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A VARIATIONAL MODEL ALLOWING BOTH SMOOTH AND SHARP PHASE BOUNDARIES IN SOLIDS

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Dedicated to Philippe Ciarlet on the occasion of his 70th birthday

Abstract. We present models for solid-solid phase transitions with surface energy that allow both smooth and sharp interfaces. The models involve the minimisation of an energy that consists of three terms: the elastic energy (a double-well potential), the smooth-interface surface energy and the sharp-interface surface energy. Existence of solutions is shown in arbitrary dimensions. The second part of the paper deals with the one-dimensional case. For the first 1D model (in which the sharp-interface energy is the same regardless of the size of the jump of the gradient), we study the regime of the parameters (one parameter represents the boundary conditions, one models the energy of the sharp interface, and the third one models the energy of the smooth interfaces) for which the minimiser presents smooth interfaces, sharp interfaces or no interfaces. We also prove that a suitable scaling of the functional Γ-converges to a pure sharp-interface model, as the parameters penalising the formation of interfaces go to zero. For the second 1D model (in which the sharp-interface energy depends on the size of the jump and can tend to zero as the jump tends to zero), we describe general properties of the minimisers, and show that their gradients have a finite number of discontinuity points.

1. **Introduction.**

Solid-solid phase transitions often lead to fine-scale mixtures of distinct phases or phase variants. For example, martensitic transformations typically lead to twinning. Many features of this microstructure have been explained by the minimisation of the energy in continuum models (see Ball and James [11]). This leads to the variational problem of minimising the elastic energy

$$I(y) := \int_\Omega W(Dy(x))\,\mathrm{d}x.$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain representing the reference configuration, N is the space dimension, $y : \Omega \to \mathbb{R}^N$ is the deformation, and $W : \mathbb{R}^{N\times N} \to [0,\infty]$ is the stored-energy function of the material (see Ball [8]). The function W is (in general) not quasiconvex and, hence, the infimum of I in a suitable Sobolev space subject to given boundary conditions may not be attained. If no minimiser exists, the minimising sequences exhibit finer and finer oscillations of the deformation gradient. In practice, the oscillations are finitely fine due to additional effects such as surface energy. Thus, the model based on elastic energy minimisation predicts many

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features of the microstructure, but not details such as its length scale. Nevertheless, it is believed (see, e.g., Kohn and Müller [28], Müller [33]) that such details can be captured by minimising the sum of elastic and surface energy.

The most common way of representing the surface energy is to add to the elastic energy a singular perturbation involving second gradients of the deformation, and a typical model for this is

$$I_\varepsilon(y) := \int_\Omega [W(Dy(x)) + \varepsilon^2 |D^2 y(x)|^2] \, \mathrm{d}x$$

(see, e.g., Müller [33], Conti and Schweizer [16]). Here $\varepsilon > 0$ is a small parameter. The interfaces that appear in the minimisers of I_ε are called *smooth* because the deformation y is in the Sobolev space $W^{2,2}$.

By analogy with the theory of Image Segmentation (see Mumford and Shah [34], De Giorgi, Carriero and Leaci [20]) or Fracture Mechanics (see Francfort and Marigo [21]), one might think that a plausible model for sharp interfaces is

$$J_\kappa(y) := \int_\Omega W(Dy(x)) \, \mathrm{d}x + \kappa \mathcal{H}^{N-1}(S_{Dy}).$$

Here $\kappa > 0$ is a small parameter, $\mathcal{H}^{N-1}$ represents the $N-1$ dimensional Hausdorff measure, and S_v denotes the approximate discontinuity set of the function v (see Ambrosio, Fusco and Pallara [6, Def. 3.63]). In fact, a similar model was proposed in Ball and James [11]. However, we argue that J_κ is not a good model. Indeed, it is easy to see that this functional is not coercive with respect to the natural norm

$$y \mapsto \|y\|_{W^{1,p}} + |Dy|_{BV}, \tag{1}$$

where $p \in (1, \infty)$ is an exponent related to the growth conditions of W, and $|\cdot|_{BV}$ denotes the BV seminorm; consider, to fix ideas, $W(F) = |F|^p$ for all $F \in \mathbb{R}^{N\times N}$, and let y_0 be a function in $W^{1,p}$ such that Dy_0 is not in BV. Then, the mollifications of y_0 (with small radius of mollification) form a family of functions with bounded energy J_κ but unbounded norm (1). In fact, one can exhibit examples for which the infimum of J_κ subject to Dirichlet boundary conditions is not attained. An explicit construction in dimension 1 is given by $\Omega = (0, 1)$, any function $W : \mathbb{R} \to [0, \infty]$ that is continuous in $[1, 2]$ and such that $W^{-1}(0) = \{1, 2\}$, and the boundary conditions are

$$y(0) = 0, \qquad y(1) = \lambda, \tag{2}$$

for a given $\lambda \in (1, 2)$. In this case, the infimum of J_κ is zero, as is easily seen from the family of functions $v_{\delta,\lambda} : (0, 1) \to \mathbb{R}$ defined by

$$v_{\delta,\lambda}(x) := \begin{cases} x & \text{if} \quad x \in (0, 2 - \lambda - \delta] \\ \dfrac{(x - 2 + \lambda + 3\delta)^2}{4\delta} + 2 - \lambda - 2\delta & \text{if} \quad x \in (2 - \lambda - \delta, 2 - \lambda + \delta) \\ 2x - 2 + \lambda & \text{if} \quad x \in [2 - \lambda + \delta, 1), \end{cases} \tag{3}$$

for each $0 < \delta < \min\{\lambda - 1, 2 - \lambda\}$. Clearly, the infimum of J_κ is not attained. In general, in arbitrary dimensions, given a candidate y for a minimiser of J_κ with $\mathcal{H}^{N-1}(S_{Dy}) > 0$, by mollification we may find a smooth deformation u with $S_{Du} = \varnothing$ and such that $\int_\Omega W(Du)$ is arbitrarily close to $\int_\Omega W(Dy)$, so that $J_\kappa(u) < J_\kappa(y)$. We are aware that if, instead of having p-growth at infinity, the function W had the property

$$W(F) = \infty \qquad \text{if} \;\; F \;\; \text{is an} \;\; N \times N \;\; \text{matrix with} \;\; \det F \le 0,$$

then a mollification of y would not necessarily be a function with finite energy. Despite all these drawbacks, a similar model to J_κ was proposed in Parry [36]. The difference was that, in his approach, the function space, the topology and the variations were different from ours, so that smoothing of the interface was not an allowed variation. In that way, he was able to prove that, in dimension 2 and under certain natural assumptions, flat shear bands (which are deformations that, of course, have jumps in their gradient) are local minimisers.

A better model that J_κ (and, in fact, closer to I_ε) is the following:

$$K_\varepsilon(y) := \int_\Omega W(Dy(x))\, \mathrm{d}x + \varepsilon V(Dy, \Omega)$$

(see, e.g., Müller [33]). Here $V(Dy, \Omega)$ is the total variation in Ω of Dy (equivalently, the BV seminorm of Dy), which is supposed to be of bounded variation (see, e.g., Ambrosio, Fusco and Pallara [6]). Functions of bounded variation, unlike Sobolev functions, can have jumps; precisely, in the model K_ε, the jumps in Dy represent the *sharp* interfaces. By using standard methods in the Calculus of Variations and the appropriate compactness and lower semicontinuity results in BV, the existence of minimisers of K_ε is easily shown; see, if necessary, the proof of Proposition 2.2 below, which can be easily adapted to prove the existence of minimisers of K_ε. The disadvantage of the model based on minimisation of K_ε is that, although minimisers y with $Dy \in BV$ exist, in general we cannot obtain minimisers y with $Dy \in SBV$; in other words, the second derivative of a minimiser may have a non-zero Cantor part, and the physical meaning of that is unclear.

We propose the following model that allows both smooth and sharp interfaces:

$$I_{\varepsilon,\kappa}(y) := \int_\Omega \left[W(Dy(x)) + \varepsilon^2 |\nabla^2 y(x)|^2 \right] \mathrm{d}x + \kappa \mathcal{H}^{N-1}(S^*_{Dy}). \tag{4}$$

Here $\varepsilon, \kappa > 0$ are small parameters. (In fact, in Section 2 we consider a more general model of which this is a special case). The set of admissible functions for $I_{\varepsilon,\kappa}$ is the set of $y \in W^{1,1}(\Omega, \mathbb{R}^N)$ such that the distributional derivative Dy of y belongs to the class of generalised special functions of bounded variation $GSBV(\Omega)^{N\times N}$, the value $I_{\varepsilon,\kappa}(y)$ is finite, and the boundary conditions are $y|_{\Gamma_1} = \bar{y}|_{\Gamma_1}$, where $\Gamma_1 \subset \partial\Omega$ is an $N-1$ rectifiable set with $\mathcal{H}^{N-1}(\Gamma_1) > 0$, and $\bar{y} \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a fixed function. In formula (4), $\nabla^2 y$ denotes the weak approximate differential of Dy, and the set S^*_{Dy} is formed by the weak approximate discontinuity points of Dy. We refer to Ambrosio, Fusco and Pallara [6, Section 4.5] for the definitions and properties of $GSBV$. The reason for choosing the class $GSBV$ instead of the more natural one SBV is purely technical, in order to be able to prove existence of solutions.

The idea is that, depending on the values of ε, κ and the boundary conditions, some materials will prefer smooth interfaces (represented by the term $|\nabla^2 y(x)|^2$), and other materials sharp interfaces (represented by the term $\mathcal{H}^{N-1}(S^*_{Dy})$).

Our model (4) was partially motivated by experiments. On the one hand, the experiments of Baele, van Tendeloo and Amelinckx [7] show interfaces in martensitic twins of Ni Mn, and the images suggest that the interfaces are atomistically sharp; as mentioned earlier, we need good models for sharp interfaces. On the other hand, experiments of Manolikas, van Tendeloo and Amelinckx [30] on $Pb_2(VO_4)_2$ suggest that the twin boundaries are joined smoothly through a boundary region where the atomic lattice is curved. Finally, Cahn [14] comments that it is not always clear to decide whether a given interface is smooth or sharp.

A possible criticism of the model (4) is that the term penalising the sharp interface only depends on the $N-1$ dimensional area of that interface, but not on the size of the jump of the gradient at the interface. The general existence result in Section 2 does allow the sharp-interface energy to depend on the size of the jump of the gradient, but does not allow this energy to tend to zero as the size of the jump tends to zero. However, in Section 5 we prove the existence of minimisers of a variant of the one-dimensional counterpart of model (4), in which the sharp-interface energy can tend to zero as the size of the jump tends to zero. We also describe general properties of the minimisers; for example, we prove that the gradients of the minimisers have only a finite number of discontinuities.

The plan of this paper is as follows. In Section 2 we prove the existence of minimisers of the model that generalises (4). Sections 3, 4 and 5 deal with the one-dimensional case. In Section 3 we compute the Γ-limit of proper scalings of the functional (4) as ε and κ go to zero; the result and techniques are very similar to the Modica-Mortola [32] model. Section 4 analyses the behaviour of the minimisers, as ε and κ go to zero and the boundary condition λ of (2) goes to one of the wells. The result is that, depending on the values of ε, κ and λ, the minimiser exhibits smooth interfaces, sharp interfaces or no interfaces at all. Finally, Section 5 introduces the variant of the model (4) explained in the paragraph above, and does a preliminary analysis of it.

2. **Existence of minimisers.** The model that we propose, which in fact generalises (4), is the following:

$$I(y) := \int_\Omega W_1(x, Dy(x), \nabla^2 y(x))\,\mathrm{d}x + \int_{S^*_{Dy}} \gamma(x, Dy^+(x), Dy^-(x), \nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{5}$$

First, we explain the meaning of each of the terms that appear in (5), and the properties that W_1 and γ should satisfy in order that the functional I represents a realistic energy. Lemma 2.1 will provide sufficient conditions for Hadamard's jump condition to hold, i.e., conditions that guarantee that Dy^+ and Dy^- are rank-one connected. Finally, in Proposition 2.2, we prove the existence of minimisers of (5) under further assumptions on W_1 and γ.

Let $N \geq 1$ be the space dimension. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set representing the reference configuration. Let $\mathcal{S} \subset SO(N)$ be the isotropy group of the material; here, $SO(N) \subset \mathbb{R}^{N\times N}$ stands for the set of orthogonal matrices with positive determinant. We are making the simplifying assumption that the isotropy group is constant throughout the material, that is to say, it does not depend on the point of Ω. If the material does not have any symmetry property, then of course the only element of $\mathcal{S}$ is the identity matrix.

Let Sym_3^N be the real $\frac{1}{2}N^2(N+1)$ dimensional vector space formed by the tensors $C = (C_{i\alpha\beta})_{i,\alpha,\beta=1}^N$ of order 3 such that $C_{i\alpha\beta} = C_{i\beta\alpha}$ for all $i, \alpha, \beta \in \{1, \ldots, N\}$. The norm of Sym_3^N is given by any of the equivalent norms on Sym_3^N regarded as a $\frac{1}{2}N^2(N+1)$ dimensional vector space over $\mathbb{R}$.

Let $W_1 : \Omega \times \mathbb{R}^{N\times N} \times \mathrm{Sym}_3^N \to [0, \infty]$ be a function. The term

$$\int_\Omega W_1(x, Dy(x), \nabla^2 y(x))\,\mathrm{d}x \tag{6}$$

will represent the contribution of the elastic energy and the smooth-surface energy, which may be coupled in a non-trivial way. This functional (6) must satisfy the frame-indifference property, which states that for every deformation y, every vector $c \in \mathbb{R}^N$ and $R \in SO(N)$, the energies of y and $\Omega \ni x \mapsto Ry(x) + c$ must coincide. This amounts to the following property of W_1:

$$\begin{aligned}
&W_1(x, F, G) = W_1(x, RF, R \odot G), \\
&\text{for a.e. } x \in \Omega, \quad \text{all} \quad F \in \mathbb{R}^{N\times N}, \quad G \in \mathrm{Sym}_3^N, \quad R \in SO(N), \\
&\text{where} \quad (R \odot G)_{i\alpha\beta} := \sum_{j=1}^N R_{ij} G_{j\alpha\beta}, \quad i, \alpha, \beta \in \{1, \ldots, N\}.
\end{aligned}$$

Similarly, as $\mathcal{S}$ is the isotropy group of the material, for all deformations $y : \Omega \to \mathbb{R}^N$, all vectors $c \in \mathbb{R}^N$ and all $S \in \mathcal{S}$, the energies of y and $\bar{y}$ must coincide, where $\bar{y} : c + S\Omega \to \mathbb{R}^N$ is the deformation defined through $\bar{y}(c + Sx) = y(x)$, for $x \in \Omega$. This amounts to the following property of W_1:

$$\begin{aligned}
&W_1(x, F, G) = W_1(x, FS, G \oslash S), \\
&\text{for a.e. } x \in \Omega, \quad \text{all} \quad F \in \mathbb{R}^{N\times N}, \quad G \in \mathrm{Sym}_3^N, \quad S \in \mathcal{S}, \\
&\text{where} \quad (G \oslash S)_{i\alpha\beta} := \sum_{\gamma,\delta=1}^N G_{i\gamma\delta} S_{\delta\beta} S_{\gamma\alpha}, \quad i, \alpha, \beta \in \{1, \ldots, N\}.
\end{aligned}$$

This analysis is standard; see, e.g., Podio-Guidugli [37].

Now we explain the term

$$\int_{S^*_{Dy}} \gamma(x, Dy^+(x), Dy^-(x), \nu(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x), \tag{7}$$

which will represent the sharp-interface energy. It is natural to assume (see, e.g., Parry [36]) that the sharp-interface energy can be represented as an integral over the jump set of the gradient whose integrand is a function depending on the values of the gradient at both sides of the jump, and the normal to the jump set. Thus, we assume the existence of a function

$$\gamma : \Omega \times \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \times \mathbb{S}^{N-1} \to [0, \infty]$$

such that the sharp-interface energy of the deformation y is (7). Here, $\mathbb{S}^{N-1}$ is the $N-1$ dimensional unit sphere in $\mathbb{R}^N$. We analyse now the conditions that γ must satisfy in order that the energy is well-defined, frame-indifferent and respect the material symmetry. This analysis follows the lines of that of Parry [36] and Ambrosio [4, Section 3]. If, in formula (7), we change the roles of Dy^+ and Dy^-, the sharp interface energy must not change, provided the normal is inverted. Therefore, the function γ has to satisfy

$$\begin{aligned}
&\gamma(x, F_1, F_2, \nu) = \gamma(x, F_2, F_1, -\nu), \\
&\text{for} \quad \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega, \quad \text{all} \quad F_1, F_2 \in \mathbb{R}^{N\times N}, \quad \nu \in \mathbb{S}^{N-1}.
\end{aligned} \tag{8}$$

We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. Frame-indifference for (7) amounts to

$$\begin{aligned}
&\gamma(x, F_1, F_2, \nu) = \gamma(x, RF_1, RF_2, \nu), \\
&\text{for} \quad \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega, \quad \text{all} \quad F_1, F_2 \in \mathbb{R}^{N\times N}, \quad R \in SO(N), \quad \nu \in \mathbb{S}^{N-1},
\end{aligned} \tag{9}$$

whereas material symmetry amounts to

$$\begin{aligned}&\gamma(x, F_1, F_2, \nu) = \gamma(x, F_1 S, F_2 S, S^T \nu),\\&\text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega, \quad \text{all } F_1, F_2 \in \mathbb{R}^{N\times N}, \quad S \in \mathcal{S}, \quad \nu \in \mathbb{S}^{N-1}.\end{aligned} \tag{10}$$

From Hadamard's jump condition, we may think that Dy^+ and Dy^- must be rank-one connected. It turns out that the validity of that assertion depends on the regularity of Dy. First we prove a version of Hadamard's jump condition; in addition, we give a meaning to Dy^+ and Dy^-. For any $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, the set $S_u \subset \Omega$ denotes the approximate discontinuity set of u as defined in Ambrosio, Fusco and Pallara [6, Def. 3.63]. Recall also the definition of countably rectifiable set, as given in [6, Def. 2.57].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be open and non-empty. Let $y \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ satisfy $Dy \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times N})$. Then the set S_{Dy} is countably $\mathcal{H}^{N-1}$-rectifiable, and there exist four Borel measurable functions*

$$Dy^+,\ Dy^- : S_{Dy} \to \mathbb{R}^{N\times N}, \qquad \nu : S_{Dy} \to \mathbb{S}^{N-1}, \qquad a : S_{Dy} \to \mathbb{R}^N \setminus \{0\}$$

such that, for $\mathcal{H}^{N-1}$-a.e. $z \in S_{Dy}$,

$$\begin{aligned}&\lim_{\varrho\to 0^+} ⨍_{\{x\in B(z,\varrho):(x-z)\cdot\nu(z)>0\}} \left|Dy(x) - Dy^+(z)\right| \,\mathrm{d}x = 0,\\&\lim_{\varrho\to 0^+} ⨍_{\{x\in B(z,\varrho):(x-z)\cdot\nu(z)<0\}} \left|Dy(x) - Dy^-(z)\right| \,\mathrm{d}x = 0.\end{aligned} \tag{11}$$

and

$$Dy^-(z) - Dy^+(z) = a(z) \otimes \nu(z). \tag{12}$$

Moreover, if for some $z \in S_{Dy}$ satisfying (11) *there also exist $A, B \in \mathbb{R}^{N\times N}$ and $\gamma \in \mathbb{S}^{N-1}$ such that*

$$\begin{aligned}0 &= \lim_{\varrho\to 0^+} ⨍_{\{x\in B(z,\varrho):(x-z)\cdot\gamma>0\}} |Dy(x) - A| \,\mathrm{d}x\\&= \lim_{\varrho\to 0^+} ⨍_{\{x\in B(z,\varrho):(x-z)\cdot\gamma<0\}} |Dy(x) - B| \,\mathrm{d}x\end{aligned}$$

then

$$(A, B, \gamma) \in \left\{(Dy^+(z), Dy^-(z), \nu(z)), (Dy^-(z), Dy^+(z), -\nu(z))\right\}; \tag{13}$$

if, in addition, there exists $\delta \in \mathbb{R}^N$ such that $B - A = \delta \otimes \gamma$, then $\delta = a(z)$.

Proof. Arguing for each open set Ω' with closure contained in Ω, and by the uniqueness property (13), we can assume that $y \in W^{1,1}(\Omega, \mathbb{R}^N)$ and $Dy \in BV(\Omega, \mathbb{R}^{N\times N})$.

Thanks to the results of De Giorgi [19] (see also Ambrosio, Fusco and Pallara [6, Prop. 3.69, Th. 3.77 and Th. 3.78]), the set S_{Dy} is countably $\mathcal{H}^{N-1}$-rectifiable, and there exist three Borel measurable functions

$$Dy^+ : S_{Dy} \to \mathbb{R}^{N\times N}, \qquad Dy^- : S_{Dy} \to \mathbb{R}^{N\times N}, \qquad \nu : S_{Dy} \to \mathbb{S}^{N-1}$$

such that, for $\mathcal{H}^{N-1}$-a.e. $z \in S_{Dy}$, equalities (11) hold.

Fix any $z \in S_{Dy}$ such that (11) holds. For every $0 < \varepsilon \le 1$, define the measurable function $u_\varepsilon : B(0,1) \to \mathbb{R}^N$ as $u_\varepsilon(x) := \varepsilon^{-1} y(z + \varepsilon x)$ for $x \in B(0,1)$. The distributional derivative Du_ε of u_ε satisfies $Du_\varepsilon(x) = Dy(z + \varepsilon x)$ a.e. $x \in B(0,1)$. We have

$$\|Du_\varepsilon\|_{L^1(B(0,1))} = \frac{1}{\varepsilon^N} \int_{B(z,\varepsilon)} |Dy(x)| \,\mathrm{d}x.$$

Thanks to (11), the family $\{Du_\varepsilon\}_{0<\varepsilon\leq 1}$ is bounded in $L^1(B(0,1),\mathbb{R}^{N\times N})$. By the Poincaré inequality, for every $0<\varepsilon\leq 1$, there exists $a_\varepsilon\in\mathbb{R}^N$ such that the family $\{u_\varepsilon+a_\varepsilon\}_{0<\varepsilon\leq 1}$ is weakly relatively compact in $BV(B(0,1),\mathbb{R}^N)$.

Define $w_0 : B(0,1)\to\mathbb{R}^N$ as

$$w_0(x) := \begin{cases} Dy^+(z) & \text{if} \quad x\cdot\nu(z)\geq 0 \\ Dy^-(z) & \text{if} \quad x\cdot\nu(z)<0, \end{cases}$$

for $x\in B(0,1)$. By (11), Du_ε converges to w_0 in $L^1(B(0,1),\mathbb{R}^{N\times N})$ as $\varepsilon\to 0^+$. Therefore, w_0 is the distributional derivative of a $BV(B(0,1),\mathbb{R}^N)$ function, in fact, of a $W^{1,\infty}(B(0,1),\mathbb{R}^N)$ function. By the classical Hadamard's jump condition (see, e.g., Ball and James [11, Prop. 1]),

$$Dy^-(z)-Dy^+(z)=\left[\left(Dy^-(z)-Dy^+(z)\right)\nu(z)\right]\otimes\nu(z).$$

Therefore, we define $a(z):=(Dy^-(z)-Dy^+(z))\,\nu(z)$. Thus, a is Borel measurable as a product of Borel measurable functions. The function a never takes the value 0, because of the definition of S_{Dy}.

Property (13) is clear (see, e.g., Ambrosio, Fusco and Pallara [6, Def. 3.67]). The uniqueness of a is a consequence of (13). □

The conclusion (12) of Lemma 2.1 is known to be false if the assumption $Dy\in BV_{\text{loc}}$ is dropped; see, e.g., Ball and James [11], Iwaniec, Verchota and Vogel [27] and Ball and Carstensen [9]. However, we do not know whether (12) is true when, instead of $Dy\in BV_{\text{loc}}$, we assume $Dy\in GSBV$, which is the natural assumption for Proposition 2.2 below.

Thus, if in the functional space where the problem is posed the assumptions of Lemma 2.1 hold, then the function γ of (7) need not be defined in the whole $\Omega\times\mathbb{R}^{N\times N}\times\mathbb{R}^{N\times N}\times\mathbb{S}^{N-1}$, but, rather, there must exist a function $\gamma_1 : \Omega\times\mathbb{R}^{N\times N}\times\mathbb{R}^N\times\mathbb{S}^{N-1}\to[0,\infty]$ such that

$$\begin{aligned}&\gamma_1(x,F,a,\nu)=\gamma(x,F,F+a\otimes\nu,\nu),\\ &\text{for } \ \mathcal{H}^{N-1}\text{-a.e. } x\in\Omega, \quad \text{all} \ \ F\in\mathbb{R}^{N\times N}, \quad a\in\mathbb{R}^N, \quad \nu\in\mathbb{S}^{N-1}.\end{aligned}$$

Of course, translating the properties of γ into properties of γ_1 is trivial. For example, in terms of γ_1, equalities (8), (9) and (10) read as

$$\begin{aligned}&\gamma_1(x,F,a,\nu)=\gamma_1(x,F+a\otimes\nu,a,-\nu)=\gamma_1(x,RF,Ra,\nu)=\gamma_1(x,FS,a,S^T\nu),\\ &\text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\Omega,\ \text{all } F\in\mathbb{R}^{N\times N},\ a\in\mathbb{R}^N,\ \nu\in\mathbb{S}^{N-1},\ R\in SO(N),\ S\in\mathcal{S}.\end{aligned}$$

In the following proposition we present a result on the existence of minimisers of the functional (5). We use the notation of Ball, Currie and Olver [10] so as to deal with functions that are polyconvex in the second gradient. For each $1\leq r\leq N$, choose a natural number N_r, and N_r functions

$$J^{r,1},\dots,J^{r,N_r} : \mathrm{Sym}_3^N\to\mathbb{R}$$

such that for every $u\in C^2(\Omega,\mathbb{R}^N)$ and $1\leq i\leq N_r$, the function $J^{r,i}\circ D^2u$ is a Jacobian of degree r of D^2u, and every Jacobian of degree r of D^2u can be written as a linear combination of

$$J^{r,1}\circ D^2u,\ \dots,\ J^{r,N_r}\circ D^2u.$$

Define $J^r : \mathrm{Sym}_3^N\to\mathbb{R}^{N_r}$ by $J^r:=(J^{r,1},\dots,J^{r,N_r})$, and $\sigma_r:=\sum_{i=1}^r N_i$. When taking about measurability of functions, $\mathcal{L}_N$ refers to $\mathbb{R}^N$-Lebesgue measurability, while $\mathcal{B}$ refers to Borel measurability.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty bounded open set with Lipschitz boundary. Let*

$$W_1 : \Omega \times \mathbb{R}^{N\times N} \times \mathrm{Sym}_3^N \to [0,\infty], \qquad \gamma : \Omega \times \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \times \mathbb{S}^{N-1} \to [0,\infty].$$

Suppose that there are functions

$$g : \Omega \times \mathbb{R}^{N\times N} \to [0,\infty], \qquad \varphi : [0,\infty) \to [0,\infty]$$

such that φ is lower semicontinuous and increasing, g is Borel measurable, for a.e. $x \in \Omega$ the function $g(x,\cdot)$ is lower semicontinuous,

$$\lim_{\substack{|F|\to\infty \\ F\in\mathbb{R}^{N\times N}}} g(x,F) = \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \tag{14}$$

and

$$W_1(x,F,G) \geq g(x,F) + \varphi(|G|), \quad \text{for a.e. } x \in \Omega, \text{ all } F \in \mathbb{R}^{N\times N},\ G \in \mathrm{Sym}_3^N. \tag{15}$$

Let assumptions C1 or C2 hold:

C1. The following assumptions hold:
- *(a) The function W_1 is $\mathcal{L}_N \times \mathcal{B}(\mathbb{R}^{N\times N} \times \mathrm{Sym}_3^N)$ measurable.*
- *(b) The function $W_1(x,\cdot,\cdot) : \mathbb{R}^{N\times N} \times \mathrm{Sym}_3^N \to [0,\infty]$ is lower semicontinuous for a.e. $x \in \Omega$.*
- *(c) The function $W_1(x,F,\cdot) : \mathrm{Sym}_3^N \to [0,\infty]$ is convex for each a.e. $x \in \Omega$ and all $F \in \mathbb{R}^{N\times N}$.*

C2. There exists $1 \leq r \leq N$ such that the following assumptions hold:
- *(a) There exists a function $\Phi : \Omega \times \mathbb{R}^{N\times N} \times \mathbb{R}^{\sigma_r} \to [0,\infty]$ such that*
$$W_1(x,F,G) = \Phi(x,F,J^1(G),\ldots,J^r(G))$$
$$\text{for a.e. } x \in \Omega, \text{ all } F \in \mathbb{R}^{N\times N},\ G \in \mathrm{Sym}_3^N.$$
- *(b) The function $\Phi(\cdot,F,J) : \Omega \to [0,\infty]$ is $\mathcal{L}_N$ measurable for every $F \in \mathbb{R}^{N\times N}$ and every $J \in \mathbb{R}^{\sigma_r}$.*
- *(c) The function $\Phi(x,\cdot,\cdot) : \mathbb{R}^{N\times N} \times \mathbb{R}^{\sigma_r} \to [0,\infty]$ is continuous for a.e. $x \in \Omega$.*
- *(d) The function $\Phi(x,F,\cdot) : \mathbb{R}^{\sigma_r} \to [0,\infty]$ is convex for a.e. $x \in \Omega$ and all $F \in \mathbb{R}^{N\times N}$.*
- *(e) There exist a function $\phi \in L^1(\Omega)$, constants $c > 0$ and $\alpha_1,\ldots,\alpha_r$ with*
$$\alpha_1 \geq 2, \qquad \alpha_r > 1; \qquad \alpha_i \geq \frac{\alpha_1}{\alpha_1 - 1} \quad \text{for } 2 \leq i \leq r-1,$$
such that
$$W_1(x,F,G) \geq \phi(x) + c\sum_{i=1}^r |J^i(G)|^{\alpha_i},$$
$$\text{for a.e. } x \in \Omega, \text{ all } F \in \mathbb{R}^{N\times N},\ G \in \mathrm{Sym}_3^N.$$

Assume that there exist a concave increasing function $\Theta : [0,\infty] \to [0,\infty]$ with $\Theta(0) > 0$ and $\Theta(\infty) = \sup_{t>0}\Theta(t)$, and a function $\psi : \Omega \times \mathbb{R}^N \to [0,\infty)$ such that

$$\gamma(x,F_1,F_2,\nu) = \Theta(|F_1 - F_2|)\psi(x,\nu), \qquad x \in \Omega, \quad F_1,F_2 \in \mathbb{R}^{N\times N}, \quad \nu \in \mathbb{S}^{N-1}, \tag{16}$$

the family of functions $\{\psi(\cdot,\nu) : \nu \in \mathbb{S}^{N-1}\}$ is equicontinuous,

$$0 < \inf_{x\in\Omega} \inf_{\nu\in\mathbb{S}^{N-1}} \psi(x,\nu) \leq \sup_{x\in\Omega} \sup_{\nu\in\mathbb{S}^{N-1}} \psi(x,\nu) < \infty, \tag{17}$$

and for each $x \in \Omega$, the function $\psi(x,\cdot)$ is convex and $\psi(x,\lambda\gamma) = |\lambda|\psi(x,\gamma)$ for all $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}^N$.

Let I be the functional of (5). Let $\bar{y} \in W^{1,1}(\Omega,\mathbb{R}^N)$ satisfy $D\bar{y} \in GSBV(\Omega)^{N\times N}$ and $I(\bar{y}) < \infty$. Let $\Gamma_1 \subset \partial\Omega$ be an $N-1$ rectifiable set with $\mathcal{H}^{N-1}(\Gamma_1) > 0$. Define

$$\mathcal{A} := \left\{ y \in W^{1,1}(\Omega,\mathbb{R}^N) : \ Dy \in GSBV(\Omega)^{N\times N}, \ I(y) < \infty, \ y|_{\Gamma_1} = \bar{y}|_{\Gamma_1} \right\}. \quad (18)$$

Then there exists a minimiser of I in $\mathcal{A}$.

In formula (5), Dy is the distributional derivative of y, the function $\nabla^2 y$ denotes the weak approximate differential of Dy (see Ambrosio, Fusco and Pallara [6, Def. 4.31, Th. 4.34]). The symmetry of $\nabla^2 y$ follows from [6, Th. 4.34(c)] and the fact that the distributional second derivative is symmetric, as well as its absolutely continuous part. The set S^*_{Dy} is formed by the weak approximate discontinuity points of Dy, the definition of which might not be standard and we recall now. By assumption Dy belongs to $GSBV(\Omega)^{N\times N}$, which is a space strictly contained in $GSBV(\Omega,\mathbb{R}^{N\times N})$; thus, every component $\partial_\alpha y_i$ of Dy belongs to $GSBV(\Omega)$. By definition, $S^*_{Dy} = \bigcup_{1\le i,\alpha\le N} S^*_{\partial_\alpha y_i}$, where for each $1 \le i,\alpha \le N$, the set $S^*_{\partial_\alpha y_i}$ denotes the weak approximate discontinuity set of $\partial_\alpha y_i$ as defined in [6, Def. 4.28]. The meaning of Dy^+, Dy^- and ν is explained by the following result, essentially due to Ambrosio [4] (see also Ambrosio, Fusco and Pallara [6, Th. 4.34, Th. 4.40]): for all $y \in \mathcal{A}$ there exist three Borel measurable functions

$$Dy^+ : S^*_{Dy} \to \mathbb{R}^{N\times N}, \quad Dy^- : S^*_{Dy} \to \mathbb{R}^{N\times N}, \quad \nu : S^*_{Dy} \to \mathbb{S}^{N-1} \quad (19)$$

such that for $\mathcal{H}^{N-1}$-a.e. $z \in S^*_{Dy}$,

$$\operatorname*{ap\,lim}_{\substack{x\to z\\ (x-z)\cdot\nu(z)>0}} Dy(x) = Dy^+(z), \qquad \operatorname*{ap\,lim}_{\substack{x\to z\\ (x-z)\cdot\nu(z)<0}} Dy(x) = Dy^-(z). \quad (20)$$

Moreover, if for some $z \in S^*_{Dy}$ satisfying (20), there also exist $A, B \in \mathbb{R}^{N\times N}$ and $\gamma \in \mathbb{S}^{N-1}$ such that

$$\operatorname*{ap\,lim}_{\substack{x\to z\\ (x-z)\cdot\gamma>0}} Dy(x) = A, \qquad \operatorname*{ap\,lim}_{\substack{x\to z\\ (x-z)\cdot\gamma<0}} Dy(x) = B. \quad (21)$$

then (13). The concept and properties of ap lim are defined in Ambrosio [4].

Proof of Proposition 2.2. The integral (7) is well-defined thanks to (21), (13) and the property $\psi(x,\nu) = \psi(x,-\nu)$ for all $\nu \in \mathbb{S}^{N-1}$ and $x \in \Omega$.

Let $\{y_j\}_{j\in\mathbb{N}}$ be a minimising sequence of I in $\mathcal{A}$. By (14), (15) and the Poincaré inequality, the sequence $\{y_j\}_{j\in\mathbb{N}}$ is weakly precompact in $W^{1,1}$. We thus extract a subsequence (not relabelled) such that y_j converges weakly in $W^{1,1}$ to some y, as $j \to \infty$. Clearly, $y|_{\Gamma_1} = \bar{y}|_{\Gamma_1}$.

Call $z_j := Dy_j$ for each $j \in \mathbb{N}$. By (15), (16) and (17),

$$\sup_{j\in\mathbb{N}} \left\{ \int_\Omega [g(x,z_j(x)) + \varphi(|\nabla z_j(x)|)]\,\mathrm{d}x + \int_{S^*_{z_j}} \Theta(|z_j^+(x) - z_j^-(x)|)\,\mathrm{d}\mathcal{H}^{N-1}(x) \right\} < \infty.$$

Thanks to Ambrosio's [4] compactness result, there exists $z \in GSBV(\Omega)^{N\times N}$ such that, up to a subsequence, z_j converges a.e. to z, and ∇z_j converges weakly in L^1_{loc} to ∇z, as $j \to \infty$. This implies $z = Dy$ and, by Vitali's convergence theorem, Dy_j converges to Dy in L^1.

If C1 holds then we apply Ioffe's [26] theorem (see also Ambrosio, Fusco and Pallara [6, Th. 5.8]) and obtain that, for every compact K contained in Ω,

$$\begin{aligned}\int_K W_1(x, Dy(x), \nabla^2 y(x))\,\mathrm{d}x &\leq \liminf_{j\to\infty} \int_K W_1(x, Dy_j(x), \nabla^2 y_j(x))\,\mathrm{d}x \\ &\leq \liminf_{j\to\infty} \int_\Omega W_1(x, Dy_j(x), \nabla^2 y_j(x))\,\mathrm{d}x.\end{aligned}$$

Moreover, by the monotone convergence theorem, letting the set K tending to Ω, we obtain

$$\int_\Omega W_1(x, Dy(x), \nabla^2 y(x))\,\mathrm{d}x \leq \liminf_{j\to\infty} \int_\Omega W_1(x, Dy_j(x), \nabla^2 y_j(x))\,\mathrm{d}x. \tag{22}$$

If C2 holds then, by the result of Fusco, Leone, Verde and March [22, Th. 4.2], we obtain that $J^i \circ \nabla^2 y_j$ converges, as $j \to \infty$, to $J^i \circ \nabla^2 y$ weakly in $L^{\alpha_i}_{\mathrm{loc}}$, for all $i \in \{1, \dots, r\}$. By polyconvexity (see, e.g., Ball, Currie and Olver [10, Th. 5.4]), we obtain (22) as well.

Finally, by the lower semicontinuity result of Ambrosio [4, Th. 3.7], we have

$$\begin{aligned}&\int_{S^*_{Dy}} \gamma(x, Dy^+(x), Dy^-(x), \nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) \\ &\qquad\leq \liminf_{j\to\infty} \int_{S^*_{Dy_j}} \gamma(x, Dy_j^+(x), Dy_j^-(x), \nu_j(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned}$$

Of course, ν is the function of (19) corresponding to y, and ν_j is that corresponding to y_j. Therefore, y is a minimiser of I in $\mathcal{A}$. $\square$

In Proposition 2.2 we have not intended to state the most general assumptions that guarantee lower semicontinuity. In this respect, see Ambrosio and Braides [5], Braides [13, Section 2.4], Kristensen [29] and Dal Maso, Fonseca, Leoni and Morini [17]. In fact, some of the results there can cover the case $\Theta(0) = 0$.

Note that the assumptions of Proposition 2.2 are compatible with the assumption

$$W_1(x, F, G) = \infty, \quad \text{a.e. } x \in \Omega, \;\; \text{all } G \in \mathrm{Sym}^N_3, \; F \in \mathbb{R}^{N\times N} \text{ such that } \det F \leq 0.$$

In that case, any $y \in \mathcal{A}$ satisfies $\det Dy > 0$ a.e.

We end this section with a comment about the set of admissible functions $\mathcal{A}$. As is well-known (see, e.g., Ambrosio, Fusco and Pallara [6, Rk. 4.27]), the class $GSBV(\Omega, \mathbb{R}^{N\times N})$ does not coincide with the class $GSBV(\Omega)^{N\times N}$. Nevertheless, it is easy to show (see, e.g., Dal Maso, Francfort and Toader [18, Prop. 2.3]) that

$$\mathcal{A} = \left\{ y \in W^{1,1}(\Omega, \mathbb{R}^N) : \; Dy \in GSBV(\Omega, \mathbb{R}^{N\times N}), \; I(y) < \infty, \; y|_{\Gamma_1} = \bar{y}|_{\Gamma_1} \right\},$$

where $\mathcal{A}$ is the set defined in (18).

3. **The one-dimensional case: Gamma-limit of the functional.** In this section, we calculate the Γ-limit of a proper scaling of the one-dimensional version of the functional (4), as $\varepsilon, \kappa \to 0$.

Let $W : (0, \infty) \to [0, \infty)$ be a continuous function satisfying $W^{-1}(0) = \{1, 2\}$ and

$$\liminf_{t\to 0^+} W(t) > 0, \qquad \liminf_{t\to\infty} W(t) > 0.$$

For every $1 < \lambda < 2$, define

$$\begin{aligned}
\mathcal{A}_\lambda &:= \big\{ y \in W^{1,1}(0,1) : \ y(0) = 0, \ y(1) = \lambda, \ y' \in SBV(0,1), \ y' > 0 \text{ a.e.}, \\
&\qquad \nabla^2 y \in L^2(0,1) \big\}, \\
\mathcal{A}_\lambda^{sm} &:= \big\{ y \in W^{2,2}(0,1) : \ y(0) = 0, \ y(1) = \lambda, \ y' > 0 \text{ a.e.} \big\}, \\
\mathcal{A}_\lambda^{sh} &:= \big\{ y \in W^{1,1}(0,1) : \ y(0) = 0, \ y(1) = \lambda, \ y' \in SBV(0,1), \ y' > 0 \text{ a.e.}, \\
&\qquad \mathcal{H}^0(S_{y'}) \geq 1 \big\}.
\end{aligned}$$

In the formulas above, y' denotes the distributional derivative of y, and $\nabla^2 y$ denotes the approximate derivative of y', which coincides a.e. with the absolutely continuous part of the second distributional derivative y'' of y. As usual, $S_{y'}$ denotes the approximate discontinuity set of y'.

Fix $1 < \lambda < 2$. For each $\varepsilon, \kappa > 0$, we define the functionals

$$I_{\varepsilon,\kappa} : \mathcal{A}_\lambda \to [0,\infty], \qquad I_\varepsilon^{sm} : \mathcal{A}_\lambda^{sm} \to [0,\infty], \qquad I_\kappa^{sh} : \mathcal{A}_\lambda^{sh} \to [0,\infty]$$

through

$$\begin{aligned}
I_{\varepsilon,\kappa}(y) &:= \int_0^1 \left[W(y'(x)) + \varepsilon^2 (\nabla^2 y(x))^2 \right] \mathrm{d}x + \kappa \mathcal{H}^0(S_{y'}), && y \in \mathcal{A}_\lambda, \\
I_\varepsilon^{sm}(y) &:= \int_0^1 \left[W(y'(x)) + \varepsilon^2 (\nabla^2 y(x))^2 \right] \mathrm{d}x, && y \in \mathcal{A}_\lambda^{sm}, \\
I_\kappa^{sh}(y) &:= \int_0^1 W(y'(x)) \, \mathrm{d}x + \kappa \mathcal{H}^0(S_{y'}), && y \in \mathcal{A}_\lambda^{sh}.
\end{aligned}$$

First, we show the existence of minimisers. Since we are in dimension 1, the proof is much simpler than in the general case of Section 2, and, in fact, we obtain more information about the minimisers; in particular, there is no need to use $GSBV$ functions. As a matter of fact, in dimension 1, the difficulty resulting from the gradient structure of the problem (see Conti and Schweizer [16]) dissapears completely. Indeed, for each $1 < \lambda < 2$ and $\varepsilon, \kappa > 0$, by the change of variables $u = y'$, the problem of minimising $I_{\varepsilon,\kappa}$ over $\mathcal{A}_\lambda$ reduces to the problem of minimising $J_{\varepsilon,\kappa}$ over $\mathcal{A}_\lambda^J$, where

$$J_{\varepsilon,\kappa}(u) := \int_0^1 \left[W(u(x)) + \varepsilon^2 (\nabla u(x))^2 \right] \mathrm{d}x + \kappa \mathcal{H}^0(S_u), \qquad u \in \mathcal{A}_\lambda^J,$$

and $\mathcal{A}_\lambda^J := \{ u \in SBV(0,1) : u > 0, \ \|u\|_{L^1(0,1)} = \lambda, \ \nabla u \in L^2(0,1) \}$. Similarly, the problem of minimising I_ε^{sm} over $\mathcal{A}_\lambda^{sm}$ reduces to the problem of minimising J_ε^{sm} over $\mathcal{A}_\lambda^{sm,J}$, and the problem of minimising I_κ^{sh} over $\mathcal{A}_\lambda^{sh}$ reduces to the problem of minimising J_κ^{sh} over $\mathcal{A}_\lambda^{sh,J}$, where

$$J_\varepsilon^{sm}(u) := \int_0^1 \left[W(u(x)) + \varepsilon^2 (\nabla u(x))^2 \right] \mathrm{d}x,$$

for $u \in \mathcal{A}_\lambda^{sm,J} := \{ u \in W^{1,2}(0,1) : u > 0, \ \|u\|_{L^1(0,1)} = \lambda \}$, and

$$J_\kappa^{sh}(u) := \int_0^1 W(u(x)) \, \mathrm{d}x + \kappa \mathcal{H}^0(S_u),$$

for $u \in \mathcal{A}_\lambda^{sh,J} := \{ u \in SBV(0,1) : u > 0, \ \|u\|_{L^1(0,1)} = \lambda, \ \mathcal{H}^0(S_u) \geq 1 \}$.

For each $1<\lambda<2$, define the function $y_{1,\lambda}:(0,1)\to\mathbb{R}$ as

$$y_{1,\lambda}(x):=\begin{cases} x & \text{if } x\in(0,2-\lambda], \\ 2x+\lambda-2 & \text{if } x\in(2-\lambda,1). \end{cases}$$

For an arbitrary $y:(0,1)\to\mathbb{R}$, define the reversal y^R of y as

$$y^R(x):=-y(1-x)+\lambda, \qquad x\in(0,1).$$

It is clear that a function y belongs to $\mathcal{A}_\lambda$ if and only if y^R belongs to $\mathcal{A}_\lambda$, and in that case, $I_{\varepsilon,\kappa}(y)=I_{\varepsilon,\kappa}(y^R)$ for each $\varepsilon,\kappa>0$. Analogous properties hold for $\mathcal{A}_\lambda^{sm}$ and $\mathcal{A}_\lambda^{sh}$.

Lemma 3.1. *Let $1<\lambda<2$ and $\varepsilon,\kappa>0$. Then there exists a minimiser of $I_{\varepsilon,\kappa}$ in $\mathcal{A}_\lambda$, and of I_ε^{sm} in $\mathcal{A}_\lambda^{sm}$. Moreover, the set of minimisers of I_κ^{sh} in $\mathcal{A}_\lambda^{sh}$ is $\{y_{1,\lambda},y_{1,\lambda}^R\}$. Finally, every minimiser of $I_{\varepsilon,\kappa}$ in $\mathcal{A}_\lambda$ is a minimiser of I_ε^{sm} in $\mathcal{A}_\lambda^{sm}$ or a minimiser of I_κ^{sh} in $\mathcal{A}_\lambda^{sh}$.*

Proof. The existence of minimisers of J_ε^{sm} over $\mathcal{A}_\lambda^{sm,J}$ follows from standard facts in the direct method of the Calculus of Variations (see, if necessary, Gurtin and Matano [25]), and this implies the existence of minimisers of I_ε^{sm} over $\mathcal{A}_\lambda^{sm}$. The rest of the assertions are obvious and can be checked by inspection. □

In the following proposition, we describe the Γ-limit of the rescaled functionals. Fix $1<\lambda<2$. In order for the functional and its Γ-limit to have the same domain, we do an extension by infinity. To be precise, define $\tilde{\mathcal{A}}_\lambda:=\mathcal{A}_\lambda\cup\mathcal{A}_\lambda^{sh}$. For each $\varepsilon,\kappa>0$ define the four functionals $\tilde{I}_{\varepsilon,\kappa},\tilde{I}_{\varepsilon,0},\tilde{I}_{0,\kappa},F_0:\tilde{\mathcal{A}}_\lambda\to[0,\infty]$ as

$$\tilde{I}_{\varepsilon,\kappa}(y):=\begin{cases} I_{\varepsilon,\kappa}(y) & \text{if } y\in\mathcal{A}_\lambda, \\ \infty & \text{otherwise,} \end{cases} \qquad \tilde{I}_{\varepsilon,0}(y):=\begin{cases} I_\varepsilon^{sm}(y) & \text{if } y\in\mathcal{A}_\lambda^{sm}, \\ \infty & \text{otherwise,} \end{cases}$$

$$\tilde{I}_{0,\kappa}(y):=\begin{cases} I_\kappa^{sh}(y) & \text{if } y\in\mathcal{A}_\lambda^{sh}, \\ \infty & \text{otherwise,} \end{cases}$$

$$F_0(y):=\begin{cases} \mathcal{H}^0(S_{y'}) & \text{if } y\in\mathcal{A}_\lambda^{sh} \text{ and } y'\in\{1,2\} \text{ a.e.}, \\ \infty & \text{otherwise,} \end{cases}$$

for $y\in\tilde{\mathcal{A}}_\lambda$. For each $\varepsilon,\kappa\geq 0$ with $(\varepsilon,\kappa)\neq 0$, define

$$E_{\varepsilon,\kappa,\lambda}:=\inf_{\tilde{\mathcal{A}}_\lambda}\tilde{I}_{\varepsilon,\kappa}. \tag{23}$$

By Lemma 3.1, for every $\varepsilon,\kappa>0$, we have $E_{0,\kappa,\lambda}=\kappa$ and $E_{\varepsilon,\kappa,\lambda}=\min\{E_{\varepsilon,0,\lambda},\kappa\}$. Moreover, using the test functions (3), we conclude that

$$E_{\varepsilon,0,\lambda}\leq\inf\{I_\varepsilon^{sm}(v_{\delta,\lambda}):0<\delta<\min\{\lambda-1,2-\lambda\}\}=2\varepsilon\|W\|_{L^1(1,2)}^{1/2}, \tag{24}$$

if $0<\varepsilon<2\|W\|_{L^1(1,2)}^{1/2}\min\{\lambda-1,2-\lambda\}$. Obviously, by considering, for each $\lambda\in(1,2)$, the function $y_{0,\lambda}:(0,1)\to(0,\infty)$ defined by

$$y_{0,\lambda}(x):=\lambda x, \qquad x\in(0,1),$$

we get that

$$E_{\varepsilon,0,\lambda}\leq W(\lambda). \tag{25}$$

Proposition 3.2. *Let $1<\lambda<2$. In the topology of $W^{1,1}$, the Γ-limit of $E_{\varepsilon,\kappa,\lambda}^{-1}\tilde{I}_{\varepsilon,\kappa}:\tilde{\mathcal{A}}_\lambda\to[0,\infty]$, as $\varepsilon,\kappa\to 0$ with $\varepsilon,\kappa\geq 0$ and $(\varepsilon,\kappa)\neq 0$, is F_0.*

Proof. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\kappa_n\}_{n\in\mathbb{N}}$ be two sequences of numbers tending to zero such that $\varepsilon_n, \kappa_n \geq 0$ and $(\varepsilon_n, \kappa_n) \neq 0$ for all $n \in \mathbb{N}$. We have to prove that the Γ-limit of $E^{-1}_{\varepsilon_n,\kappa_n,\lambda}\tilde{I}_{\varepsilon_n,\kappa_n}$ is F_0.

If $\kappa_n = 0$ for all $n \in \mathbb{N}$, the result was proved by Modica [31]. If $\varepsilon_n = 0$ for all $n \in \mathbb{N}$, the result is obvious.

Suppose that $\varepsilon_n > 0$ and $0 < \kappa_n \leq E_{\varepsilon_n,0,\lambda}$ for all $n \in \mathbb{N}$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\tilde{\mathcal{A}}_\lambda$ converging to a function y in $W^{1,1}$. If $W(y') = 0$ then, by the results of Modica [31],

$$\mathcal{H}^0(S_{y'}) \leq \liminf_{n\to\infty} E^{-1}_{\varepsilon_n,0,\lambda} I^{sm}_{\varepsilon_n}(y_n) \leq \liminf_{n\to\infty} E^{-1}_{\varepsilon_n,\kappa_n,\lambda}\tilde{I}_{\varepsilon_n,\kappa_n}(y_n).$$

If $W(y') \neq 0$ then

$$\lim_{n\to\infty} E^{-1}_{\varepsilon_n,\kappa_n,\lambda}\tilde{I}_{\varepsilon_n,\kappa_n}(y_n) \geq \lim_{n\to\infty} \kappa_n^{-1}\int_0^1 W(y_n') = \infty.$$

This proves the lim inf inequality. For the lim sup inequality, let $y \in W^{1,1}$. If $W(y') \neq 0$ then $F_0(y) = \infty$ and there is nothing to prove. If $W(y') = 0$ then $\nabla^2 y = 0$ a.e. and $F_0(y) = \mathcal{H}^0(S_{y'}) = E^{-1}_{\varepsilon_n,\kappa_n,\lambda}\tilde{I}_{\varepsilon_n,\kappa_n}(y)$ for all $n \in \mathbb{N}$.

Suppose that $\varepsilon_n > 0$ and $E_{\varepsilon_n,0,\lambda} < \kappa_n$ for all $n \in \mathbb{N}$. Then, the lim inf and the lim sup inequalities are direct consequences of Modica [31].

The analyses of the cases above prove that for each two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\kappa_n\}_{n\in\mathbb{N}}$ tending to zero such that $\varepsilon_n, \kappa_n \geq 0$ with $(\varepsilon_n, \kappa_n) \neq 0$ for all $n \in \mathbb{N}$, the lim inf and the lim sup inequalities hold. □

Fix $1 < \lambda < 2$. The asymptotic behaviour of the minimum energy $E_{\varepsilon,\kappa,\lambda}$ is given by $E_{\varepsilon,\kappa,\lambda} = \min\{E_{\varepsilon,0,\lambda}, \kappa\}$ for all $\varepsilon, \kappa > 0$, and $\lim_{\varepsilon\to 0^+} E_{\varepsilon,0,\lambda}/\varepsilon = 2\|\sqrt{W}\|_{L^1(1,2)}$, whose proof can be found in Modica [31].

The following result shows the $W^{1,1}$ compactness of sequences whose rescaled energy is uniformly bounded. Its proof follows that of Modica and Mortola [32] (see also Modica [31] or Alberti [1, §4.5]).

Proposition 3.3. *Assume that there exist $t_0 > 2$, and $c_1, c_2 > 0$ such that*

$$W(t) \geq c_1(t - t_0) - c_2, \qquad t \in (t_0, \infty). \tag{26}$$

Fix $1 < \lambda < 2$. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\kappa_n\}_{n\in\mathbb{N}}$ be two sequences tending to zero such that $\varepsilon_n > 0$ and $\kappa_n \geq 0$ for all $n \in \mathbb{N}$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\tilde{\mathcal{A}}_\lambda$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{E_{\varepsilon_n,\kappa_n,\lambda}}\tilde{I}_{\varepsilon_n,\kappa_n}(y_n) < \infty.$$

Then $\{y_n\}_{n\in\mathbb{N}}$ is precompact in $W^{1,1}(0,1)$.

Proof. If $\kappa_n = 0$ for infinitely many $n \in \mathbb{N}$, the result was essentially proved by Modica and Mortola [32] (see also Alberti [1]).

Therefore, we can suppose that $\varepsilon_n, \kappa_n > 0$ for all $n \in \mathbb{N}$. It suffices to prove that if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of functions in $\mathcal{A}^J_\lambda$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{E_{\varepsilon_n,\kappa_n,\lambda}} J_{\varepsilon_n,\kappa_n}(u_n) < \infty,$$

then $\{u_n\}_{n\in\mathbb{N}}$ is precompact in L^1. Fix any $s_0 \in (0,1)$. For each $n \in \mathbb{N}$ define $v_n := \min\{\max\{s_0, u_n\}, t_0\}$; note that v_n does not necessarily belong to $\mathcal{A}^J_\lambda$. We have

$$\sup_{n\in\mathbb{N}} \frac{1}{E_{\varepsilon_n,\kappa_n,\lambda}} J_{\varepsilon_n,\kappa_n}(v_n) < \infty.$$

In addition, from Lemma 3.1 we see that $\lim_{n\to\infty} E_{\varepsilon_n,\kappa_n,\lambda} = 0$. Hence, thanks also to (26),

$$\lim_{n\to\infty} \left[\|W(u_n)\|_{L^1} + |\{x \in (0,1) : u_n(x) < s_0 \text{ or } u_n(x) > t_0\}| + \|v_n - u_n\|_{L^1}\right] = 0.$$

Therefore, it suffices to show that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is precompact in L^1. By Lemma 3.1, $\sup_{n\in\mathbb{N}}[\mathcal{H}^0(S_{v_n}) + \|v_n\|_{L^\infty}] < \infty$.

Let H be a primitive of $2\sqrt{W}$. Then H is of class C^1 in $[1,2]$, and, hence, by the chain rule for BV functions (see, e.g., Ambrosio, Fusco and Pallara [6, Th. 3.96]), $H \circ v_n \in BV(0,1)$ and $\nabla(H \circ v_n) = 2\sqrt{W(v_n)}\nabla v_n$. Recall that ∇ represents the absolutely continuous part of the distributional derivative, which coincides with the a.e. derivative. We apply now the usual Modica-Mortola [32] argument:

$$\begin{aligned} \int_0^1 2\sqrt{W(v_n)}\nabla v_n &\le \int_0^1 \left[\frac{W(v_n)}{E_{\varepsilon_n,\kappa_n,\lambda}} + E_{\varepsilon_n,\kappa_n,\lambda}(\nabla v_n)^2\right] \\ &\le \frac{1}{E_{\varepsilon_n,\kappa_n,\lambda}} \int_0^1 \left[W(v_n) + E^2_{\varepsilon_n,0,\lambda}(\nabla v_n)^2\right]. \end{aligned}$$

Applying (24), we obtain that

$$\sup_{n\in\mathbb{N}} \left[\|\nabla(H\circ v_n)\|_{L^1} + \|H\circ v_n\|_{L^\infty} + \mathcal{H}^0(S_{H\circ v_n})\right] < \infty.$$

Therefore, $\{H \circ v_n\}_{n\in\mathbb{N}}$ is bounded in BV and hence precompact in L^1. Since H admits a continuous inverse, then $\{v_n\}_{n\in\mathbb{N}}$ is also precompact in L^1. □

Note that assumption (26) is satisfied by the typical double-well potential W. See Figure 1.

4. **The one-dimensional case: behaviour of minimisers.** In this section, we suppose $N = 1$, and under further assumptions on W, we describe the behaviour of the minimisers of $I_{\varepsilon,\kappa}$ in $\mathcal{A}_\lambda$ as ε and κ tend to 0, and λ tends to one of the wells of W (either 1 or 2).

The following general conditions on the elastic stored-energy function W will be assumed throughout this section:

$$\begin{aligned} &W : (0,\infty) \to [0,\infty) \;\text{ is of class }\; C^5, \\ &W^{-1}(0) = \{1,2\}, \qquad W'(1) = W'(2) = 0, \\ &\text{there exist }\; 1 < t_1 < t_2 < 2 \;\text{ such that }\; W''|_{(0,t_1)\cup(t_2,\infty)} > 0, \quad W''_{(t_1,t_2)} < 0, \\ &\qquad\qquad \limsup_{t\to 0^+} W'(t) < W'(t_2) \quad\text{and}\quad \liminf_{t\to\infty} W'(t) > W'(t_1). \end{aligned} \tag{27}$$

The graph of a typical W is depicted in Figure 1, together with its first and second derivative. Note that assumptions (27) imply (26). In fact, not all results of this section require all assumptions (27), but only some of their consequences. Nevertheless, for simplicity of the exposition, we have decided to assume (27) all through the section.

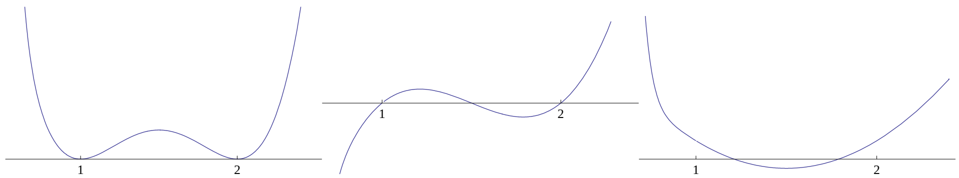

FIGURE 1. Graphs of W, W' and W''.

Lemma 4.1. *For each $1 < \lambda < 2$ and $\varepsilon > 0$, let $P_{\varepsilon,\lambda}$ be the following set of equations for u:*

$$P_{\varepsilon,\lambda} \equiv \begin{cases} 2\varepsilon^2 u''(x) = W'(u(x)) - \int_0^1 W'(u(t))\,\mathrm{d}t, & x \in [0,1], \\ u'(0) = u'(1) = 0, \qquad \int_0^1 u(t)\,\mathrm{d}t = \lambda, & u \in C^2([0,1]). \end{cases}$$

Let u be a minimiser of J_ε^{sm} in $\mathcal{A}_\lambda^{sm,J}$. Then u solves $P_{\varepsilon,\lambda}$, and $u(x) \in [r_1, r_2]$ for all $x \in [0,1]$, for some constants $r_2 > r_1 > 0$ depending only on W.

Proof. See Gurtin and Matano [25]. □

When the boundary condition λ is kept fixed, the asymptotic behaviour of the minimisers of J_ε^{sm} in $\mathcal{A}_\lambda^{sm,J}$ as $\varepsilon \to 0$ was described by Carr, Gurtin and Slemrod [15] and Modica [31].

In the following result, we describe the energy of the minimisers of I_ε^{sm} in $\mathcal{A}_\lambda^{sm}$ as a function of ε. For each $\varepsilon, \kappa \geq 0$ with $(\varepsilon, \kappa) \neq 0$, let $M_{\varepsilon,\kappa,\lambda}$ be the set of minimisers of $\tilde{I}_{\varepsilon,\kappa}$ in $\tilde{\mathcal{A}}_\lambda$, and recall the definition of $E_{\varepsilon,\kappa,\lambda}$ given in (23).

Lemma 4.2. *For every $\lambda \in (1,2)$ there exists a unique $\varepsilon^1(\lambda) \in (0,\infty)$ such that*

$$y_{0,\lambda} \notin M_{\varepsilon,0,\lambda} \text{ if } 0 < \varepsilon < \varepsilon^1(\lambda); \quad \{y_{0,\lambda}\} = M_{\varepsilon,0,\lambda} \text{ if } \varepsilon > \varepsilon^1(\lambda); \quad y_{0,\lambda} \in M_{\varepsilon^1(\lambda),0,\lambda}. \tag{28}$$

We thus define the function $\varepsilon^1 : (1,2) \to (0,\infty)$. In addition, the function

$$\varepsilon \mapsto E_{\varepsilon,0,\lambda} \text{ is strictly increasing in } (0, \varepsilon^1(\lambda)], \text{ and constantly } W(\lambda) \text{ in } [\varepsilon^1(\lambda), \infty). \tag{29}$$

Moreover,

$$\sup_{\lambda \in (1,2)} \varepsilon^1(\lambda) < \infty, \qquad \lim_{\lambda \to 1^+} \varepsilon^1(\lambda) = \lim_{\lambda \to 2^-} \varepsilon^1(\lambda) = 0.$$

Proof. Let $1 < \lambda < 2$. For each $0 < \varepsilon_1 < \varepsilon_2$ and $y \in \mathcal{A}_\lambda^{sm}$ we have $I_{\varepsilon_1}^{sm}(y) \leq I_{\varepsilon_2}^{sm}(y)$, so $E_{\varepsilon_1,0,\lambda} \leq E_{\varepsilon_2,0,\lambda}$; if, in addition, $E_{\varepsilon_1,0,\lambda} = E_{\varepsilon_2,0,\lambda}$ then for all $y_2 \in M_{\varepsilon_2,0,\lambda}$ we have

$$\begin{aligned} E_{\varepsilon_2,0,\lambda} = E_{\varepsilon_1,0,\lambda} \leq I_{\varepsilon_1}^{sm}(y_2) &= \int_0^1 \left[W(y_2') + \varepsilon_1^2 (y_2'')^2\right] \\ &\leq \int_0^1 \left[W(y_2') + \varepsilon_2^2 (y_2'')^2\right] = I_{\varepsilon_2}^{sm}(y_2) = E_{\varepsilon_2,0,\lambda}. \end{aligned}$$

Therefore, all the inequalities above are equalities, and in particular, $y_2'' = 0$ a.e. This shows $M_{\varepsilon_2,0,\lambda} = \{y_{0,\lambda}\}$.

Now let $0 < \varepsilon_1 < \varepsilon_2$ and suppose that $y_{0,\lambda} \in M_{\varepsilon_1,0,\lambda}$. Then

$$E_{\varepsilon_2,0,\lambda} \leq I_{\varepsilon_2}^{sm}(y_{0,\lambda}) = W(\lambda) = E_{\varepsilon_1,0,\lambda}.$$

By the result of the paragraph above, we get $M_{\varepsilon_2,0,\lambda} = \{y_{0,\lambda}\}$.

The two paragraphs above show that there exists a unique $\varepsilon^1(\lambda) \in [0,\infty]$ such that (28) and (29). Now, by (24), $\lim_{\varepsilon\to 0^+} E_{\varepsilon,0,\lambda} = 0$. Therefore, $\varepsilon^1(\lambda) \neq 0$.

Now we prove the inequality $\sup_{\lambda\in(1,2)} \varepsilon^1(\lambda) < \infty$. For each $1 < \lambda < 2$ and $\varepsilon > 0$, let $P_{\varepsilon,\lambda}$ be equation described in Lemma 4.1. We are going to prove the following facts:

P1. If u is a non-constant solution of $P_{\varepsilon,\lambda}$ then the function $v : [0,1] \to \mathbb{R}$ defined by
$$v(x) := \begin{cases} u(2x) & \text{if} \quad x \in [0,1/2] \\ u(2(1-x)) & \text{if} \quad x \in (1/2,1] \end{cases} \tag{30}$$
is a non-monotonic solution of $P_{\varepsilon/2,\lambda}$.

P2. There exists a constant $C > 0$, depending on W, such that every solution u of $P_{\varepsilon,\lambda}$ satisfies $\|u\|_\infty \le C$.

P3. For each $M > 0$ there exists $t > 0$, depending on M and W, such that for each $\varepsilon > t$, we have that every $u \in \mathcal{A}_\lambda^{sm,J}$ with $\|u\|_\infty \le M$ satisfies that the second variation $\delta^2 J_\varepsilon^{sm}(u,\cdot)$ is positive definite on the set of $v \in W^{1,2}(0,1)$ such that $\int_0^1 v = 0$.

P4. If u is a non-monotonic solution of $P_{\varepsilon,\lambda}$, then there exists $v \in W^{1,2}(0,1)$ with $\int_0^1 v = 0$ such that $\delta^2 J_\varepsilon^{sm}(u,v) < 0$.

Property P1 is immediate.

The proof of Property P2 follows an argument similar to Alikakos and McKinney [2] that we reproduce now. Let u solve $P_{\varepsilon,\lambda}$. Let $x_1, x_2 \in [0,1]$ be such that
$$u(x_1) = \min_{x\in[0,1]} u(x), \qquad u(x_2) = \max_{x\in[0,1]} u(x).$$
If $x_1 \in (0,1)$ then clearly $u''(x_1) \ge 0$. If $x_1 \in \{0,1\}$ then, as $u'(0) = u'(1) = 0$, we still have $u''(x_1) \ge 0$. Likewise, $u''(x_2) \le 0$. Therefore, we obtain from $P_{\varepsilon,\lambda}$,
$$W'(u(x_2)) = 2\varepsilon^2 u''(x_2) + \int_0^1 W'(u(t))\,\mathrm{d}t \le 2\varepsilon^2 u''(x_1) + \int_0^1 W'(u(t))\,\mathrm{d}t = W'(u(x_1)).$$
Thanks to (27), we conclude that $u(x_1)$ is a priori bounded from below and $u(x_2)$ is a priori bounded from above by constants depending only on W.

Property P3 follows from the Poincaré inequality. Indeed, recall the expression of second variation:
$$\delta^2 J_\varepsilon^{sm}(u_1,v_1) = \int_0^1 \left[\varepsilon^2 (v_1')^2 + W''(u_1) v_1^2\right], \qquad u_1, v_1 \in W^{1,2}(0,1).$$
Then, for all $u \in \mathcal{A}_\lambda^{sm,J}$ with $\|u\|_\infty \le M$ and $v \in W^{1,2}(0,1)$ with $\int_0^1 v = 0$,
$$\delta^2 J_\varepsilon^{sm}(u,v) \ge \varepsilon^2 \|v'\|_{L^2}^2 - c_1 \|v\|_{L^2}^2 \ge c_2 \left(\varepsilon^2 - c_3\right) \|v\|_{W^{1,2}}^2,$$
for some constants $c_1, c_2, c_3 > 0$ depending only on W and M.

Property P4 was proved in Carr, Gurtin and Slemrod [15, Th. 8.2].

Now we show how Properties P1–P4 imply $\sup_{\lambda\in(1,2)} \varepsilon^1(\lambda) < \infty$. Let $1 < \lambda < 2$ and $0 < \varepsilon < \varepsilon^1(\lambda)$. Let $y \in M_{\varepsilon,0,\lambda}$ and call $u = y'$. By (28), $y \neq y_{0,\lambda}$ and, hence, by Lemma 4.1, u is a non-constant solution of $P_{\varepsilon,\lambda}$. By P1, the function v defined in (30) is a non-monotonic solution of $P_{\varepsilon/2,\lambda}$. By P2, v is a priori bounded in L^∞ by a constant depending only on W. By P4, there exists $v_1 \in W^{1,2}(0,1)$ such that $\int_0^1 v_1 = 0$ and $\delta^2 J_{\varepsilon/2}^{sm}(v,v_1) < 0$. By P3, $\varepsilon/2$ is a priori bounded from above by a constant depending only on W. Hence, $\sup_{\lambda\in(1,2)} \varepsilon^1(\lambda)$ is a finite constant depending only on W.

Finally, the equalities $\lim_{\lambda\to 1^+}\varepsilon^1(\lambda)=\lim_{\lambda\to 2^-}\varepsilon^1(\lambda)=0$ were showed by Carr, Gurtin and Slemrod [15, Th. 7.1]. □

Lemma 4.3. *The function ε^1 is continuous. The function $(\varepsilon,\lambda)\mapsto E_{\varepsilon,0,\lambda}$ is continuous in $(0,\infty)\times(1,2)$. In addition,*

$$\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\sup_{\lambda\in(1,2)}E_{\varepsilon,0,\lambda}\le 2\|W\|^{1/2}_{L^1(1,2)}. \tag{31}$$

Proof. Let $\{(\varepsilon_j,\lambda_j)\}_{j\in\mathbb{N}}$ be a sequence in $(0,\infty)\times(1,2)$ converging to $(\varepsilon,\lambda)\in(0,\infty)\times(1,2)$. For each $j\in\mathbb{N}$, let $y_j\in M_{\varepsilon_j,0,\lambda_j}$. Then $E_{\varepsilon,0,\lambda}\le I^{sm}_\varepsilon(\frac{\lambda}{\lambda_j}y_j)$ for all $j\in\mathbb{N}$. Let $\{j_k\}_{k\in\mathbb{N}}$ be a subsequence such that $\liminf_{j\to\infty}E_{\varepsilon_j,0,\lambda_j}=\lim_{k\to\infty}E_{\varepsilon_{j_k},0,\lambda_{j_k}}$. Now, by (25),

$$\sup_{j\in\mathbb{N}}\|y_j''\|_{L^2}<\infty. \tag{32}$$

By the boundary conditions, the Poincaré inequality and the compact embeddings of Sobolev maps, for a subsequence (not relabelled), we have that the sequence $\{y'_{j_k}\}_{k\in\mathbb{N}}$ converges a.e. By Lemma 4.1, $y_j'(x)\in[r_1,r_2]$ for all $x\in[0,1]$ and $j\in\mathbb{N}$. Therefore,

$$\lim_{k\to\infty}\|W(\frac{\lambda}{\lambda_{j_k}}y'_{j_k})-W(y'_{j_k})\|_{L^1}=0. \tag{33}$$

Equations (32) and (33) show that $E_{\varepsilon,0,\lambda}\le\liminf_{j\to\infty}E_{\varepsilon_j,0,\lambda_j}$. Now let $y\in M_{\varepsilon,0,\lambda}$. Then, an analogous argument shows that, for a subsequence $\{j_k\}_{k\in\mathbb{N}}$,

$$\limsup_{j\to\infty}E_{\varepsilon_j,0,\lambda_j}\le\lim_{k\to\infty}I^{sm}_{\varepsilon_{j_k}}(\frac{\lambda_{j_k}}{\lambda}y)=E_{\varepsilon,0,\lambda}.$$

Now we show (31). Using (27), we find that, for each $\varepsilon>0$ small enough,

$$\begin{aligned}\sup_{\lambda\in(1,2)}E_{\varepsilon,0,\lambda}=\max\Big\{&W(1+2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon),W(2-2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon),\\ &\max\{E_{\varepsilon,0,\lambda}:\lambda\in[1+2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon,2-2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon]\}\Big\}.\end{aligned}$$

Again by (27) and a Taylor expansion,

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}W(1+2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon)=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}W(2-2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon)=0,$$

whereas by (24),

$$\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\max\{E_{\varepsilon,0,\lambda}:\lambda\in[1+2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon,2-2^{-1}\|W\|^{-1/2}_{L^1(1,2)}\varepsilon]\}\le 2\|W\|^{1/2}_{L^1(1,2)}.$$

This shows (31).

Now let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence in $(1,2)$ converging to $\lambda\in(1,2)$. Take a subsequence (not relabelled) such that the inferior limit of $\varepsilon^1(\lambda_j)$ as $j\to\infty$ is in fact the limit. By (28) and (31) we have $\lim_{j\to\infty}\varepsilon^1(\lambda_j)>0$. By (28) and the continuity property proved above, we have

$$W(\lambda)=\lim_{j\to\infty}W(\lambda_j)=\lim_{j\to\infty}E_{\varepsilon^1(\lambda_j),0,\lambda_j}=E_{\lim_{j\to\infty}\varepsilon^1(\lambda_j),0,\lambda}. \tag{34}$$

Therefore, by (28), $\varepsilon^1(\lambda)\le\lim_{j\to\infty}\varepsilon^1(\lambda_j)$. This proves the lower semicontinuity of ε^1. If it were not continuous, by Lemma 4.2, there would exist a $\delta>0$, a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ and two numbers $\lambda\in(1,2)$ and $\varepsilon\ge 0$ verifying $\varepsilon^1(\lambda_j)\le\varepsilon^1(\lambda)-\delta$ for all $j\in\mathbb{N}$ and such that $\lim_{j\to\infty}\lambda_j=\lambda$ and $\lim_{j\to\infty}\varepsilon^1(\lambda_j)=\varepsilon$. By (29), for each $j\in\mathbb{N}$ we have $E_{\varepsilon^1(\lambda_j),0,\lambda}\le E_{\varepsilon^1(\lambda)-\delta,0,\lambda}$. If $\varepsilon=0$, by (28) and (31) we obtain, arguing as in (34), that $W(\lambda)=0$, which contradicts $\lambda\in(1,2)$. If $\varepsilon>0$, arguing as in (34), we obtain that $W(\lambda)=\lim_{j\to\infty}E_{\varepsilon^1(\lambda_j),0,\lambda}$. So $W(\lambda)\le E_{\varepsilon^1(\lambda)-\delta,0,\lambda}$, and this contradicts (29). $\square$

In fact, arguing as in Modica [31], one can prove that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\sup_{\lambda\in(1,2)}E_{\varepsilon,0,\lambda}=2\|\sqrt{W}\|_{L^1(1,2)}. \tag{35}$$

Lemma 4.4. *Define $\mathcal{U}_W:=\{(\kappa,\lambda)\in\mathbb{R}\times(1,2):0<\kappa<W(\lambda)\}$. For every $(\kappa,\lambda)\in\mathcal{U}_W$ there exists a unique $\varepsilon^2(\kappa,\lambda)>0$ such that*

$$E_{\varepsilon,0,\lambda}<\kappa \quad \text{if}\ \ \varepsilon<\varepsilon^2(\kappa,\lambda);\qquad E_{\varepsilon,0,\lambda}>\kappa \quad \text{if}\ \ \varepsilon>\varepsilon^2(\kappa,\lambda);\qquad E_{\varepsilon^2(\kappa,\lambda),0,\lambda}=\kappa.$$

The function $\varepsilon^2:\mathcal{U}_W\to(0,\infty)$ thus defined is continuous. Moreover, $\varepsilon^2(\kappa,\lambda)<\varepsilon^1(\lambda)$ for all $(\kappa,\lambda)\in\mathcal{U}_W$, and

$$\lim_{\kappa\to 0^+}\sup_{\lambda\in(1,2)\cap W^{-1}(\kappa,\infty)}\varepsilon^2(\kappa,\lambda)=0. \tag{36}$$

Furthermore, for every $1<\lambda_0<2$,

$$\lim_{\substack{(\kappa,\lambda)\to(W(\lambda_0),\lambda_0)\\(\kappa,\lambda)\in\mathcal{U}_W}}\varepsilon^2(\kappa,\lambda)=\varepsilon^1(\lambda_0).$$

Proof. Let $(\kappa,\lambda)\in\mathcal{U}_W$. By Lemmas 4.2 and 4.3, the number $\varepsilon^2(\kappa,\lambda)$ is well-defined, and, in addition, $\varepsilon^2(\kappa,\lambda)<\varepsilon^1(\lambda)$.

Now we prove (36). Were this inequality false, there would exist a $\delta>0$ and two sequences $\{\kappa_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$0<\kappa_j<W(\lambda_j),\quad 1<\lambda_j<2,\quad \varepsilon^2(\kappa_j,\lambda_j)>\delta,\qquad \text{for all}\ \ j\in\mathbb{N} \tag{37}$$

with $\lim_{j\to\infty}\kappa_j=0$ and $\lim_{j\to\infty}\lambda_j=\lambda_0$, for some $\lambda_0\in[1,2]$. As $\varepsilon^2(\kappa_j,\lambda_j)<\varepsilon^1(\lambda_j)$ for all $j\in\mathbb{N}$, by Lemma 4.2, necessarily $\lambda_0\in(1,2)$. By (29) and (37), we have $E_{\delta,0,\lambda_j}<\kappa_j$ for all $j\in\mathbb{N}$, so $\lim_{j\to\infty}E_{\delta,0,\lambda_j}=0$. By Lemma 3.1, we have $E_{\delta,0,\lambda_0}>0$, and this is a contradiction with Lemma 4.3.

Now let $\{(\kappa_j,\lambda_j)\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{U}_W$ converging to some $(\kappa,\lambda)\in\mathcal{U}_W$. By Lemma 4.2, the sequence $\{\varepsilon^2(\kappa_j,\lambda_j)\}_{j\in\mathbb{N}}$ is bounded. Take any convergent subsequence (not relabelled). If $\lim_{j\to\infty}\varepsilon^2(\kappa_j,\lambda_j)=0$, we arrive at a contradiction with (35). If not, by Lemma 4.3,

$$\kappa=\lim_{j\to\infty}\kappa_j=\lim_{j\to\infty}E_{\varepsilon^2(\kappa_j,\lambda_j),0,\lambda_j}=E_{\lim_{j\to\infty}\varepsilon^2(\kappa_j,\lambda_j),0,\lambda},$$

and, hence, $\lim_{j\to\infty}\varepsilon^2(\kappa_j,\lambda_j)=\varepsilon^2(\kappa,\lambda)$.

Finally, let $1<\lambda_0<2$, and let $\{(\kappa_j,\lambda_j)\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{U}_W$ whose limit is $(W(\lambda_0),\lambda_0)$ and such that $\lim_{j\to\infty}\varepsilon^2(\kappa_j,\lambda_j)=\varepsilon_0$ for some $0\le\varepsilon_0\le\sup_{\lambda\in(1,2)}\varepsilon^1(\lambda)$. If $\varepsilon_0=0$ we arrive at a contradiction with (35). Therefore, $\varepsilon_0>0$. Arguing as in the paragraph above, we obtain that $E_{\varepsilon_0,0,\lambda_0}=W(\lambda_0)$. By Lemma 4.2, $\varepsilon_0\ge\varepsilon^1(\lambda_0)$, but since $\varepsilon^2(\kappa_j,\lambda_j)<\varepsilon^1(\lambda_j)$ for all $j\in\mathbb{N}$, we conclude that $\varepsilon_0=\varepsilon^1(\lambda_0)$. $\square$

The following proposition is a restatement of one of the main results of Carr, Gurtin and Slemrod [15]; it states the uniqueness (modulo reversal) of the minimisers of I_ε^{sm}.

Proposition 4.5. *For each $\lambda\in(1,2)$ there exists $e_\lambda\in(0,\varepsilon^1(\lambda)]$ such that for every $\varepsilon\in(0,e_\lambda)$, we have $M_{\varepsilon,0,\lambda}=\{y_{2,\varepsilon,\lambda},y_{2,\varepsilon,\lambda}^R\}$ for some $y_{2,\varepsilon,\lambda}\in W^{2,2}(0,1)$ with $y_{2,\varepsilon,\lambda}\ne y_{2,\varepsilon,\lambda}^R$.*

Proof. Everything was proved in Carr, Gurtin and Slemrod [15, Th. 9.1], except perhaps the fact that $y_{2,\varepsilon,\lambda} \neq y^R_{2,\varepsilon,\lambda}$. By Carr, Gurtin and Slemrod [15, Th. 8.2], $y'_{2,\varepsilon,\lambda}$ is monotone, so if $y_{2,\varepsilon,\lambda} = y^R_{2,\varepsilon,\lambda}$ then $y'_{2,\varepsilon,\lambda}$ would be constant and hence $y_{2,\varepsilon,\lambda} = y_{0,\lambda}$; so by Lemma 4.2, $\varepsilon \geq \varepsilon^1(\lambda)$. □

Although in Proposition 4.5 there is a natural choice of e_λ, namely, the supremum of the numbers satisfying that property, we do not make any special choice, and, in what follows, for each $\lambda \in (1,2)$ we fix any e_λ. By Lemmas 3.1 and 4.4, and Proposition 4.5, we have that for each $(\kappa,\lambda) \in \mathcal{U}_W$ and $\varepsilon \in (0, e_\lambda)$,

$$M_{\varepsilon,\kappa,\lambda} = \{y_{1,\lambda}, y^R_{1,\lambda}\} \text{ if } \varepsilon > \varepsilon^2(\kappa,\lambda); \qquad M_{\varepsilon,\kappa,\lambda} = \{y_{2,\varepsilon,\lambda}, y^R_{2,\varepsilon,\lambda}\} \text{ if } \varepsilon < \varepsilon^2(\kappa,\lambda);$$
$$M_{\varepsilon^2(\kappa,\lambda),\kappa,\lambda} = \{y_{1,\lambda}, y^R_{1,\lambda}, y_{2,\varepsilon,\lambda}, y^R_{2,\varepsilon,\lambda}\}.$$

In the following lemma we define some functions and a set, and prove some of their properties. We will need those definitions the statement of Theorem 4.7 below.

Lemma 4.6. *Define* $V_1, V_2 : (0, \|W\|_{L^\infty(1,2)}) \to (1,2)$ *and* $f_1 : (0, \|W\|_{L^\infty(1,2)}) \to (0, \max_{\lambda\in(1,2)} \varepsilon^1(\lambda))$ *as*

$$V_1(\kappa) := \min W^{-1}(\kappa) \cap (1,2), \qquad V_2(\kappa) := \max W^{-1}(\kappa) \cap (1,2),$$
$$f_1(\kappa) := \max\left\{ \sup_{\lambda\in(V_1(\kappa),V_2(\kappa))} \varepsilon^2(\kappa,\lambda), \max_{\lambda\in(1,V_1(\kappa)]\cup[V_2(\kappa),2)} \varepsilon^1(\lambda) \right\},$$

for each $0 < \kappa < \|W\|_{L^\infty(1,2)}$. *Then* V_1 *is an increasing homeomorphism,* V_2 *is a decreasing homeomorphism, and* f_1 *is continuous. Moreover,* $V_1(\kappa) < V_2(\kappa)$ *for all* $0 < \kappa < \|W\|_{L^\infty(1,2)}$, *and*

$$\lim_{\kappa\to0^+} \frac{(V_1(\kappa)-1)^2}{\kappa} = \frac{2}{W''(1)}, \quad \lim_{\kappa\to0^+} \frac{(2-V_2(\kappa))^2}{\kappa} = \frac{2}{W''(2)}, \quad \lim_{\kappa\to0^+} f_1(\kappa) = 0.$$

Define $T_1, T_2 : (0, \max_{\lambda\in(1,2)} \varepsilon^1(\lambda)) \to (1,2)$ *as*

$$T_1(\varepsilon) := \min(\varepsilon^1)^{-1}(\varepsilon), \quad T_2(\varepsilon) := \max(\varepsilon^1)^{-1}(\varepsilon), \qquad 0 < \varepsilon < \max_{\lambda\in(1,2)} \varepsilon^1(\lambda),$$

and $f_2 : (0,\infty) \to (0, \|W\|_{L^\infty(1,2)}]$ *as*

$$f_2(\varepsilon) := \max_{\lambda\in(1,2)} E_{\varepsilon,0,\lambda}, \qquad \varepsilon > 0.$$

Then T_1, T_2 *and* f_2 *are continuous. For all* $0 < \varepsilon < \max_{\lambda\in(1,2)} \varepsilon^1(\lambda)$ *we have* $T_1(\varepsilon) < T_2(\varepsilon)$. *Moreover,*

$$\lim_{\varepsilon\to0^+} T_1(\varepsilon) = 1, \qquad \lim_{\varepsilon\to0^+} T_2(\varepsilon) = 2, \qquad \lim_{\varepsilon\to0^+} \frac{f_2(\varepsilon)}{\varepsilon} = 2\|\sqrt{W}\|_{L^1(1,2)}.$$

Define $\mathcal{V}$ *as the set of* $(\varepsilon,\kappa) \in (0,\infty)^2$ *such that there exists* $\lambda_0 \in (1,2)$ *with* $\varepsilon < e_{\lambda_0}$ *and* $E_{\varepsilon,0,\lambda_0} < \kappa < f_2(\varepsilon)$. *Then* $\mathcal{V}$ *is open, and for all* $0 < \varepsilon_0 < \sup_{\lambda\in(1,2)} e_\lambda$ *and* $0 < \kappa_0 < f_2(\varepsilon_0)$, *there exists* $a > 0$ *such that* $(\varepsilon, \kappa_0) \in \mathcal{V}$ *for all* $0 < \varepsilon < a$.

Proof. Assumptions (27) imply that there exists $s_0 \in (1,2)$ such that W is an increasing homeomorphism from $[1, s_0]$ to $[0, \|W\|_{L^\infty(1,2)}]$, and a decreasing homeomorphism from $[s_0, 2]$ to $[0, \|W\|_{L^\infty(1,2)}]$. This and a Taylor expansion imply all the properties of V_1 and V_2. The continuity of f_1 then follows from Lemmas 4.2, 4.3 and 4.4, because they imply that for each $\kappa \in (0, \|W\|_{L^\infty(1,2)})$ there exists $\lambda \in (V_1(\kappa), V_2(\kappa))$ such that $f_1(\kappa) = \varepsilon^2(\kappa,\lambda)$ or there exists $\lambda \in (1, V_1(\kappa)] \cup [V_2(\kappa), 2)$

such that $f_1(\kappa) = \varepsilon^1(\lambda)$. The limit $\lim_{\kappa\to 0^+} f_1(\kappa) = 0$ follows from Lemma 4.2, (36) and the properties of V_1 and V_2.

All the properties of the functions T_1 and T_2 follow from Lemmas 4.2 and 4.3. The properties of f_2 follow from Lemma 4.3, (25) and (35).

The set $\mathcal{V}$ is open because of the continuity of f_2 and of the map described in Lemma 4.3. Let $0 < \varepsilon_0 < \sup_{\lambda\in(1,2)} e_\lambda$ and $0 < \kappa_0 < f_2(\varepsilon_0)$. By (25), there exists $\lambda_0 \in (1, 2)$ such that $E_{\varepsilon_0,0,\lambda_0} < \kappa_0$. Let $0 < \varepsilon_1 < \min\{\varepsilon_0, e_{\lambda_0}\}$. By Lemma 4.2, $E_{\varepsilon_1,0,\lambda_0} \le E_{\varepsilon_0,0,\lambda_0} < \kappa_0$. Thus $(\varepsilon_1, \kappa_0) \in \mathcal{V}$. □

In the following theorem, we fix $\varepsilon, \kappa > 0$, and move λ from 1 to 2. Our purpose is to analyse the regime of the values of the parameter λ for which the minimisers of $I_{\varepsilon,\kappa}$ in $\mathcal{A}_\lambda$ present smooth interfaces, sharp interfaces, or no interfaces at all.

Theorem 4.7. *The following assertions hold true:*

S1. Let $\varepsilon > \sup_{\lambda\in(1,2)} \varepsilon^1(\lambda)$ and $\kappa > f_2(\varepsilon)$. Then $M_{\varepsilon,\kappa,\lambda} = \{y_{0,\lambda}\}$ for all $\lambda \in (1, 2)$.

S2. Let $0 < \kappa < \|W\|_{L^\infty(1,2)}$ and $\varepsilon > f_1(\kappa)$. Then

$$M_{\varepsilon,\kappa,\lambda} = \{y_{0,\lambda}\} \quad \textit{if} \quad \lambda \in (1, V_1(\kappa)) \cup (V_2(\kappa), 2);$$
$$M_{\varepsilon,\kappa,\lambda} = \{y_{0,\lambda}, y_{1,\lambda}, y^R_{1,\lambda}\} \quad \textit{if} \quad \lambda \in \{V_1(\kappa), V_2(\kappa)\};$$
$$M_{\varepsilon,\kappa,\lambda} = \{y_{1,\lambda}, y^R_{1,\lambda}\} \quad \textit{if} \quad \lambda \in (V_1(\kappa), V_2(\kappa)).$$

S3. Let $0 < \varepsilon < \sup_{\lambda\in(1,2)} e_\lambda$ and $\kappa > f_2(\varepsilon)$. Then

$$M_{\varepsilon,\kappa,\lambda} = \{y_{0,\lambda}\} \quad \textit{if} \quad \lambda \in (1, T_1(\varepsilon)) \cup (T_2(\varepsilon), 2);$$
$$y_{0,\lambda} \in M_{\varepsilon,\kappa,\lambda} \quad \textit{if} \quad \lambda \in \{T_1(\varepsilon), T_2(\varepsilon)\};$$
$$M_{\varepsilon,\kappa,\lambda} = M_{\varepsilon,0,\lambda} \quad \textit{if} \quad \lambda \in [T_1(\varepsilon), T_2(\varepsilon)];$$
$$\textit{there exists } \lambda \in (T_1(\varepsilon), T_2(\varepsilon)) \textit{ such that } M_{\varepsilon,\kappa,\lambda} = \{y_{2,\varepsilon,\lambda}, y^R_{2,\varepsilon,\lambda}\}.$$

S4. Let $(\varepsilon, \kappa) \in \mathcal{V}$. Then

$$M_{\varepsilon,\kappa,\lambda} = \{y_{0,\lambda}\} \quad \textit{if} \quad \lambda \in (1, \min\{T_1(\varepsilon), V_1(\kappa)\}) \cup (\max\{T_2(\varepsilon), V_2(\kappa)\}, 2);$$
$$\textit{there exists } \lambda \in (1, 2) \textit{ such that } M_{\varepsilon,\kappa,\lambda} = \{y_{2,\varepsilon,\lambda}, y^R_{2,\varepsilon,\lambda}\};$$
$$\textit{the set of } \lambda \in (1, 2) \textit{ such that } M_{\varepsilon,\kappa,\lambda} = \{y_{1,\lambda}, y^R_{1,\lambda}\} \textit{ is open and non-empty.}$$

Proof. Assertion S1 follows from Lemmas 3.1 and 4.2. Assertion S2 follows from Lemmas 3.1, 4.2 and 4.4. Assertion S3 follows from Lemmas 3.1 and 4.2 and Proposition 4.5. Assertion S4 follows from Lemmas 3.1, 4.2 and 4.3 and Proposition 4.5. □

The relevance of Theorem 4.7, from the point of view of the experiments, is that it predicts that, according to the boundary condition, the same material can present no interfaces, sharp interfaces or smooth interfaces. Unfortunately, we are not aware of experiments in which this behaviour is observed. In Figure 2 we represent typical scenarios that meet each of the conditions S1–S4 of Theorem 4.7.

Of course, the applicability of Theorem 4.7 depends on the knowledge of the functions and the set defined in Lemma 4.6. In that lemma, a very preliminary description is given. In this respect, we mention that, still in dimension 1 and when W is a quartic polynomial, Grinfeld and Novick-Cohen [23, 24] have done an analysis of the solutions of the equation $P_{\varepsilon,\lambda}$ of Lemma 4.1. As a by-product (and bearing in mind the equivalence between minimising J^{sm}_ε in $\mathcal{A}^{sm,J}_\lambda$, and minimising

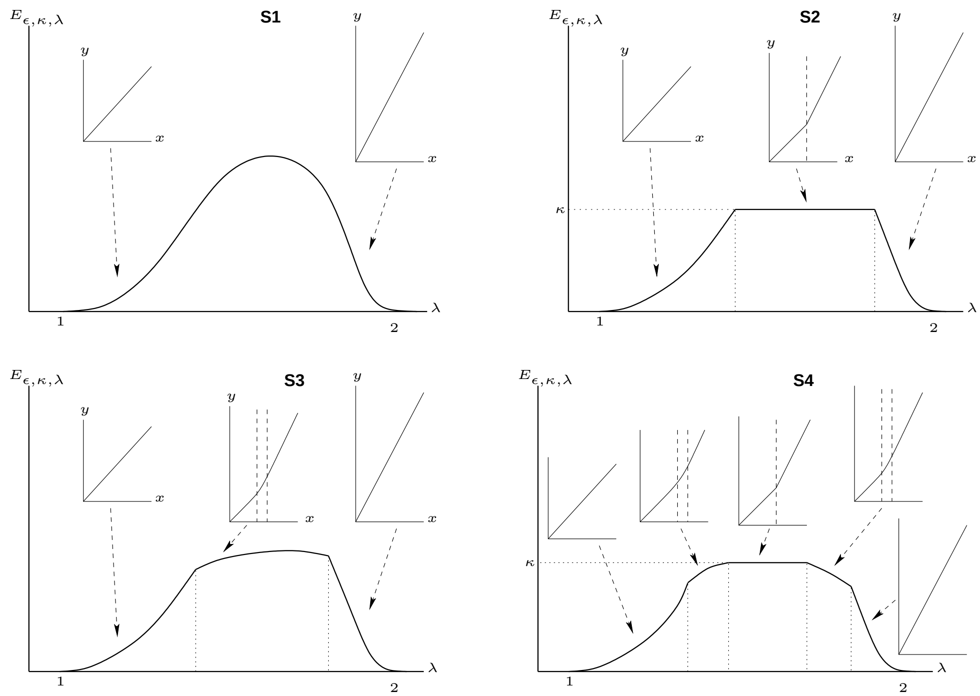

FIGURE 2. Representation of each of the scenarios S1–S4 described in Theorem 4.7.

I_ε^{sm} in $\mathcal{A}_\lambda^{sm}$), they have computed the regime of the parameters ε and λ for which the global minimisers of I_ε^{sm} in $\mathcal{A}_\lambda^{sm}$ present smooth interfaces or no interfaces at all. From that analysis, including the parameter κ does not add any difficulty, and, hence (when W is a quartic polynomial), one can easily deduce an improved version of Theorem 4.7, in which some of the functions defined in Lemma 4.6 can be computed explicitly, and the set $M_{\varepsilon,\kappa,\lambda}$ of minimisers can be described totally.

5. **A further model.** Still in dimension one, in this section we propose a variant of the model of Sections 3 and 4, and do a preliminary analysis of it. In the model of Sections 3 and 4, the sharp-interface energy is the same regardless of the size of the jump of u, but a more realistic model is one in which the sharp-interface energy depends on the size of the jump in u and can tend to zero as the jump in u tends to zero. Thus, following the general notation of Section 3, we propose the model based on minimisation of the functional $J : \mathcal{A}_\lambda^J \to \mathbb{R}$ defined by

$$J(u) := \int_0^1 \left[W(u(x)) + \varepsilon^2(\nabla u(x))^2\right] \mathrm{d}x + \int_{S_u} \psi([u])\,\mathrm{d}\mathcal{H}^0, \qquad u \in \mathcal{A}_\lambda^J, \tag{38}$$

where, as in Section 3, $\mathcal{A}_\lambda^J := \{u \in SBV(0,1) : u > 0,\ \|u\|_{L^1(0,1)} = \lambda,\ \nabla u \in L^2(0,1)\}$. Here $\psi : \mathbb{R} \to \mathbb{R}$ is a function that measures the energy of the jump, where, for any $u \in SBV(0,1)$ and $x \in (0,1)$, the jump $[u](x)$ of u at x is the well-defined quantity

$$[u](x) := \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} u(s)\,\mathrm{d}s - \lim_{h\to 0^+} \frac{1}{h}\int_{x-h}^{x} u(s)\,\mathrm{d}s.$$

Naturally, we have that $S_u = \{x \in (0,1) : [u](x) \neq 0\}$. Of course, an equivalent way of expressing $\int_{S_u} \psi([u]) \, \mathrm{d}\mathcal{H}^0$ is $\sum_{x \in S_u} \psi([u](x))$.

We now present the result on the existence and properties of minimisers of J.

Theorem 5.1. *Let $\varepsilon > 0$. Let $W : (0,\infty) \to [0,\infty)$ be a function of class C^1 satisfying $\lim_{t \to 0^+} W(t) = \infty$ and suppose that there exist r_1, r_2 with $0 < r_1 < r_2$ such that $-\infty < \sup_{(0,r_i]} W' = \inf_{[r_i,\infty)} W' < \infty$ for $i \in \{1,2\}$. Let $\psi : \mathbb{R} \to [0,\infty)$ be a continuous even function of class C^1 in $(0,\infty)$, non-decreasing in $(0,\infty)$, and such that*

$$\lim_{t \to 0} \frac{\psi(t)}{|t|} = \infty$$

and

$$\psi(a+b) \leq \psi(a) + \psi(b), \qquad a, b \in \mathbb{R}. \tag{39}$$

Let $\lambda \in (r_1, r_2)$.

Then there exists a minimiser of the functional $J : \mathcal{A}_\lambda^J \to \mathbb{R}$ defined in (38). Moreover, any minimiser u satisfies:

(i) *$u \in [r_1, r_2]$ a.e.*
(ii) *S_u is finite.*
(iii) *∇u is continuous and in SBV,*

$$W'(u) - 2\varepsilon^2 \nabla^2 u = c \tag{40}$$

for some constant $c \in \mathbb{R}$, $\nabla u(0) = \nabla u(1) = 0$ and $2\varepsilon^2 \nabla u(z) = \psi'([u](z))$ for all $z \in S_u$.

(iv) *$c = \int_0^1 W'(u) \, \mathrm{d}x$ and*

$$W(u) - \varepsilon^2 (\nabla u)^2 - cu = d, \tag{41}$$

for some constant $d \in \mathbb{R}$.

Proof. The proof is divided into several steps.

Proof of (i). We prove that

$$\text{if } u \in \mathcal{A}_\lambda^J \text{ satisfies } \left|u^{-1}(0,r_1)\right| + \left|u^{-1}(r_2,\infty)\right| > 0 \text{ then } J(u) > \inf J_{\mathcal{A}_\lambda^J}. \tag{42}$$

This in particular implies (i).

In order to prove (42), we reproduce an argument due to Gurtin and Matano [25]. For every $u \in \mathcal{A}_\lambda^J$ and every $0 < a \leq b$, define $u_{a,b} \in SBV(0,1)$ as

$$u_{a,b}(x) := \begin{cases} a & \text{if} \quad x \in (0,1) \quad \text{with} \quad u(x) < a, \\ u(x) & \text{if} \quad x \in (0,1) \quad \text{with} \quad a \leq u(x) \leq b, \\ b & \text{if} \quad x \in (0,1) \quad \text{with} \quad u(x) > b. \end{cases}$$

Define $g : \{(x_1, x_2) \in [0,\infty]^2 : x_1 \leq x_2\} \to \mathbb{R}$ as

$$\begin{aligned} g(a,b) &= \int_0^1 u_{a,b} \, \mathrm{d}x && \text{if } \ 0 < a \leq b < \infty, \\ g(a,\infty) &= a|u^{-1}(0,a)| + \int_{u^{-1}([a,\infty))} u \, \mathrm{d}x && \text{if } \ a \in [0,\infty], \\ g(0,b) &= \int_{u^{-1}((0,b])} u \, \mathrm{d}x + b|u^{-1}(b,\infty)| && \text{if } \ b \in [0,\infty]. \end{aligned}$$

By Lebesgue's dominated convergence theorem, g is continuous.

Define $\alpha := \sup_{(0,r_1]} W'$ and $W_1 : (0,\infty) \to \mathbb{R}$ as $W_1(x) := W(x) - \alpha x$ for $x > 0$. Then $\sup_{(0,r_1]} W_1' = 0 \leq \inf_{[\lambda,\infty)} W_1'$.

Suppose that $u \in \mathcal{A}^J_\lambda$ satisfies $|u^{-1}(0, r_1)| > 0$. Then $g(r_1, \infty) > \lambda$ and $g(r_1, \lambda) < \lambda$. Hence there exists $t_1 \in (\lambda, \infty)$ such that $g(r_1, t_1) = \lambda$; consequently, $u_{r_1,t_1} \in \mathcal{A}^J_\lambda$. Moreover, $W_1(u_{r_1,t_1}) \le W_1(u)$ and

$$\varepsilon^2 \int_0^1 (\nabla u_{r_1,t_1})^2 \,\mathrm{d}x + \int_{S_{u_{r_1,t_1}}} \psi([u_{r_1,t_1}]) \,\mathrm{d}\mathcal{H}^0 < \varepsilon^2 \int_0^1 (\nabla u)^2 \,\mathrm{d}x + \int_{S_u} \psi([u]) \,\mathrm{d}\mathcal{H}^0.$$

Therefore, $J(u_{r_1,t_1}) < J(u)$, since

$$\int_0^1 W_1(u) \,\mathrm{d}x = \int_0^1 W(u) \,\mathrm{d}x - \alpha\lambda.$$

Thus, u is not a minimiser of J in $\mathcal{A}^J_\lambda$,

Analogously, if $u \in \mathcal{A}^J_\lambda$ satisfies $|u^{-1}(r_2, \infty)| > 0$ then u is not a minimiser of J in $\mathcal{A}^J_\lambda$. This proves (42) and, hence, that any minimizer u of J satisfies (i).

Proof of existence. Define $\mathcal{A} := \{u \in \mathcal{A}^J_\lambda : u \in [r_1, r_2] \text{ a.e.}\}$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimising sequence of J in $\mathcal{A}$. By Ambrosio's [3] compactness result in SBV and standard compactness results in BV and L^∞, there exists a non-negative function $u \in SBV(0,1)$ such that $\|u\|_{L^1} = \lambda$ and $u \in [r_1, r_2]$ a.e., and for a subsequence (not relabelled), u_j converges to u weakly* in BV, weakly* in L^∞, and also almost everywhere. By Fatou's lemma, $\int_0^1 W(u) \le \liminf_{j\to\infty} \int_0^1 W(u_j)$; in particular, $u > 0$ almost everywhere. In addition, by the lower semicontinuity theorem in SBV of Braides [13, Th. 2.10], we get that

$$\int_0^1 \varepsilon^2 (\nabla u)^2 \,\mathrm{d}x + \int_{S_u} \psi([u]) \,\mathrm{d}\mathcal{H}^0 \le \liminf_{j\to\infty} \int_0^1 \varepsilon^2 (\nabla u_j)^2 \,\mathrm{d}x + \int_{S_{u_j}} \psi([u_j]) \,\mathrm{d}\mathcal{H}^0.$$

Thus, u is a minimiser of J in $\mathcal{A}$. By (42), u is a minimiser of J in $\mathcal{A}^J_\lambda$.

From now on until the end of the proof, we fix a minimiser u of J in $\mathcal{A}^J_\lambda$.

Proof of (ii). The result is obvious if $\psi(0) > 0$, so we will assume that $\psi(0) = 0$, which implies

$$\lim_{t\to 0} |\psi'(t)| = \infty. \tag{43}$$

Let $x_0 \in S_u$ and define $\varphi_{x_0} : (0,1) \to \mathbb{R}$ as

$$\varphi_{x_0}(x) := \begin{cases} 1 - x_0 & \text{if} \quad x \in (0, x_0), \\ -x_0 & \text{if} \quad x \in [x_0, 1). \end{cases}$$

Thanks to (i), for all $|t| < r_1$ we have $u + t\varphi_{x_0} \in \mathcal{A}^J_\lambda$; hence the derivative of the function $\mathbb{R} \ni t \mapsto J(u + t\varphi_{x_0})$ at 0 equals zero, that is to say,

$$(1 - x_0) \int_0^{x_0} W'(u) \,\mathrm{d}x - x_0 \int_{x_0}^1 W'(u) \,\mathrm{d}x - \psi'([u](x_0)) = 0. \tag{44}$$

Therefore, by (i),

$$|\psi'([u](x_0))| \le 2x_0(1 - x_0)\|W'\|_{L^\infty([r_1,r_2])} \le \frac{1}{2}\|W'\|_{L^\infty([r_1,r_2])}. \tag{45}$$

Thanks to (43), there exists a constant $\delta > 0$ depending only on W', ψ' such that $|[u](x_0)| \ge \delta$. Finally,

$$\infty > \sum_{z\in S_u} \psi([u](z)) \ge \mathcal{H}^0(S_u)\psi(\delta),$$

so $\mathcal{H}^0(S_u) < \infty$.

Proof of (iii). By standard regularity theory applied to each connected component of $(0,1) \setminus S_u$, we obtain that, restricted to each connected component of $(0,1) \setminus S_u$, the function ∇u is of class C^1, and the derivative of ∇u coincides with $\nabla^2 u$. Let $\varphi \in C^\infty([0,1])$ satisfy $\int_0^1 \varphi = 0$. As in the proof of (ii), we calculate the first variation of J at u in the direction of φ and obtain

$$\int_0^1 [W'(u) - 2\varepsilon^2 \nabla^2 u]\varphi \, \mathrm{d}x - 2\varepsilon^2 \nabla u(0)\varphi(0) - 2\varepsilon^2 \sum_{z \in S_u} [\nabla u](z)\varphi(z) + 2\varepsilon^2 \nabla u(1)\varphi(1) = 0.$$

This implies (40) for some constant c, that ∇u is continuous, and that $\nabla u(0) = \nabla u(1) = 0$. This and (44) imply that $2\varepsilon^2 \nabla u(z) = \psi'([u](z))$ for all $z \in S_u$.

Proof of (iv). The formula for c follows by integrating (40) and using the continuity of ∇u.

To prove (41) let $\varphi \in C_c^\infty(0,1)$. Since for $|t|$ sufficiently small the map $s \mapsto s + t\varphi(s)$ is strictly monotone, it is invertible. Therefore, for each $x \in [0,1]$ there is a unique solution $s = s_t(x)$ to

$$s + t\varphi(s) = x$$

and by the implicit function theorem $s_t(x)$ is smooth in x and t. Let

$$f(t) := \int_0^1 u(s_t(x)) \, \mathrm{d}x.$$

Then $f(0) = \lambda$ and

$$f(t) = \int_0^1 u(s)(1 + t\varphi'(s)) \, \mathrm{d}s. \tag{46}$$

For $|t|$ small, define the function $v_t : (0,1) \to \mathbb{R}$ by

$$v_t(x) := \frac{\lambda}{f(t)} u(s_t(x)), \qquad x \in (0,1).$$

Clearly $v_t \in \mathcal{A}_\lambda^J$ and

$$J(v_t) = \int_0^1 \left[W\left(\frac{\lambda}{f(t)} u(s)\right)(1 + t\varphi'(s)) + \frac{\varepsilon^2 \lambda^2}{f(t)^2} \cdot \frac{(\nabla u(s))^2}{1 + t\varphi'(s)} \right] \mathrm{d}s$$
$$+ \sum_{z \in S_u} \psi\left(\frac{\lambda}{f(t)}[u](z)\right).$$

Setting $\frac{\mathrm{d}}{\mathrm{d}t} J(v_t)|_{t=0} = 0$ we obtain

$$0 = \int_0^1 \left(W(u(s)) - \varepsilon^2 (\nabla u(s))^2\right) \varphi'(s) \, \mathrm{d}s$$
$$- \lambda^{-1} f'(0) \left(\int_0^1 (W'(u(s))u(s) + 2\varepsilon^2 (\nabla u(s))^2) \, \mathrm{d}s + \sum_{z \in S_u} \psi'([u](z))[u](z) \right).$$

Since $f'(0) = \int_0^1 u(s)\varphi'(s) \, \mathrm{d}s$ from (46), we deduce that

$$\int_0^1 \left(W(u(s)) - \varepsilon^2 (\nabla u(s))^2 - c_0 u(s)\right) \varphi'(s) \, \mathrm{d}s = 0,$$

where

$$c_0 := \lambda^{-1} \left(\int_0^1 (W'(u(s))u(s) + 2\varepsilon^2 (\nabla u(s))^2) \, \mathrm{d}s + \sum_{z \in S_u} \psi'([u](z))[u](z) \right),$$

and so $W(u) - \varepsilon^2(\nabla u)^2 - c_0 u = d$, for some constant d. Calculating c_0 using (iii) we find that $c_0 = c$. □

We finish the paper with some comments on Theorem 5.1.

A function $\psi : \mathbb{R} \to [0, \infty]$ such that (39) is called *subadditive*. As is well known (see Bouchitté and Buttazzo [12]), subadditivity plays an essential role in the lower semicontinuity of functionals defined on surface integrals, in particular of integrals defined on the jump set of a BV function. See also Braides [13] for more results on lower semicontinuity under subadditivity, as well as for examples of subadditive functions; for instance, it is easy to check that any concave function is subadditive.

The interesting case in Theorem 5.1 occurs when $\psi(0) = 0$, since this is the main difference with the model of Sections 3 and 4. A realistic example of a function W satisfying the conditions of Theorem 5.1 is given by any function W such that (27); see Fig. 1. Examples of functions ψ satisfying the conditions of Theorem 5.1 and $\psi(0) = 0$ are

$$\psi_1(t) := \kappa|t|^\alpha, \quad \psi_2(t) := \frac{\kappa|t|^\alpha}{1 + |t|^\alpha}, \quad \psi_3(t) := \kappa|t| \log\left(1 + \frac{1}{|t|}\right), \qquad t \in \mathbb{R},$$

for some $\kappa > 0$ and some $\alpha \in (0, 1)$, where log stands for the natural logarithm. Another possibility is a concave function ψ satisfying $\psi(0) = 0$, and which is strictly increasing in $(0, 1)$, and constant in $[1, \infty)$. Although it is possibly more realistic to assume that ψ is bounded (as ψ_2 and ψ_3 are), this is not an essential requirement due to the a priori bound of Theorem 5.1(i). The function ψ_3 is reminiscent of the interfacial energy function used in dislocation models of crystal grain boundaries, which is of the form

$$\psi_4(t) := \kappa|t| \left(A - \log|t|\right), \qquad t \in \mathbb{R},$$

for some constant $A > 0$, where t represents the angle of rotation between two grain boundaries; see Nabarro [35] and Shockley and Read [38]. Indeed, the function ψ_4 is usually used for relatively small values of t, and one can easily check that

$$\lim_{t \to 0} \frac{\psi_3(t)}{\psi_4(t)} = 1.$$

Note that in the case when $\psi'(t) > 0$ for all $t > 0$ there are no piecewise constant minimisers that are not constant, since for such a minimiser we would have $2\varepsilon^2 \nabla u(z) = \psi'([u](z)) = 0$ for all $z \in S_u$. This contrasts with the behaviour of the minimisers for the model in Sections 3 and 4.

Under the assumptions of Theorem 5.1, we have been unable to prove that minimisers are either smooth or have only one discontinuity. Nevertheless, the proof of (ii) provides us with an upper bound on the number of discontinuity points of a minimiser (in fact, of a critical point) u in terms of its energy $J(u)$, which can be used to give some necessary conditions on ε, W and ψ to guarantee that any minimiser is smooth or has only one discontinuity. To illustrate this, we only work out the particular case when $\psi(t) = \kappa|t|^\alpha$ for $t \in \mathbb{R}$, where $\kappa > 0$ and $\alpha \in (0, 1)$ are given. Following the general notation of the proof, let u be a minimiser of J in $\mathcal{A}_\lambda^J$ and $x_0 \in S_u$. We then have from (45) that

$$|[u](x_0)| \geq \left(\frac{1}{2\kappa\alpha} \|W'\|_{L^\infty([r_1, r_2])}\right)^{\frac{-1}{1-\alpha}}. \tag{47}$$

Assume additionally that $\lambda \in (1,2)$ and that W satisfies the general conditions of Section 3, i.e., that $W^{-1}(0) = \{1,2\}$ and $\liminf_{t\to\infty} W(t) > 0$. Recall now the definition of $E_{\varepsilon,0,\lambda}$ given by (23), as well as the estimates (24), (25), (31) and (35). Note that the definition of $E_{\varepsilon,0,\lambda}$ does make sense for the functional J of (38). By using as a test function the derivative of $y_{1,\lambda}$ (see Section 3), we get that $\inf_{\mathcal{A}^J_\lambda} J \leq \psi(1) = \kappa$. Therefore, as in the proof of (ii) and using (47),

$$\min\{E_{\varepsilon,0,\lambda}, \kappa\} \geq J(u) \geq \mathcal{H}^0(S_u)\kappa \left(\frac{1}{2\kappa\alpha} \|W'\|_{L^\infty([r_1,r_2])} \right)^{\frac{-\alpha}{1-\alpha}}.$$

Consequently, if

$$\frac{\min\{E_{\varepsilon,0,\lambda}, \kappa\}}{\kappa} \left(\frac{1}{2\kappa\alpha} \|W'\|_{L^\infty([r_1,r_2])} \right)^{\frac{\alpha}{1-\alpha}} < 2$$

then minimisers are smooth or have only one discontinuity, whereas if

$$\frac{\min\{E_{\varepsilon,0,\lambda}, \kappa\}}{\kappa} \left(\frac{1}{2\kappa\alpha} \|W'\|_{L^\infty([r_1,r_2])} \right)^{\frac{\alpha}{1-\alpha}} < 1$$

then minimisers are smooth, but this criterion is not optimal.

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