PACIFIC JOURNAL OF MATHEMATICS
Vol. 116, No 1, 1985

REMARKS ON THE PAPER 'BASIC CALCULUS OF VARIATIONS'

J. M. BALL

We show that a condition studied in E. Silverman's paper is not, as claimed, necessary for lower semicontinuity of multiple integrals in the calculus of variations.

The purpose of this note is to show that a condition studied in [7] is not, as claimed, a necessary condition for lower semicontinuity of multiple integrals in the calculus of variations. To keep things simple we consider integrals of the form

$$I_F(y) = \int_G F(y'(x))\,dx,$$

where $G \subset \mathbf{R}^k$ is a bounded domain, $y\colon G \to \mathbf{R}^N$, $y'(x) = (\partial y^i/\partial x^\alpha)$, and $F\colon M^{N\times k} \to \mathbf{R}$ is continuous. Here $M^{N\times k}$ denotes the linear space of real $N \times k$ matrices. We suppose throughout that $K \geq 2$, $N \geq 2$. In [7] F is called *T-convex* if there exists a convex function f, defined on $\mathbf{R}^r$, $r = \binom{N+k}{k} - 1$, such that

$$F(p) = f(\tau(p)) \quad \text{for all}\, p \in M^{N\times k},$$

where $\tau(p)$ denotes the minors of p of all orders j, $1 \leq j \leq \min(k, N)$, arranged in some prescribed order. T-convexity of F was studied in [**1, 2, 3**] under a different name, *polyconvexity*, which we shall use in the remainder of this note, and it is equivalent to a condition introduced earlier by Morrey [**4**, p. 49]. (These papers contain lower semicontinuity and existence theorems for polyconvex integrands of the same type as given in [**7**, §§4–7].) Let us say that I_F is lsc if $I_F(y) \leq \liminf_{j\to\infty} I_F(y_j)$ whenever $y_j \to y$ uniformly on G with $\sup_{x,\bar{x}\in G}|y_j(x) - y_j(\bar{x})| \leq C < \infty$ for all j. (Equivalently, if G has sufficiently regular boundary then I_F is lsc if and only if I_F is sequentially weak* lower semicontinuous on the Sobolev space $W^{1,\infty}(G; \mathbf{R}^n)$.) A consequence of [**7**, Theorem 3.6] is that I_F lsc implies F polyconvex; that this conclusion is false was pointed out implicitly by Morrey [**4**, p. 26]. Morrey's remark is based on an example due to Terpstra [**8**] of a quadratic form

$$Q(p) = \sum_{\substack{1\leq i,j\leq N\\ 1\leq \alpha,\beta\leq k}} a_{i\alpha j\beta}\, p_{i\alpha}\, p_{j\beta}$$

having constant coefficients $a_{i\alpha j\beta}$ and with the properties

(i) (rank 1 convexity) $Q(\lambda \otimes \mu) \geq 0$ for all $\lambda \in \mathbf{R}^N$, $\mu \in \mathbf{R}^k$,

(ii) there is no linear combination $\tilde{Q}(p)$ of 2×2 minors of p such that

$$Q(p) \geq \tilde{Q}(p) \quad \text{for all } p \in M^{N\times k}.$$

Terpstra showed that such quadratic forms exist if and only if $k \geq 3$ and $N \geq 3$. By Morrey [4, Theorem 5.2] I_Q is lsc if and only if Q satisfies (i). But if Q satisfies (ii) then Q is not polyconvex; more generally, we have the following proposition.

PROPOSITION. *Let $F(p) = Q(p)$ in a neighbourhood of $p = 0$. If Q satisfies* (ii) *then F is not polyconvex.*

Proof. Suppose F is polyconvex. By the convexity of f there exists $\theta \in \mathbf{R}^r$ such that

$$F(p) = f(\tau(p)) \geq f(0) + \langle \theta, \tau(p) \rangle \quad \text{for all } p \in M^{N\times k}.$$

We write $\langle \theta, \tau(p) \rangle = \sum_{j=1}^{\min(k,N)} \tilde{Q}_j(p)$, where each $\tilde{Q}_j(p)$ is a linear combination of $j \times j$ minors of p. Note that $F(0) = f(0) = 0$. For any p and for $|t|$ sufficiently small we thus have

$$F(tp) = t^2 Q(p) \geq \sum_{j=1}^{\min(k,N)} t^j \tilde{Q}_j(p).$$

Dividing by $|t|$ and letting $t \to 0$ we see that $\tilde{Q}_1(p) \equiv 0$. Dividing by t^2 and letting $t \to 0$ we obtain $Q(p) \geq \tilde{Q}_2(p)$, contradicting (ii). □

Of course any Q satisfying (i) and (ii) is not bounded below. However, applying the proposition to $F(p) = \max\{-1, Q(p)\}$ we see that if Q satisfies (i), (ii) then $G(p) = \max\{0, 1 + Q(p)\}$ is nonnegative, I_G is lsc (it is the maximum of two lsc functionals), but G is not polyconvex.

The proof of Theorem 3.6 in [7] consists of first showing (Lemma 3.4, Corollary 3.5) that I_F lsc implies F polyconvex in the special case when $N \geq k$ and F depends only on minors of maximal order k. This part of the proof does not appear to be complete. The general case is then reduced to the special one by adjoining new variables $\xi\colon G \to \mathbf{R}^k$ such that

$$F(y') = h\left(\tau\begin{pmatrix} \xi' \\ y' \end{pmatrix}\right)$$

for some function h depending only on kth order minors of the $(N + k) \times k$ matrix $\binom{\xi'}{y'}$; however, such a function h does not in general exist, since all kth order minors of $\binom{\xi'}{y'}$ can be zero without determining y'.

The example of Terpstra is neither explicit nor elementary, and being written in German is inaccessible to some. Recently D. Serre [5, 6] has provided an explicit example, namely

$$Q_\varepsilon(p) = H(p) - \varepsilon \sum_{i,\alpha=1}^{3} (p_{i\alpha})^2,$$

$$H(p) = (p_{11} - p_{23} - p_{32})^2 + (p_{12} - p_{31} + p_{13})^2 + (p_{21} - p_{13} - p_{31})^2 + (p_{22})^2 + (p_{33})^2,$$

where $N = k = 3$ and $\varepsilon > 0$ is sufficiently small. To keep this note self-contained we now give a direct proof, following Serre [6], that Q_ε satisfies (i) and (ii). First we note that $H(\lambda \otimes \mu) = 0$ implies that

$$\lambda_1\mu_1 - \lambda_2\mu_3 - \lambda_3\mu_2 = \lambda_1\mu_2 - \lambda_3\mu_1 + \lambda_1\mu_3 = \lambda_2\mu_1 - \lambda_1\mu_3 - \lambda_3\mu_1 = \lambda_2\mu_2 = \lambda_3\mu_3 = 0,$$

and hence that $\lambda = 0$ or $\mu = 0$. Thus $\inf_{|\lambda|=|\mu|=1} H(\lambda \otimes \mu) \overset{\text{def}}{=} \varepsilon_0$ is positive and (i) follows for $\varepsilon \leq \varepsilon_0$. Suppose for contradiction that

$$Q_\varepsilon(p) \geq \tilde{Q}(p) = -\sum_{1 \leq i, \alpha \leq 3} A_{i\alpha}(\operatorname{adj} p)_{i\alpha} \quad \text{for all } p,$$

where $A \in M^{3\times 3}$ is constant. Consider p of the form

$$p = \begin{pmatrix} b+d & a-c & c \\ a+c & 0 & d \\ a & b & 0 \end{pmatrix},$$

so that

$$\operatorname{adj} p = \begin{pmatrix} -bd & bc & d(a-c) \\ ad & -ac & c(a+c) - d(b+d) \\ b(a+c) & a(a-c) - b(b+d) & c^2 - a^2 \end{pmatrix}.$$

For such p we have $H(p) = 0$ and thus

$$\sum_{1 \leq i, \alpha \leq 3} A_{i\alpha}(\operatorname{adj} p)_{i\alpha} - \varepsilon(a^2 + b^2 + c^2 + d^2) \geq 0.$$

The left-hand side is a quadratic form in a, b, c, d given explicitly by

$$a^2(A_{32} - A_{33} - \varepsilon) + b^2(-A_{32} - \varepsilon) + c^2(A_{23} + A_{33} - \varepsilon) + d^2(-A_{23} - \varepsilon) + (\text{terms in } ab, ac, ad, bc, bd, cd).$$

For this sum to be nonnegative the coefficients of a^2, b^2, c^2, d^2 must be nonnegative. But the sum of these coefficients is -4ε, a contradiction.

Acknowledgment. I am grateful to R. V. Kohn and D. Serre for their comments.

References

[1] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal., **63** (1977), 337–403.

[2] _____, *On the calculus of Variations and Sequentially Weakly Continuous Maps*, in "*Ordinary and Partial Differential Equations*" (W. N. Everitt and B. D. Sleeman, Eds.), Lecture Notes in Mathematics, Vol. 564, pp. 13–25, Springer, Berlin/New York, 1976.

[3] J. M. Ball, J. C. Currie and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Functional Anal., **41** (1981), 135–174.

[4] C. B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math., **2** (1952), 25–53.

[5] D. Serre, *Condition de Legendre-Hadamard; espaces de matrices des rang* $\neq 1$, C. R. Acad. Sci. Paris Sér I Math., **293** (1981), 23–26.

[6] _____, *Formes quadratiques et calcul des variations*, Journal de Mathématiques Pures et Appliquées, **62** (1983), 177–196.

[7] E. Silverman, *Basic calculus of variations*, Pacific J. Math., **104** (1983), 471–482.

[8] F. J. Terpstra, *Die darstellung biquadratischer formen als summen von quadraten mit anwendung auf die variationsrechnung*, Math. Ann., **116** (1938), 166–180.

Received April 29, 1983 and in revised form September 9, 1983. Research supported by a U. K. Science and Engineering Research Council Senior Fellowship and by the Mathematical Sciences Research Institute, Berkeley, California.

Heriot-Watt University
Edinburgh, EH 14 4AS
Scotland