International School of Liquid Crystals
Erice, 7-10 October 2018

# An exterior problem for liquid crystals

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# Oseen-Frank theory

Free energy $I(\mathbf{n}) = \int_\Omega W(\mathbf{n}, \nabla\mathbf{n})\, d\mathbf{x}$,

$$2W(\mathbf{n},\nabla\mathbf{n}) = K_1(\mathrm{div}\,\mathbf{n})^2 + K_2(\mathbf{n}\cdot\mathrm{curl}\,\mathbf{n})^2 + K_3|\mathbf{n}\wedge\mathrm{curl}\,\mathbf{n}|^2 \\ +(K_2+K_4)(\mathrm{tr}(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2).$$

$\mathbf{n}(\mathbf{x}) \in S^2$ (unit sphere) is the *director.*

2D case: $\mathbf{n}(\mathbf{x}) = (n_1(x_1,x_2), n_2(x_1,x_2), 0)$

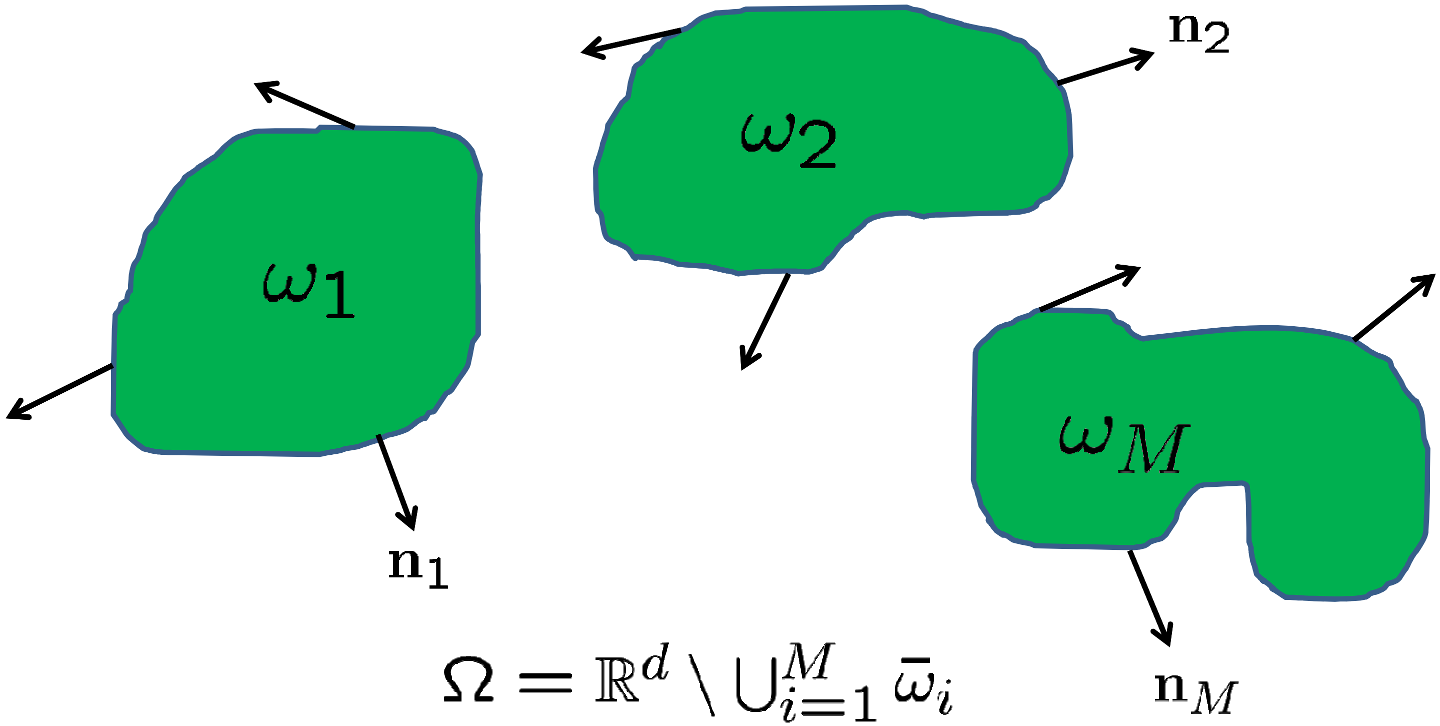
$\mathrm{curl}\,\mathbf{n} = (0, 0, n_{2,1} - n_{1,2}),\ \mathbf{n}\cdot\mathrm{curl}\,\mathbf{n} = 0$

$|\mathbf{n}\wedge\mathrm{curl}\,\mathbf{n}|^2 = |\mathrm{curl}\,\mathbf{n}|^2 = (n_{2,1} - n_{1,2})^2$

$\mathrm{tr}\,(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2 = 0$

# Exterior problem for Oseen-Frank theory

$\omega_i \subset \mathbb{R}^d$ disjoint, bounded, open, with sufficiently smooth boundaries $\partial\omega_i$.




$$\Omega = \mathbb{R}^d \setminus \cup_{i=1}^{M} \bar{\omega}_i$$

What can we say about equilibrium configurations $\mathbf{n}$ in $\Omega$ satisfying the boundary conditions $\mathbf{n}|_{\partial\omega_i} = \mathbf{n}_i$, $i = 1, \ldots, M$, in particular about their behaviour at $\infty$?

It would be nice to be able to handle the case $d = 3$ for general elastic constants and allow the $\omega_i$ to move. However we will assume that the $\omega_i$ are fixed, that $d = 2$, and that we are in the one-constant case

$$K_1 = K_2 = K_3 = 2,\ K_4 = 0.$$

Why is the one-constant $d = 2$ case so much simpler?

First, we can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, writing $z = x_1 + ix_2$ for $\mathbf{x} = (x_1, x_2)$, and identifying $\mathbf{n}(\mathbf{x})$ with

$$\tilde{\mathbf{n}}(z) = n_1(x_1, x_2) + in_2(x_1, x_2) = e^{i\Phi(\mathbf{x})}.$$

The Euler-Lagrange equation for the one-constant energy

$$I(\mathbf{n}) = \int_\Omega |\nabla \mathbf{n}|^2 \, d\mathbf{x},$$

$\mathbf{n} : \Omega \to S^2$, is the *harmonic map* equation

$$\Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} = \mathbf{0},$$

which is equivalent to

$$\Delta \tilde{\mathbf{n}} + |\nabla \mathbf{n}|^2 \tilde{\mathbf{n}} = i e^{i\Phi} \Delta \Phi = \mathbf{0},$$

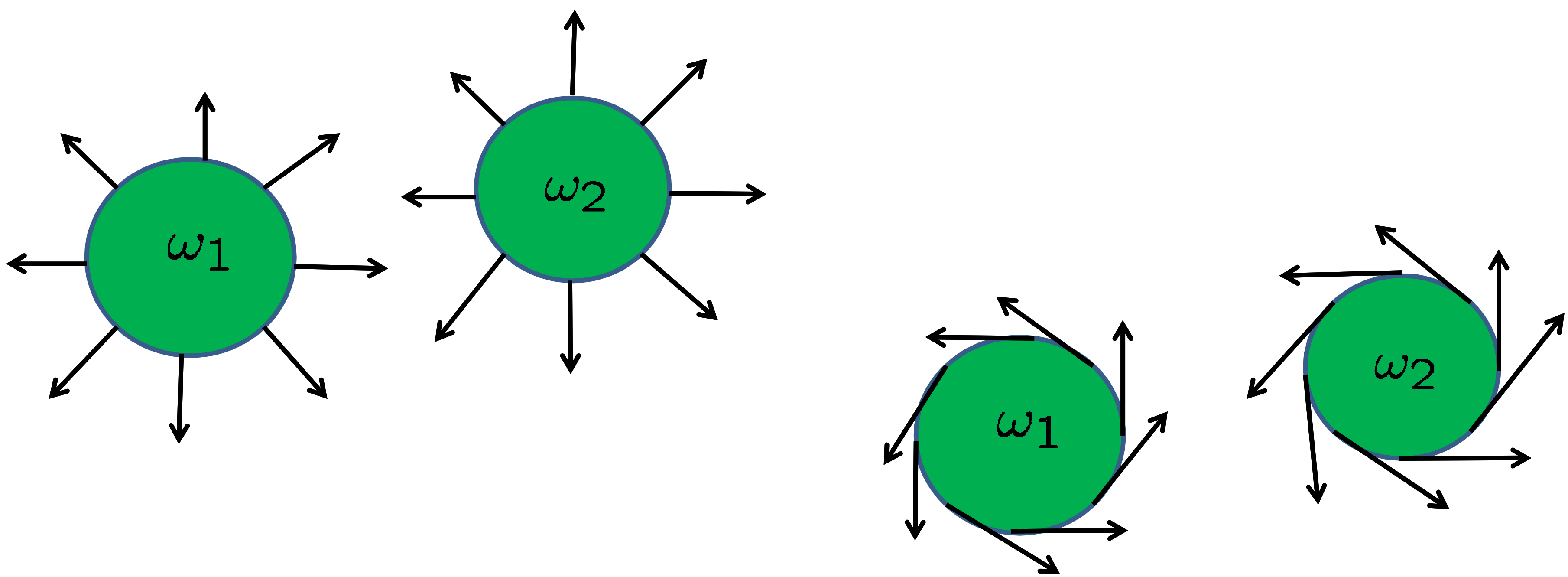
that is to the linear Laplace equation

$$\Delta \Phi = 0.$$

In particular, equilibrium solutions of locally finite energy are smooth (no defects).

Also, if $\tilde{\mathbf{n}} = \tilde{\mathbf{n}}(z), \tilde{\mathbf{m}} = \tilde{\mathbf{m}}(z)$ are equilibrium solutions so is $\tilde{\mathbf{n}}(z)\tilde{\mathbf{m}}(z)$ as a product of complex numbers.

One interesting consequence is that any constant rotation, or rotation plus reflection, of an equilibrium is also an equilibrium.



In polar coordinates $(r, \theta)$ Laplace's equation $\Delta\Phi = 0$ becomes

$$\Phi_{ss} + \Phi_{\theta\theta} = 0,$$

where $s = \ln r$.

One solution is $\Phi = \theta$, giving the 2D hedgehog

$$e^{i\theta} = \frac{z}{|z|},$$

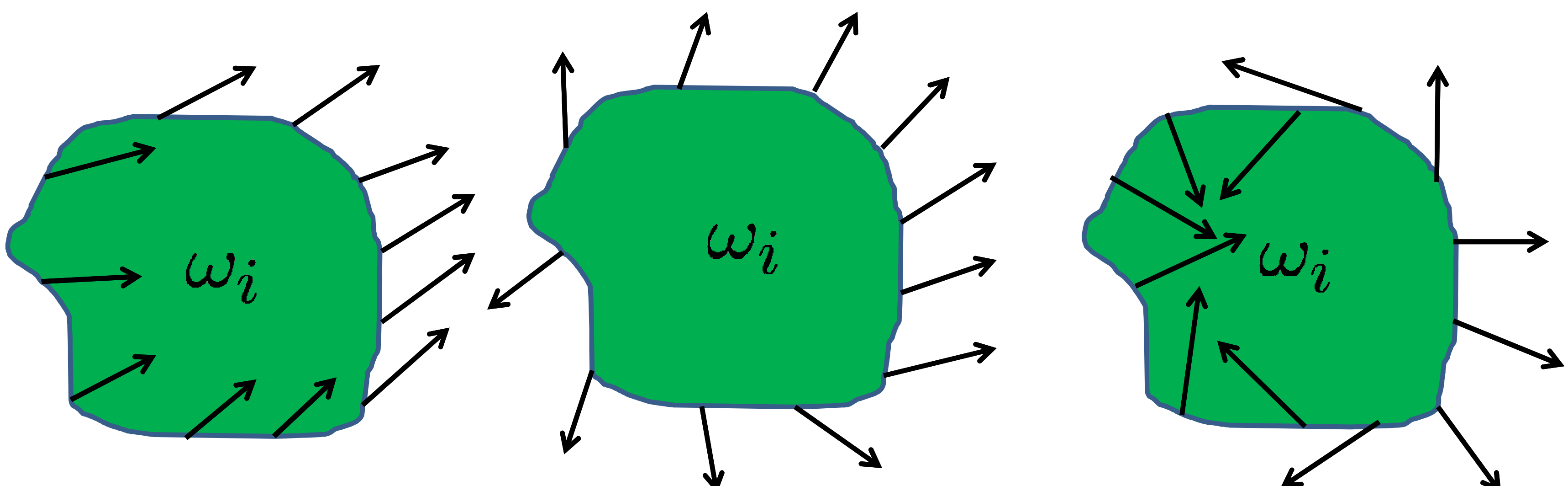
which has finite energy only for bounded domains not containing the origin.

Note that this solution, while solving Laplace's equation locally, is not a globally defined smooth solution due to the jump of $2\pi$ in $\theta$ as we circle the origin. How can this be remedied?

# Definition of degree

If we smoothly parametrize $\partial\omega_i$ by $t \in [0, 2\pi)$, then can write $\mathbf{n}_i(t) = (\cos\Phi(t), \sin\Phi(t))$ with $\Phi$ smooth and define

$$\begin{aligned} \deg \mathbf{n}_i &= \frac{1}{2\pi}(\Phi(2\pi) - \Phi(0)) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Phi'(t)\, dt. \end{aligned}$$




$\deg \mathbf{n}_i = 0$ $\deg \mathbf{n}_i = 1$ $\deg \mathbf{n}_i = 2$

**Carbou's trick.** Pick $\mathbf{a}_i \in \omega_i$, $i = 1, \ldots, M$.
(For convenience we choose $\mathbf{a}_1 = \mathbf{0}$ to be the origin of polar coordinates.)

Let $d_i = \deg \mathbf{n}_i$. Then we can write any equilibrium solution $\tilde{\mathbf{n}}$ as

$$\tilde{\mathbf{n}}(z) = \left(\frac{z - \mathbf{a}_i}{|z - \mathbf{a}_i|}\right)^{d_1} \cdots \left(\frac{z - \mathbf{a}_M}{|z - \mathbf{a}_M|}\right)^{d_M} e^{i\varphi(z)},$$

where $\varphi$ is a smooth solution of $\Delta\varphi = 0$ in $\Omega$.

Rough proof:

$$\tilde{\mathbf{n}}(z) \left(\frac{z - \mathbf{a}_i}{|z - \mathbf{a}_i|}\right)^{-d_1} \cdots \left(\frac{z - \mathbf{a}_M}{|z - \mathbf{a}_M|}\right)^{-d_M}$$

is an equilibrium solution of degree zero.

Let $B_R = B(\mathbf{0}, R)$, so that $\cup_{i=1}^M \bar{\omega}_i \subset B_{R_0}$ for some $R_0$, and

$$X = \{\mathbf{n} : \Omega \to S^2 : \int_{\Omega \cap B_R} |\nabla \mathbf{n}|^2 \, d\mathbf{x} < \infty \text{ for all } R > R_0, \mathbf{n}|_{\partial\omega_i} = \mathbf{n}_i\}.$$

Let $k = \sum_{i=1}^M d_i$, where $d_i = \deg \mathbf{n}_i$.

$X$ can be split into countably many *homotopy classes* (topologically distinct families of $\mathbf{n}$, so that director fields in two different homotopy classes cannot be continuously deformed one to the other). The different homotopy classes correspond to adding different multiples of $2\pi$ to the boundary values for $\varphi$ on each $\partial\omega_i$.

**Theorem.** Let $k = \sum_{i=1}^{M} \deg \mathbf{n}_i$. There is a unique minimizer $\mathbf{n}_C$ of

$$E(\mathbf{n}) = \int_{\Omega} \left( |\nabla \mathbf{n}|^2 - \frac{k^2}{|\mathbf{x}|^2} \right) d\mathbf{x},$$

in each homotopy class $C$, $\mathbf{n}_C : \bar{\Omega} \to S^1$ is a smooth harmonic map and

$$|\mathbf{n}_C(\mathbf{x}) - \mathbf{n}_C^{\infty}(\mathbf{x})| \leq \frac{C_0}{r},$$

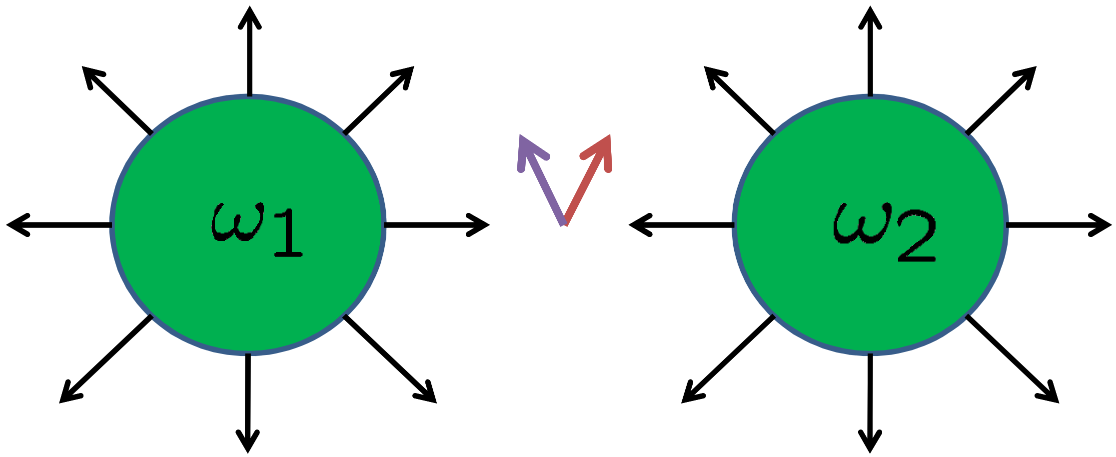
for some constant $C_0 > 0$, where

$$\mathbf{n}_C^{\infty}(\mathbf{x}) = (\cos(k\theta + \beta_C), \sin(k\theta + \beta_C))$$

and $\beta_C \in \mathbb{R}$. In each homotopy class $C$ there is also a harmonic map $\hat{\mathbf{n}}_C$ with $E(\hat{\mathbf{n}}_C) = +\infty$.

Furthermore $E$ attains a minimum $\mathbf{n}^*$ in $X$, but $\mathbf{n}^*$ is not in general unique.

Counterexample for uniqueness of $\mathbf{n}^*$.




Remarks

1. The nonuniqueness in this example might go away if we formulated everything in terms of *line-fields* $\mathbf{n} \otimes \mathbf{n}$. These can be handled by the trick of squaring $\tilde{\mathbf{n}}$, i.e. looking at the director field $\tilde{\mathbf{n}}^2$.

2. In the two-constant $d = 2$ case there can be radial solutions that are periodic in $\ln r$, so the asymptotic behaviour is much more complicated.