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GLOBAL ATTRACTORS FOR DAMPED SEMILINEAR WAVE EQUATIONS

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Abstract. The existence of a global attractor in the natural energy space is proved for the semilinear wave equation $u_{tt} + \beta u_t - \Delta u + f(u) = 0$ on a bounded domain $\Omega \subset \mathbf{R}^n$ with Dirichlet boundary conditions. The nonlinear term f is supposed to satisfy an exponential growth condition for $n = 2$, and for $n \geq 3$ the growth condition $|f(u)| \leq c_0(|u|^\gamma + 1)$, where $1 \leq \gamma \leq \frac{n}{n-2}$. No Lipschitz condition on f is assumed, leading to presumed nonuniqueness of solutions with given initial data. The asymptotic compactness of the corresponding generalized semiflow is proved using an auxiliary functional. The system is shown to possess Kneser's property, which implies the connectedness of the attractor.

In the case $n \geq 3$ and $\gamma > \frac{n}{n-2}$ the existence of a global attractor is proved under the (unproved) assumption that every weak solution satisfies the energy equation.

Dedicated to M.I. Vishik on the occasion of his 80^{th} birthday

1. **Introduction.** Let $\Omega \subset \mathbf{R}^n$, $n \geq 1$, be bounded and open with boundary $\partial\Omega$. Consider the damped semilinear wave equation

$$u_{tt} + \beta u_t - \Delta u + f(u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

with boundary condition

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

and initial conditions

$$u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \tag{1.3}$$

where $\beta > 0$ is a constant and $f : \mathbf{R} \to \mathbf{R}$ is continuous. Suppose that f satisfies the sign condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{1.4}$$

where λ_1 is the first eigenvalue of $-\Delta$ with the boundary condition (1.2). If $n \geq 3$ suppose in addition that f satisfies the growth condition

$$|f(u)| \leq c_0(|u|^{\frac{n}{n-2}} + 1), \tag{1.5}$$

where $c_0 > 0$ is a constant, while if $n = 2$ suppose that

$$|f(u)| \leq \exp\theta(u), \tag{1.6}$$

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for some continuous function θ satisfying

$$\lim_{|u|\to\infty} \frac{\theta(u)}{u^2} = 0. \tag{1.7}$$

No additional growth condition is needed if $n = 1$.

Let $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix}$ and

$$V(\varphi) = \int_\Omega \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + F(u) \right) dx,$$

where $F' = f$. Then solutions of the above problem satisfy

$$\frac{d}{dt} V(\varphi(t)) = -\beta \int_\Omega u_t^2 \, dx. \tag{1.8}$$

The main purpose of this paper is to give a proof of the following theorem.

Theorem 1.1. *Under the above hypotheses,* (1.1), (1.2) *possesses a connected global attractor* A *in the space* $X = H_0^1(\Omega) \times L^2(\Omega)$. *For each complete orbit* ξ *in* A *the* α *and* ω *limit sets of* ξ *are connected subsets of the set* Z *of rest points on which* V *is constant. If* Z *is totally disconnected then the limits*

$$z_- = \lim_{t\to-\infty} \xi(t), \;\; z_+ = \lim_{t\to\infty} \xi(t)$$

exist and z_-, z_+ *are rest points; furthermore,* $\varphi(t)$ *tends to a rest point in* X *as* $t \to \infty$ *for every solution* φ.

There is a large literature on the asymptotic behaviour of solutions to (1.1), (1.2). The earliest work on the convergence of solutions to rest points as $t \to \infty$ for nonlinearities f allowing multiple rest points seems to be that of Ball [6] and Webb [66]. In [6] weak convergence methods were used in a similar spirit to this paper to show, for example, that if $n \geq 3$ and (1.5) holds then every solution has a nonempty ω limit set consisting entirely of rest points. Without the hypothesis that the set of rest points is totally disconnected, convergence of solutions to a unique rest point still holds if $n = 1$ (Hale & Raugel [27]) or, for arbitrary n, if f is analytic and satisfies suitable growth conditions (Jendoubi [34], Haraux & Jendoubi [30, 31]). On the other hand counterexamples for nonanalytic $f = f(x, u)$ have recently been given by Jendoubi & Poláčik [35] (see also Poláčik [54]).

The existence of a global attractor for (1.1), (1.2) was proved by Hale [26] and Haraux [29] for f satisfying for $n \geq 3$ the growth condition

$$f(u) \leq c_0(|u|^\gamma + 1), \tag{1.9}$$

with $1 \leq \gamma < \frac{n}{n-2}$. For the case $n = 2$, Hale & Raugel [28] proved the existence of the attractor under an exponential growth condition of the type (1.6) (such a condition previously appearing in the work of Gallouet [20]). The existence of the attractor in the critical case $\gamma = \frac{n}{n-2}$ was first proved by Babin & Vishik [3], and then more generally by Arrieta, Carvalho & Hale [1]. For other treatments see Chepyzhov & Vishik [12], Ladyzhenskaya [44], Raugel [55] and Temam [60]. In all these works f is assumed to be at least locally Lipschitz with a growth condition on the Lipschitz constant. The price for dropping such an assumption, as in Theorem 1.1, is that uniqueness of solutions is no longer to be expected (see Remark 5.2 below). The only previous work on attractors for (1.1) under hypotheses that are not known to imply uniqueness of solutions seems to be that of Babin & Vishik [2], who for $n = 3$ proved the existence of an attractor, but in the weak topology of X, for f satisfying (1.5), $f(u)u \geq -C$, and a weakened Lipschitz condition.

In order to handle nonuniqueness of solutions, we use the framework of generalized semiflows developed in Ball [7]. This is one of several related approaches (the earliest being apparently that of Barbashin [9]), one being that used by Babin & Vishik [2], that are discussed in [7]. See Caraballo, Marín-Rubio & Robinson [11] for a comparison between parts of the theory in [7] and the related work of Melnik & Valero [48, 49], which is more adapted to differential inclusions. The necessary definitions and results from [7] are given in Section 2, where the opportunity is taken to clarify the relation between the theory and that of Hale [25] and Ladyzhenskaya [44], which was not described adequately in [7].

To establish that the system (1.1), (1.2) generates a generalized semiflow G there are two possible main approaches, either to write the system in the semilinear form

$$\dot{\varphi} = A\varphi + \mathcal{F}(\varphi), \tag{1.10}$$

for suitable operators A and $\mathcal{F}$, and use the variation of constants formula in the Banach space X (see Section 3), or to use the Galerkin method, as described in the book of Lions [45]. Under the growth conditions (1.5)-(1.7) these approaches are essentially equivalent (see, for example, Proposition 3.4). We choose to use the method based on (1.10), making use of results in Ball [6] which allow one to handle the case when $\mathcal{F} : X \to X$ is sequentially weakly conntinuous but not compact. We do this for two main reasons. Firstly, the proof of the energy equation (1.8) is more straightforward (for the analogous calculation for a weak solution constructed using the Galerkin method see Lions [45, pp22-25] and also Babin & Vishik [2, pp406-407]). Secondly, the method leads to a natural proof of *Kneser's property*, that the set of points in X reached after a fixed time $t \geq 0$ starting from given initial data is connected (see Section 5). By a result in [7] Kneser's property implies the connectedness of the global attractor A. Although there are isolated proofs of Kneser's property in the literature for semilinear parabolic equations (see Kaminogo [36], Kaminogo & Kikuchi [37], Kikuchi [42], Kikuchi & Nakagiri [43]), this seems to be the first time this issue has been considered for a semilinear wave equation.

When Ω is the whole of $\mathbf{R}^n$ or a compact n-dimensional Riemannian manifold without boundary the existence of a global attractor has been proved for classes of f satisfying (1.9) with $1 \leq \gamma < \frac{n+2}{n-2}$ by Lopes [46], Feireisl [17, 18] and Kapitanski [38] using estimates of Strichartz [59] type. These results would suggest that for Dirichlet boundary conditions the critical exponent for the existence of an attractor is $\gamma = \frac{n+2}{n-2}$ rather than $\gamma = \frac{n}{n-2}$. However this has not been proved, and indeed there is no indication that there is any critical exponent. In fact for any $\gamma \geq 1$ we can prove the global existence of a weak solution to (1.1), (1.2) under appropriate supplementary conditions on f using the Galerkin method. Unfortunately, it is not known whether the energy equation holds for $\gamma > \frac{n}{n-2}$. However under the (unproved) assumption that *all weak solutions satisfy the energy equation* the existence of a global attractor in the appropriate energy space can be proved for any $\gamma \geq 1$ under some mild supplementary conditions (see Theorem 4.4). The situation is similar to that for the three-dimensional Navier-Stokes equations of incompressible flow, for which it is proved in Ball [7] that there is a global attractor in the usual Hilbert space H of divergence-free L^2 velocity fields under the (similarly unproved) assumption that every weak solution satisfying a certain energy inequality is continuous in time with values in H. We recall that for the Navier-Stokes equations Sell [58] proved the existence of a global attractor without any unproved hypotheses on solutions, but in a space of trajectories and not in H (for later work in the same spirit see Chepyzhov & Vishik [12]).

The key idea of the paper is to use an auxiliary functional to prove asymptotic compactness of G. For the system (1.1), (1.2) this functional is given by

$$I(\varphi) = V(\varphi) + \frac{\beta}{2}\int_\Omega uu_t\,dx,$$

which formally satisfies the equation

$$\frac{d}{dt}I(\varphi) = -\beta I(\varphi) + H(u),$$

where

$$H(u) = \beta\int_\Omega (F(u) - \frac{1}{2}uf(u))\,dx.$$

Whereas (as pointed out, for example, in Raugel [55]) this method does not always work, it has now been used in a variety of applications (see Cabral, Rosa & Temam [10], Dai & Guo [13], Ghidaglia [21], Goubet [22], Goubet & Moise [23], Goubet & Rosa [24], Karachalios & Stavrakakis [41], Karachalios & Zographopoulos [39], Lu & B. Wang [47], Moise & Rosa [50], Moise, Rosa & X. Wang [52, 51], Rosa [56, 57], Temam [60], B. Wang [62, 63], X. Wang [65], B. Wang & Lange [64]). The method was originally outlined in a lecture given at Oberwolfach in 1992, but it has unfortunately taken me 10 years to write it up. The proof of Kneser's property in Section 5 is new, but otherwise the treatment in the paper follows that of the Oberwolfach lecture, with of course the addition of various details.

Dimensionality and related properties of the attractor are not considered in this paper, and for this the reader is referred to Temam [60], Eden, Milani & Nicolaenko [16], Eden, Foias, Nicolaenko & Temam [14], Karachalios & Stavrakakis [40], Eden & Kalantarov [15], Huang, Yi & Yin [33] and Zhou [68, 69].

2. **Global attractors for generalized semiflows.** We begin by summarizing some definitions and results from [7] that we shall use.

Let X be a metric space (not necessarily complete) with metric d. If $C \subset X$ and $b \in X$ we set $\rho(b,C) := \inf_{c\in C} d(b,c)$, and if $B \subset X, C \subset X$ we set $\operatorname{dist}(B,C) := \sup_{b\in B}\rho(b,C)$.

Definition 2.1. A *generalized semiflow* G on X is a family of maps $\varphi : [0,\infty) \to X$ (called *solutions*) satisfying the hypotheses:

(H1) (*Existence*) For each $z \in X$ there exists at least one $\varphi \in G$ with $\varphi(0) = z$.

(H2) (*Translates of solutions are solutions*) If $\varphi \in G$ and $\tau \geq 0$, then $\varphi^\tau \in G$, where $\varphi^\tau(t) := \varphi(t+\tau)$, $t \in [0,\infty)$.

(H3) (*Concatenation*) If $\varphi, \psi \in G$, $t \geq 0$, with $\psi(0) = \varphi(t)$ then $\theta \in G$, where

$$\theta(\tau) := \begin{cases} \varphi(\tau) & \text{for } 0 \leq \tau \leq t, \\ \psi(\tau - t) & \text{for } t < \tau. \end{cases}$$

(H4) (*Upper-semicontinuity with respect to initial data*) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Let G be a generalized semiflow and let $E \subset X$. Define for $t \geq 0$

$$T(t)E = \{\varphi(t) : \varphi \in G \text{ with } \varphi(0) \in E\}, \tag{2.1}$$

so that $T(t) : 2^X \to 2^X$, where 2^X is the space of all subsets of X. We make use of the following continuity hypotheses for a generalized semiflow G.

(C1) Each $\varphi \in G$ is continuous from $(0, \infty)$ to X.

(C2) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ uniformly for t in compact subsets of $(0, \infty)$.

(C3) Each $\varphi \in G$ is continuous from $[0, \infty)$ to X.

(C4) If $\varphi_j \in G$ with $\varphi_j(0) \to z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ uniformly for t in compact subsets of $[0, \infty)$.

If X is a Banach space with dual space X^*, we will also need the analogous property to (C4) in the weak topology, namely:

(C4w) If $\varphi_j \in G$ with $\varphi_j(0) \rightharpoonup z$ then there exist a subsequence φ_μ of φ_j and $\varphi \in G$ with $\varphi(0) = z$ such that $\langle \varphi_\mu, \theta \rangle \to \langle \varphi, \theta \rangle$ uniformly for t in compact subsets of $[0, \infty)$, for any $\theta \in X^*$.

The *positive orbit* of $\varphi \in G$ is the set $\gamma^+(\varphi) = \{\varphi(t) : t \geq 0\}$. If $E \subset X$ then the *positive orbit* of E is the set $\gamma^+(E) = \bigcup_{t \geq 0} T(t)E$

The $\omega-$*limit set* of $\varphi \in G$ is the set

$$\omega(\varphi) = \{z \in X : \varphi(t_j) \to z \text{ for some sequence } t_j \to \infty\}.$$

A *complete orbit* is a map $\xi : \mathbf{R} \to X$ such that for any $s \in \mathbf{R}$, $\xi^s \in G$. If ξ is a complete orbit then the α-*limit set* of ξ is the set

$$\alpha(\xi) = \{z \in X : \xi(t_j) \to z \text{ for some sequence } t_j \to -\infty\}.$$

If $E \subset X$ the $\omega-$*limit set* of E is the set

$$\omega(E) = \{z \in X : \text{there exist } \varphi_j \in G \text{ with } \varphi_j(0) \in E,\ \varphi_j(0) \text{ bounded, and a sequence } t_j \to \infty \text{ with } \varphi_j(t_j) \to z\}.$$

The subset $A \subset X$ *attracts* a set E if $\text{dist}(T(t)E, A) \to 0$ as $t \to \infty$.

We say that A is *positively invariant* if $T(t)A \subset A$ for all $t \geq 0$, and that A is *invariant* if $T(t)A = A$ for all $t \geq 0$.

The subset A is a *global attractor* if A is compact, invariant, and attracts all bounded sets.

The generalized semiflow G is *eventually bounded* if given any bounded $B \subset X$ there exists $\tau \geq 0$ with $\gamma^\tau(B)$ bounded.

G is *point dissipative* if there is a bounded set B_0 such that for any $\varphi \in G$ $\varphi(t) \in B_0$ for all sufficiently large t.

G is *asymptotically compact* if for any sequence $\varphi_j \in G$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to \infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.

Proposition 2.1. *Let G be asymptotically compact. Then G is eventually bounded.*

Theorem 2.2. *A generalized semiflow G has a global attractor if and only if G is point dissipative and asymptotically compact. The global attractor A is unique and given by*

$$A = \bigcup \{\omega(B) : B \text{ a bounded subset of } X\} = \omega(X). \tag{2.2}$$

Furthermore A is the maximal compact invariant subset of X.

Theorem 2.2 generalizes corresponding results for semiflows due to Hale [25] and Ladyzhenskaya [44]. They prove the existence of a global attractor for semiflows that are asymptotically smooth (in the sense of Hale), equivalently of class AK

in the sense of Ladyzhenskaya), and for which the positive orbits of bounded sets are bounded. Correspondingly we say that the generalized flow G is *asymptotically smooth* if whenever B is nonempty, bounded and positively invariant, there exists a compact set K which attracts B, and that G is *of class AK* if whenever B is bounded with $\gamma^\tau(B)$ bounded for some $\tau \geq 0$ and $\varphi_k \in G$, $\varphi_k(0) \in B$, $t_k \to \infty$, then $\varphi_k(t_k)$ is relatively compact. The equivalences in the following proposition clarify the relationship between the hypotheses of Hale and Ladyzhenskaya and those of Theorem 2.2 (a relationship that was not fairly described in [7]).

Proposition 2.3. (i) *G is asymptotically smooth if and only if G is of class AK.*
(ii) *G is asymptotically compact if and only if G is asymptotically smooth and eventually bounded.*

Proof. The proofs are straightforward. As an illustration suppose that G is of class AK. We show that G is asymptotically smooth. For $B \subset X$ nonempty, bounded and positively invariant define $K = \{z \in X : \varphi_k(t_k) \to z$ for some $\varphi_k \in G$ with $\varphi_k(0) \in B$ and $t_k \to \infty\}$. Let $z_r \in K$, $r = 1, 2, \ldots$. Then given r there exists $t_r \geq r$ and $\varphi_r \in G$ with $\varphi_r(0) \in B$ and $d(\varphi_r(t_r), z_r) < \frac{1}{r}$. Since G is of class AK, there exists a subsequence r' of r with $\varphi_{r'}(t_{r'}) \to y$ for some y, and by definition $y \in K$. Hence $z_{r'} \to y$. Thus K is compact and it is easily seen that K attracts B. □

Proposition 2.4. *Let G be asymptotically compact and satisfy* (C1). *If $\varphi \in G$ then $\omega(\varphi)$ is connected. If ψ is a complete orbit then $\alpha(\psi)$ is connected.*

We say that G has *Kneser's property* if $T(\tau)\{z\}$ is connected for all $z \in X, \tau \geq 0$.

Theorem 2.5. *Let G be asymptotically compact and satisfy* (C1). *If G has Kneser's property and if $E \subset X$ is connected then $\omega(E)$ is connected.*

Remark 2.1. In the statement of Theorem 2.5 in [7] the hypothesis that G be asymptotically compact was used in the proof but accidentally omitted from the statement. Without this hypothesis $\omega(E)$ need not be connected. For example, if G is the semiflow generated on $\mathbf{R}^2$ by the ordinary differential equations

$$\dot{x} = x(1 - x^2), \quad \dot{y} = x^2 - 1,$$

which has integral curves given by $x = ae^{-y}, a \in \mathbf{R}$, then $\omega([-1, 1] \times \{0\}) = \{-1\} \times (-\infty, 0] \cup \{1\} \times (-\infty, 0]$.

Corollary 2.6. *Let X be connected, and let G satisfy* (C1) *and have Kneser's property. If A is a global attractor then A is connected.*

A complete orbit $\xi \in G$ is *stationary* if $\xi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in X$. Each such z is called a *rest point*. We denote the set of rest points of G by $Z(G)$.

We say that $V : X \to \mathbf{R}$ is a *Lyapunov function* for G provided

(*i*) V is continuous,
(*ii*) $V(\varphi(t)) \leq V(\varphi(s))$ whenever $\varphi \in G$ and $t \geq s \geq 0$.
(*iii*) if $V(\xi(t)) =$ constant for some complete orbit ξ and all $t \in \mathbf{R}$ then ξ is stationary.

Theorem 2.7. *Let G be asymptotically compact, let* (C1) *hold, and suppose there exists a Lyapunov function V for G. Suppose further that $Z(G)$ is bounded. Then G is point dissipative, so that there exists a global attractor A. For each complete orbit ξ lying in A the limit sets $\alpha(\xi), \omega(\xi)$ are connected subsets of $Z(G)$ on which*

V is constant. If $Z(G)$ is totally disconnected (in particular, if $Z(G)$ is countable) the limits

$$z_- = \lim_{t\to-\infty} \xi(t), \;\; z_+ = \lim_{t\to\infty} \xi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to \infty$ for every $\varphi \in G$.

3. **The generalized semiflow generated by the semilinear wave equation.**

3.1. **Weak solutions.** We set $v = u_t$ and write (1.1)-(1.2) in the form

$$\dot{\varphi} = A\varphi + \mathcal{F}(\varphi), \tag{3.1}$$

where

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ \Delta & -\beta \end{pmatrix}, \; \mathcal{F}(\varphi) = \begin{pmatrix} 0 \\ -f(u) \end{pmatrix}. \tag{3.2}$$

We denote by e^{At} the strongly continuous group of bounded linear operators generated by A on $X = H^1_0 \times L^2$, where $H^1_0 = H^1_0(\Omega), L^p = L^p(\Omega)$. If $u, \bar{u} \in H^1_0$, $v, \bar{v} \in L^2$, we write

$$(\nabla u, \nabla \bar{u}) = \int_\Omega \nabla u \cdot \nabla \bar{u}\; dx, \quad (v, \bar{v}) = \int_\Omega v\bar{v}\, dx,$$

with corresponding norms $\|\nabla u\| = (\nabla u, \nabla u)^{\frac{1}{2}}$, $\|v\| = (v, v)^{\frac{1}{2}}$. We regard X as a Hilbert space with inner product

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \rangle = (\nabla u, \nabla \bar{u}) + (v, \bar{v}),$$

and identify X with its dual. Note that the domain $D(A)$ of A is given by

$$D(A) = \{ \begin{pmatrix} u \\ v \end{pmatrix} : u, v \in H^1_0, \Delta u \in L^2 \}.$$

Lemma 3.1. *The adjoint A^* of A is given by*

$$A^* = -\begin{pmatrix} 0 & 1 \\ \Delta & \beta \end{pmatrix},$$

with

$$D(A^*) = \{ \begin{pmatrix} \chi \\ \psi \end{pmatrix} : \chi, \psi \in H^1_0, \; \Delta\chi \in L^2 \}.$$

Proof. From the definition of the adjoint, $\begin{pmatrix} \chi \\ \psi \end{pmatrix} \in D(A^*)$ and $A^* \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ if and only if

$$\langle \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} \chi \\ \psi \end{pmatrix}, \begin{pmatrix} v \\ \Delta u - \beta v \end{pmatrix} \rangle \qquad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in D(A),$$

which holds if and only if

$$(\nabla p, \nabla u) = (\psi, \Delta u) \qquad \text{for all } u \in H^1_0 \text{ with } \Delta u \in L^2, \tag{3.3}$$

$$(\nabla \chi, \nabla v) = (q + \beta\psi, v) \qquad \text{for all } v \in H^1_0. \tag{3.4}$$

Now approximating p by C^∞_0 functions we have that

$$(\nabla p, \nabla u) = -(p, \Delta u) \qquad \text{for all } u \in H^1_0 \text{ with } \Delta u \in L^2,$$

and so (3.3) is equivalent to

$$(p+\psi, \Delta u) = 0.$$

Solving $\Delta u = p+\psi$ for $u \in H_0^1$ we deduce that (3.3) holds if and only if $p = -\psi \in H_0^1$.

But (3.4) holds if and only if $\Delta\chi \in L^2$ and $q = -\Delta\chi - \beta\psi$.

□

Lemma 3.2. *f satisfies* (1.6),(1.7) *if and only if, given any $M > 0$ there exists $C_M > 0$ such that*

$$|f(u)| \leq C_M \exp\left(\frac{u^2}{M}\right) \qquad \textit{for all } u. \tag{3.5}$$

Proof. If f satisfies (1.6) then by (1.7) for any given $M > 0$ there exists $k_M > 0$ such that $\theta(u) \leq u^2/M$ for $|u| > k_M$. Thus

$$\theta(u) \leq a_M + \frac{u^2}{M},$$

where $a_M = \max_{|u|\leq k_M} \theta(u)$, and so

$$|f(u)| \leq \exp\left(a_M + \frac{u^2}{M}\right) = C_M \exp\left(\frac{u^2}{M}\right).$$

Conversely, suppose (3.5) holds for all $M > 0$. We may suppose that $C_M \to \infty$ as $M \to \infty$. Let

$$\theta(u) = \inf_{M=1,2,\ldots} \left(\ln C_M + \frac{u^2}{M}\right).$$

Since $\ln C_M + u^2/M \to \infty$ as $M \to \infty$ uniformly on compact sets, on any compact set θ is a minimum of a finite number of continuous functions. Hence θ is continuous. It is easily seen that (1.7) holds, while (1.6) follows since $\ln|f(u)| \leq \theta(u)$. □

Lemma 3.3. *$\mathcal{F} : X \to X$ and is sequentially weakly continuous and continuous. If $n = 1, 2$ then $\mathcal{F}$ is compact, that is $w_j \rightharpoonup w$ in X implies $\mathcal{F}(w_j) \to \mathcal{F}(w)$ in X.*

Proof. We must show that $f : H_0^1 \to L^2$ and is sequentially weakly continuous and continuous, and compact for $n = 1, 2$.

First let $n \geq 3$. Since H_0^1 is continuously embedded in $L^{\frac{2n}{n-2}}$, by (1.5) f maps bounded sets in H_0^1 to bounded sets in L^2. Let $u_j \rightharpoonup u$ in H_0^1. Then $f(u_j)$ is bounded in L^2 and so a subsequence (not relabelled) converges weakly in L^2 to some χ. But by the compactness of the embedding of H_0^1 in L^2 we may assume that $u_j \to u$ a.e.. Hence, since f is continuous, $f(u_j) \to f(u)$ a.e., from which it follows by standard arguments (for example, using Lusin's or Mazur's theorem) that $\chi = f(u)$ and that the whole sequence $f(u_j) \rightharpoonup f(u)$ in L^2. Thus f is sequentially weakly continuous.

If $u_j \to u$ strongly in H_0^1 then in addition by (1.5) we have that $f(u_j)^2$ is bounded above by the sequence $2c_0^2(1 + |u_j|^{\frac{2n}{n-2}})$, which is strongly convergent in L^1. Hence by a version of the dominated convergence theorem (cf. Ball & Marsden [8, Lemma 4.8]), $\int_\Omega f(u_j)^2 dx \to \int_\Omega f(u)^2 dx$ and so $f(u_j) \to f(u)$ in L^2.

Now let $n = 2$. If $u_j \rightharpoonup u$ in H_0^1 then by Trudinger's inequality [61]

$$\sup_j \int_\Omega \exp(\alpha {u_j}^2)\, dx < \infty$$

for $\alpha > 0$ sufficiently small. Thus, by Lemma 3.2, $f(u_j)$ is bounded in L^p for any $1 \le p < \infty$ and so by the same reasoning as for $n \ge 3$, $f(u_j) \to f(u)$ in L^2. If $n = 1$ then $u_j \rightharpoonup u$ in H_0^1 implies $f(u_j) \to f(u)$ uniformly. $\square$

Definition 3.1. Let $-\infty < t_0 < t_1 < \infty$. A map $\varphi \in C([t_0, t_1]; X)$ is a weak solution of (3.1) on $[t_0, t_1]$ if $\mathcal{F}(\varphi) \in L^1((t_0, t_1); X)$ and if for each $\theta \in D(A^*)$ the function $\langle \varphi(t), \theta \rangle$ is absolutely continuous on $[t_0, t_1]$ and satisfies

$$\frac{d}{dt}\langle \varphi(t), \theta \rangle = \langle \varphi(t), A^*\theta \rangle + \langle \mathcal{F}(\varphi(t)), \theta \rangle$$

for a.e. $t \in [t_0, t_1]$. φ is a weak solution on $[t_0, \infty)$ if it is a weak solution on $[t_0, t_1]$ for all $t_1 > t_0$.

The definition is equivalent to the standard concept of a weak solution to (1.1), (1.2). In the following result u_t denotes the distributional derivative with respect to t of u considered as an element of $L^2(Q)$, where $Q = \Omega \times (t_0, t_1)$.

Proposition 3.4. $\varphi = \begin{pmatrix} u \\ v \end{pmatrix}$ *is a weak solution of* (3.1) *on* $[t_0, t_1]$ *if and only if* $u \in C([t_0, t_1]; H_0^1), v = u_t \in C([t_0, t_1]; L^2)$*, and for each* $\psi \in H_0^1$*,* $(u_t, \psi) \in C^1([t_0, t_1])$ *with*

$$\frac{d}{dt}(u_t, \psi) + (\nabla u, \nabla \psi) + \beta(u_t, \psi) + (f(u), \psi) = 0 \qquad \textit{for all } t \in [t_0, t_1].$$

Proof. By Lemma 3.3 for any weak solution φ we have that $\langle \varphi(t), \theta \rangle \in C^1([t_0, t_1])$ for any $\theta \in D(A^*)$. Thus by Lemma 3.1, φ is a weak solution on $[t_0, t_1]$ if and only if $u \in C([t_0, t_1]; H_0^1), v \in C([t_0, t_1]; L^2)$ and for each $\chi \in H_0^1$ with $\Delta\chi \in L^2$, $(\nabla u, \nabla \chi) \in C^1([t_0, t_1])$ with

$$\frac{d}{dt}(\nabla u, \nabla \chi) = -(v, \Delta\chi) \qquad \text{in } [t_0, t_1], \tag{3.6}$$

and for all $\psi \in H_0^1, (v, \psi) \in C^1([t_0, t_1])$ with

$$\frac{d}{dt}(v, \psi) + (\nabla u, \nabla \psi) + \beta(v, \psi) + (f(u), \psi) = 0 \qquad \text{in } [t_0, t_1]. \tag{3.7}$$

Now since

$$(\nabla u, \nabla \chi) = -(u, \Delta\chi),$$

and given $\rho \in L^2$ we may solve $\Delta\chi = \rho$ for $\chi \in H_0^1$, (3.6) is equivalent to

$$\frac{d}{dt}(u, \rho) = (v, \rho) \qquad \text{in } [t_0, t_1]. \tag{3.8}$$

for all $\rho \in L^2$. Hence if $\eta \in C_0^\infty(Q)$ has the form $\eta(x,t) = \sigma(t)\rho(x)$ with $\sigma \in C_0^\infty(t_0, t_1), \rho \in C_0^\infty(\Omega)$, then

$$\int_Q \eta_t u \, dx \, dt = -\int_Q \eta v \, dx \, dt. \tag{3.9}$$

Since the linear span of such functions η is dense in $C_0^\infty(Q)$ (see, for example, Friedlander [19, Theorem 4.3.1, p 44]) we have that (3.9) holds for all $\eta \in C_0^\infty(Q)$, and so $v = u_t$, which in turn implies (3.8) for $u \in C([t_0, t_1]; H_0^1), v = u_t \in C([t_0, t_1]; L^2)$. The result now follows from (3.7). $\square$

Remark 3.1. Equivalently, φ is a weak solution if and only if $u \in C([t_0, t_1]; H_0^1), u_t \in C([t_0, t_1]; L^2), u_{tt} \in C([t_0, t_1]; H^{-1})$ and

$$u_{tt} + \beta u_t - \Delta u + f(u) = 0 \qquad \text{in } \mathcal{D}'(Q),$$

where $H^{-1} = H^{-1}(\Omega)$ denotes the dual space of H_0^1.

Weak solutions are the same as mild solutions, that is solutions to the variation of constants formula. This follows immediately from Ball [5] (see also Balakrishnan [4]).

Proposition 3.5. *A function $\varphi : [t_0, t_1] \to X$ is a weak solution of* (3.1) *on $[t_0, t_1]$ if and only if $\mathcal{F}(\varphi(\cdot)) \in L^1((t_0, t_1); X)$ and φ satisfies the variation of constants formula*

$$\varphi(t) = e^{A(t-t_0)}\varphi(t_0) + \int_{t_0}^{t} e^{A(t-s)}\mathcal{F}(\varphi(s))\, ds \tag{3.10}$$

for all $t \in [t_0, t_1]$.

3.2. **The generalized semiflow.**

Theorem 3.6. *Given any $\varphi_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$, there exists at least one weak solution $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix}$ of* (3.1) *with $\varphi(0) = \varphi_0$ on some interval $[0, T]$, $T > 0$, and any such weak solution can be extended to a weak solution on $[0, \infty)$. The family of all weak solutions $\varphi : [0, \infty) \to X$ is a generalized semiflow on X satisfying the continuity conditions* (C4w) *and* (C4)*. For each weak solution $V(\varphi(\cdot)) \in C^1([0, \infty))$, with*

$$\frac{d}{dt}V(\varphi(t)) = -\beta\|u_t\|^2, \qquad t \in [0, \infty), \tag{3.11}$$

and $(u, u_t) \in C^1([0, \infty))$ with

$$\frac{d}{dt}(u, u_t) = \|u_t\|^2 - \beta(u, u_t) - \|\nabla u\|^2 - (f(u), u), \qquad t \in [0, \infty). \tag{3.12}$$

Proof. The theorem is essentially proved in Ball [6, Section 5], but we give the main points for the convenience of the reader. For the local existence we use [6, Theorem 5.9], which has as its main hypothesis the sequential weak continuity of $\mathcal{F}$, established in Lemma 3.3; this result guarantees that given $\varphi_0 \in X$ there is at least one weak solution φ with $\varphi(0) = \varphi_0$ defined on a maximal interval $[0, t_{\max})$, where $0 < t_{\max} \leq \infty$, and that for any such weak solution with $t_{\max} < \infty$

$$\int_0^{t_{\max}} \|\mathcal{F}(\varphi(t))\|_X \, dt = \infty. \tag{3.13}$$

To show that $t_{\max} = \infty$ we first derive the energy equation (3.11). To this end consider the functional

$$E(u) = \int_\Omega F(u(x))\, dx.$$

We first claim that $E : H_0^1 \to \mathbf{R}$ is sequentially weakly continuous. Let $u_j \rightharpoonup u$ in H_0^1. If $n \geq 3$ we have from (1.5) that

$$F(u) \leq C(|u|^{\frac{2n-2}{n-2}} + 1),$$

so that $|F(u_j)|$ is bounded above by a strongly convergent sequence in L^1, while if $n \leq 2$ then $F(u_j)$ is bounded in L^p for any $1 \leq p < \infty$. Hence $E(u_j) \to E(u)$.

Next we claim that $E \in C^1(H_0^1;\mathbf{R})$ with derivative $E'(u)(\psi) = (f(u),\psi)$ for $u,\psi \in H_0^1$. In fact E is Gateaux differentiable with the indicated derivative since

$$\begin{aligned}\frac{E(u+\tau\psi)-E(u)}{\tau} &= \frac{1}{\tau}\int_\Omega\int_0^1 \frac{d}{ds}F(u+s\tau\psi)ds\,dx\\ &= \int_\Omega\int_0^1 f(u+s\tau\psi)\psi\,ds\,dx,\end{aligned}$$

and we can pass to the limit $\tau \to 0$ using the dominated convergence theorem and the hypotheses (1.5),(1.6). To show that E is C^1 it then suffices (cf. Zeidler [67, Proposition 4.8 p137]) to prove that $E' : H_0^1 \to H^{-1}$ is continuous. Let $u_j \to u$ in H_0^1. Then if $n \geq 3$,

$$\begin{aligned}\|E'(u_j)-E'(u)\|_{H^{-1}} &\leq \sup_{\|\psi\|_{H_0^1}\leq 1} \|f(u_j)-f(u)\|_{L^{\frac{2n}{n+2}}}\|\psi\|_{L^{\frac{2n}{n-2}}}\\ &\leq C\|f(u_j)-f(u)\|_{L^{\frac{2n}{n+2}}},\end{aligned}$$

where here and below C denotes a generic constant, and since

$$|f(u_j)-f(u)|^{\frac{2n}{n+2}} \leq C(1+|u_j|^{\frac{2n^2}{n^2-4}}+|u|^{\frac{2n^2}{n^2-4}}), \tag{3.14}$$

and the right-hand side of (3.14) is strongly convergent in L^1, we have that $E'(u_j) \to E'(u)$ in H^{-1} as required. If $n \leq 2$ then we obtain the same conclusion since $f(u_j) \to f(u)$ strongly in L^p for all $1 \leq p < \infty$. Since E is C^1 so is

$$V(\varphi) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla u\|^2 + E(u),$$

with derivative

$$V'(\varphi)\begin{pmatrix}\psi\\ \chi\end{pmatrix} = (v,\chi)+(\nabla u,\nabla\psi)+(f(u),\psi). \tag{3.15}$$

It follows from (3.15) that

$$\langle V'(\varphi), A\varphi+\mathcal{F}(\varphi)\rangle = -\beta\|v\|^2, \tag{3.16}$$

for all $\varphi \in D(A)$. Now let $T>0$, define $g(t)=\mathcal{F}(\varphi(t))$, and let $g_j \in C^1([0,T];X)$ with $g_j \to g$ in $C([0,T];X)$. Let $\varphi_{0j} \in D(A)$ with $\varphi_{0j} \to \varphi_0$ in X, and define $\varphi_j \in C([0,T];X)$ by

$$\varphi_j(t) = e^{At}\varphi_{0j} + \int_0^t e^{A(t-s)}g_j(s)\,ds.$$

Then (cf. Pazy [53, Corollary 2.5 p107]) $\varphi_j(t) \in D(A)$, $\varphi_j \in C^1([0,T];X)$ and $\dot\varphi_j(t) = A\varphi_j(t)+g_j(t)$ for all $t \in [0,T]$. A simple estimation shows that $\varphi_j \to \varphi$ in $C([0,T];X)$. Thus using (3.16)

$$\begin{aligned}V(\varphi_j(t))-V(\varphi_{0j}) &= \int_0^t \langle V'(\varphi_j(s)), A\varphi_j(s)+g_j(s)\rangle\,ds \qquad (3.17)\\ &= -\beta\int_0^t \|v_j(s)\|^2\,ds + \int_0^t \langle V'(\varphi_j(s)), g_j(s)-\mathcal{F}(\varphi_j(s))\rangle\,ds,\end{aligned}$$

where $\varphi_j = \begin{pmatrix}u_j\\ v_j\end{pmatrix}$. Since V is C^1 we can pass to the limit in (3.17) to obtain

$$V(\varphi(t)) - V(\varphi(0)) = -\beta\int_0^1 \|v(s)\|^2\,ds,$$

from which (3.11) follows. The same method establishes (3.12).

From the energy equation it follows that $V(\varphi(t))$ is uniformly bounded on $[0, t_{\max})$. Since from (1.4)

$$F(u) \geq -\frac{\lambda}{2}u^2 + a$$

for some $\lambda < \lambda_1$ and some a, we can write

$$\begin{aligned} 2V(\varphi(t)) &= \int_\Omega [u_t^2 + (1 - \frac{\lambda}{\lambda_1})|\nabla u|^2 + \frac{\lambda}{\lambda_1}(|\nabla u|^2 - \lambda_1 u^2) + 2F(u) + \lambda u^2]\,dx \\ &\geq \int_\Omega [u_t^2 + (1 - \frac{\lambda}{\lambda_1})|\nabla u|^2 + 2a]\,dx. \end{aligned} \tag{3.18}$$

Hence $\|\varphi(t)\|_X$ is uniformly bounded on $[0, t_{\max})$. But since $\mathcal{F}$ maps bounded sets to bounded sets, this implies by (3.13) that $t_{\max} = \infty$.

We have thus proved that condition (H1) in the definition of a generalized semiflow holds. The conditions (H2), (H3) follow immediately from the definition of a solution, and so it remains to establish (C4w) and (C4) (which implies (H4)).

Let $\varphi_j \in G$ with $\varphi_j(0) \rightharpoonup z$, and let $T > 0$, $\theta \in X^*$. We show that $\langle \varphi_j, \theta \rangle$ is equicontinuous on $[0, T]$. Let $0 \leq t \leq t + \tau \leq T$. Then

$$\begin{aligned} \langle \varphi_j(t+\tau) - \varphi_j(t), \theta \rangle = \langle (e^{A(t+\tau)} - e^{At})\varphi_j(0), \theta \rangle + \int_t^{t+\tau} \langle e^{A(t+\tau-s)}\mathcal{F}(\varphi_j(s)), \theta \rangle\,ds \\ + \int_0^t \langle (e^{A(t+\tau-s)} - e^{A(t-s)})\mathcal{F}(\varphi_j(s)), \theta \rangle\,ds. \end{aligned} \tag{3.19}$$

Now by [6, Lemma 5.11] if $w_j \rightharpoonup w$ in X and $t_j \to t$ in $[0, T]$ then $e^{At_j}w_j \rightharpoonup e^{At}w$ in X, from which it follows that given $\varepsilon > 0, M > 0$ there exists $\delta > 0$ such that $|\langle (e^{A(t+\tau)} - e^{At})w, \theta \rangle| \leq \varepsilon$ whenever $\|w\|_X \leq M$, $t, t+\tau \in [0, T]$ with $0 \leq \tau \leq \delta$. Since $\|\varphi_j(s)\|_X$, $\|\mathcal{F}(\varphi_j(s))\|_X$ are uniformly bounded for $s \in [0, T]$ independently of j, the equicontinuity then follows easily from (3.19). Thus by [6, Lemma 5.12] there exist a subsequence φ_μ of φ_j and a weakly continuous map $\varphi : [0, T] \to X$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \rightharpoonup \varphi(t)$ for each $t \in [0, T]$. Given $\theta \in X^*$ it is now easy to pass to the limit in the equation

$$\langle \varphi_\mu(t), \theta \rangle = \langle e^{At}\varphi_\mu(0), \theta \rangle + \int_0^t \langle e^{A(t-s)}\mathcal{F}(\varphi_\mu(s)), \theta \rangle\,ds$$

using the sequential weak continuity of $\mathcal{F}$ to show that φ is a weak solution. Taking $T = 1, 2, \ldots$ and choosing an appropriate diagonal sequence we thus obtain (C4w).

Now suppose that $\varphi_j(0) \to z$ strongly. Let φ_μ, φ be as in (C4w), so that in particular $\varphi(0) = z$. Since V is continuous, $V(\varphi_j(0)) \to V(\varphi(0))$. Now let $t_\mu \to t$ in $[0, T]$. Since E is sequentially weakly continuous, V is sequentially weakly lower semicontinuous on X. Hence, writing $\varphi_\mu = \begin{pmatrix} u_\mu \\ u_{\mu t} \end{pmatrix}$, $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix}$, we obtain

$$V(\varphi(t)) \leq \liminf_{\mu\to\infty} V(\varphi_\mu(t_\mu)), \qquad \int_0^t \|u_t\|^2 ds \leq \liminf_{\mu\to\infty} \int_0^t \|u_{\mu t}\|^2 ds, \tag{3.20}$$

and hence

$$V(\varphi(t)) + \beta\int_0^t \|u_t\|^2 ds \leq \liminf_{\mu\to\infty} \left(V(\varphi_\mu(t_\mu)) + \beta\int_0^{t_\mu} \|u_{\mu t}\|^2 ds \right). \tag{3.21}$$

Since by the energy equation both sides of (3.21) equal $V(\varphi(0))$, it follows from (3.20) that $V(\varphi_\mu(t_\mu)) \to V(\varphi(t))$, and thus that $\|\varphi_\mu(t_\mu))\|_X^2 \to \|\varphi(t)\|_X^2$, which together with the weak convergence implies that $\varphi_\mu(t_\mu) \to \varphi(t)$ strongly in X. Thus (C4) holds. □

Remark 3.2. In fact Theorem 3.6 remains valid if (1.4) is replaced by the weaker condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} \geq -k \tag{3.22}$$

for some $k > 0$. For it follows from (3.22) that

$$F(u) \geq -\frac{k'}{2}u^2 + c \tag{3.23}$$

for constants $k' > k$ and c, and hence if $\|\varphi(0)\|_X \leq M$ and $T > 0$ we have

$$\begin{aligned} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + c_1 - \frac{k'}{2}\|u\|^2 &\leq V(\varphi(t)) \\ &\leq V(\varphi(0)) \leq C(M) < \infty \end{aligned} \tag{3.24}$$

for all $t \in [0, T]$, where $c_1, C(M)$ are constants. Thus

$$\|u_t\|^2 \leq D(M)(1 + \|u\|)^2$$

on $[0, T]$, and so, using

$$u(\cdot, t) - u(\cdot, 0) = \int_0^t u_t(\cdot, s)\, ds,$$

we have that

$$(1 + \|u\|)(t) \leq 1 + \lambda_1^{-\frac{1}{2}} M^{\frac{1}{2}} + D(M)^{\frac{1}{2}} \int_0^t (1 + \|u\|)(s)\, ds.$$

Applying Gronwall's inequality we deduce that for suitable constants $\|u\|(t) \leq D_1(M) < \infty$ on $[0, T]$, and hence from (3.24) that $\|\varphi(t)\|_X \leq C_1(M) < \infty$ on $[0, T]$. With this estimate in hand the proof goes through with only minor modifications. However we need the stronger condition (1.4) to prove the existence of a global attractor; see Remark 4.1 below.

4. **Asymptotic compactness and the existence of a global attractor.**

4.1. **Proof of Theorem 1.1.** We prove Theorem 1.1, with the exception of the assertion that A is connected, which is deferred to Section 5. In order to apply Theorem 2.7 we need to verify its various hypotheses. We first note that $z = \begin{pmatrix} u \\ v \end{pmatrix}$ is a rest point of G if and only if

$$\langle z, A^*\theta \rangle + \langle \mathcal{F}(z), \theta \rangle = 0$$

for all $\theta \in D(A^*)$, that is, by a lemma in Ball [5], $z \in D(A)$ and $Az + \mathcal{F}(z) = 0$. This is equivalent to $v = 0$ and $u \in H_0^1$ with $\Delta u \in L^2$ and

$$-\Delta u + f(u) = 0. \tag{4.1}$$

Since V is continuous and by (3.11) nonincreasing along solutions, in order to prove that V is a Lyapunov function we just have to show that property (iii) in the definition holds. But if $V(\xi(t))$ is constant for $\xi = \begin{pmatrix} u \\ u_t \end{pmatrix}$ then we have $u_t = 0$ for all t and hence, from Proposition 3.4, ξ is a rest point.

The proof of the following result contains the key idea of the paper.

Proposition 4.1. *G is asymptotically compact.*

Proof. Consider the functional

$$I(\varphi) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\beta}{2}(u, u_t) + E(u).$$

By Theorem 3.6,

$$\frac{d}{dt}I(\varphi) = -\beta I(\varphi) + H(u),$$

where

$$H(u) = \beta \int_\Omega (F(u) - \frac{1}{2}uf(u))\,dx.$$

Hence

$$\frac{d}{dt}[e^{\beta t}I(\varphi(t))] = e^{\beta t}H(u),$$

and so, given any $M > 0$,

$$I(\varphi(M)) = e^{-\beta M}I(\varphi(0)) + \int_0^M e^{\beta(t-M)}H(u)\,dt. \tag{4.2}$$

Now let $\varphi_j \in G$ with $\varphi_j(0)$ bounded, and let $t_j \to \infty$. From the energy equation $V(\varphi_j(t_j))$ is bounded, and thus (see (3.18)) so is $\varphi_j(t_j)$. Thus we may assume that $\varphi_j(t_j) \rightharpoonup \chi$, and that also $\varphi_j(t_j - M) \rightharpoonup \chi_{-M}$, for some $\chi, \chi_{-M} \in X$. By (C4w) we can further assume that there exists $\bar{\varphi} = \begin{pmatrix} u \\ u_t \end{pmatrix} \in G$ with

$$\varphi_j(t_j + t - M) \rightharpoonup \bar{\varphi}(t),$$

where $\bar{\varphi}(0) = \chi_{-M}$, $\bar{\varphi}(M) = \chi$. We apply (4.2) to $\varphi_j(t_j + t - M) = \begin{pmatrix} u_j \\ u_{jt} \end{pmatrix}(t)$. Thus

$$I(\varphi_j(t_j)) = e^{-\beta M}I(\varphi_j(t_j - M)) + \int_0^M e^{\beta(t-M)}H(u_j)\,dt.$$

Since by the energy equation and our growth hypotheses, $H(u_j)(t)$ is uniformly bounded on $[0, M]$ with $H(u_j)(t) \to H(u)(t)$, we have by (4.2) applied to $\bar{\varphi}$ that

$$\begin{aligned} \lim_{j\to\infty} \int_0^M e^{\beta(t-M)}H(u_j)\,dt &= \int_0^M e^{\beta(t-M)}H(u)\,dt \\ &= I(\bar{\varphi}(M)) - e^{-\beta M}I(\varphi(0)). \end{aligned}$$

Hence

$$\limsup_{j\to\infty} I(\varphi_j(t_j)) \leq Ce^{-\beta M} + I(\bar{\varphi}(M)) - e^{-\beta M}I(\bar{\varphi}(0)).$$

Since I is sequentially weakly lower semicontinuous, letting $M \to \infty$ we deduce that

$$\limsup_{j\to\infty} I(\varphi_j(t_j)) \leq I(\chi) \leq \liminf_{j\to\infty} I(\varphi_j(t_j)),$$

and so $I(\varphi_j(t_j)) \to I(\chi)$. Therefore $\|\varphi_j(t_j)\|_X \to \|\chi\|_X$ and hence $\varphi_j(t_j) \to \chi$ strongly. Thus G is asymptotically compact. □

Lemma 4.2. *The set Z of rest points is bounded in X.*

Proof. If $z = \begin{pmatrix} u \\ 0 \end{pmatrix}$ is a rest point, then by (4.1)

$$\begin{aligned} 0 = \|\nabla u\|^2 + (u, f(u)) &\geq \|\nabla u\|^2 - \lambda \|u\|^2 + c \\ &\geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|\nabla u\|^2 + c, \end{aligned}$$

where $\lambda < \lambda_1$ and c are constants independent of u. □

This completes the verification of the hypotheses of Theorem 2.7, and hence the proof of Theorem 1.1 with the exception of the connectedness of the attractor.

Remark 4.1. If (1.4) is replaced by the weaker condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} \geq -\lambda_1$$

then in general there does not exist a global attractor. For example, let $f(u) = -\lambda_1 u + c$, where $c \neq 0$, and let $\omega_1 \in H_0^1$ be the positive eigenfunction of $-\Delta$ corresponding to the eigenvalue λ_1. Then, for any solution u of (1.1), $g(t) = (u, \omega_1)(t)$ satisfies

$$\ddot{g} + \beta \dot{g} + d = 0, \tag{4.3}$$

where $d = c(1, \omega_1) \neq 0$, and all solutions of (4.3) are unbounded.

4.2. **Existence of an attractor under weaker growth conditions.** We now consider the question of whether a global attractor exists for dimensions $n \geq 3$ for nonlinearities not satisfying the growth condition (1.5). Suppose that f satisfies (1.4) and that

$$|f(u)| \leq c_0(|u|^\gamma + 1), \tag{4.4}$$

where $c_0 > 0$ and $1 \leq \gamma < \infty$. If $\gamma > \frac{n+2}{n-2}$ we assume additionally that

$$F(u) \geq c_1 |u|^{\gamma+1} - c_2, \tag{4.5}$$

where $c_1 > 0$, c_2 are constants. Let $Y_\gamma = H_0^1 \cap L^{\gamma+1}$. Then $Y_\gamma^* = H^{-1} + L^{\frac{\gamma+1}{\gamma}}$. Note that $Y_\gamma = H_0^1$ if $\gamma \leq \frac{n+2}{n-2}$. Set $X_\gamma = Y_\gamma \times L^2$. We now no longer have that $f : H_0^1 \to L^2$, and so the methods of Section 3 do not apply. However, we can still prove the existence of a weak solution to (1.1)-(1.3) using the Galerkin method (see Lions [45, pp1-27]).

Theorem 4.3. *Let f satisfy* (1.4) *and* (4.4)-(4.5). *Given* $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X_\gamma$ *there exists a function u with* $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix} : [0, \infty) \to X_\gamma$ *weakly continuous such that* $u_{tt} : [0, \infty) \to Y_\gamma^*$ *is weakly continuous,* $\varphi(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ *and*

$$(u_{tt}, \psi) + \beta(u_t, \psi) + (\nabla u, \nabla \psi) + (f(u), \psi) = 0, \quad t \in [0, \infty), \tag{4.6}$$

for all $\psi \in Y_\gamma$.

Remark 4.2. In the first term of (4.6) $(\cdot, \cdot)$ is the continuous extension of the L^2 inner product to $Y_\gamma^* \times Y_\gamma$.

Proof. Following Lions [45], and using the hypotheses (1.4), (4.4), (4.5) in the obvious way, we obtain $u \in L^\infty((0,\infty);Y_\gamma)$ with $u_t \in L^\infty((0,\infty);L^2)$, $u_{tt} \in L^\infty((0,\infty);Y_\gamma^*)$ satisfying (4.6) in the sense of distributions on $(0,\infty)$ for every $\psi \in Y_\gamma$. By modifying u on a set of t measure zero (see [45, Lemma 1.2]) we can suppose that $u : [0,\infty) \to L^2$ is continuous, and similarly that $u_t : [0,\infty) \to Y_\gamma^*$ is continuous, with $u(0) = u_0$, $u_t(0) = u_1$. Since $u \in L^\infty((0,\infty);Y_\gamma)$, $u_t \in L^\infty((0,\infty);L^2)$, it follows easily that $\varphi : [0,\infty) \to X_\gamma$ is weakly continuous, and the weak continuity of u_{tt} then follows from (4.6). □

It is not known whether the weak solution given by Theorem 4.3 satisfies the energy equation (3.11). However, we now assume that it does, specifically that the following (unproved) condition holds.

(E) Every weakly continuous solution $\varphi : [0,\infty) \to X_\gamma$ in the sense of (4.6) satisfies the energy equation

$$V(\varphi(t)) + \beta \int_0^t \|u_t(s)\|^2 ds = V(\varphi(0)), \qquad t \in [0,\infty).$$

We also make use of the following further conditions on f.

$$\liminf_{|u|\to\infty} \frac{\frac{1}{2}uf(u) - F(u)}{|u|^{\gamma+1}} \geq 0, \tag{4.7}$$

$$\liminf_{|u|\to\infty} \left(\frac{f(u)}{u} - c_3|u|^{\gamma-1} \right) > -\lambda_1, \tag{4.8}$$

where $c_3 > 0$ is a constant.

We note that the hypotheses (1.4), (4.4), (4.5), (4.7), (4.8) are all satisfied for f of the form

$$f(u) = \kappa|u|^{\gamma-1}u + h(u),$$

where $\kappa > 0$ and h is continuous with $|h(u)| \leq C(|u|^\rho + 1)$ for constants C and $\rho \in [1,\gamma)$.

Theorem 4.4. *Let f satisfy* (1.4) *and* (4.4)-(4.5). *Under the assumption* (E) *the family of weakly continuous solutions $\varphi : [0,\infty) \to X_\gamma$ to* (1.1) *in the sense of* (4.6) *forms a generalized semiflow G on X_γ satisfying the continuity conditions* (C4w) *and* (C4).

Suppose further that if $\gamma = \frac{n+2}{n-2}$ then (4.7) *holds, while if $\gamma > \frac{n+2}{n-2}$ then both* (4.7) *and* (4.8) *hold. Then G possesses a global attractor A in X_γ. For each complete orbit ξ in A the α and ω limit sets of ξ are connected subsets of the set $Z \subset X_\gamma$ of rest points on which V is constant. If Z is totally disconnected then the limits*

$$z_- = \lim_{t\to-\infty} \xi(t), \ z_+ = \lim_{t\to\infty} \xi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point in X_γ as $t \to \infty$ for every solution φ.

Proof. First suppose that $\varphi_j = \begin{pmatrix} u_j \\ u_{jt} \end{pmatrix} \in G$ with $\varphi_j(0) \rightharpoonup z = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$. By the energy equation and (4.5) we have that $\|\varphi_j(t)\|_{X_\gamma} \leq M < \infty$ for all j and $t \geq 0$. A similar argument to that in the proof of Theorem 4.3 shows that there is a subsequence φ_μ such that $\varphi_\mu \stackrel{*}{\rightharpoonup} \varphi$ in $L^\infty((0,\infty);X_\gamma)$ for some $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix} \in G$ with $\varphi(0) = z$. Now let $t_\mu \to t$. Since

$$u_\mu(t_\mu) - u_\mu(0) = \int_0^{t_\mu} u_{\mu t}(s)\, ds$$

we have that

$$u_\mu(t_\mu) \rightharpoonup u(0) + \int_0^t u_t(s)\, ds = u(t)$$

in L^2, and since $u_\mu(t_\mu)$ is bounded in Y_γ this implies that $u_\mu(t_\mu) \rightharpoonup u(t)$ in Y_γ. Similarly, using the equation

$$(u_{\mu t}(t_\mu), v) - (u_{\mu t}(0), v) = \int_0^{t_\mu} (u_{\mu tt}(s), v)\, ds$$

for all $v \in Y_\gamma$, we deduce that $u_{\mu t}(t_\mu) \rightharpoonup u_t(t)$ in L^2. Hence (C4w) holds. Then we obtain (C4) in the same way as in Theorem 3.6.

Next we note that the identity (3.12) holds. For this it suffices to show that

$$(u, u_t)(t_1) - (u, u_t)(t_0) = \int_{t_0}^{t_1} [\|u_t\|^2 + (u_{tt}, u)]\, dt \tag{4.9}$$

for any $0 < t_0 < t_1 < \infty$. Now since for any $T > 0$ we have $u \in C([0,T];Y_\gamma)$, $u_t \in C([0,T]);L^2)$, $u_{tt} \in C([0,T];Y_\gamma^*)$, we can mollify u with respect to t to obtain a sequence $u^{(k)} \in C^2([t_0,t_1];H_0^1)$ with $u^{(k)} \to u$ in $C([t_0,t_1];Y_\gamma)$, $u_t^{(k)} \to u_t$ in $C([t_0,t_1]);L^2)$ and $u_{tt}^{(k)} \to u_{tt}$ in $C([t_0,t_1];Y_\gamma^*)$. Then (4.9) follows by passing to the limit in the same identity for $u^{(k)}$.

With (3.12) in hand, the proof of Proposition 4.1 goes through with a slight modification. In fact the argument only requires that $-H$ and E be sequentially weakly lower semicontinuous on Y_γ. For E this follows from (3.23), Fatou's lemma and the compactness of the embedding of H_0^1 in L^2. If $\gamma < \frac{n+2}{n-2}$ then H is sequentially weakly continuous on H_0^1. If $\gamma \geq \frac{n+2}{n-2}$ then by (4.7) we have for any $\varepsilon > 0$ that

$$\frac{1}{2}uf(u) - F(u) + \varepsilon|u|^{\gamma+1} \geq M(\varepsilon)$$

for some constant $M(\varepsilon)$. Hence if $u_j \rightharpoonup u$ in Y_γ, by Fatou's lemma we have that

$$\begin{aligned} \liminf_{j\to\infty} -H(u_j) + C\varepsilon &\geq \liminf_{j\to\infty} \int_\Omega (\frac{1}{2}u_j f(u_j) - F(u_j) + \varepsilon|u_j|^{\gamma+1})\, dx \\ &\geq -H(u) + \varepsilon\int_\Omega |u|^{\gamma+1}\, dx, \end{aligned}$$

from which the required lower semicontinuity follows by letting $\varepsilon \to 0$.

Finally the proof of Lemma 4.2 is easily modified using (4.8) to show that the set of rest points is bounded in X_γ. Hence the existence and properties of the attractor follow from Theorem 2.7. □

5. **Kneser's property and connectedness of the attractor.** We complete the proof of Theorem 1.1 by showing that Kneser's property holds.

Theorem 5.1. *Suppose that the hypotheses of Theorem* 3.6 *hold. Then* $T(\tau)\{z\}$ *is connected for every* $z \in X$ *and* $\tau \geq 0$.

Proof. We adapt the standard proof of Kneser's theorem for ordinary differential equations (cf. Hartman [32]). Let $n \geq 3$. We claim that that there is a sequence $f_k \in C^1(\mathbf{R})$ with $f_k \to f$ in $C([0,T])$ for any $T > 0$, $\sup_{u\in\mathbf{R}} |f_k'(u)| < \infty$ for each k, and satisfying

$$|f_k(u)| \leq c(|u|^{\frac{n}{n-2}} + 1), \tag{5.1}$$

$$F_k(u) \geq -\frac{\lambda}{2}u^2 + d, \tag{5.2}$$

for constants $c > 0$, $\lambda < \lambda_1$, d independent of k, where $F_k(u) = \int_0^u f_k(r)\,dr$.

To see this, define for $k = 1, 2, ...$

$$f^k(u) = \begin{cases} f(k) & \text{if} \quad u > k, \\ f(u) & \text{if} \quad |u| \leq k, \\ f(-k) & \text{if} \quad u < -k, \end{cases} \tag{5.3}$$

Then $f^k \leq c_0(|u|^{\frac{n}{n-2}} + 1)$, and $F^k(u) = \int_0^u f^k(r)\,dr$ satisfies $F^k(u) \geq -\frac{\lambda}{2}u^2 + d_1$ for some $0 < \lambda < \lambda_1$ and d_1 independent of k. Let ρ_ε, $0 < \varepsilon < 1$, be a mollifier and let $f^{k,\varepsilon} = \rho_\varepsilon * f^k$. Since $f^{k,\varepsilon}(u) = f^k(u)$ for $|u| \geq k+1$, given k there exists a sufficiently small $\varepsilon_k > 0$ such that $\sup_{u\in\mathbf{R}} |f^{k,\varepsilon_k}(u) - f^k(u)| < \frac{1}{k}$. Then $f_k = f^{k,\varepsilon_k}$ has the required properties, since

$$\begin{aligned} F_k(u) &= F^k(u) + \int_0^u (f_k(r) - f^k(r))\,dr \\ &\geq -\frac{\lambda}{2}u^2 + d_1 - \frac{k+1}{k}. \end{aligned}$$

Note that $f_k : H_0^1 \to L^2$ satisfies the global Lipschitz condition

$$\|f_k(p) - f_k(q)\| \leq C_k \|p - q\|_{H_0^1} \qquad \text{for all } p, q \in H_0^1.$$

Note also that if $u \in C([0,T]; H_0^1)$ then $f_k(u) \to f(u)$ in $C([0,T]; L^2)$. To see this let $t_k \to t$ in $[0,T]$. We have that $u(\cdot, t_k) \to u(\cdot, t)$ in $L^{\frac{2n}{n-2}}$, and in particular may assume that $u(x, t_k) \to u(x,t)$ for a.e. $x \in \Omega$. Then

$$\lim_{k\to\infty} \int_\Omega |f_k(u(x,t_k)) - f(u(x,t))|^2\,dx = 0, \tag{5.4}$$

since the integrand is by (5.1) bounded above by a strongly convergent sequence in L^1.

If $n \leq 2$ then (5.2), (5.3), (5.4) still hold, where in the case $n = 2$ we use Lemma 3.2 and Trudinger's inequality as in the proof of Lemma 3.3.

Let $z = \begin{pmatrix} u_o \\ u_1 \end{pmatrix} \in X$ let $\tau > 0$ snd suppose that the compact set $T(\tau)\{z\}$ is not connected. Then $T(\tau)\{z\} = A_1 \cup A_2$ for disjoint compact subsets A_1, A_2 of X. Let U_1, U_2 be disjoint open neighbourhoods of A_1, A_2 respectively and let $\varphi = \begin{pmatrix} u \\ u_t \end{pmatrix}$, $\tilde{\varphi} = \begin{pmatrix} \tilde{u} \\ \tilde{u}_t \end{pmatrix}$ be two solutions in G with $\varphi(0) = \tilde{\varphi}(0) = z$ and $\varphi(\tau) \in U_1$, $\tilde{\varphi}(\tau) \in U_2$. For $\lambda \in [0,1]$ consider the equation

$$w_{tt} + \beta w - \Delta w + f_k(w) = g_k(\lambda, t), \tag{5.5}$$

where

$$g_k(\lambda, t) = \lambda(f_k(u) - f(u)) + (1-\lambda)(f_k(\tilde{u}) - f(\tilde{u})),$$

which we may write in the form

$$\eta(t) = e^{At}z + \int_0^t e^{A(t-s)}(\mathcal{F}_k(\eta(s)) + h_k(s))\,ds, \tag{5.6}$$

where

$$\eta = \begin{pmatrix} w \\ w_t \end{pmatrix}, \quad \mathcal{F}_k(\eta) = \begin{pmatrix} 0 \\ -f_k(w) \end{pmatrix}, \tag{5.7}$$

and

$$h_k(t) = \begin{pmatrix} 0 \\ g_k(\lambda, t) \end{pmatrix}.$$

Since $h_k \in C([0,\tau] : X)$ and $\mathcal{F}_k : X \to X$ is Lipschitz, (5.6) has a unique solution defined on some interval $[0,\tau_1]$, where $\tau_1 > 0$. Furthermore, the argument in the proof of Theorem 3.6 shows that the energy equation

$$V_k(\eta(t)) - V_k(z) = -\beta \int_0^t \|w_t(s)\|^2\,ds + \int_0^t (g_k(\lambda, s), w_t(s))\,ds \tag{5.8}$$

holds on the interval of existence, where

$$V_k(\eta) = \frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|\nabla w\|^2 + \int_\Omega F_k(w)\,dx.$$

From (5.8) we deduce that

$$V_k(\eta(t)) - V_k(z) \le \frac{1}{2\beta}\int_0^t \|g_k(\lambda, s)\|^2 ds,$$

Also it is easily proved that

$$\lim_{k\to\infty} V_k(z) = V(z). \tag{5.9}$$

Hence an argument similar to that in the proof of Theorem 3.6 shows that the solution exists on $[0,\tau]$ and $\|\eta(t)\|_X \le M$ for all $t \in [0,\tau]$, where M is a constant independent of k and λ. Furthermore the solution $\eta(t)$ depends continuously on λ for each $t \in [0,\tau]$. Since by uniqueness $\eta = \varphi$ for $\lambda = 1$ an $\eta = \tilde{\varphi}$ for $\lambda = 0$, it follows that for each k there exists a $\lambda_k \in [0,1]$ such that the corresponding solution η_k of (5.6) satisfies $\eta_k(\tau) \notin U_1 \cup U_2$.

We now let $k \to \infty$. Since the solutions η_k are uniformly bounded in $C([0,\tau]; X)$, we can argue as in the proof of Theorem 3.6, using the fact that $g_k(\lambda_k, \cdot) \to 0$ in $C([0,\tau]; L^2)$, to deduce that there exists a subsequence η_μ such that $\eta_\mu(t) \rightharpoonup \eta_\infty(t)$ for each $t \in [0,\tau]$ for some $\eta_\infty \in G$ with $\eta_\infty(0) = z$. Since

$$\lim_{k\to\infty}\left(V_k(\eta_k(\tau)) - V_k(z) + \beta\int_0^\tau \|w_{kt}\|^2 ds\right) = V(\eta_\infty(\tau)) - V(z) + \beta\int_0^\tau \|w_{\infty t}\|^2 ds,$$

and since (5.9) holds, it follows in the now familiar way that in fact $\eta_\mu(\tau) \to \eta_\infty(\tau)$ strongly in X, so that $\eta_\infty(\tau) \notin U_1 \cup U_2$ and hence $\eta_\infty(\tau) \notin T(\tau)\{z\}$, a contradiction. □

Remark 5.1. It is not obvious how to extend this result to f satisfying weaker growth hypotheses under the *a priori* assumption (E), in particular because it is not clear how to define the last integral in (5.8).

Remark 5.2. Presumably nonunique solutions to (1.1), (1.2) abound under our growth hypotheses, even for smooth f behaving sufficiently irregularly at infinity. However it does not seem obvious how to give an example for the boundary conditions (1.2). For Neumann boundary conditions examples of nonuniqueness can easily be constructed using x-independent solutions satisfying the ordinary differential equation $\ddot{u} + \beta\dot{u} + f(u) = 0$ for suitable non-Lipschitz f and initial data $u(0) = \dot{u}(0) = 0$. For example, if we let $f(u) = -6u^{\frac{1}{3}} - 3\beta u^{\frac{2}{3}}$ for $0 \leq u \leq 1$ then we have the solutions $u(t) = 0$ and $\tilde{u}(t) = t^3$ for $t > 0$ sufficiently small.

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