Springer Lecture Notes in Mathematics, vol. 564, 13–25.

(5) Ball, J. M. Remarques sur l'existence et la régularité des solutions d'élastostatique non-lineáire. (In the Press).

(6) Ball, J. M. In preparation.

(7) Ericksen, J. L. Equilibrium of bars. *J. of Elasticity* **5** (1975), 191–201.

(8) Ericksen, J. L. Special topics in elastostatics. In *Advances in Applied Mechanics* **17** (1977) New York.

(9) Giusti, E. and Miranda, M. Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcola delle variazioni. *Boll. Unione Mat. Ital. Ser.* 4. **1** (1968), 219–226.

(10) Hadamard, J. *Leçons sur la propagation des ondes* (Paris, Hermann, 1903).

(11) Hartman, P. *Ordinary differential equations* (New York, John Wiley, 1964).

(12) Hartman, P. and Olech, C. On global asymptotic stability of solutions of ordinary differential equations. *Trans. Amer. Math. Soc.* **104** (1962), 154–178.

(13) Knowles, J. K. and Sternberg, E. On the ellipticity of the equations of nonlinear elastostatics for a special material. *J. Elasticity* **5** (1975), 341–362.

(14) Knowles, J. K. and Sternberg, E. On the failure of ellipticity of the equations for finite elastic plane strain. *Arch. Rational Mech. Anal.* **63** (1977), 321–326.

(15) Knowles, J. K. and Sternberg, E. On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elasticity. *J. Elasticity* **9** (1978), 329–380.

(16) Necas, J. Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity. In *Theory of non-linear operators* (Berlin, Akademie-Verlag, 1977).

(17) Olech, C. On the global stability of autonomous systems in the plane. *Contributions to Differential Equations* **1** (1963), 389–400.

(18) Palais, R. S. Critical point theory and the minimax principle. *Proc. Symp. Pure Math.* **15**, Amer. Math. Soc. Providence, R. I. (1970), 185–212.

(19) Palais, R. S. and Smale, S. A generalized Morse theory. *Bull. Amer. Math. Soc.* **70** (1964), 165–172.

(20) Rivlin, R. S. Large elastic deformations of isotropic materials. II. Some uniqueness theorems for pure homogeneous deformations. *Phil. Trans. Roy. Soc. London* A **240** (1948), 491–508.

(21) Rivlin, R. S. Stability of pure homogeneous deformations of an elastic cube under dead loading. *Quart. Appl. Math.* **32** (1974), 265–271.

(22) Rockafellar, R. T. *Convex analysis* (Princeton University Press, 1970).

(23) Sawyers, K. and Rivlin, R. S. Bifurcation conditions for a thick elastic plate under thrust. *Int. J. Solids and Structures* **10** (1974), 483–501.

(24) Truesdell, C. and Noll, W. The non-linear field theories of mechanics. In *Handbuch der Physik*, ed. S. Flugge, vol. 3, part 3, 1–590 (Berlin and New York, Springer-Verlag, 1965).