MIGSAA course 2020

Hyperbolic equations

John Ball

Heriot-Watt University and

Maxwell Institute for Mathematical Sciences

jb101@hw.ac.uk

Plan of course

Lectures 1-8 The linear wave equation
Lectures 9-14 Semilinear hyperbolic equations
Lectures 15-20 (Pieter Blue)

Reading material

L. Bers, F. John, M. Schechter, *Partial Differential Equations*, AMS, 1964.

A. Bressan, *Hyperbolic Systems of Conservation Laws*, OUP, 2000.

T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, OUP, 1998.

C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd edition, Springer, 2010.

L.C. Evans, *Partial Differential Equations*, 2nd edition, AMS, 2010.

G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, 1976.

J.K. Hale, *Asymptotic behavior of dissipative systems*, AMS, 1988.

S. Helgason, *The Radon transform*, 2nd edition, Springer, 1999.

D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 1981.

E. Hille and R. Phillips, *Functional Analysis and Semi-groups*, AMS, 1957.

M. Ikawa, *Hyperbolic Partial Differential Equations and Wave Phenomena*, AMS, 2000.
F. John, *Partial Differential Equations*, 4th edition, Springer, 1982.
O. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, CUP, 1991.
P. Lax, *Hyperbolic Partial Differential Equations*, AMS, 2006
J-L. Lions, *Quelques méthodes de resolution des problèmes aux limites nonlinéaires*, Dunod, 1969.
B. Makarov and A. Podkorytov, *Real Analysis: Measures, Integrals and Applications*, Springer, 2013.
A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.
W. Strauss, *Partial Differential Equations: an introduction*, Wiley, 1992.
F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, 1975.
K. Yosida, *Functional Analysis*, Springer, 1980.

The linear wave equation

In the scalar case $u = u(x,t) \in \mathbb{R}$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ open, $t \in \mathbb{R}$, this is the equation

$$u_{tt} = c^2 \Delta u,$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, and $c > 0$ is the constant *wave speed*.

(If $\mathbf{u}(x,t)$ is a vector solution of $\mathbf{u}_{tt} = c^2 \Delta \mathbf{u}$ then each component u_i of $\mathbf{u}$ satisfies the scalar equation.)

By rescaling t (letting $\tau = ct$) we may assume that $c = 1$. Hence from now on we consider the equation

$$u_{tt} = \Delta u.$$

We seek solutions satisfying *initial conditions*

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),$$

and suitable boundary conditions (e.g. $u|_{\partial\Omega} = 0$ if Ω is bounded) or decay conditions as $|x| \to \infty$.

Sample applications:

$n = 3$ electromagnetic,acoustic and seismic waves

$n = 2$ transverse waves on a membrane

$n = 1$ vibrating string

There are many more complicated linear and non-linear models of wave phenomena (e.g. surface waves on water) for which the linear wave equation is a prototype.

In general, the way to solve linear PDE is to give a formula for solutions.

Recall how this is done for the 1D wave equation

$$u_{tt} = u_{xx}, \quad x \in \Omega = \mathbb{R},$$

with initial conditions $u(x, 0) = u_0(x), u_t(x, 0) = u_1(x)$.

Assuming u is a C^2 function of x and t, change variables to $\xi = x + t, \eta = x - t$ (the lines $x \pm t =$const. giving the *characteristics* of the wave equation). Then the equation becomes $u_{\xi\eta} = 0$ with general solution $u = f(\xi) + g(\eta)$ for arbitrary functions f, g.

Applying the initial data we have

$$u_0(x) = f(x) + g(x),\ u_1(x) = f'(x) - g'(x),$$

from which we obtain *d'Alembert's solution*

$$u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)ds.$$

Note that $u(x,t)$ is uniquely determined by the values of the initial data in the interval $[x-t, x+t]$, showing that there is a finite speed of propagation of disturbances.

This solution is meaningful even when u_0, u_1 are not smooth, suggesting that a weaker definition of solution may be possible and relevant for rough initial data.

In order to define such weak solutions, recall that for $E \subset \mathbb{R}^s$ open $L^1_{\text{loc}}(E)$ denotes the set of measurable functions $v : E \to \mathbb{R}$ such that $v \in L^1(A)$ for all compact $A \subset E$.

A function $u \in L^1_{\text{loc}}(E)$ has a *weak derivative* $\frac{\partial u}{\partial x_i} \in L^1_{\text{loc}}(E)$ if

$$\int_E u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_E \frac{\partial u}{\partial x_i} \varphi \, dx \text{ for all } \varphi \in C_0^\infty(E).$$

If the weak derivative $\frac{\partial u}{\partial x_i}$ exists then it is unique.

Denote by $W^{1,1}_{\text{loc}}(E)$ the set of $u \in L^1_{\text{loc}}(E)$ with weak derivatives $\frac{\partial u}{\partial x_i} \in L^1_{\text{loc}}(E)$ for $i = 1, \ldots, s$.

We consider for $T > 0$ the wave equation

$$u_{tt} = \Delta u, \quad (x,t) \in \mathbb{R}^n \times [0,T], \qquad (1)$$

with initial data

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad (2)$$

where $u_0 \in W^{1,1}_{\text{loc}}(\mathbb{R}^n), u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Definition. A function $u = u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ is a *weak solution* of (1), (2) if $u, u_t \in C([0,T]; L^1(B(0,R)))$ for any $R > 0$, with $u(\cdot,0) = u_0(\cdot), u_t(\cdot,0) = u_1(\cdot)$, and for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have that

$$(u,\varphi)(t) := \int_{\mathbb{R}^n} u(x,t)\varphi(x)\,dx \in C^2([0,T])$$

with

$$\frac{d^2}{dt^2}(u,\varphi) = (u, \Delta\varphi) \text{ for } t \in [0,T].$$

Let us check that $u(x,t)$ given by d'Alembert's formula is indeed a weak solution.

We can write d'Alembert's formula as

$$u(x,t) = f(x+t) + g(x-t),$$

where

$$f(\tau) = \frac{1}{2}\left(u_0(\tau) + \int_0^\tau u_1(s)\,ds\right),$$

$$g(\tau) = \frac{1}{2}\left(u_0(\tau) - \int_0^\tau u_1(s)\,ds\right).$$

Then $f, g \in W^{1,1}_{\text{loc}}(\mathbb{R})$ with weak derivatives $f' = \frac{1}{2}(u_{0x} + u_1)$, $g' = \frac{1}{2}(u_{0x} - u_1) \in L^1_{\text{loc}}(\mathbb{R})$ respectively, and

$$u_t(x,t) = \frac{1}{2}(f'(x+t) - g'(x-t)).$$

By the continuity of translates in L^1 we have that $u, u_t : [0,T] \to L^1(-R,R)$ are continuous for any R with the correct initial values.

Finally, let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}(u,\varphi)(t) &= \int_{\mathbb{R}} u(x,t)\varphi(x)\,dx \\ &= \int_{\mathbb{R}} (f(x)\varphi(x-t) + g(x)\varphi(x+t))\,dx,\end{aligned}$$

is smooth in t with second time derivative

$$\begin{aligned}\frac{d^2}{dt^2}(u,\varphi) &= \int_{\mathbb{R}} (f(x)\varphi_{xx}(x-t) + g(x)\varphi_{xx}(x+t))\,dx, \\ &= (u,\varphi_{xx})\end{aligned}$$

as required.

But is this the unique weak solution?

To prove this we have to show that if u is a weak solution with initial data $u_0 = u_1 = 0$ then $u = 0$.

We no longer have the regularity to use the characteristics argument, so what to do?

We will smooth the solution so that we can use characteristics.

Let $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, be a mollifier, i.e. $\rho_\varepsilon \geq 0$, $\operatorname{supp} \rho_\varepsilon \subset\subset B(0, \varepsilon)$, $\int_{\mathbf{R}^n} \rho_\varepsilon(x)\, dx = 1$.

Given a weak solution u with zero initial data let $u_\varepsilon = \rho_\varepsilon * u$, i.e.

$$u_\varepsilon(x,t) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) u(y,t)\, dy.$$

We claim that u_ε is a C^2 solution of the wave equation with zero initial data. In the case $n = 1$ the d'Alembert formula implies that $u_\varepsilon = 0$. But $u_\varepsilon(\cdot, t) \to u(\cdot, t)$ in $L^1(B(0,R))$ as $\varepsilon \to 0$, so that $u = 0$.

Note first that, for each t, u_ε is smooth in x. Hence

$$\begin{aligned}\Delta u_\varepsilon(x,t) &= \int_{\mathbb{R}^n} \Delta_x \rho_\varepsilon(x-y)u(y,t)\,dy \\ &= \int_{\mathbb{R}^n} \Delta_y \rho_\varepsilon(x-y)u(y,t)\,dy \\ &= \frac{d^2}{dt^2}\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)u(y,t)\,dy \\ &= \frac{\partial^2}{\partial t^2}u_\varepsilon(x,t),\end{aligned}$$

so that u_ε satisfies the wave equation.

To check the regularity, first note that $u_\varepsilon(x,t)$ is continuous in (x,t) since $u \in C([0,T];L^1(B(0,R)))$ for every R. The same argument shows that both $u_{\varepsilon t}$ and any x-derivative of u_ε are continuous in (x,t). Finally, the formula for Δu_ε shows that $\frac{\partial^2 u_\varepsilon}{\partial t^2}$ is also continuous in x,t, so that u_ε is C^2.

Solving the wave equation in $\mathbb{R}^n, n > 1$.

Again we want to solve

$$u_{tt} = \Delta u, \; x \in \mathbb{R}^n, t \geq 0,$$

with initial conditions

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x).$$

Assuming $u \in C^2$ we try to find an explicit solution using Poisson's method of spherical means.

The spherical mean of a function $u(x), x \in \mathbb{R}^n$, is its average over the sphere $S(x,r)$ with centre x and radius $r > 0$:

$$\begin{aligned} M_u(x,r) &= \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} u(y)\, dS_y \\ &= \frac{1}{\omega_n} \int_{S^{n-1}} u(x + r\xi)\, dS_\xi, \end{aligned}$$

where $\omega_n = \mathcal{H}^{n-1}(S^{n-1})$.

The second line above is meaningful for all $r \in \mathbb{R}$, and not just for $r > 0$, so we use it to extend the definition to all r. Note that then $M_u(x,0) = u(x)$ and $M_u(x,r) = M_u(x,-r)$.

Theorem 1 For $u \in C^2$ the spherical mean satisfies the Darboux equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_u(x,r) = \Delta_x M_u(x,r).$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial r} M_u(x,r) &= \frac{1}{\omega_n} \int_{S^{n-1}} \frac{\partial u}{\partial x_i}(x + r\xi)\xi_i \, dS \\
&= \frac{r}{\omega_n} \int_{B(0,1)} \Delta_x u(x + r\xi) \, d\xi \\
&= \frac{r^{1-n}}{\omega_n} \Delta_x \int_{B(x,r)} u(y) \, dy,
\end{aligned}$$

where in the second line we used the divergence theorem.

Hence

$$\frac{\partial}{\partial r} M_u(x,r) = \frac{r^{1-n}}{\omega_n} \Delta_x \int_0^r \int_{S(x,\rho)} u(y)\, dS_y\, d\rho$$
$$= r^{1-n} \Delta_x \int_0^r \rho^{n-1} M_u(x,\rho)\, d\rho,$$

and so

$$\frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r} M_u(x,r) \right) = \Delta_x r^{n-1} M_u(x,r),$$

giving the result. □

Now let $u = u(x,t), x \in \mathbb{R}^n, t \geq 0$ be a C^2 solution of the wave equation and form the spherical mean with respect to x

$$M_u(x,r,t) = \frac{1}{\omega_n} \int_{S^{n-1}} u(x + r\xi, t)\, dS_\xi.$$

Then

$$\begin{aligned} \Delta_x M_u &= \frac{1}{\omega_n} \int_{S^{n-1}} \Delta_x u(x + r\xi, t)\, dS_\xi \\ &= \frac{\partial^2}{\partial t^2} \frac{1}{\omega_n} \int_{S^{n-1}} u(x + r\xi, t)\, dS_\xi \\ &= \frac{\partial^2}{\partial t^2} M_u. \end{aligned}$$

Hence, by the Darboux equation, we find that M_u satisfies the *Euler-Poisson-Darboux equation*

$$\frac{\partial^2}{\partial t^2} M_u = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_u,$$

which we need to solve with the initial data

$$M_u = M_{u_0}(x, r), \;\; \frac{\partial}{\partial t} M_u = M_{u_1}(x, r)$$

at $t = 0$.

The easiest case is when $n = 3$, when we obtain

$$\frac{\partial^2}{\partial t^2}(rM_u) = \left(r\frac{\partial^2}{\partial r^2} M_u + 2\frac{\partial}{\partial r} M_u \right) = \frac{\partial^2}{\partial r^2}(rM_u).$$

Hence rM_u satisfies the 1D wave equation with respect to r, t with initial data

$$rM_u = rM_{u_0}(x,r), \ \frac{\partial}{\partial t} rM_u = rM_{u_1}(x,r)$$

for $t = 0$.

Applying D'Alembert's formula we deduce that

$$\begin{aligned} rM_u(x,r,t) &= \frac{1}{2}\Bigg((r+t)M_{u_0}(x,r+t) \\ &\quad +(r-t)M_{u_0}(x,r-t) + \int_{r-t}^{r+t} \xi M_{u_1}(x,\xi)\, d\xi\Bigg), \end{aligned}$$

and we need to pass to the limit $r \to 0$ to recover $u(x,t)$.

Using the fact that $M_{u_0}(x,r)$ and $M_{u_1}(x,r)$ are even in r we have that

$$
\begin{aligned}
M_u(x,r,t) &= \frac{1}{2r}\left((t+r)M_{u_0}(x,t+r) - (t-r)M_{u_0}(x,t-r)\right) \\
& \frac{1}{2r}\int_{t-r}^{t+r} \xi M_{u_1}(x,\xi)\, d\xi,
\end{aligned}
$$

so that passing to the limit $r \to 0$ we obtain

$$u(x,t) = tM_{u_1}(x,t) + \frac{\partial}{\partial t}(tM_{u_0}(x,t)).$$

Hence we arrive at *Kirchhoff's solution of the 3D wave equation*

$$\boxed{u(x,t) = \frac{1}{4\pi t}\int_{S(x,t)} u_1(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S(x,t)} u_0(y)\, dS_y\right).}$$

This reduction to the solution of a 1D wave equation works in any odd dimension, but the details are more complicated (see Evans, John). However it doesn't work in even dimensions, in particular for $n = 2$.

However, to get explicit formulae for even dimensions $n = 2k$ we can add an extra coordinate and apply the formula for the odd dimension $2k + 1$. This is *Hadamard's method of descent*, which we now apply to the case $n = 2$.

Thus we apply Kirchhoff's formula to initial data

$$u_0(x_1, x_2),\ u_1(x_1, x_2)$$

which do not depend on x_3.

Note that for any function $f(x_1, x_2)$

$$\int_{S(x,t)} f(y)\, dS_y = 2t \int_{B(x,t)} \frac{f(y)}{\sqrt{t^2 - |y-x|^2}} dy,$$

where $B(x,t) = \{y \in \mathbb{R}^2 : |x-y| < t\}$ and $f(y) = f(y_1, y_2)$.

Hence the solution for $n = 2$ is given by *Poisson's solution to the 2D wave equation*

$$\boxed{\begin{aligned} u(x,t) \;=\; & \frac{1}{2\pi} \int_{B(x,t)} \frac{u_1(y)}{\sqrt{t^2-|y-x|^2}} dy \\ & + \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{B(x,t)} \frac{u_0(y)}{\sqrt{t^2-|y-x|^2}} dy. \end{aligned}}$$

From Kirchhoff's and Poisson's formula various important conclusions can be drawn.

Both formulae shown that $u(x,t)$ depends only on the values of the initial data in the ball $B(x,t) \subset \mathbb{R}^n$. This is the *principle of causality* and is true in any dimension. Intuitively, since the wave speed is 1, disturbances due to the initial data outside this ball take longer than time t to reach x.

But there is an important difference between the formulae.

This difference is that Kirchhoff's formula shows that for $n = 3$ the solution $u(x, t)$ depends only on the data and its derivatives on the sphere $S(x, t)$, while Poisson's formula shows that for $n = 2$ the solution $u(x, t)$ depends on the initial data in the whole ball $B(x, t)$.

Thus when $n = 3$ disturbances propagate at *exactly the wave speed*. In particular, if the initial data is supported in a small ball $B(0, \rho)$ then the solution $u(x, t)$ at the point x is zero except for times in the interval $[|x| - \rho, |x| + \rho]$.

This is *Huyghen's Principle* which is fundamental to human experience, since it means that sharp light and sound signals can be received, enabling us e.g. to hear music as it is played and watch movies.

On the other hand, Huyghen's principle is not valid in 2D. The effect of a disturbance initially concentrated in $B(0, \rho)$ and reaching a point x will never die out completely.

It also follows from Kirchhoff's formula that there can be a pointwise loss of derivatives in the solution with respect to the initial data.

To see this more explicitly we can consider *spherical waves*. These are special solutions of the wave equation of the form

$$u(x,t) = \frac{1}{r}(f(r+t) + g(r-t)).$$

Following the chapter by F. John in Bers, John & Schechter, choose $f(r) = g(r) = \frac{1}{2} r\varphi(r)$ with φ even. Then we have the solution

$$u(x,t) = \begin{cases} \frac{1}{2}(\varphi(t+r) + \varphi(t-r)) & x \neq 0 \\ + \frac{t}{2r}(\varphi(t+r) - \varphi(t-r)) & \\ \varphi(t) + t\varphi'(t) & x = 0, \end{cases}$$

with initial data $u_0(x) = \varphi(r)$, $u_1(x) = 0$, from which the loss of a derivative (due to focussing) follows.

Indeed if φ vanishes in a neighbourhood of the origin then $\varphi \in C^k$ iff $u_0 \in C^k$, but then u is not in general C^k in t at $x = 0$.

The Sobolev spaces $H^1(\Omega), H_0^1(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be open.

Definition. The Sobolev space $H^1(\Omega)$ is the set of $u \in L^2(\Omega)$ with weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$ for $i = 1, \ldots, n$.

Recall that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi\, dx \text{ for all } \varphi \in C_0^\infty(\Omega).$$

$H^1(\Omega)$ is a Hilbert space under the inner product

$$\langle u, v \rangle = \int_\Omega (u \cdot v + \nabla u \cdot \nabla v)\ dx.$$

Definition. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

If Ω is bounded with Lipschitz boundary, then the *trace* $\mathrm{tr}\, u$ of u on $\partial\Omega$ is well defined (as a function in $L^2(\partial\Omega)$) and $H_0^1(\Omega) = \{u \in H^1(\Omega) : \mathrm{tr}\, u = 0\}$.

If Ω is bounded then the Poincaré inequality implies that

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u(x)|^2 dx$$

defines an equivalent norm on $H_0^1(\Omega)$.

Also $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$.

Fourier transforms and series

If $u \in L^1(\mathbb{R}^n)$ (u complex) its *Fourier transform* is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, dx$$

and its *inverse Fourier transform* by

$$\check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\xi) e^{ix\cdot\xi}\, d\xi$$

Plancherel's Theorem. If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, given $u \in L^2(\mathbb{R}^n)$ there is a sequence $\varphi^{(j)} \in C_0^\infty(\mathbb{R}^n)$ with $\varphi^{(j)} \to u$ in $L^2(\mathbb{R}^n)$.

Hence by Plancherel's theorem $\widehat{\varphi}^{(j)}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and so converges in $L^2(\mathbb{R}^n)$ to some limit, which does not depend on the approximating sequence, and which we define to be $\hat{u}$. The inverse Fourier transform $\check{u}$ of $u \in L^2(\mathbb{R}^n)$ is defined similarly.

If $\Omega \subset \mathbb{R}^n$ is a bounded open set, then (see e.g. Evans) there exists an orthonormal basis $\{\omega_j\}_{j=1}^{\infty}$ of (real) $L^2(\Omega)$ consisting of the eigenfunctions $\omega_j \in H_0^1(\Omega)$ of $-\Delta$, that is of solutions to

$$\begin{aligned} -\Delta\omega_j &= \lambda_j\omega_j \text{ in } \Omega \\ \omega_j|_{\partial\Omega} &= 0, \end{aligned}$$

with corresponding real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, i.e. where $\lambda_j \to \infty$ as $j \to \infty$.

Thus if $(u,v) = \int_\Omega u(x)v(x)\,dx$ denotes the inner product of $u,v \in L^2(\Omega)$ we have that $(\omega_j,\omega_k) = \delta_{jk}$, and any $f \in L^2(\Omega)$ has a unique Fourier expansion

$$f = \sum_{j=1}^{\infty} (f,\omega_j)\omega_j$$

which is convergent in $L^2(\Omega)$.

Also we have *Parseval's Theorem*

$$\|f\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} (f,\omega_j)^2.$$

If $f \in H^1_0(\Omega)$ then the series is convergent in $H^1_0(\Omega)$ and

$$\|f\|^2_{H^1_0} := \int_\Omega |\nabla f|^2 dx = \sum_{j=1}^{\infty} \lambda_j (f,\omega_j)^2.$$

Solution of wave equation by Fourier methods

(i) *The wave equation in* $\mathbb{R}^n$ *via Fourier Transforms.*

Consider the wave equation

$$u_{tt} = \Delta u, \text{ for } x \in \mathbb{R}^n, t > 0, \quad (3)$$

with initial data

$$u = u_0, \ u_t = u_1 \text{ for } t = 0. \quad (4)$$

Formally we have, taking Fourier transforms in x,

$$\hat{u}_{tt} + |\xi|^2 \hat{u} = 0 \text{ for } t > 0,$$

with initial data

$$\hat{u} = \hat{u}_0, \ \hat{u}_t = \hat{u}_1 \text{ for } t = 0.$$

So

$$\widehat{u}(\xi,t) = A(\xi)e^{i|\xi|t} + B(\xi)e^{-i|\xi|t},$$

with

$$A(\xi) + B(\xi) = \widehat{u}_0(\xi),\ A(\xi) - B(\xi) = -\frac{i}{|\xi|}\widehat{u}_1(\xi).$$

Hence

$$A(\xi) = \frac{1}{2}\left(\widehat{u}_0(\xi) - \frac{i}{|\xi|}\widehat{u}_1(\xi)\right),\ B = \frac{1}{2}\left(\widehat{u}_0(\xi) + \frac{i}{|\xi|}\widehat{u}_1(\xi)\right)$$

and so

$$\widehat{u}(\xi,t) = \widehat{u}_0(\xi)\cos(|\xi|t) + \widehat{u}_1(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

Taking the inverse Fourier transform we thus obtain

$$\boxed{u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\hat{u}_0(\xi) \cos(|\xi|t) + \hat{u}_1(\xi) \frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi. \quad (*)}$$

We slightly strengthen the previous definition of a weak solution:

Definition. A function $u = u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ is a *weak solution* of (3), (4) on $[0,T]$ if $u \in C([0,T]; H^1_0(\mathbb{R}^n))$, $u_t \in C([0,T]; L^2(\mathbb{R}^n))$ with $u(\cdot,0) = u_0(\cdot), u_t(\cdot,0) = u_1(\cdot)$, and for all $\varphi \in C^\infty_0(\mathbb{R}^n)$ we have that

$$(u,\varphi)(t) := \int_{\mathbb{R}^n} u(x,t)\varphi(x)\, dx \in C^2([0,T])$$

with

$$\frac{d^2}{dt^2}(u,\varphi) = (u, \Delta\varphi) \text{ for } t \in [0,T].$$

Theorem 2 If $u_0 \in H_0^1(\mathbb{R}^n), u_1 \in L^2(\mathbb{R}^n)$ then (*) is the unique weak solution of the wave equation with initial data u_0, u_1, and the energy equation

$$\int_{\mathbb{R}^n} \left(|\nabla u(x,t)|^2 + u_t^2(x,t)\right) dx = \int_{\mathbb{R}^n} (|\nabla u_0(x)|^2 + u_1^2(x))\, dx$$

holds for all $t \geq 0$.

Proof. We first note that

$$u_t(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-\hat{u}_0(\xi)|\xi| \sin(|\xi|t) + \hat{u}_1(\xi) \cos(|\xi|t))\, e^{ix\cdot\xi}\, d\xi.$$

Therefore $u(\cdot,0) = u_0, u_t(\cdot,0) = u_1$.

It also follows that $u : [0,T] \to H_0^1(\mathbb{R}^n)$ and $u_t : [0,T] \to L^2(\mathbb{R}^n)$ are continuous.

(For example, taking for simplicity $u_1 = 0$, we then have that

$$u_t(x,s) - u_t(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} -\hat{u}_0(\xi)|\xi|(\sin(|\xi|s) - \sin(|\xi|t))e^{ix\cdot\xi}\, d\xi,$$

so that by Plancherel's theorem

$$\|u_t(\cdot,s) - u_t(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2|\xi|^2|\sin(|\xi|s) - \sin(|\xi|t)|^2 d\xi,$$

and the continuity follows since $u_0 \in H^1_0(\mathbb{R}^n)$ and hence $|\hat{u}_0(\xi)|^2|\xi|^2 \in L^1(\mathbb{R}^n)$.)

Now suppose that $u_0, u_1 \in C^\infty_0$. Then it is easy to check that $u(x,t)$ given by (*) is a smooth solution.

Also, again by Plancherel,

$$\begin{aligned}
\int_{\mathbb{R}^n}(|\nabla u(x,t)|^2+|u_t(x,t)|^2)\,dx &= \int_{\mathbb{R}^n}(|\widehat{\nabla u}(\xi,t)|^2+|\hat{u}_t(\xi,t)|^2)d\xi \\
&= \int_{\mathbb{R}^n}(|\xi|^2|\hat{u}(\xi,t)|^2+|\hat{u}_t(\xi,t)|^2)\,d\xi \\
&= \int_{\mathbb{R}^n}(|\xi|^2|\hat{u}_0(\xi)|^2+|\hat{u}_1(\xi)|^2)\,d\xi \\
&= \int_{\mathbb{R}^n}(|\nabla u_0(x)|^2+|u_1(x)|^2)\,dx,
\end{aligned}$$

so that the energy equation holds.

Given $u_0 \in H_0^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$, let $u_0^{(j)}, u_1^{(j)} \in C_0^\infty(\mathbb{R}^n)$ with $u_0^{(j)} \to u_0$ in $H_0^1(\mathbb{R}^n)$, $u_1^{(j)} \to u_1$ in $L^2(\mathbb{R}^n)$, and denote by $u^{(j)}(x,t)$ the solution given by (*) with initial data $u_0^{(j)}, u_1^{(j)}$.

From the formula $u^{(j)}$ is a Cauchy sequence in $C([0,T];L^2(\mathbb{R}^n))$, while by the energy equation $\nabla u^{(j)}$ and $u_t^{(j)}$ are Cauchy sequences in $C([0,T];L^2(\mathbb{R}^n)^n)$ and $C([0,T];L^2(\mathbb{R}^n))$ respectively. Hence the limit u is given by (*) and satisfies the energy equation.

Thus if $\varphi \in C_0^\infty(\mathbb{R}^n)$ we can pass to the limit in the equation

$$(u_t^{(j)},\varphi)(t) - (u_t^{(j)},\varphi)(s) = \int_s^t (u^{(j)},\Delta\varphi)(\tau)d\tau$$

to deduce that $(u,\varphi) \in C^2([0,T])$ and

$$\frac{d^2}{dt^2}(u,\varphi) = (u,\Delta\varphi),$$

so that u is a weak solution.

To complete the proof, by the previous mollification argument we just need to show that the unique C^2 solution with zero initial data is zero. This follows from Kirchhoff's formula (for $n = 3$). $\square$

Remark: It seems that to prove uniqueness you need something implying a finite speed of propagation. Treves (Theorem 13.1) gives a more general existence and uniqueness result for u_0, u_1 merely distributions, which for the uniqueness makes use of an analysis of the fundamental solution of the wave equation.

Another representation of the solution is as a superposition of plane waves via *Radon transforms* (see Lax, Helgason). We just give the formula for n odd.

The Radon transform of a function f is defined by

$$\tilde{f}(\omega, s) = \int_{\{y \cdot \omega = s\}} f(y)\, d\mathcal{H}^{n-1} y.$$

Then the solution is

$$u(x,t) = \int_{S^{n-1}} h(\omega, x \cdot \omega - t)\, dS_\omega,$$

where

$$h(\omega, s) = \partial_s^{n-1} \tilde{u}_0(\omega, s) - \partial_s^{n-2} \tilde{u}_1(\omega, s).$$

(ii) *The wave equation in a bounded domain via Fourier series.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We solve the wave equation

$$\begin{cases} u_{tt} = \Delta u \text{ for } x \in \Omega, t > 0, \\ u|_{\partial\Omega} = 0 \end{cases} \quad (5)$$

and initial conditions

$$u = u_0,\ u_t = u_1 \text{ for } t = 0. \quad (6)$$

where $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$.

Definition. u is a weak solution of (5), (6) on $[0,T]$ if $u \in C([0,T]; H_0^1(\Omega))$, $u_t \in C([0,T]; L^2(\Omega))$, with $u(\cdot,0) = u_0, u_t(\cdot,0) = u_1$ and for any $\varphi \in C_0^\infty(\Omega)$ we have that $(u,\varphi) \in C^2([0,T])$ with

$$\frac{d^2}{dt^2}(u,\varphi) = (u, \Delta\varphi).$$

Remark. In the last equation we can equivalently write $(u,\Delta\varphi) = -(\nabla u, \nabla\varphi) := -\int_\Omega \nabla u \cdot \nabla\varphi \, dx$. Also, since this equation can equivalently be written as

$$(u_t,\varphi)(t) - (u_t,\varphi)(s) = -\int_s^t (\nabla u, \nabla\varphi)(\tau)\, d\tau,$$

we get the same equation for any $\varphi \in H_0^1(\Omega)$, since $C_0^\infty(\Omega)$ is by definition dense in $H_0^1(\Omega)$.

Theorem 3 The unique weak solution of (5),(6) is given by

$$u(x,t) = \sum_{j=1}^{\infty} \left(u_{0j} \cos(\sqrt{\lambda_j} t) + u_{1j} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} \right) \omega_j,$$

where $u_{0j} = (u_0, \omega_j), u_{1j} = (u_1, \omega_j)$, and the energy equation

$$\int_\Omega (|\nabla u(x,t)|^2 + u_t^2(x,t))\, dx = \int_\Omega (|\nabla u_0(x)|^2 + u_1^2(x))\, dx$$

holds for all $t \in [0,T]$.

Proof. Let u be a weak solution. Then

$$u(\cdot,t) = \sum_{j=1}^{\infty} u_j(t) \omega_j, \quad t \in [0,T]$$

where $u_j(t) = (u(\cdot,t), \omega_j)$.

Taking $\varphi = \omega_j$ we see that $u_j \in C^2([0,T])$ and

$$\ddot{u}_j = -\lambda_j u_j,$$

with initial conditions $u_j(0) = u_{0j}, \dot{u}_j(0) = u_{1j}$.

Solving this ODE gives the solution as in the theorem, proving uniqueness.

Since $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$ we have that

$$\int_\Omega |\nabla u_0|^2 dx = \sum_{j=1}^{\infty} \lambda_j u_{0j}^2 < \infty, \ \|u_1\|_2^2 = \sum_{j=1}^{\infty} u_{1j}^2 < \infty.$$

Let

$$u_N(x,t) = \sum_{j=1}^{N} u_j(t)\omega_j(x).$$

Then u_N is a weak solution (in fact smooth in $\Omega \times [0,T]$) and

$$\int_\Omega (|\nabla(u_M - u_N)(x,t)|^2 + |(u_M - u_N)_t(x,t)|^2)\, dx = \sum_{j=N}^{M} (\lambda_j u_{0j}^2 + u_{1j}^2),$$

so that u_N, u_{Nt} are Cauchy sequences in $C([0,T]; H_0^1(\Omega))$, $C([0,T]; L^2(\Omega))$ respectively.

So $u \in C([0,T]; H_0^1(\Omega)), u_t \in C([0,T]; L^2(\Omega))$, and passing to the limit $N \to \infty$ in

$$(u_{Nt}, \varphi)(t) - (u_{Nt}, \varphi)(s) = \int_s^t (u_N, \Delta\varphi)(\tau)\, d\tau$$

we deduce that u is a weak solution.

Finally, we can pass to the limit $N \to \infty$ in the energy equation

$$\int_\Omega (|\nabla u_N(x,t)|^2 + u_{Nt}(x,t)^2)\, dx = \sum_{j=1}^{N} (\lambda_j u_{0j}^2 + u_{1j}^2)$$

for u_N to get the energy equation for u. $\square$

Remark. Let $X = H_0^1(\Omega) \times L^2(\Omega)$. Note that the energy equation implies that the solution $\{u, u_t\} \in C([0,T]; X)$ depends continuously on the initial data $\{u_0, u_1\}$ in X. The same is true when $\Omega = \mathbb{R}^n$ (with a separate calculation to check continuity in $C([0,T]; L^2(\Omega))$).

Semiflows and linear semigroups

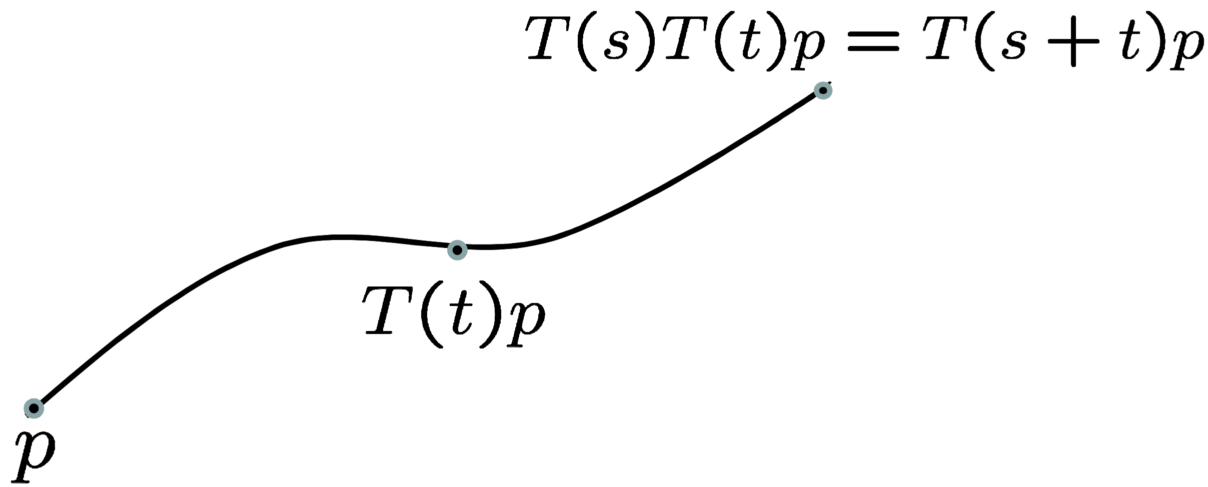
Definition. A *semiflow* $\{T(t)\}_{t\geq 0}$ on a metric space (X, d) is a family of continuous maps $T(t) : X \to X$ satisfying
(i) $T(0) =$ identity,
(ii) $T(s+t) = T(s)T(t)$ for all $s \geq 0, t \geq 0$,
(iii) for each $p \in X$ the map $t \mapsto T(t)p$ is continuous from $[0, \infty) \to X$.

(Alternatively (*nonlinear*) *semigroup* or *dynamical system*.)

Interpretation: $T(t)p$ is the state at time t of an autonomous system with initial data $p \in X$.

$T(t)$ continuous expresses continuity with respect to the initial data.

Condition (ii) is a statement of uniqueness of solutions for given initial data.



Note, however, that there is no assumption of backwards uniqueness (or backwards existence).

It is possible to consider weaker versions of (iii), for example that for each p the map $t \mapsto T(t)p$ is strongly measurable from $[0, \infty) \to X$, and surprisingly this implies that $t \mapsto T(t)p$ is continuous from $(0, \infty) \to X$ (see JB Proc. AMS 1976). Another similar example of the semigroup property (ii) strengthening continuity properties is:

Theorem 4 (Chernoff & Marsden, Bull. AMS 1970) If $\{T(t)\}_{t \geq 0}$ is a semiflow on X, then the map $(t, p) \mapsto T(t)p$ is continuous from $(0, \infty) \times X \to X$.

Proof. Let $p_j \to p$ in X. Let $0 < a < b < \infty$, and for $\varepsilon > 0, m = 1, 2, \ldots$, set

$$S_{m,\varepsilon} = \{t \in [a,b] : d(T(t)p_j, T(t)p) \leq \varepsilon \text{ for all } j \geq m\}.$$

By (iii) $S_{m,\varepsilon}$ is closed, and by the continuity of $T(t)$

$$\bigcup_{m=1}^{\infty} S_{m,\varepsilon} = [a,b].$$

By the Baire Category Theorem, some $S_{r,\varepsilon}$ contains an open interval.

Since we may apply this argument to any $[a,b] \subset (0,\infty)$ there exists a dense open subset S_ε of $(0,\infty)$ such that if $t_0 \in S_\varepsilon$ there exists an open neighbourhood $N_\varepsilon(t_0)$ of t_0 and $r_\varepsilon(t_0)$ such that $d(T(t)p_j, T(t)p) \leq \varepsilon$ whenever $j \geq r_\varepsilon(t_0)$, $t \in N_\varepsilon(t_0)$.

Let

$$K = \bigcap_{i=1}^{\infty} S_{1/i}.$$

Clearly $T(t_j)p_j \to T(t)p$ whenever $t_j \to t$ and $t \in K$. Again by the Baire Category Theorem, K is dense in $(0, \infty)$.

Now let $t > 0$ be arbitrary and $t_j \to t$. Let $t_1 \in K$, $0 < t_1 < t$. Then $T(t_1 + t_j - t)p_j \to T(t_1)p$ and so

$$T(t_j)p_j = T(t-t_1)T(t_1+t_j-t)p_j \to T(t-t_1)T(t_1)p = T(t)p,$$

as required. $\square$

Now suppose that X is a real Banach space, and that each $T(t) : X \to X$ is a bounded linear operator. Then $\{T(t)\}_{t\geq 0}$ is called a C^0-*semigroup*.

We have shown that weak solutions to the linear wave equation generate a C^0-semigroup on the Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega)$ when either Ω is bounded open, or $\Omega = \mathbb{R}^n$.

We associate with every C^0-semigroup a corresponding linear differential equation

$$\dot{w} = Aw,$$

where A is an (in general unbounded) linear operator on X

Definition. The *infinitesimal generator* A of the C^0–semigroup $\{T(t)\}_{t\geq 0}$ is the linear operator

$$Aw = \lim_{t\to 0+} \frac{T(t)w - w}{t} \quad (w \in D(A)),$$

with domain $D(A)$ consisting of those $w \in X$ for which the limit exists in X.

Theorem 5 If $w \in D(A)$ then
(a) $T(t)w \in D(A)$ for all $t \geq 0$,
(b) $AT(t)w = T(t)Aw$ for all $t \geq 0$,
(c) the map $t \mapsto T(t)w$ belongs to $C^1([0,\infty);X)$ with derivative $\frac{d}{dt}T(t)w$=AT(t)w,
(d) $z(t) = T(t)w$ is the unique function in $C([0,T];X)$ with $z(0) = w$ which for all $t > 0$ belongs to $D(A)$, is differentiable in t, and satisfies $\dot{z}(t) = Az(t)$.

Proof. Let $w \in D(A)$. Then

$$\begin{aligned}
\lim_{s\to 0+} \frac{T(s)T(t)w - T(t)w}{s} &= \lim_{s\to 0+} \frac{T(t)T(s)w - T(t)w}{s} \\
&= T(t) \lim_{s\to 0+} \frac{T(s)w - w}{s} \\
&= T(t)Aw,
\end{aligned}$$

where in the first line we have used the semigroup property (ii) and in the second that $T(t)$ is bounded. This proves (a) and (b).

Now let $t > 0, h > 0$ and note that

$$\frac{T(t+h)w - T(t)w}{h} - T(t)Aw = T(t)\left(\frac{T(h)w - w}{h} - Aw\right).$$

Hence the derivative from the right of $T(t)w$ exists and equals $T(t)Aw$.

Also

$$\frac{T(t-h)w - T(t)w}{-h} - T(t)Aw = T(t-h)\left(\frac{T(h)w - w}{h} - T(h)Aw\right),$$

and so, using the joint continuity of $T(s)w$ in $s > 0$ and w, we get that the derivative from the left of $T(t)w$ also exists and equals $T(t)Aw$. Using (b) and (iii) we get (c).

To prove (d) let $v(s) = T(t-s)z(s)$. Then by (b),(c)

$$\begin{aligned} \frac{dv(s)}{ds} &= T(t-s)\dot{z}(s) - \frac{d}{dt}T(t-s)z(s) \\ &= T(t-s)Az(s) - AT(t-s)z(s) \\ &= 0. \end{aligned}$$

Hence $v(s)$ is constant for $0 < s \le t$, and (using again the joint continuity) $v(0) = v(t)$, that is $z(t) = T(t)w$ as required. $\square$

Theorem 6
(a) $D(A)$ is dense in X, and
(b) A is a closed operator (i.e. the graph $\{(w, Aw) : w \in D(A)\}$ is closed in $X \times X$).

Proof. If $w \in X$ then $z(t) = \frac{1}{t}\int_0^t T(s)w\,ds$ converges to w as $t \to 0+$, so it suffices to show that $z(t) \in D(A)$ for $t > 0$. (Here the integral is a Bochner integral in the Banach space X.)

But

$$
\begin{aligned}
t \cdot \frac{T(\tau)z(t) - z(t)}{\tau} &= \frac{1}{\tau}\left(T(\tau)\left(\int_0^t T(s)w\,ds\right) - \int_0^t T(s)w\,ds\right) \\
&= \frac{1}{\tau}\int_0^t (T(\tau+s)w - T(s)w)\,ds \\
&= \frac{1}{\tau}\left(\int_\tau^{\tau+t} T(s)w\,ds - \int_0^t T(s)w\,ds\right) \\
&= \frac{1}{\tau}\left(\int_t^{t+\tau} T(s)w\,ds - \int_0^\tau T(s)w\,ds\right) \\
&\to T(t)w - w \text{ as } \tau \to 0+,
\end{aligned}
$$

so that $z(t) \in D(A)$ with $Az(t) = \frac{T(t)w-w}{t}$.

To prove A closed, let $w_j \in D(A)$ and $w_j \to w, Aw_j \to v$ as $j \to \infty$.

Then

$$T(t)w_j - w_j = \int_0^t T(s)Aw_j\,ds,$$

and letting $j \to \infty$ we obtain

$$T(t)w - w = \int_0^t T(s)v\,ds,$$

so that

$$\lim_{t\to 0+} \frac{T(t)w - w}{t} = v,$$

proving that $w \in D(A)$ and $v = Aw$. $\square$

The *Hille-Yosida Theorem* gives necessary and sufficient conditions for a densely defined, closed linear operator A to generate a C^0-semigroup. It is most useful and easiest to state for the case of *contraction semigroups*.

Definition. A C^0-semigroup $\{T(t)\}_{t\geq 0}$ is a *contraction semigroup* if the operator norm $\|T(t)\| \leq 1$ for all $t \geq 0$, i.e.

$$\|T(t)w\| \leq \|w\| \text{ for all } t \geq 0.$$

(Thus the wave equation generates a contraction semigroup on $X = H_0^1(\Omega) \times L^2(\Omega)$ when $\Omega \subset \mathbb{R}^n$ is bounded open.)

Definition. The *resolvent set* $\rho(A)$ of a closed linear operator A on X with domain $D(A)$ is the set of real λ such that

$$\lambda \mathbf{1} - A : D(A) \to X$$

is one-to-one and onto. If $\lambda \in \rho(A)$ then the *resolvent operator* $R_\lambda : X \to X$ is defined by

$$R_\lambda w = (\lambda \mathbf{1} - A)^{-1} w.$$

Lemma 7 If $\lambda \in \rho(A)$ then R_λ is a bounded linear operator.

Proof. We show that the graph of R_λ is closed, so that the result follows from the closed graph theorem.

So let $w_j \to w, R_\lambda w_j \to r$ in X. Then

$$(\lambda \mathbf{1} - A) R_\lambda w_j = w_j$$

so that $A R_\lambda w_j \to \lambda r - w$. Since A is closed this implies that $r \in D(A)$ and $Ar = \lambda r - w$. Hence $r = R_\lambda w$ as required. □

Lemma 8 If $\lambda, \mu \in \rho(A)$ then

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \quad R_\lambda R_\mu = R_\mu R_\lambda.$$

Proof. We have that

$$\begin{aligned}
(\mu \mathbf{1} - A)(\lambda \mathbf{1} - A)(R_\lambda - R_\mu) &= (\mu \mathbf{1} - A)(\mathbf{1} - (\lambda \mathbf{1} - A) R_\mu) \\
&= (\mu \mathbf{1} - A)(\mathbf{1} - (\mu \mathbf{1} - A + (\lambda - \mu)) R_\mu) \\
&= (\mu \mathbf{1} - A)(\mu - \lambda) R_\mu \\
&= (\mu - \lambda) \mathbf{1}.
\end{aligned}$$

Multiplying by $R_\lambda R_\mu$ gives the first identity, which implies the second.

Lemma 9 If A is the generator of the C^0 contraction semigroup $\{T(t)\}_{t\geq 0}$ then $\lambda \in \rho(A)$ for all $\lambda > 0$, and for $w \in X$

$$R_\lambda w = \int_0^\infty e^{-\lambda t} T(t) w \, dt, \tag{7}$$

(so that R_λ is the Laplace transform of the semigroup).

Proof. Denote by

$$L_\lambda w = \int_0^\infty e^{-\lambda t} T(t) w \, dt$$

the integral in (7). The integral exists because $\|T(t)w\| \leq \|w\|$.

We have that for $h > 0$

$$\begin{aligned}\frac{T(h)L_\lambda w - L_\lambda w}{h} &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t}(T(t+h)w - T(t)w)\,dt\right) \\ &= -\frac{1}{h}\int_0^h e^{-\lambda(t-h)}T(t)w\,dt + \\ &\quad \frac{1}{h}\int_0^\infty (e^{-\lambda(t-h)} - e^{-\lambda t})T(t)w\,dt \\ &= -e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda t}T(t)w\,dt \\ &\quad + \left(\frac{e^{\lambda h}-1}{h}\right)\int_0^\infty e^{-\lambda t}T(t)w\,dt \\ &\longrightarrow -w + \lambda L_\lambda w \text{ as } h \to 0+\,.\end{aligned}$$

Hence $L_\lambda w \in D(A)$ and $AL_\lambda w = -w + \lambda L_\lambda w$, so that $(\lambda\mathbf{1} - A)L_\lambda w = w$. In particular $\lambda\mathbf{1} - A$ is onto.

Now let $u \in D(A)$ and set $f(t) = e^{-\lambda t}T(t)u$. Then $(f(t), Af(t)) \in G(A)$, for all t, where G is the graph of A, which is a closed linear subspace of $X \times X$. Hence $\int_0^\infty (f(t), Af(t))\, dt \in G(A)$, which implies that $A \int_0^\infty f(t)\, dt = \int_0^\infty Af(t)\, dt$.

Thus

$$\begin{aligned} AL_\lambda u &= A \int_0^\infty e^{-\lambda t} T(t) u\, dt \\ &= \int_0^\infty e^{-\lambda t} A T(t) u\, dt \\ &= \int_0^\infty e^{-\lambda t} T(t) A u\, dt = L_\lambda A u, \end{aligned}$$

so that $L_\lambda(\lambda \mathbf{1} - A)u = u$ (since we previously showed that $AL_\lambda u = -u + \lambda L_\lambda u$). Hence $\lambda \mathbf{1} - A$ is also one-to-one, so that $\lambda \in \rho(A)$ and $R_\lambda w = L_\lambda w$ for all $w \in X$ as required. □

Theorem 10 (Hille-Yosida) Let A be a closed, densely defined, linear operator on X. Then A is the generator of a C^0 contraction semigroup if and only if

$$(0,\infty) \subset \rho(A) \text{ and } \|R_\lambda\| \le \frac{1}{\lambda} \text{ for all } \lambda > 0.$$

(Thus the issue of solving the dynamic problem $\dot{w} = Aw$ is reduced to studying the static problem $\lambda w - Aw = f$.)

Proof. Lemma 9 shows that the conditions are necessary, because it implies that

$$\|R_\lambda w\| \le \int_0^\infty e^{-\lambda t}\|w\|\,dt = \frac{1}{\lambda}\|w\|.$$

To prove sufficiency, we construct a contraction semigroup generated by A using the *Yosida approximation* to A defined for $\lambda > 0$ by

$$A_\lambda := -\lambda \mathbf{1} + \lambda^2 R_\lambda = \lambda A R_\lambda.$$

We first claim that if $u \in D(A)$ then $A_\lambda u \to Au$ as $\lambda \to \infty$.

To see this note that since

$$\lambda R_\lambda u - u = A R_\lambda u = R_\lambda A u,$$

$$\|\lambda R_\lambda u - u\| \leq \|R_\lambda\| \cdot \|Au\| \leq \frac{1}{\lambda}\|Au\| \to 0.$$

Since for $w \in X$

$$\|\lambda R_\lambda w - w\| \leq \|\lambda R_\lambda u - u\| + \|\lambda R_\lambda\| \cdot \|w - u\| + \|w - u\|,$$

and $\|\lambda R_\lambda\| \leq 1$ with $D(A)$ dense it follows that

$$\lambda R_\lambda w \to w \text{ as } \lambda \to \infty \text{ for all } w \in X.$$

Hence, setting $w = Au$ we have that

$$A_\lambda u = \lambda A R_\lambda u = \lambda R_\lambda A u \to Au,$$

as claimed.

We now use the fact that A_λ is a bounded linear operator to define

$$T_\lambda(t) = e^{A_\lambda t} = e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda^2 t)^j}{j!} R_\lambda^j.$$

Since $\|R_\lambda\| \le \frac{1}{\lambda}$,

$$\|T_\lambda(t)\| \le e^{-\lambda t} \sum_{j=0}^{\infty} \frac{\lambda^{2j} t^j}{j!} \|R_\lambda\|^j \le e^{-\lambda t} \sum_{j=0}^{\infty} \frac{\lambda^j t^j}{j!} = 1.$$

Hence $\{T_\lambda(t)\}_{t\ge 0}$ is a C^0 contraction semigroup, with infinitesimal generator A_λ.

Next we show that as $\lambda \to \infty$ we have $T_\lambda(t)w \to T(t)w$ for all $w \in X$, where $\{T(t)\}_{t \geq 0}$ is a contraction semigroup.

To this end let $\lambda > 0, \mu > 0$ and note that by Lemma 8, $A_\lambda A_\mu = A_\mu A_\lambda$, so that

$$A_\mu T_\lambda(t) = T_\lambda(t) A_\mu \text{ for all } t \geq 0.$$

Therefore, for $u \in D(A)$,

$$\begin{aligned} T_\lambda(t)u - T_\mu(t)u &= \int_0^t \frac{d}{ds}(T_\mu(t-s)T_\lambda(s)u)\,ds \\ &= \int_0^t T_\mu(t-s)T_\lambda(s)(A_\lambda u - A_\mu u)\,ds. \end{aligned}$$

Therefore

$$\|T_\lambda(t)u - T_\mu(t)u\| \le t\|A_\lambda u - A_\mu u\| \to 0 \text{ as } \lambda, \mu \to \infty,$$

so that

$$T(t)u := \lim_{\lambda\to\infty} T_\lambda(t)u$$

exists for each $t \ge 0$.

Since $\|T_\lambda(t)\| \le 1$, if $w \in X$ we have

$$\|T_\lambda(t)w - T_\mu(t)w\| \le \|T_\lambda(t)u - T_\mu(t)u\| + 2\|w - u\|,$$

so that, since $D(A)$ is dense, the limit

$$T(t)w := \lim_{\lambda\to\infty} T_\lambda(t)w$$

exists for all $w \in X$, uniformly on compact subsets of $[0, \infty)$, and $\{T(t)\}_{t\ge0}$ is a contraction semigroup.

Denote by B the generator of $\{T(t)\}_{t\geq 0}$. It remains to show that $B = A$.

If $u \in D(A)$ we have that

$$T_\lambda(t)u - u = \int_0^t T_\lambda(s)A_\lambda u\, ds,$$

and since

$$\|T_\lambda(s)A_\lambda u - T(s)Au\| \leq \|A_\lambda u - Au\| + \|(T_\lambda(s) - T(s))Au\| \to 0$$

we deduce that

$$T(t)u - u = \int_0^t T(s)Au\, ds.$$

Hence $u \in D(B)$ and $Bu = Au$. Suppose that there exists $z \in D(B) \setminus D(A)$.

Then, since $\lambda\mathbf{1} - A$ is onto, $(\lambda\mathbf{1} - B)z = (\lambda\mathbf{1} - A)u$ for some $u \in D(A)$. Hence

$$(\lambda\mathbf{1} - B)(z - u) = 0,$$

implying that $z = u \in D(A)$ (since $\lambda\mathbf{1} - B$ is one-to-one). This contradiction implies that $B = A$, completing the proof of the theorem. $\square$

Corollary 11 A closed densely defined linear operator A on X is the generator of a C^0–semigroup $\{T(t)\}_{t\geq 0}$ satisfying

$$\|T(t)\| \leq e^{\omega t}$$

if and only if $(\omega, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \dfrac{1}{\lambda - \omega}$ for $\lambda > \omega$.

Proof. Apply the theorem to $S(t) = e^{-\omega t}T(t)$. $\square$

The wave equation via the Hille-Yosida theorem.

We write the wave equation

$$u_{tt} = \Delta u$$

as

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u_t \\ \Delta u \end{pmatrix},$$

or, setting $v = u_t$,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

Let $\Omega \subset \mathbb{R}^n$ be open, $X = H_0^1(\Omega) \times L^2(\Omega)$.

Lemma 12

$$D(A) = \{\begin{pmatrix} u \\ v \end{pmatrix} \in X : \Delta u \in L^2(\Omega), v \in H_0^1(\Omega)\}$$

is dense in X, and $A : D(A) \to X$ is closed.

Proof. $D(A)$ is dense since $C_0^\infty(\Omega) \times C_0^\infty(\Omega) \subset D(A)$.

Let $\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix}, \begin{pmatrix} v^{(j)} \\ \Delta u^{(j)} \end{pmatrix}$ be convergent in $X \times X$, so that $u^{(j)} \to u \in H_0^1(\Omega), v^{(j)} \to v \in L^2(\Omega)$, $v^{(j)} \to \bar{v} \in H_0^1(\Omega), \Delta u^{(j)} \to z \in L^2(\Omega)$.

Then $\bar{v} = v$ and since

$$\int_\Omega \Delta u^{(j)} \varphi \, dx = -\int_\Omega \nabla u^{(j)} \cdot \nabla \varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$ we have that

$$\int_\Omega z\varphi \, dx = -\int_\Omega \nabla u \cdot \nabla \varphi \, dx,$$

so that $z = \Delta u$ and the graph of A is closed. □

Theorem 13 A is the generator of a C^0-semigroup $\{T(t)\}_{t\geq 0}$ on X, and the energy equation

$$E(T(t)w) = E(w), \;\; t \geq 0$$

is satisfied for all $w = \begin{pmatrix} u \\ v \end{pmatrix} \in X$, where

$$E(w) := \int_\Omega (|\nabla u(x)|^2 + v(x)^2) \, dx.$$

Proof. We first show that $(0,\infty) \subset \rho(A)$. Thus we need to prove that for any $\lambda > 0$ and $f \in H_0^1(\Omega)$, $g \in L^2(\Omega)$ there exists a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in X$ with $\Delta u \in L^2(\Omega)$, $v \in H_0^1(\Omega)$ to

$$\begin{aligned} \lambda u - v &= f, \\ \lambda v - \Delta u &= g. \end{aligned}$$

Since $v = \lambda u - f$ we just need to show that there is a unique solution $u \in H_0^1(\Omega)$ to

$$\lambda^2 u - \Delta u - (\lambda f + g) = 0.$$

The existence follows by minimization of the functional

$$I(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda^2}{2}u^2 - (\lambda f + g)u \right) dx$$

over $H_0^1(\Omega)$. We sketch the standard argument.

First note that I is bounded below, since $\lambda > 0$ and thus

$$\frac{\lambda^2}{4}u^2 - (\lambda f + g)u \geq -\frac{1}{\lambda^2}(\lambda f + g)^2,$$

and the RHS is integrable since $f, g \in L^2(\Omega)$.

Let $l = \inf_{H_0^1(\Omega)} I$, so that $0 \geq l > -\infty$ (since $0 \in H_0^1(\Omega)$) and let $u^{(j)}$ be a minimizing sequence.

Then

$$\begin{aligned}\int_\Omega\left(\frac{1}{2}|\nabla u^{(j)}|^2+\frac{\lambda^2}{2}u^{(j)2}\right)dx &\leq \int_\Omega(\lambda f+g)u^{(j)}\,dx\\ &\leq \int_\Omega\left(\frac{\lambda^2 u^{(j)2}}{4}+\frac{(\lambda f+g)^2}{\lambda^2}\right)dx,\end{aligned}$$

and so

$$\int_\Omega\left(\frac{1}{2}|\nabla u^{(j)}|^2+\frac{\lambda^2}{4}u^{(j)2}\right)dx\leq c<\infty,$$

so that $u^{(j)}$ is bounded in $H_0^1(\Omega)$.

Hence a subsequence $u^{(j_k)}\rightharpoonup u$ in $H_0^1(\Omega)$, i.e.

$$\int_\Omega u^{(j_k)}v\,dx\to\int_\Omega uv\,dx,\int_\Omega\nabla u^{(j_k)}\cdot\nabla v\,dx\to\int_\Omega\nabla u\cdot\nabla v\,dx$$

for all $v\in H_0^1(\Omega)$ (because any bounded sequence in a Hilbert space has a weakly convergent subsequence).

Therefore

$$\begin{aligned}
I(u^{(j_k)}) &= \int_\Omega \Big(\frac{1}{2}(|\nabla u^{(j_k)} - \nabla u|^2 + 2\nabla u^{(j_k)} \cdot \nabla u - |\nabla u|^2) \\
&\quad + \frac{\lambda^2}{2}((u^{(j_k)} - u)^2 + 2u^{(j_k)} u - u^2) - (\lambda f + g) u^{(j_k)}\Big)\, dx \\
&\geq \int_\Omega \Big(\nabla u^{(j_k)} \cdot \nabla u - \frac{1}{2}|\nabla u|^2 \\
&\qquad + \lambda^2 (u^{(j_k)} u - \frac{1}{2}u^2) - (\lambda f + g) u^{(j_k)}\Big)\, dx \\
&\longrightarrow I(u).
\end{aligned}$$

Hence $I(u) = l$ and u is a minimizer.

Therefore

$$\begin{aligned}
\lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} &= \int_\Omega \Big((\nabla u \cdot \nabla v) + \lambda^2 u \cdot v - (\lambda f + g) v\Big)\, dx \\
&= 0.
\end{aligned}$$

Hence $\Delta u \in L^2(\Omega)$ and

$$-\Delta u + \lambda^2 u - (\lambda f + g) = 0,$$

and the uniqueness follows by noting that the difference w between two solutions satisfies

$$\int_\Omega (|\nabla w|^2 + \lambda^2 w^2)\, dx = 0.$$

We will apply Corollary 11; to prove the resolvent estimate, note that

$$R_\lambda \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Therefore

$$\begin{aligned} \lambda u - v &= f \\ \lambda \nabla u - \nabla v &= \nabla f \\ \lambda v - \Delta u &= g. \end{aligned}$$

Taking the inner product of these equations with $\nabla u, u, v$ respectively and adding, we obtain

$$\begin{aligned} \lambda \int_\Omega (u^2 + |\nabla u|^2 + v^2)\, dx &= (v, u) \\ &\quad + \int_\Omega (fu + \nabla f \cdot \nabla u + gv)\, dx. \end{aligned}$$

But $(v, u) \leq \frac{1}{2} \int_\Omega (u^2 + |\nabla u|^2 + v^2)\, dx$.

Hence

$$(\lambda-\frac{1}{2})\int_\Omega(u^2+|\nabla u|^2+v^2)\,dx \le \int_\Omega(f,\nabla f,g)\cdot(u,\nabla u,g)\,dx,$$

from which the estimate

$$\|R_\lambda\| \le \frac{1}{\lambda-\frac{1}{2}} \text{ for } \lambda > \frac{1}{2}$$

follows.

It remains to show that the energy equation holds. To this end we note that

$$E(w) := \int_\Omega(|\nabla u(x)|^2 + v(x)^2)\,dx$$

is a C^1 function of $w \in X$, and that if $w \in D(A)$ then

$$E'(w)(Aw) = 2\int_\Omega(\nabla u\cdot\nabla v - v\cdot\Delta u)\,dx = 0.$$

But by Theorem 5(c), $t \mapsto T(t)w$ is C^1 for $w \in D(A)$ with derivative $AT(t)w$.

Thus if $w \in D(A)$ then $t \mapsto E(T(t)w)$ is C^1 with derivative

$$E'(T(t)w)(AT(t)w) = 0.$$

Hence $E(T(t)w) = E(w)$ for all $t \geq 0$, $w \in D(A)$, and thus also for $w \in X$. $\square$

Remarks. (i) The theorem gives the extra information that for Ω unbounded $\|u(\cdot,t)\|_{L^2(\Omega)} \leq e^{t/2}\|(u_0,u_1)\|_X$.

In fact for unbounded Ω it is possible for $\|u(\cdot,t)\|_{L^2(\Omega)} \to \infty$ as $t \to \infty$.

For example, in the case $\Omega = \mathbb{R}^n$ with $u_0 = 0$ we have that

$$\widehat{u}(\xi,t) = \widehat{u}_1(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

For $\varepsilon > 0$ let

$$\widehat{u}_1(\xi) = \begin{cases} r^{-\frac{n}{2}+\varepsilon} & r \in [0,1) \\ 0 & r \geq 1. \end{cases}$$

Then

$$\int_{\mathbb{R}^n} |\hat{u}_1|^2 d\xi = \omega_n \int_0^1 r^{-1+2\varepsilon} dr < \infty,$$

so that $u_1 \in L^2(\mathbb{R}^n)$ and is real and radially symmetric.

But

$$\begin{aligned} \|\hat{u}\|^2_{L^2(\mathbb{R}^n)} &= \omega_n \int_0^1 r^{2\varepsilon-3} \sin^2(rt)\, dr \\ &= t^{2(1-\varepsilon)} \omega_n \int_0^t s^{2\varepsilon-1} \left(\frac{\sin s}{s}\right)^2 ds. \end{aligned}$$

(ii) We also have the regularity result that if $u_0 \in H^1_0(\Omega)$, $\Delta u_0 \in L^2(\Omega)$ and $u_1 \in H^1_0(\Omega)$ then $\Delta u \in C([0,T]; L^2(\Omega))$, $u_t \in C([0,T]; H^1_0(\Omega))$. (This could alternatively be approached via an energy method.)

Weak solutions for linear semigroups

Let X be a real Banach with dual space X^*. We denote the action of $v \in X^*$ on $w \in X$ by $\langle w, v \rangle$.

Let A be a closed linear operator on X with dense domain $D(A)$. Define $D(A^*)$ to be the set of those $v \in X^*$ for which there exists $v^* \in X^*$ such that

$$\langle w, v^* \rangle = \langle Aw, v \rangle \text{ for all } w \in D(A).$$

Note that, since $D(A)$ is dense, if $v \in D(A^*)$ then v^* is unique. We then define the *adjoint* $A^* : D(A^*) \to X^*$ of A by $A^* v = v^*$, so that

$$\langle w, A^* v \rangle = \langle Aw, v \rangle \text{ for all } w \in D(A).$$

Lemma 14 A^* is closed.

Proof. Let $v^{(j)} \in D(A^*)$ with $v^{(j)} \to v$, $A^* v^{(j)} \to z$ in X^*. Then

$$\langle w, A^* v^{(j)} \rangle = \langle Aw, v^{(j)} \rangle \text{ for all } w \in D(A),$$

so that passing to the limit we get

$$\langle w, z \rangle = \langle Aw, v \rangle \text{ for all } w \in D(A).$$

Therefore $v \in D(A^*)$ and $A^* v =$z as required. $\square$

$D(A^*)$ is not in general dense. For example, let $X = L^1(0,1)$ and $A = \frac{d}{dx}$, so that $D(A) = W^{1,1}(0,1)$. Then $X^* = L^\infty(0,1)$ with

$$\langle w, v \rangle = \int_0^1 wv\,dx.$$

Then we need

$$\int_0^1 wA^*v\,dx = \int_0^1 \frac{dw}{dx}v\,dx \text{ for all } w \in W^{1,1}(0,1),$$

which implies that $A^*v = -\frac{dv}{dx}$ with $D(A^*) = W_0^{1,\infty}(0,1)$.

But $\overline{D(A^*)} \subset C([0,1])$ so that $D(A^*)$ is not dense.

Lemma 15 Let $w, z \in X$ satisfy

$$\langle z, v\rangle = \langle w, A^* v\rangle \text{ for all } v \in D(A^*).$$

Then $w \in D(A)$ and $z = Aw$.

Proof. Suppose not. Then $(w, z) \notin G(A)$, where $G(A) \subset X \times X$ denotes the graph of A, which is closed by hypothesis. Hence by the Hahn-Banach theorem there exist $v, v^* \in X^*$ with

$$\langle u, v^*\rangle + \langle Au, v\rangle = 0 \text{ for all } u \in D(A)$$

and $\langle w, v^*\rangle + \langle z, v\rangle \neq 0$.

But this implies that $v \in D(A^*)$ and $v^* = -A^* v$. Hence $\langle z, v\rangle \neq \langle w, A^* v\rangle$, a contradiction. □

Corollary 16 If X is reflexive then $D(A^*)$ is dense in X^*.

Proof. If $\overline{D(A^*)} \neq X^*$ then by Hahn-Banach, since X is reflexive, there exists a nonzero $z \in X$ with $\langle z, v \rangle = 0$ for all $v \in D(A^*)$. But then

$$\langle z, v \rangle = \langle 0, A^* v \rangle \text{ for all } v \in D(A^*),$$

so that by the lemma $z = A^* 0 = 0$, a contradiction. □

Lemma 17 For any C^0–semigroup $\{T(t)\}_{t\geq 0}$ there exists constants $M \geq 1, \omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Proof. For each $w \in X$ we have that

$$\sup_{t\in[0,1]} \|T(t)w\| < \infty.$$

Hence by the uniform boundedness theorem

$$\sup_{t\in[0,1]} \|T(t)\| \leq M < \infty,$$

where $M \geq 1$.

Writing any $t \geq 0$ in the form $t = m + s$ where m is a nonnegative integer and $s \in [0, 1)$ we deduce that

$$\|T(t)\| = \|T(1)^m T(s)\| \leq M^{m+1} \leq M^{t+1} = Me^{(\ln M)t}.$$

$\square$

Corollary 18 $(t, w) \mapsto T(t)w$ is continuous from $[0, \infty) \times X$ to X.

Proof. Let $t_j \to t$, $w_j \to w$. Then

$$T(t_j)w_j - T(t)w = T(t_j)(w_j - w) + T(t_j)w - T(t)w$$

and $\|T(t_j)(w_j - w)\| \leq Me^{\omega t_j}\|w_j - w\| \to 0$. $\square$

We consider for $\tau > 0$ the equation

$$\dot{w}(t) = Aw(t) + f(t), \quad t \in (0, \tau], \tag{8}$$

where $f \in L^1(0, \tau; X)$ and $A : D(A) \to X$ is closed with dense domain $D(A)$.

Definition. A function $w \in C([0, \tau]; X)$ is a weak solution of (8) on $[0, \tau]$ if for every $v \in D(A^*)$ the function $\langle w(t), v \rangle$ is absolutely continuous on $[0, \tau]$ and

$$\frac{d}{dt}\langle w(t), v \rangle = \langle w(t), A^* v \rangle + \langle f(t), v \rangle$$

for a.e. $t \in (0, \tau)$.

Theorem 19 There exists for each $p \in X$ a unique weak solution w to (8) satisfying $w(0) = p$ if and only if A is the generator of a C^0−semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X, and in this case

$$w(t) = T(t)p + \int_0^t T(t-s)f(s)\,ds, \quad t \in [0, \tau]. \qquad (9)$$

(solutions of (9) are often called *mild solutions* of (8))

Proof. Let A generate the C^0−semigroup $\{T(t)\}_{t\geq 0}$. If $w \in D(A), v \in D(A^*)$ we have by Theorem 5 that for $t \geq 0$

$$\begin{aligned} \langle T(t)w, v\rangle &= \langle p, v\rangle + \langle \int_0^t AT(s)w\,ds, v\rangle \\ &= \langle p, v\rangle + \int_0^t \langle T(s)w, A^*v\rangle\,ds. \end{aligned}$$

Since $D(A)$ is dense the second equality holds for $w \in X$, and thus $\langle T(t)w, v\rangle$ is absolutely continuous on $[0, \tau]$ with derivative

$$\frac{d}{dt}\langle T(t)w, v\rangle = \langle T(t)w, A^*v\rangle \text{ for a.e. } t \in [0, \tau].$$

Let $w(t)$ be given by (9). Then $w \in C([0, \tau]; X)$, and

$$\langle w(t), v\rangle = \langle T(t)p, v\rangle + \int_0^t \langle T(t-s)f(s), v\rangle \, ds.$$

Suppose first that $f \in C([0, \tau]; X)$ and calculate the time derivative of the second term.

For $h \neq 0$ we have that

$$\frac{1}{h}\left(\int_0^{t+h}\langle T(t+h-s)f(s),v\rangle\,ds-\int_0^t\langle T(t-s)f(s),v\rangle\,ds\right)$$
$$=\frac{1}{h}\int_0^t\langle T(t+h-s)f(s)-T(t-s)f(s),v\rangle\,ds$$
$$+\frac{1}{h}\int_t^{t+h}\langle T(t+h-s)f(s),v\rangle\,ds$$
$$\longrightarrow\int_0^t\langle T(t-s)f(s),A^*v\rangle\,ds+\langle f(t),v\rangle,$$

where we used Corollary 18.

Therefore

$$\frac{d}{dt}\langle w(t),v\rangle=\langle w(t),A^*v\rangle+\langle f(t),v\rangle$$

and so w is a weak solution.

If $f \in L^1(0,\tau);X)$ there exists a sequence $f_j \in C([0,\tau];X)$ with $f_j \to f$ in $L^1(0,\tau;X)$.

Let

$$w_j(t) = T(t)p + \int_0^t T(t-s)f_j(s)\,ds.$$

Then

$$\|w_j(t) - w(t)\| \le Me^{\omega\tau}\int_0^t \|f_j(s) - f(s)\|\,ds,$$

so that $w_j \to w$ in $C([0,\tau];X)$.

But we have that

$$\langle w_j(t), v\rangle = \langle p, v\rangle + \int_0^t (\langle w_j(s), A^* v\rangle + \langle f_j(s), v\rangle)\, ds,\ t \in [0, \tau],$$

so that passing to the limit we deduce that w is a weak solution.

To show that w is unique, let $u = w - \bar{w}$ be the difference of two solutions. Then

$$\langle u(t), v\rangle = \langle \int_0^t u(s)\, ds, A^* v\rangle$$

for all $v \in D(A^*)$, so that by Lemma 15, $z(t) := \int_0^t u(s)\, ds \in D(A)$ and $\dot{z}(t) = Az(t)$. Hence by Theorem 5(d) $z = 0$ and $w = \bar{w}$.

Conversely let A be such that there is a unique weak solution for each initial data p, and for $t \in [0,\tau]$ define $T(t)p = w(t) - W(t), t \in [0,\tau]$, where W is the unique weak solution corresponding to zero initial data. If $t \geq 0$ then $t = m\tau + s$ where m is a nonnegative integer and $s \in [0,\tau)$ and we define $T(t)p = T(\tau)^m T(s)p$.

The map $\theta : X \to C([0,\tau]; X)$ defined by $\theta(p) = T(\cdot)p$ has closed graph, and hence $\{T(t)\}_{t \geq 0}$ is a C^0-semigroup.

Let B be the generator of $\{T(t)\}_{t\geq 0}$, and let $p \in D(B)$. Then if $v \in D(A^*)$

$$\frac{d}{dt}\langle T(t)p, v\rangle|_{t=0+} = \langle Bp, v\rangle = \langle p, A^*v\rangle,$$

so that by Lemma 15 $p \in D(A)$ and $Ap = B$.

So it remains to prove that $D(A) \subset D(B)$. To this end note that Lemma 15 implies that if $p \in D(A)$ then $\int_0^t T(s)p\,ds$ and $\int_0^t T(s)Ap\,ds$ belong to $D(A)$ and

$$\begin{aligned} T(t)p &= p + A\int_0^t T(s)p\,ds \\ T(t)Ap &= Ap + A\int_0^t T(s)Ap\,ds. \end{aligned}$$

Define

$$z(t) = \int_0^t T(s)Ap\,ds - \int_0^t AT(s)p\,ds.$$

Then $z \in C([0,\tau];X)$ and $z(0) = 0$. For $v \in D(A^*)$ we have that

$$\begin{aligned}\frac{d}{dt}\langle z(t), v\rangle &= \langle T(t)Ap, v\rangle - \langle AT(t)p, v\rangle \\ &= \langle Ap + A\int_0^t T(s)Ap\,ds, v\rangle \\ &\quad -\langle p + A\int_0^t T(s)p\,ds, A^*v\rangle \\ &= \langle z(t), A^*v\rangle.\end{aligned}$$

Thus z is a weak solution of $\dot{z} = Az$ with $z(0) = 0$, and the hypotheses imply that $z = 0$.

Hence

$$\begin{aligned}
\lim_{h \to 0+} \frac{T(h)p - p}{h} &= \lim_{h \to 0+} \frac{1}{h} A \int_0^h T(s)p\, ds \\
&= \lim_{h \to 0+} \frac{1}{h} \int_0^h T(s)Ap\, ds \\
&= Ap,
\end{aligned}$$

so that $p \in D(B)$ and $Bp = Ap$. $\square$

The adjoint of the wave operator

For $\Omega \subset \mathbb{R}^n$ open we have $X = H_0^1(\Omega) \times L^2(\Omega)$ with inner product

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \rangle = \int_\Omega (up + \nabla u \cdot \nabla p + vq)\, dx.$$

$$A = \begin{pmatrix} 0 & \mathbf{1} \\ \Delta & 0 \end{pmatrix},$$

with

$$D(A) = \{ \begin{pmatrix} u \\ v \end{pmatrix} \in X : \Delta u \in L^2(\Omega), v \in H_0^1(\Omega) \}$$

$$\begin{pmatrix} \chi \\ \psi \end{pmatrix} \in D(A^*) \text{ and } A^* \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

iff $\begin{pmatrix} \chi \\ \psi \end{pmatrix} \in X$ and

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \rangle = \langle \begin{pmatrix} v \\ \Delta u \end{pmatrix}, \begin{pmatrix} \chi \\ \psi \end{pmatrix} \rangle \text{ for all } \begin{pmatrix} u \\ v \end{pmatrix} \in D(A),$$

that is

$$\int_\Omega (pu + \nabla p \cdot \nabla u + qv)\, dx$$
$$= \int_\Omega (\chi v + \nabla \chi \cdot \nabla v + \psi \Delta u)\, dx$$

for all $u, v \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$.

Hence we have the two equations:

$$\int_\Omega (\chi v + \nabla\chi \cdot \nabla v)\, dx = \int_\Omega qv\, dx \qquad (10)$$

for all $v \in H_0^1(\Omega)$, and

$$\int_\Omega (pu + \nabla p \cdot \nabla u)\, dx = \int_\Omega \psi \Delta u\, dx \qquad (11)$$

for all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$.

(10) implies that, for $q \in L^2(\Omega)$, χ is the unique solution of

$$-\Delta\chi + \chi = q \text{ for } \chi \in H_0^1(\Omega),$$

which we write as $q = (\mathbf{1} - \Delta)\chi$.

To handle (11) is more tricky. Given $p \in H_0^1(\Omega)$ we first note that, approximating p by C_0^∞ functions,

$$\int_\Omega \nabla p \cdot \nabla u \, dx = -\int_\Omega p \Delta u \, dx,$$

so that we can write (11) as

$$\int_\Omega pu \, dx = \int_\Omega (p + \psi) \Delta u \, dx$$

for all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$.

Now let $v \in H_0^1(\Omega)$ be the unique solution of

$$-\Delta v + v = \psi,$$

which we write as $v = (1 - \Delta)^{-1}\psi$.

Then we have that

$$-\int_\Omega v\Delta u\,dx + \int_\Omega vu\,dx = \int_\Omega \psi u\,dx,$$

which combined with (11) gives

$$\int_\Omega (-\Delta u + u)(p + \psi - v)\,dx = 0.$$

Now let $u \in H_0^1(\Omega)$ be the unique solution of

$$-\Delta u + u = p + \psi - v.$$

Hence

$$\int_\Omega (p + \psi - v)^2 dx = 0$$

and so $v = p + \psi$ and $p = [(1 - \Delta)^{-1} - 1]\psi$.

We have proved

Theorem 20 The adjoint of A is given by

$$A^* = \begin{pmatrix} 0 & (1-\Delta)^{-1} - 1 \\ 1-\Delta & 0 \end{pmatrix},$$

with

$$D(A^*) = \{\begin{pmatrix} \chi \\ \psi \end{pmatrix} \in X : \Delta\chi \in L^2(\Omega), \psi \in H_0^1(\Omega)\}.$$

Now let us see what the definition of weak solution means: we have that $w = \begin{pmatrix} u \\ v \end{pmatrix}$ is such that for any $V = \begin{pmatrix} \chi \\ \psi \end{pmatrix} \in D(A^*)$ we have that $\langle w, V\rangle$ is differentiable with derivative

$$\frac{d}{dt}\langle w(t), V\rangle = \langle w(t), A^*V\rangle.$$

That is

$$\frac{d}{dt}\int_\Omega (u\chi + \nabla u\cdot\nabla\chi + v\psi)\,dx$$
$$= \int_\Omega (up + \nabla u\cdot\nabla p + vq)\,dx,$$

i.e.

$$\frac{d}{dt}\int_\Omega (uq + v\psi)\,dx = \int_\Omega (-\nabla\psi\cdot\nabla u + vq)\,dx$$

for all $\psi \in H_0^1(\Omega)$ and $q = \chi - \Delta\chi$ with $\chi \in H_0^1(\Omega)$ and $\Delta\chi \in L^2(\Omega)$.

Equivalently,

$$\frac{d}{dt}\int_\Omega uq\,dx = \int_\Omega vq\,dx, \quad \frac{d}{dt}\int_\Omega v\psi\,dx = -\int_\Omega \nabla\psi\cdot\nabla u\,dx.$$

The first equation gives that, for all $t \in [0, \tau]$,

$$z(x,t) := u(x,t) - u(x,0) + \int_0^t v(x,s), ds$$

satisfies

$$\int_\Omega z(x,t)(\chi(x) - \Delta\chi(x))\,dx = 0$$

for all $\chi \in H_0^1(\Omega)$ with $\Delta\chi \in L^2(\Omega)$. Choosing χ to solve $-\Delta\chi + \chi = z(\cdot,t)$ we deduce that $z(\cdot,t) = 0$, so that $v = u_t$.

Then the second equation gives that

$$\frac{d}{dt}\int_\Omega u_t\psi\,dx = -\int_\Omega \nabla u \cdot \nabla\psi\,dx \text{ for all } \psi \in H_0^1(\Omega),$$

recovering our previous definition of weak solution, and hence proving uniqueness.

Lemma 21 The range of Δ on $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, i.e. if $z \in L^2(\Omega)$ satisfies

$$\int_\Omega z \Delta u \, dx = 0$$

for all $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ then $z = 0$.

Proof. It would be nice to find $u \in H_0^1(\Omega)$ with $\Delta u = z$, when we could deduce $z = 0$ immediately. However in general (for Ω unbounded, e.g. $\Omega = \mathbb{R}^n$) there is no solution $u \in H_0^1(\Omega)$.

Instead, for $\varepsilon > 0$ let u_ε be the unique solution in $H_0^1(\Omega)$ to

$$-\Delta u_\varepsilon + \varepsilon u_\varepsilon = z.$$

Then

$$\begin{aligned}\int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2)\,dx &= \int_\Omega z u_\varepsilon\,dx \\ &\leq \frac{1}{2}\int_\Omega (\varepsilon u_\varepsilon^2 + \frac{1}{\varepsilon} z^2)\,dx,\end{aligned}$$

so that $\varepsilon u_\varepsilon$ is bounded in $L^2(\Omega)$ and

$$\lim_{\varepsilon\to 0+} \int_\Omega |\nabla(\varepsilon u_\varepsilon)|^2 dx = 0.$$

Hence $\varepsilon u_\varepsilon$ is bounded in $H_0^1(\Omega)$ and thus we may assume that $\varepsilon u_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$, and $\nabla v = 0$, hence $v = 0$, and so

$$0 = -\int_\Omega z \Delta u_\varepsilon\,dx = \int_\Omega z(z - \varepsilon u_\varepsilon)\,dx \to \int_\Omega z^2\,dx.$$

□

Theorem 22

(i) $A^* : D(A^*) \to X = X^*$ is one-to-one.

(ii) $R(A) = \{Az : z \in D(A)\}$ is dense in X.

(iii) There is no nontrivial linear constant of motion, that is there is no nonzero $z \in X$ such that $\langle T(t)p, z\rangle = \langle p, z\rangle$ for all $t \geq 0$ and $p \in D(A)$.

Proof. If (ii) were false there would exist a nonzero $\begin{pmatrix} w \\ z \end{pmatrix} \in X$ with $\langle \begin{pmatrix} w \\ z \end{pmatrix}, A \begin{pmatrix} u \\ v \end{pmatrix} \rangle = 0$ for all $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$, that is

$$\int_\Omega (wv + \nabla w \cdot \nabla v + z\Delta u)\, dx = 0$$

for all $u, v \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$.

Taking first $u = 0$ we find that $-w + \Delta w = 0$, whence

$$\int_\Omega (|\nabla w|^2 + w^2)\, dx = 0$$

and $w = 0$. Then $z = 0$ by Lemma 21.

Now suppose $A^* \begin{pmatrix} \chi \\ \psi \end{pmatrix} = 0$, for some $\begin{pmatrix} \chi \\ \psi \end{pmatrix} \in D(A^*)$. Then

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, A^* \begin{pmatrix} \chi \\ \psi \end{pmatrix} \rangle = 0$$

for all $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$, so that by (ii) $\chi = \psi = 0$. Hence A^* is one-to-one.

Finally, if $\langle T(t)p, z\rangle$ is constant in t for all $p \in D(A)$ then

$$0 = \frac{d}{dt}\langle T(t)p, z\rangle_{t=0} = \langle Ap, z\rangle$$

for all $p \in D(A)$, so that $z = 0$ by (ii). $\square$

Resolving the puzzle over Kirchhoff's formula

Recall Kirchhoff's formula for a C^2 solution of the wave equation for $n = 3$ and initial data $u(\cdot, 0) = u_0$, $u_t(\cdot, 0) = u_1$, namely

$$u(x,t) = \frac{1}{4\pi t}\int_{S(x,t)} u_1(y)\,dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S(x,t)} u_0(y)\,dS_y\right).$$

Supposing $u_0 = 0$ we have that

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi t}\int_{S(x,t)} u_1(y)\,dS_y \\ &= \frac{t}{4\pi}\int_{S^2} u_1(x+zt)\,dS_z \end{aligned}$$

Hence, formally we have

$$\begin{aligned} u_t(x,t) &= \frac{1}{t}u(x,t) + \frac{t}{4\pi}\int_{S^2} \nabla u_1(x+zt)\cdot z\, dS_z \\ &= \frac{1}{t}u(x,t) + \frac{t^2}{4\pi}\Delta_x \int_B u_1(x+zt)\, dz, \end{aligned}$$

where $B = B(0,1)$.

However we know that for $u_1 \in L^2(\mathbb{R}^3)$ we have that $u_t(\cdot,t) \in L^2(\mathbb{R}^3)$ for each $t \geq 0$, and so this seems to suggest that for each $w \in L^2(\mathbb{R}^3)$

$$w_B(x) := \int_B w(x+z)\, dz$$

satisfies $\Delta w_B \in L^2(\mathbb{R}^3)$.

Surprisingly this is in fact true, and we have essentially proved it, since we can approximate u_1 in $L^2(\mathbb{R}^3)$ by functions $u_{1j} \in C_0^\infty(\mathbb{R}^3)$. Denoting the solution with initial data $u_0 = 0, u_{1j}$ by $u_j(x,t)$ we have that u_j is smooth (this follows for example from the formula for the solution in terms of Fourier transforms) and hence, for each $t > 0$, $\Delta \int_B u_{1j}(x+zt)\,dz$ converges in $L^2(\mathbb{R}^3)$ to some function $v(\cdot,t)$. Multiplying by a function $\varphi \in C_0^\infty(\mathbb{R}^3)$ and integrating we obtain

$$\int_{\mathbb{R}^3} v(x,t)\varphi(x)\,dx = \int_B u_1(x+zt)\,dz\,\Delta\varphi(x)\,dx$$

as required. However let us give a more direct and general proof.

Theorem 23 Let $w \in L^2(\mathbb{R}^n)$. Then w_B defined by

$$w_B(x) = \int_B w(x+z)\,dz$$

belongs to $H^{\frac{n+1}{2}}(\mathbb{R}^n)$.

Lemma 24 The Fourier transform $\widehat{\chi}_B$ of the characteristic function χ_B of the unit ball B satisfies

$$\widehat{\chi}_B(\xi) = C_n \frac{\cos(|\xi| - \frac{\pi}{4}(n+1))}{|\xi|^{\frac{n+1}{2}}} + O\left(\frac{1}{|\xi|^{\frac{n}{2}+1}}\right),$$

as $|\xi| \to \infty$ for some positive constant C_n.

Proof. Since B is invariant under rotations, writing $\xi = r\theta$ with $r = |\xi| > 0, \theta \in S^{n-1}$, we see that

$$(2\pi)^{\frac{n}{2}}\widehat{\chi}_B(\xi) = \int_B e^{-ir\theta\cdot x}\,dx = \int_B e^{-irR\theta\cdot x}\,dx = \int_B e^{-irx_n}\,dx,$$

where we chose $R \in SO(n)$ with $R\theta = e_n$.

But

$$\begin{aligned}\int_B e^{-irx_n}dx &= \int_{-1}^{1} e^{-irx_n}\int_{|x'|^2\leq 1-x_n^2} dx'\,dx_n \\ &= \frac{\omega_{n-1}}{n-1}\int_{-1}^{1} e^{-irx_n}(1-x_n^2)^{\frac{n-1}{2}}\,dx_n.\end{aligned}$$

The result then follows for example from Makarov & Podkorytov pp 605-606. We do the calculation for $n = 3$.

For $n = 3$

$$\int_B e^{-irx_3}\, dx_3 \;=\; \pi \int_{-1}^{1} \cos(rt)(1-t^2)\, dt$$
$$\;=\; 4\pi \left(-\frac{\cos r}{r^2} + \frac{\sin r}{r^3}\right),$$

so that

$$\widehat{\chi}_B(\xi) = \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} \left(-\frac{\cos|\xi|}{|\xi|^2} + \frac{\sin|\xi|}{|\xi|^3}\right).$$

$\square$

Proof of Theorem 23. We note that

$$w_B(x) = \int_{\mathbb{R}^n} \chi_B(z) w(x-z)\, dz = (w * \chi_B)(x).$$

Hence $\widehat{w_B}(\xi) = (2\pi)^{\frac{n}{2}}\hat{w}(\xi)\hat{\chi}_B(\xi)$, so that $w_B \in H^\alpha(\mathbb{R}^n)$ iff

$$(1 + |\xi|^\alpha)\hat{w}(\xi)\hat{\chi}_B(\xi) \in L^2(\mathbb{R}^n),$$

which holds provided $(1+|\xi|^\alpha)\hat{\chi}_B(\xi) \in L^\infty(\mathbb{R}^n)$, and the result follows from Lemma 25. □

Remark. Because of results of Herz (Annals of Math. 1962) the same results holds if B is replaced by a bounded convex set $C \subset \mathbb{R}^n$ with sufficiently smooth boundary having everywhere positive Gaussian curvature. However if B is replaced by the unit cube $Q = (-1,1)^n$ then $w_Q(x) = \int_Q w(x+\xi)\, d\xi$ has less regularity than w_B.

In fact

$$\hat{\chi}_Q(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_Q e^{-i\xi\cdot x}\, dx = \frac{2^n}{(2\pi)^{\frac{n}{2}}} \prod_{j=1}^{n} \frac{\sin \xi_j}{\xi_j}.$$

Hence if $\alpha \geq 0$ then $(1 + |\xi|^\alpha)\hat{\chi}_Q(\xi) \in L^\infty(\mathbb{R}^n)$ iff $\alpha \leq 1$. Hence $w_Q \in H^1(\mathbb{R}^n)$ but in general $w_Q \notin H^\alpha(\mathbb{R}^n)$ for $\alpha > 1$.

Semilinear equations

Let X be a real Banach space, and A be the generator of a C^0–semigroup of bounded linear operators on X, which we henceforth denote by $\{e^{At}\}_{t\geq 0}$. We consider the semilinear equation

$$\dot{w} = Aw + f(w) \tag{12}$$

with initial data

$$w(0) = p \in X,$$

where $f : X \to X$ is *locally Lipschitz*, that is for any $M > 0$ there exists a constant $C_M > 0$ such that

$$\|f(w_1) - f(w_2)\| \leq C_M \|w_1 - w_2\| \text{ for } \|w_1\| \leq M, \|w_2\| \leq M.$$

Definition. A function $w \in C([0,\tau];X)$ is a weak solution of (12) on the interval $[0,\tau], \tau > 0$, if for each $v \in D(A^*)$ the function $\langle w(t), v\rangle$ is absolutely continuous on $[0,\tau]$ and

$$\frac{d}{dt}\langle w(t), v\rangle = \langle w(t), A^*v\rangle + \langle f(w(t)), v\rangle \text{ for all } t \in [0,\tau].$$

To prove existence we will make use of the contraction mapping theorem with parameters.

Theorem 25 Let (E,d) be a complete metric space and Λ a metric space of parameters, and let $F : E \times \Lambda \to E$ be such that
(i) F is a uniform contraction, i.e. there exists $k \in (0,1)$ such that $d(F(u,\lambda), F(v,\lambda)) \leq kd(u,v)$ for all $u, v \in E$, $\lambda \in \Lambda$.
(ii) for each $u \in E$, $F(u,\lambda)$ is continuous in λ.
Then for each $\lambda \in \Lambda$ there exists a unique fixed point $u^*(\lambda) \in E$, i.e. $F(u^*(\lambda),\lambda) = u^*(\lambda)$, and $u^*(\lambda)$ depends continuously on λ.

Proof of Theorem 25. The existence of $u^*(\lambda)$ follows from the Banach Fixed Point Theorem, and we just need to show that $u^*(\lambda)$ depends continuously on λ. But if $\lambda_j \to \lambda$ then

$$\begin{aligned} d(u^*(\lambda_j), u^*(\lambda)) &= d(F(u^*(\lambda_j), \lambda_j), F(u^*(\lambda), \lambda)) \\ &\leq d(F(u^*(\lambda_j), \lambda_j), F(u^*(\lambda), \lambda_j)) \\ &\quad + d(F(u^*(\lambda), \lambda_j), F(u^*(\lambda), \lambda)) \\ &\leq k d(u^*(\lambda_j), u^*(\lambda)) + o(1), \end{aligned}$$

and, since $k < 1$, $d(u^*(\lambda_j), u^*(\lambda)) \to 0$. $\Box$

Theorem 26 For any $p \in X$ there exists a unique weak solution w to (12) satisfying $w(0) = p$ defined on a maximal interval of existence $[0, t_{\max})$, $t_{\max} > 0$, and

$$w(t) = e^{At}p + \int_0^t e^{A(t-s)} f(w(s))\, ds \qquad (13)$$

for $t \in [0, t_{\max})$. If $t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}-} \|w(t)\| = \infty \text{ and } \int_0^{t_{\max}} \|f(w(t))\|\, dt = \infty.$$

The solution $w(\cdot, p)$ depends continuously on p. More precisely, if the solution $w(\cdot, p)$ exists on the interval $[0, \tau]$ and $p_j \to p$ in X, then for large enough j the solution $w(\cdot, p_j)$ exists on $[0, \tau]$ and $w(\cdot, p_j) \to w(\cdot, p)$ in $C([0, \tau]; X)$.

Proof. By Theorem 19 w is a weak solution on $[0,\tau]$ with $w(0)=p$ iff w satisfies (13) for $t\in[0,\tau]$.

Step 1 (local existence). We show that given $M>0$ there exists $T_M>0$ such that if $\|p\|\leq M$ then there is a unique solution $w(\cdot,p)$ of (13) on $[0,T_M]$, and that if $p_j\to p$, $\|p_j\|\leq M$, then $w(\cdot,p_j)\to w(\cdot,p)$ in $C([0,T_M];X)$. Recall that by Lemma 17

$$\|e^{At}\|\leq K \text{ for all } t\in[0,1]$$

for some $K\geq 1$, and let $T_M=\min(\frac{M}{C_{2KM}2KM+\|f(0)\|},1)$.

Let $E=\{w\in C([0,T_M];X):\ \big|w\big|\ \leq 2KM\}$, where $\big|w\big|:=\max_{t\in[0,T_M]}\|w(t)\|$. Then E is a closed subset of $C([0,T_M];X)$ and hence is complete with respect to the metric $d(w_1,w_2)=\big|w_1-w_2\big|$.

For $\|p\| \leq M$ and $w \in E$ define

$$T(w,p)(t) = e^{At}p + \int_0^t e^{A(t-s)} f(w(s))\, ds.$$

Then $T(w,p) \in C([0,T_M];X)$ and for $w \in E$, $t \in [0,T_M]$

$$\begin{aligned}\|T(w,p)(t)\| &\leq KM + K\int_0^t (\|f(w(s)) - f(0)\| + \|f(0)\|)\, ds \\ &\leq KM + KT_M(C_{2KM}2KM + \|f(0)\|) \leq 2KM,\end{aligned}$$

so that $T(\cdot,p) : E \to E$.

Furthermore, if $w_1, w_2 \in E$ and $t \in [0,T_M]$ then

$$\begin{aligned}\|T(w_1,p)(t) - T(w_2,p)(t)\| &\leq \int_0^{T_M} K\|f(w_1(s)) - f(w_2(s))\|\, ds \\ &\leq T_M K C_{2KM} \Big| w_1 - w_2 \Big|,\end{aligned}$$

and $T_M K C_{2KM} \leq \frac{1}{2}$, so that $T(\cdot,p)$ is a uniform contraction.

Also, since

$$\|T(w,p)(t) - T(w,q)(t)\| \le K\|p-q\|,$$

$p \mapsto T(w,p)$ is continuous for $\|p\| \le M$. Hence by Theorem 25 there exists a unique fixed point $w(\cdot,p) \in E$ which depends continuously on p for $\|p\| \le M$.

Step 2 (definition of t_{max} *and uniqueness).*

Define for $p \in X$

$t_{\text{max}} = \sup\{\tau > 0 : \exists$ a solution w on $[0,\tau]$ with $w(0) = p\}$.

By Step 1 $t_{\text{max}} > 0$.

Let $p \in X$ and suppose for contradiction that there are two solutions $w \neq \tilde{w}$ defined on $[0, \tau]$, $\tau > 0$, with $w(0) = \tilde{w}(0) = p$.

Let $t_0 = \inf\{t \in [0, \tau] : w(t) \neq \tilde{w}(t)\}$. Then by Step 1 $t_0 > 0$, and clearly $t_0 < \tau$. Since $w(t) = \tilde{w}(t)$ for all $t \in [0, t_0)$, by the continuity of solutions $w(t_0) = \tilde{w}(t_0)$. Hence we can apply Step 1 with initial data $w(t_0)$ to conclude that there exists a unique solution $\widehat{w}$ of

$$\widehat{w}(t) = e^{At} w(t_0) + \int_0^t e^{A(t-s)} f(\widehat{w}(s))\, ds$$

on some interval $[0, \varepsilon]$, $\varepsilon > 0$.

$$
\begin{aligned}
\text{But } w(t+t_0) &= e^{A(t+t_0)} + \int_0^{t+t_0} e^{A(t+t_0-s)} f(w(s))\,ds \\
&= e^{At}\left(w(t_0) - \int_0^{t_0} e^{A(t_0-s)} f(w(s))\,ds\right) \\
&\qquad + \int_0^{t+t_0} e^{A(t+t_0-s)} f(w(s))\,ds \\
&= e^{At} w(t_0) + \int_{t_0}^{t+t_0} e^{A(t+t_0-s)} f(w(s))\,ds \\
&= e^{At} w(t_0) + \int_0^{t} e^{A(t+t_0-s)} f(w(t_0+s))\,ds.
\end{aligned}
$$

Hence $w(t+t_0) = \tilde{w}(t+t_0) = \widehat{w}(t)$ for $t \in [0, \varepsilon]$, and this contradiction proves uniqueness.

Step 3 (continuous dependence). Let w be a solution on $[0, \tau]$ with $w(0) = p$ and choose $M > \max_{t\in[0,\tau]} \|w(t)\|$. Let $p_j \to p$. Then we can assume that $\|p_j\| \le M$ for all j and so by Step 1 a solution w_j with $w_j(0) = p_j$ exists on $[0, T_M]$ for each j, and $w_j \to w$ in $C([0, T_M]; X)$.

If $\tau \leq T_M$ we are done. If $\tau > T_M$ then $w_j(T_M) \to w(T_M)$, and repeating the argument with initial data $w_j(T_M)$ we have that w_j is defined on $[0, 2T_M]$ and $w_j \to w$ in $C([T_M, 2T_M]; X)$, and hence in $C([0, 2T_M]; X)$. After N such steps, where $(N-1)T_M < \tau \leq NT_M$, we obtain $w_j \to w$ in $C([0, \tau]; X)$ as required.

Step 4 (blow-up if $t_{\text{max}} < \infty$).

Let $t_{\text{max}} < \infty$ and assume for contradiction that there exists a sequence $t_j \to t_{\text{max}}-$ with $\|w(t_j)\|$ bounded. Then by Step 1 with initial data $w(t_j)$ we get existence of a solution on $[t_j, t_j + \varepsilon]$ for some $\varepsilon > 0$ independent of j, contradicting the definition of t_{max}.

Hence

$$\lim_{t \to t_{\max}} \|w(t)\| = \infty.$$

Since $\|e^{At}\| \le K < \infty$ for $t \in [0, t_{\max}]$ we have that

$$\|w(t)\| \le K\|p\| + K \int_0^{t_{\max}} \|f(w(s))\| \, ds,$$

and so

$$\int_0^{t_{\max}} \|f(w(t))\| \, dt = \infty. \qquad \square$$

We now show that the under suitable hypotheses the weak solution satisfies an energy identity.

Theorem 27 Suppose that there exist functions $V \in C^1(X), h \in C(X)$ such that

$$\langle Aw + f(w), V'(w)\rangle = h(w) \text{ for all } w \in D(A).$$

Then the weak solution w in Theorem 26 satisfies

$$V(w(t)) = V(p) + \int_0^t h(w(s))\,ds \text{ for all } t \in [0, t_{\max}).$$

Proof. Let $0 < \tau < t_{\max}$ and for $t \in [0,\tau]$ let $F(t) = f(w(t))$. Then $F \in C([0,\tau];X)$ and so there exists a sequence $F_j \in C^1([0,\tau];X)$ with $F_j \to F$ in $C([0,\tau];X)$. Also let $p_j \to p$ in X with $p_j \in D(A)$ for all j.

Define

$$w_j(t) = e^{At}p_j + \int_0^t e^{A(t-s)}F_j(s)\,ds.$$

Then

$$w_j(t) = e^{At}p_j + \int_0^t e^{As}F_j(t-s)\,ds$$

so that $w_j \in C^1([0,\tau];X)$.

But if $v \in D(A^*)$ then

$$\langle \dot{w}_j(t), v\rangle = \langle w_j(t), A^*v\rangle + \langle F_j(t), v\rangle,$$

so that by Lemma 15 we have $w_j(t) \in D(A)$ and

$$\dot{w}_j(t) = Aw_j(t) + F_j(t).$$

Hence

$$
\begin{aligned}
V(w_j(t)) &= V(p_j) + \int_0^t \langle Aw_j(s) + F_j(s), V'(w_j(s)) \rangle \, ds \\
&= V(p_j) + \int_0^t \langle Aw_j(s) + f(w_j(s)), V'(w_j(s)) \rangle \, ds \\
&\quad + \int_0^t \langle F_j(s) - F(s), V'(w_j(s)) \rangle \, ds \\
&\quad + \int_0^t \langle f(w(s)) - f(w_j(s)), V'(w_j(s)) \rangle \, ds.
\end{aligned}
$$

But

$$
\|w_j(t) - w(t)\| \leq K \left(\|p_j - p\| + \int_0^t \|F_j(t) - F(t)\| \, ds \right),
$$

so that $w_j \to w$ in $C([0, \tau]; X)$

Since V is C^1 we thus have that

$$V(w_j(t)) = V(p_j) + \int_0^t h(w_j(s))\, ds + o(1),$$

and passing to the limit we obtain

$$V(w(t)) = V(p) + \int_0^t h(w(s))\, ds, \text{ for } t \in [0, \tau]$$

as required. $\square$

Note that $D(A)$ is a Banach space under the norm

$$\|w\|_{D(A)} = \|w\| + \|Aw\|,$$

and that $\{e^{At}\}_{t \geq 0}$ is a C^0-semigroup on $D(A)$.

Theorem 28 (see Pazy, Thm 6.1.6) Let X be reflexive. Then if $p \in D(A)$ the unique weak solution w in Theorem 25 belongs to $C^1([0, t_{\max}); X) \cap C([0, t_{\max}); D(A))$ and satisfies

$$\dot{w}(t) = Aw(t) + f(w(t)), \text{ for all } t \in [0, t_{\max}).$$

Proof. Let w be a solution on $[0, \tau]$, $\tau > 0$. For $t \in [0, \tau)$ and sufficiently small $h > 0$ we have that

$$\begin{aligned} \frac{1}{h}(w(t+h) - w(t)) \; &= \; e^{At}\left(\frac{e^{Ah} - \mathbf{1}}{h}\right)p \\ &+ \frac{1}{h}\left(\int_0^{t+h} e^{A(t+h-s)} f(w(s))\, ds - \int_0^t e^{A(t-s)} f(w(s))\, ds\right) \end{aligned}$$

$$= \; e^{At}\left(\frac{e^{Ah}-1}{h}\right)p+\frac{1}{h}\int_0^h e^{A(t+h-s)}f(w(s))\,ds$$
$$+\frac{1}{h}\int_0^t e^{A(t-s)}(f(w(s+h))-f(w(s)))\,ds$$

The sum of the first two terms tends to $e^{At}(Ap+f(p))$ as $h\to 0$. Hence
$$\|\frac{w(t+h)-w(t)}{h}\|\leq C+K\int_0^t\|\frac{w(s+h)-w(s)}{h}\|\,ds,\; t\in[0,\tau),$$
so that by Gronwall's inequality
$$\|\frac{w(t+h)-w(t)}{h}\|\leq Ce^{Kt},\; t\in[0,\tau).$$

Hence, since X is reflexive for fixed t there is a sequence $h_j \to 0$ such that as $j \to \infty$

$$\frac{w(t+h_j) - w(t)}{h_j} \rightharpoonup z(t) \text{ in } X.$$

But for any $v \in D(A^*)$

$$\langle w(t), v\rangle = \langle p, v\rangle + \int_0^t (\langle w(s), A^*v\rangle + \langle f(w(s)), v\rangle), \, ds,$$

and so

$$\begin{aligned} \langle z(t), v\rangle &= \lim_{j\to\infty} \frac{1}{h_j} \int_t^{t+h_j} (\langle w(s), A^*v\rangle + \langle f(w(s)), v\rangle), \, ds, \\ &= \langle w(t), A^*v\rangle + \langle f(w(t)), v\rangle. \end{aligned}$$

Hence by Lemma 15, $w(t) \in D(A)$ and

$$z(t) = Aw(t) + f(w(t)).$$

Also $\langle z(t), v\rangle$ is continuous in t for each $v \in D(A^*)$ with $\|z(t)\| \leq Ce^{Kt}$ for $t \in [0, \tau)$. Since by Corollary 16 $D(A^*)$ is dense, it follows that $z : [0, \tau) \to X$ is weakly continuous, hence separably valued, and thus strongly measurable and integrable. Therefore

$$\langle w(t) - p - \int_0^t z(s)\, ds, v\rangle = 0$$

for all $v \in D(A^*)$, so that

$$w(t) = p + \int_0^t z(s)\, ds \text{ for all } t \in [0, \tau),$$

giving the result. □

Remark. This gives a different proof of Theorem 27 in the case X reflexive, since we have that, for $p \in D(A)$, $V(w(\cdot)) \in C^1([0, t_{\mathsf{max}}); X)$ with

$$\frac{d}{dt}V(w(t)) = \langle Aw(t)+f(w(t)), V'(w(t))\rangle = h(w(t)),$$

so that

$$V(w(t)) = V(p)+\int_0^t h(w(s))\, ds \text{ for all } t \in [0, t_{\mathsf{max}}),$$

and if $p \in X$ we can approximate by $p_j \in D(A)$.

Let us apply these results to the semilinear wave equation

$$u_{tt} - \Delta u + g(u) = 0,$$

which we write in the form

$$\dot{w} = Aw + f(w), \qquad (14)$$

where $w = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ and $f(w) = \begin{pmatrix} 0 \\ -g(u) \end{pmatrix}$.

Let $\Omega \subset \mathbb{R}^n$ be open and $X = H_0^1(\Omega) \times L^2(\Omega)$, so that

$$D(A) = \{\begin{pmatrix} u \\ v \end{pmatrix} \in X : \Delta u \in L^2(\Omega), v \in H_0^1(\Omega)\}.$$

We assume that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and that if $n > 1$, for some constant $C > 0$,

$$|g(u) - g(\tilde{u})| \le C(1 + |u|^{p-1} + |\tilde{u}|^{p-1})|u - \tilde{u}| \text{ for all } u, \tilde{u},$$

where $1 \le p \le \frac{n}{n-2}$ if $n \ge 3$, $p \ge 1$ arbitrary if $n = 2$. If $\mathcal{L}^n(\Omega) = \infty$ we assume additionally that $g(0) = 0$.

We claim that $f : X \to X$ and is locally Lipschitz, equivalently $g : H_0^1(\Omega) \to L^2(\Omega)$ and is locally Lipschitz.

Recall that $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$, where $2 \le q \le \frac{2n}{n-2}$ if $n \ge 3$, $q \ge 2$ arbitrary if $n = 2$, and that if $n = 1$ $H_0^1(\Omega)$ is continuously embedded in $C_B(\Omega)$, the space of bounded continuous functions on Ω with the sup norm.

Hence if $n \geq 3$, for example, if $\|u\|_{H_0^1}, \|\tilde{u}\|_{H_0^1} \leq M$,

$$\begin{aligned}
\|g(u) - g(\tilde{u})\|_2 &\leq C\|(1 + |u|^{p-1} + |\tilde{u}|^{p-1})(u - \tilde{u})\|_2 \\
&\leq C\|(1 + |u|^{\frac{2}{n-2}} + |\tilde{u}|^{\frac{2}{n-2}})(u - \tilde{u})\|_2 \\
&\leq C(\|u - \tilde{u}\|_2 + \|(|u|^{\frac{2}{n-2}} + |\tilde{u}|^{\frac{2}{n-2}})(u - \tilde{u})\|_2) \\
&\leq C(\|u - \tilde{u}\|_2 + M^{\frac{2}{n-2}}\|u - \tilde{u}\|_{\frac{2n}{n-2}}) \\
&\leq C(M)\|u - \tilde{u}\|_{H_0^1},
\end{aligned}$$

where the constants change from line to line.

Setting $\tilde{u} = 0$ we get in particular that

$$\|g(u)\|_2 \leq \|g(0)\|_2 + C(M)\|u\|_{H_0^1},$$

so that (since $g(0) = 0$ when $\mathcal{L}^n(\Omega) = \infty$) $g : H_0^1(\Omega) \to L^2(\Omega)$ and is locally Lipschitz, as required.

Hence, from Theorems 26 and 28 we obtain

Theorem 29 Given $p = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$ there exists a unique maximally defined weak solution $w = \begin{pmatrix} u \\ u_t \end{pmatrix} \in C([0, t_{\max}); X)$ of (14) with $w(0) = p$, depending continuously on p, and if $t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}-} (\|u(\cdot, t)\|_{H_0^1} + \|u_t(\cdot, t)\|_2) = \infty,$$

$$\int_0^{t_{\max}} \|g(u(\cdot, t))\|_2 \, dt = \infty.$$

If in addition $\Delta u_0 \in L^2(\Omega), u_1 \in H_0^1(\Omega)$, then $\Delta u \in C([0, t_{\max}); L^2(\Omega))$ and $u_t \in C([0, t_{\max}); H_0^1(\Omega))$.

Now let

$$V(w) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |v|^2)\, dx + \int_\Omega G(u)\, dx,$$

where $G(u) := \int_0^u g(z)\, dz$.

Note that (again taking the case $n \geq 3$)

$$\begin{aligned} |G(u)| &= \left| \int_0^u g(s)\, ds \right| \\ &\leq \int_0^{|u|} \left(|g(0)| + C(1 + |s|^{\frac{2}{n-2}})|s| \right) ds \\ &\leq |u||g(0)| + C\left(|u|^2 + |u|^{\frac{2n-2}{n-2}} \right), \end{aligned}$$

so that $V : X \to \mathbb{R}$ and is continuous (because if $u_j \to u \in H_0^1(\Omega)$ then $|G(u_j)|$ is bounded above by a strongly convergent sequence in $L^1(\Omega)$).

Next note that

$$
\begin{aligned}
|G(\tilde{u}) - G(u) - g(u)(\tilde{u} - u)| &= \left| \int_u^{\tilde{u}} (g(s) - g(u))\, ds \right| \\
&\leq C \left| \int_u^{\tilde{u}} \left| 1 + |s|^{\frac{2}{n-2}} + |u|^{\frac{2}{n-2}} \right| \cdot |s - u|\, ds \right| \\
&\leq C \left(1 + |u|^{\frac{2}{n-2}} + |\tilde{u}|^{\frac{2}{n-2}} \right) \left| \int_u^{\tilde{u}} |s - u|\, ds \right| \\
&\leq C \left(1 + |u|^{\frac{2}{n-2}} + |\tilde{u}|^{\frac{2}{n-2}} \right) |\tilde{u} - u|^2
\end{aligned}
$$

so that

$$
\left| \int_\Omega G(\tilde{u})\, dx - \int_\Omega G(u)\, dx - \int_\Omega g(u)(\tilde{u} - u)\, dx \right| \leq C \|\tilde{u} - u\|^2_{\frac{2n}{n-2}},
$$

for $\|u\|_{H_0^1}, \|\tilde{u}\|_{H_0^1} \leq M$.

Hence $E(u) = \int_\Omega G(u)\,dx$ is differentiable with derivative

$$E'(u)(v) = \int_\Omega g(u)v\,dx.$$

But if $u_j \to u$ in $H_0^1(\Omega)$ then

$$\begin{aligned}
&\left|\int_\Omega (g(u_j) - g(u))v\,dx\right| \\
&\qquad \le C\int_\Omega (1 + |u_j|^{\frac{2}{n-2}} + |u|^{\frac{2}{n-2}})|u_j - u||v|\,dx, \\
&\qquad \le C\left(\|u_j - u\|_2\|v\|_2 + \left(\|u_j\|^{\frac{2}{n-2}}_{\frac{n}{n-2}} + \|u\|^{\frac{2}{n-2}}_{\frac{n}{n-2}}\right)\|u_j - u\|_{\frac{2n}{n-2}}\|v\|_{\frac{2n}{n-2}}\right) \\
&\qquad \le C\|u_j - u\|_{H_0^1}\|v\|_{H_0^1},
\end{aligned}$$

so that $E'(u_j) \to E'(u)$ in $(H_0^1(\Omega))^*$.

Hence V is C^1 with

$$\langle \begin{pmatrix} \chi \\ \psi \end{pmatrix}, V'(w)\rangle = (\nabla u, \nabla\chi) + (v, \psi) + (g(u), \chi),$$

Therefore if $w = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ then

$$\langle Aw + f(w), V'(w) \rangle$$
$$= (\nabla u, \nabla v) + (v, \Delta u - g(u)) + (g(u), v) = 0.$$

So from Theorem 27 we deduce

Theorem 30 The unique weak solution $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$ in Theorem 29 satisfies the energy identity

$$V(w(t)) = V(p) \text{ for all } t \in [0, t_{\max}).$$

Under an additional hypothesis, the energy identity implies boundedness of $\|w(t)\|$ independent of $t \geq 0$, so that by Theorem 29 $t_{\max} = \infty$ and we have global existence.

Theorem 31 In addition to the hypotheses of Theorem 29, assume that for some constants $k > 0$ and $a \leq 0$

$$G(u) + ku^2 \geq a \text{ for all } u.$$

Then $t_{\text{max}} = \infty$, and setting $T(t)p = w(t)$ we have that weak solutions of (14) generate a semiflow $\{T(t)\}_{t \geq 0}$ on X with $(t, p) \mapsto T(t)p$ continuous from $[0, \infty) \times X \to X$.

Proof. First recall that in the case $\mathcal{L}^n(\Omega) = \infty$ we have assumed that $g(0) = 0$. Hence $G'(0) = 0$ and by our growth assumption $G' = g$ is locally Lipschitz. Therefore by increasing k we may assume that $a = 0$.

From the energy equation we have that

$$\frac{1}{2}(\|u_t(\cdot,t)\|_2^2 + \|\nabla u(\cdot,t)\|_2^2) + a\mathcal{L}^n(\Omega) - k\|u(\cdot,t)\|_2^2$$
$$\leq V(w(t)) = V(w(0))$$

so that

$$\|u_t(\cdot,t)\|_2^2 \leq d(1 + \|u(\cdot,t)\|_2)^2$$

for $t \in [0, t_{\max})$ and some $d > 0$.

But $u(\cdot,t) - u_0 = \int_0^t u_t(\cdot,s)\,ds$, so that

$$1 + \|u(\cdot,t)\|_2 \leq 1 + \|u_0\|_2 + d^{\frac{1}{2}} \int_0^t (1 + \|u(\cdot,s)\|_2)\,ds.$$

By Gronwall's inequality $\|u(\cdot,t)\|_2$ is bounded on compact intervals of $[0, t_{\text{max}})$, and thus so is $\|w(t)\|_X$, giving $t_{\text{max}} = \infty$. That weak solutions generate a semiflow with the joint continuity property then follows from Theorem 26. □

Remark. Since the equation

$$u_{tt} - \Delta u + g(u) = 0$$

is invariant under changing t to $-t$, we can also solve it for $t \leq 0$, so that $\{T(t)\}_{t\in\mathbb{R}}$ is a *flow* (or *group*).

The fact that $\{e^{At}\}_{t\in\mathbb{R}}$ is a group of continuous linear operators can either be proved by checking that $-A$ generates a C^0–semigroup, or by verifying that

$$e^{-At} := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} e^{At} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

satisfies $e^{-At}e^{At} = \mathbf{1}$ by calculating

$$\frac{d}{dt}(e^{-At}e^{At}p) = 0.$$

Under still stronger conditions $\|w(t)\|_X$ is bounded for all $t \geq 0$. To this end define

$$\lambda_1(\Omega) = \inf_{u\in H_0^1(\Omega), \|u\|_2=1} \int_\Omega |\nabla u|^2 \, dx.$$

Notice that if $\mathcal{L}^n(\Omega) < \infty$ then, because in that case the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, the minimum is attained and $\lambda_1(\Omega) > 0$ (and $\lambda_1(\Omega)$ is the smallest eigenvalue of $-\Delta$ in $H_0^1(\Omega)$).

But $\lambda_1(\Omega) > 0$ for any domain for which the Poincaré inequality holds, for example any domain of finite width. On the other hand $\lambda_1(\Omega) = 0$, for example, if $\Omega = \mathbb{R}^n$.

Now suppose that there exist $\lambda < \lambda_1(\Omega)$ and $a \leq 0$, with $a = 0$ if $\mathcal{L}^n(\Omega) = \infty$, such that

$$G(u) + \frac{\lambda}{2}u^2 \geq a \text{ for all } u.$$

Then setting $\varepsilon = \frac{\lambda_1(\Omega)-\lambda}{\lambda_1(\Omega)+1} > 0$ we obtain

$$
\begin{aligned}
\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + G(u)\right) dx \\
&\geq \int_\Omega \left(\frac{1}{2}\varepsilon|\nabla u|^2 + \frac{1}{2}(1-\varepsilon)\lambda_1(\Omega)u^2 - \frac{\lambda}{2}u^2 + a\right) dx \\
&= \frac{\varepsilon}{2}\int_\Omega (|\nabla u|^2 + u^2)\, dx + a\mathcal{L}^n(\Omega),
\end{aligned}
$$

so that $\|w(t)\|_X$ is bounded for $t \in \mathbb{R}$.

For example for $n = 3$ we can take $g(u) = u^3 - u$, and we have global existence for any Ω, and boundedness of $\|w(t)\|_X$ if $\lambda_1(\Omega) > 1$, for example for a domain Ω of finite width $< \pi$.

Finite-time blow-up

Consider, for $n = 3$ and $\Omega \subset \mathbb{R}^3$ open with $\mathcal{L}^3(\Omega) < \infty$, the equation

$$u_{tt} - \Delta u = u^3,$$

so that $g(u) = -u^3$.

By Theorem 29, if $p = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X = H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique weak solution $w(t)$ with $w(0) = p$ defined on a maximal interval of existence $[0, t_{\max})$ with $t_{\max} > 0$.

We show that it can happen that $t_{\max} < \infty$ and study the behaviour of $w(t)$ when $t \to t_{\max}-$.

Given $p \in X$ we know that the energy equation $V(w(t)) = V(p)$ holds for $t \in [0, t_{\max})$, where

$$V(w) = \frac{1}{2}(\|\nabla u\|^2 + \|u_t\|^2) - \frac{1}{4}\int_\Omega u^4\,dx.$$

Suppose for contradiction that $t_{\max} = \infty$ and let

$$F(t) = \|u(\cdot,t)\|_2^2.$$

Then $F \in C^2([0,\infty))$ and

$$\begin{aligned} \ddot{F}(t) &= -2\|\nabla u(\cdot,t)\|_2^2 + 2\|u_t(\cdot,t)\|_2^2 + 2\int_\Omega u(x,t)^4\,dx \\ &= 4\|u_t(\cdot,t)\|_2 + \int_\Omega u(x,t)^4\,dx - 4V(p) \\ &\geq kF(t)^2 - 4V(p), \end{aligned}$$

where $k = (\mathcal{L}^3(\Omega))^{-1}$ (justified e.g. by first assuming $p \in D(A)$ and approximating).

Now let us assume that $V(p) \leq 0$ and that if $V(p) = 0$ then $(u_0, u_1) > 0$.

If $V(p) < 0$ then $\dot{F}(t) > 0$ for large enough t, so that we may assume that $(u_0, u_1) > 0$. Hence

$$\ddot{F}(t)\dot{F}(t) \geq kF(t)^2\dot{F}(t),$$

and $\dot{F}(t) \geq \dot{F}(0)$, implying that $F(t) \to \infty$ as $t \to \infty$, and that

$$\dot{F}(t)^2 \geq \frac{2k}{3}F(t)^3 + C$$

for some constant C.

Fix s sufficiently large for $\frac{2k}{3}F(s)^3 + C > 0$.

Then for $t \geq s$

$$\frac{d}{dt} \int_{F(s)}^{F(t)} \frac{dF}{\sqrt{C + 2kF^3}} \geq 1,$$

so that

$$\int_{F(s)}^{F(t)} \frac{dF}{\sqrt{C + 2kF^3}} \geq t - s$$

for all $t \geq s$. But

$$\int_{F(s)}^{\infty} \frac{dF}{\sqrt{C + 2kF^3}} < \infty,$$

a contradiction.

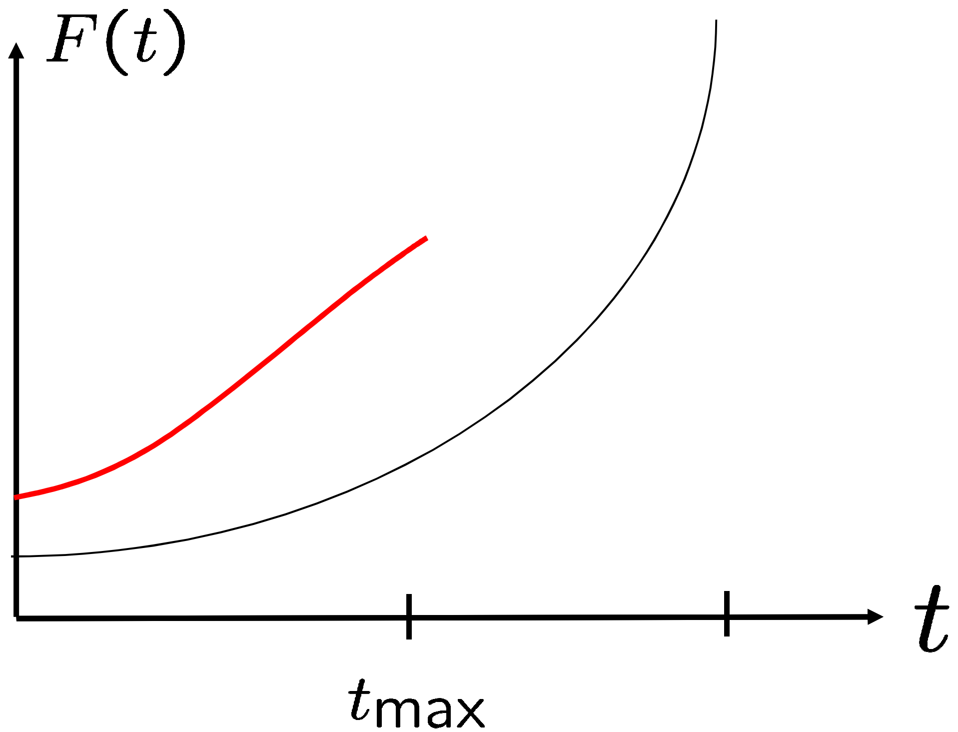
Theorem 32 If $V(p) \leq 0$, with $(u_0, u_1) > 0$ if $V(p) = 0$, then the weak solution $w(t)$ has maximal interval of existence $[0, t_{\max})$ with $t_{\max} < \infty$ and

$$\lim_{t \to t_{\max}-} (\|\nabla u(\cdot,t)\|_2^2 + \|u_t(\cdot,t)\|_2^2) = \lim_{t \to t_{\max}-} \|u(\cdot,t)\|_4 = \infty,$$

$$\int_0^{t_{\max}} \|u(\cdot,t)\|_6^3 \, dt = \infty.$$

Proof. We have already shown that $t_{\max} < \infty$ and the other assertions follow from Theorem 29, the fact that $\|\nabla u\|_2$ is an equivalent norm on $H_0^1(\Omega)$, and the energy equation.

Remark. The argument to prove $t_{\max} < \infty$ was based on establishing a differential inequality for $F(t)$ whose solutions blow up in finite time. *However this doesn't prove that* $\lim_{t \to t_{\max}-} \|u(\cdot, t)\|_2 = \infty$ (and it isn't clear whether or not this is true). All it does is to give an upper bound for $t_{\max}$.



The following example shows this is a real issue. Consider for $\Omega \in \mathbb{R}^n$ bounded and open the equation

$$u_t = \Delta u - \left(\int_\Omega u^2\, dx\right) u, \; x \in \Omega, t > 0,$$

with boundary condition $u|_{\partial\Omega} = 0$ and initial condition $u(x,0) = u_0(x)$.

Letting $X = L^2(\Omega)$, $A = \Delta$ with

$$D(A) = \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\},$$

and defining $f : X \to X$ by $f(v) = -\left(\int_\Omega v^2\, dx\right) v$ we can apply Theorem 26 to deduce that given $u_0 \in X$ there exists a unique weak solution $u \in C([0, t_{\max}); X)$ with $u(0) = u_0$.

In fact it can be shown that $u(t) \in D(A)$ for $t > 0$ (smoothing of the initial data as for the heat equation).

Now consider the corresponding *backwards equation*

$$v_t = -\Delta v + \left(\int_\Omega v^2\,dx\right) v$$

with the same boundary conditions.

For the forwards equation choose $u_0 \in X \backslash D(A)$. Thus the solution v to the backwards equation with initial data $w(\tau)$, where $0 < \tau < t_{\max}$ has maximal interval of existence $[0, \tau]$, since if it were longer then by the smoothing of the forwards equation we would have $u_0 \in D(A)$.

Furthermore

$$\lim_{t \to \tau} \|v(t)\|_2 = \|u_0\|_2 < \infty.$$

However, we will show that for v we have $t_{\max} < \infty$ by a “blow-up” argument.

Indeed, letting $F(t) = \|v(t)\|_2^2$ we have that

$$\dot{F}(t) \geq 2F(t)^2,$$

from which it follows that

$$F(t) \geq \frac{1}{\frac{1}{F(0)} - 2t},$$

and the RHS blows up as $t \to \frac{1}{2F(0)}-$.

Commentary

1. There are many variants of the semigroup method applied to semilinear equations corresponding especially to 'parabolic cases' in which $\{e^{At}\}_{t\geq 0}$ is smoothing for $t > 0$. These are described, for example, in Pazy and Henry.

2. There is a vast literature on semilinear wave equations treated by PDE methods (see e.g. Tao), mostly for the case $\Omega = \mathbb{R}^n$, when the critical power in the nonlinearity $g(u) = |u|^{p-1}u$ is found to be $p = \frac{n+2}{n-2}$, and not $p = \frac{n}{n-2}$ (thus, for $n = 3$, $p = 5$ rather than $p = 3$).

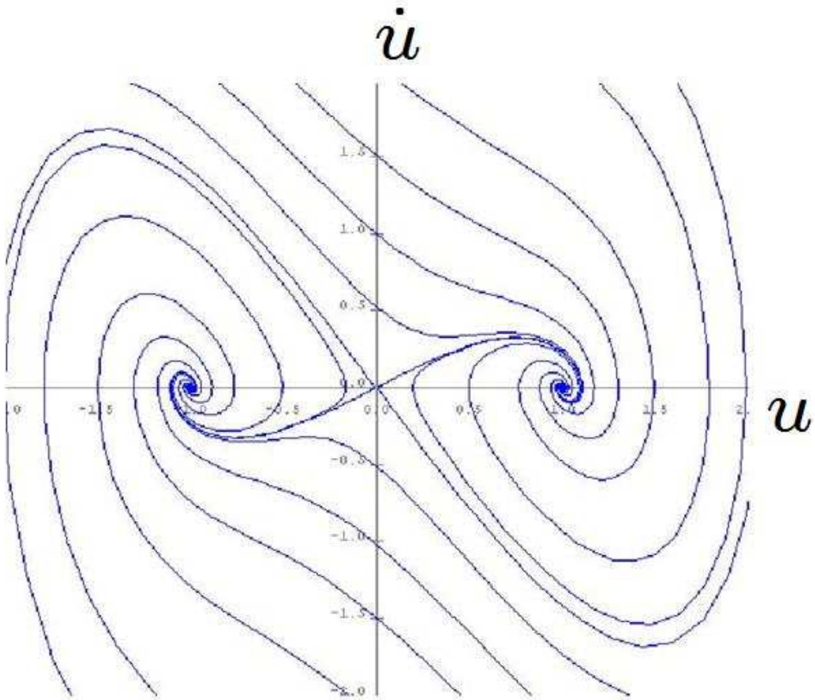
2. contd. Then we get global existence, uniqueness and energy conservation results for $2 \leq p \leq \frac{n+2}{n-2}$ using Strichartz space-time estimates. Whether this can in general be made to work for different, in particular bounded, Ω is the object of current research, and has been done for smooth bounded domains in $\mathbb{R}^3$ by Burq, Lebeau, Planchon JAMS 2008 (see also Ibrahim & Jrad, JDE 2011 and Blair, Smith & Sogge, Analyse Nonlinéaire 2009 for results for $n = 2$, $n = 4$ respectively).

Finite-time blow-up for the focussing case $g(u) = -|u|^{p-1}u$, and details of limiting profiles as $t \to t_{\max}-$ are also studied in many papers (e.g. by Kenig, Merle, Zaag ...). Two types of blow-up are possible: Type I in which the energy norm blows up, and Type II when it doesn't and there is a concentration effect.

Finally, the case $g(u) = |u|^{p-1}u$ can be studied for $p > \frac{n+2}{n-2}$ via the Galerkin method (see e.g. J.-L. Lions). Then we get existence of a weak solution, but uniqueness and energy conservation are (after many years) still almost virgin territory.

Damped hyperbolic equations and the approach to equilibrium

Motivating example. $\ddot{u} + \dot{u} + u^3 - u = 0$



Rest points
$u = 0, \pm 1, \dot{u} = 0$

Every solution converges to a rest point as $t \to \infty$.

There is a global attractor consisting of the restpoints and unstable manifold of 0.

Lyapunov function

$$V(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \frac{1}{4}(u^2 - 1)^2, \qquad \dot{V} = -\dot{u}^2$$

Asymptotic behaviour of semiflows

Let $\{T(t)\}_{t\geq 0}$ be a semiflow on the metric space (X,d). The *positive orbit* of $p \in X$ is the set

$$\gamma^+(p) = \{T(t)p : t \geq 0\}.$$

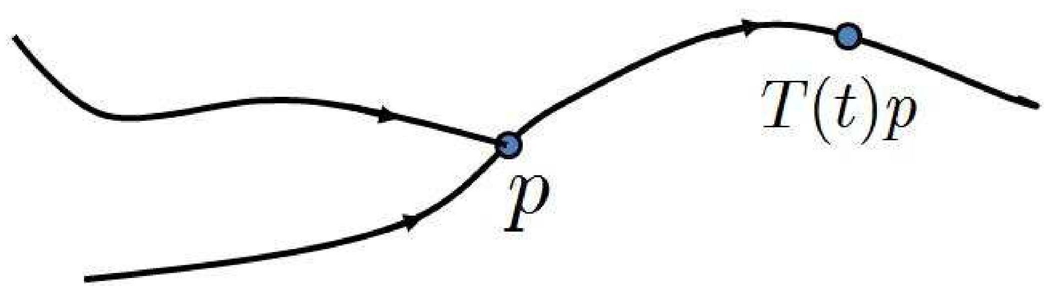
The $\omega-$*limit set* of p is the set

$$\begin{aligned} \omega(p) &= \{\chi \in X : T(t_j)p \to \chi \\ &\qquad \text{for some sequence } t_j \to \infty\} \\ &= \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} T(\tau)p}. \end{aligned}$$

A map $\psi : \mathbb{R} \to X$ is a *complete orbit* if

$$\psi(t+s) = T(t)\psi(s) \text{ for all } s \in \mathbb{R}, t \geq 0.$$

Note that we do not assume backwards uniqueness, so there might be more than one complete orbit passing through a point $p \in X$.



If ψ is a complete orbit then the α-*limit set* of ψ is the set

$$\begin{aligned}\alpha(\psi) &= \{\chi \in X : \psi(t_j) \to \chi \\ &\qquad \text{for some sequence } t_j \to -\infty\} \\ &= \bigcap_{t \le 0} \overline{\bigcup_{\tau \le t} \psi(\tau)}.\end{aligned}$$

If $E \subset X$, $t \geq 0$, we set

$$T(t)E = \{T(t)p : p \in E\}.$$

A subset $E \subset X$ is *positively invariant* if $T(t)E \subset E$ for all $t \geq 0$, and *invariant* if $T(t)E = E$ for all $t \geq 0$.

Note that if E invariant then there is a complete orbit contained in E passing through any point of E.

Indeed if $p \in E$ then there exist $p_{-1} \in E$ with $T(1)p_{-1} = p$, $p_{-2} \in E$ with $T(1)p_{-2} = p_{-1}$, and so on, so that

$$\psi(t) = \begin{cases} T(t)p, & t \geq 0 \\ T(t+i)p_{-i}, & t \in [-i, -i+1), i = 1, 2, \ldots \end{cases}$$

defines a complete orbit passing through p.

Theorem 33

(i) Let $\gamma^+(p)$ be relatively compact. Then $\omega(p)$ is nonempty, compact, invariant and connected. As $t \to \infty$,

$$\text{dist}\,(T(t)p, \omega(p)) \to 0,$$

where $\text{dist}\,(q, E) := \inf_{\chi \in E} d(q, \chi)$.

(ii) Let ψ be a complete orbit with $\{\psi(t) : t \leq 0\}$ relatively compact. Then $\alpha(\psi)$ is nonempty, compact, invariant and connected, and as $t \to -\infty$

$$\text{dist}\,(\psi(t), \alpha(\psi)) \to 0.$$

We prove (i). The proof of (ii) is similar. That $\omega(p)$ is nonempty is clear. Since $\omega(p)$ is the intersection of compact sets, it is compact.

To prove the invariance, let $\chi \in \omega(p)$. Then $T(t_j)p \to \chi$ for some sequence $t_j \to \infty$. If $t \geq 0$ then, since $T(t)$ is continuous,

$$T(t+t_j)p = T(t)T(t_j)p \to T(t)\chi$$

and so $T(t)\omega(p) \subset \omega(p)$.

Also $\{T(t_j - t)p\}$ is relatively compact, and so

$$T(t_{j_k} - t)p \to q \in X$$

for some subsequence $\{t_{j_k}\}$.

Therefore

$$T(t_{j_k})p = T(t)T(t_{j_k} - t)p \to T(t)q = \chi.$$

Hence $T(t)\omega(p) \supset \omega(p)$ and so $\omega(p)$ is invariant.

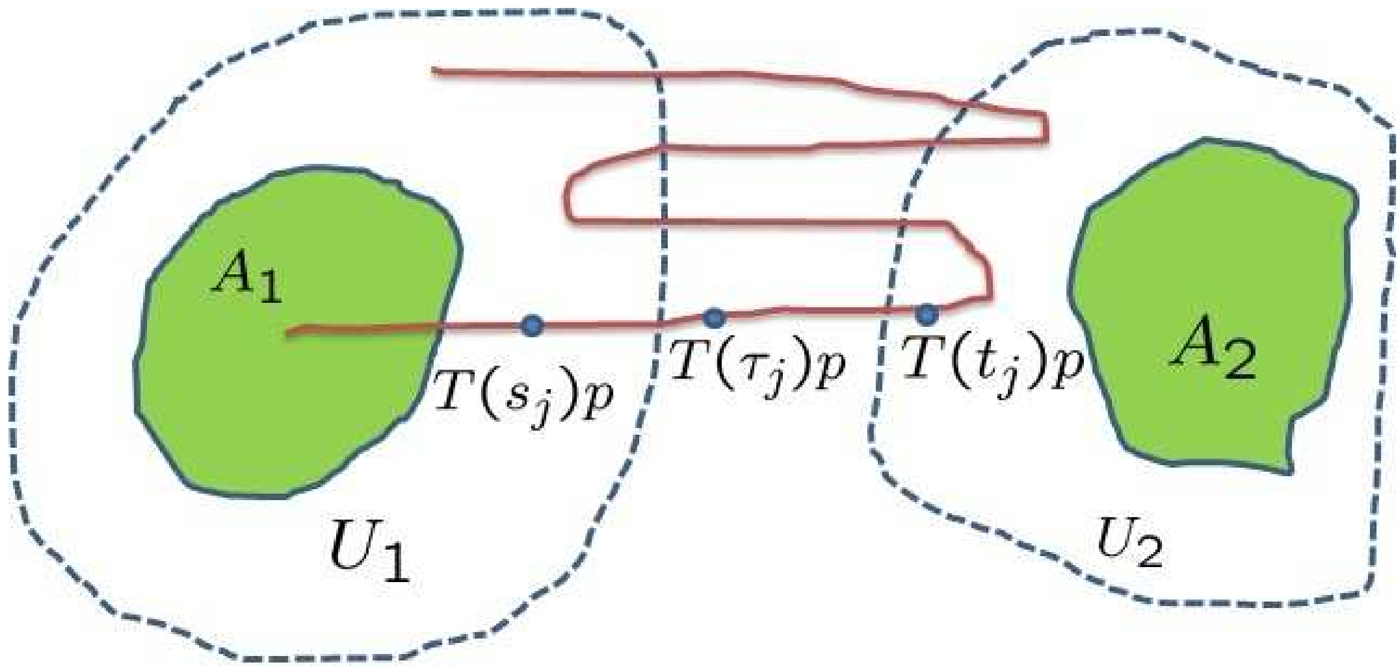
If $\operatorname{dist}(T(t)p, \omega(p)) \not\to 0$ as $t \to \infty$, then there exist $\varepsilon > 0$ and a sequence $t_j \to \infty$ such that $d(T(t_j)p, z) \geq \varepsilon$ for all $z \in \omega(p)$. But a subsequence $T(t_{j_k})p \to \chi \in \omega(p)$, a contradiction.

Suppose $\omega(p)$ is not connected. Then $\omega(p) = A_1 \cup A_2$ with A_1, A_2 nonempty disjoint compact sets.

(Indeed, by the definition of connectedness we can write $\omega(p) = V_1 \cup V_2$ with $\bar{V_1} \cap V_2 = V_1 \cap \bar{V_2} = \emptyset$, and since $\omega(p)$ is closed we have $\omega(p) = \bar{V_1} \cup \bar{V_2}$. Thus $\bar{V_1} \cap \bar{V_2} = \emptyset$ (because if $x \in \bar{V_1} \cap \bar{V_2}$ then $x \in V_1 \cup V_2$) and we can set $A_1 = \bar{V_1}, A_2 = \bar{V_2}$. Since A_1, A_2 are closed subsets of a compact set, they are themselves compact.)

Let U_1, U_2 be disjoint open sets with $A_1 \subset U_1, A_2 \subset U_2$. We can take, for example, $U_i = \{q \in X : \text{dist}\,(q, A_i) < \varepsilon\}$ for $\varepsilon > 0$ sufficiently small.

Then there exist sequences $s_j > t_j$ with $t_j \to \infty$ such that $T(s_j)p \in U_1, T(t_j)p \in U_2$ and hence, by the continuity in t, there exists $\tau_j \in (t_j, s_j)$ with $T(\tau_j)p \notin U_1 \cup U_2$.



Hence by the relative compactness of $\gamma^+(p)$ there exists some $\chi \in \omega(p) \backslash (A_1 \cup A_2)$, a contradiction. $\Box$

A point $z \in X$ is a *rest point* if $T(t)z = z$ for all $t \geq 0$. The set Z of rest points is closed.

Definition.
A function $V : X \to \mathbb{R}$ is a *Lyapunov function* if
(i) V is continuous,
(ii) $V(T(t)p) \leq V(p)$ for all $p \in X, t \geq 0$,
(iii) If $V(\psi(t)) = c$ for some complete orbit ψ, all $t \in \mathbb{R}$ and some constant c, then $\psi(t) = z$ for all $t \in \mathbb{R}$ for some rest point z.

(Note that (ii) implies that $V(T(t)p) \leq V(T(s)p)$ for all $t \geq s \geq 0$, since $V(T(t)p) = V(T(t-s)T(s)p) \leq V(T(s)p)$.)

Theorem 34 (LaSalle Invariance Principle) Let V be a Lyapunov function, and let $p \in X$ with $\gamma^+(p)$ relatively compact. Then $\omega(p)$ consists only of rest points. If the only nonempty connected subsets of Z are single points (for example, if there are only a finite number of rest points) then $\omega(p) = \{z\}$ for some rest point z, and $T(t)p \to z$ as $t \to \infty$.

Proof. Since V is continuous and $\gamma^+(p)$ is relatively compact, $V(T(t)p)$ is bounded below for $t \geq 0$. But $t \mapsto V(T(t)p)$ is nonincreasing, and so

$$V(T(t)p) \to c \text{ as } t \to \infty$$

for some constant c.

Let $z \in \omega(p)$. Then, since $\omega(p)$ is invariant, $z = \psi(0)$ for a complete orbit ψ contained in $\omega(p)$. Hence $V(\psi(t)) = c$ for all $t \in \mathbb{R}$, and so by (iii) z is a rest point.

If the only nonempty connected subsets of Z are single points then since $\omega(p)$ is connected, $\omega(p) = z$ for some rest point, so that $T(t)p \to z$ as $t \to \infty$ by Theorem 33. □

Definitions. The rest point z is (*Lyapunov*) *stable* if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $p \in B(z, \delta)$ then $T(t)p \in B(z, \varepsilon)$ for all $t \geq 0$. The rest point z is *unstable* if it is not stable. The rest point z is *asymptotically stable* if z is stable and there exists $\rho > 0$ such that $p \in B(z, \rho)$ implies $T(t)p \to z$ as $t \to \infty$.

If the rest point z is asymptotically stable then clearly z is *isolated*, that is there is some $\varepsilon > 0$ such that z is the only rest point in $B(z, \varepsilon)$.

Theorem 35 Let z be an isolated rest point, let V be a Lyapunov function, let $\gamma^+(p)$ be relatively compact for any p with $\gamma^+(p)$ bounded, and suppose that for all $\delta > 0$ sufficiently small

$$\inf_{d(p,z)=\delta} V(p) > V(z) \quad \text{(Existence of a potential well)}$$

Then z is asymptotically stable.

Proof. Suppose z is not stable. Then there exist $\varepsilon > 0$, $p_j \to z$, $t_j \geq 0$ with $d(T(t_j)p_j, z) \geq \varepsilon$.

We can suppose that ε is small enough such that

$$c_\varepsilon := \inf_{d(p,z)=\frac{\varepsilon}{2}} V(p) > V(z),$$

and such that z is the only rest point in $\overline{B(z,\varepsilon)}$.

Let j be sufficiently large. Then since V is continuous, $V(p_j) < c_\varepsilon$. By the continuity of $t \mapsto T(t)p_j$ there exists $\tau_j \in (0, t_j)$ with $d(T(\tau_j)p_j, z) = \frac{\varepsilon}{2}$, and thus

$$c_\varepsilon \leq V(T(\tau_j)p_j) \leq V(p_j) < c_\varepsilon,$$

a contradiction.

By the stability, given $\varepsilon > 0$ there exists $\rho > 0$ such that if $d(p, z) < \rho$ then $T(t)p \in \overline{B(z, \varepsilon)}$ for all $t \geq 0$. Then $\gamma^{+}(p)$ is bounded, and so by the assumption of the theorem relatively compact. Thus, by Theorem 44, $\omega(p) \subset Z \cap \overline{B(z, \varepsilon)}$ and so $\omega(p) = \{z\}$ and $T(t)p \to z$ as $t \to \infty$. $\square$

Remark. If $X = \mathbb{R}^n$ then the existence of a potential well is equivalent to the condition that z is a strict local minimizer of V, i.e. that there exists $\varepsilon > 0$ such that $V(p) > V(z)$ if $0 < d(p, z) \leq \varepsilon$. This follows easily from the fact that the sphere $S(z, \varepsilon)$ is compact, so that V attains a minimum on $S(z, \varepsilon) = \{p : d(p, z) = \varepsilon\}$.

But if X is a metric space whose spheres $S(z, \varepsilon)$ are not compact (as is the case for infinite-dimensional normed vector spaces) then the existence of a potential well is a stronger condition than being a strict local minimizer. If we just assumed that z was a strict local minimizer then the danger would be that orbits could leak out of balls by going into higher and higher dimensions.

Theorem 36 Let V be a Lyapunov function and suppose that $\gamma^+(p)$ is relatively compact for any p with $\gamma^+(p)$ bounded. Let z be an isolated rest point which is not a local minimizer of V (i.e. for any $\varepsilon > 0$ there is a point p with $d(p, z) < \varepsilon$ and $V(p) < V(z)$). Then z is unstable.

Proof. Let $\varepsilon > 0$ be sufficiently small so that z is the only rest point in $\overline{B(z,\varepsilon)}$. Suppose for contradiction that z is stable. Then there exists $\delta > 0$ such that $d(p,z) < \delta$ implies $d(T(t)p,z) < \varepsilon$ for all $t \geq 0$. But since z is not a local minimizer there exists p with $d(p,z) < \delta$ and $V(p) < V(z)$.

Since $\gamma^+(p) \subset \overline{B(z,\varepsilon)}$, $\gamma^+(p)$ is by assumption relatively compact. Hence by the invariance principle there exist a sequence $t_j \to \infty$ and a rest point $\tilde{z} = \lim_{j\to\infty} T(t_j)p$ in $\omega(p)$ with $\tilde{z} \in \overline{B(z,\varepsilon)}$. But $V(\tilde{z}) = \lim_{j\to\infty} V(T(t_j)p) < V(z)$. Hence $\tilde{z} \neq z$, a contradiction. $\Box$

The *region of attraction* of a rest point z is the set

$$A(z) = \{p \in X : T(t)p \to z \text{ as } t \to \infty\}.$$

Theorem 37 $A(z)$ is connected.

Proof. Suppose not, so that $A(z) = U \cup V$ with U, V nonempty and $U \cap \bar{V} = \bar{U} \cap V = \emptyset$. Let $p \in U, q \in V$. For any $t \geq 0$, $T(t)p \in A(z)$. Let $S = \{t \geq 0 : T(t)p \in U\}$. Let $t_j \in S$, $t_j \to t$. Then $T(t)p = \lim_{j\to\infty} T(t_j)p \in \bar{U}$ and so $T(t)p \in U$. Hence S is closed in $[0, \infty)$. Similarly S is open, and thus $\gamma^+(p) \subset U$. Similarly $\gamma^+(q) \subset V$. But $z \in \bar{U}$, hence $z \notin V$. Similarly $z \notin U$. But $z \in A(z) = U \cup V$, a contradiction.

Theorem 38 If z is an asymptotically stable rest point then $A(z)$ is open.

Proof. By assumption there exists $\rho > 0$ such that $r \in B(z, \rho)$ implies $T(t)r \to z$ as $t \to \infty$. Let $p \in A(z)$. Then there exists $s > 0$ such that $d(T(s)p, z) < \rho$. Hence by the continuity of $T(s)$ there exists $\sigma > 0$ such that $d(p, q) < \sigma$ implies $d(T(s)q, z) \leq d(T(s)q, T(s)p) + d(T(s)p, z) < \rho$, so that by asymptotic stability $T(t)q \to z$ as $t \to \infty$ and hence $q \in A(z)$. $\square$

Definitions. The semiflow $\{T(t)\}_{t\geq 0}$ is *asymptotically compact* if for any bounded sequence $\{p_j\}$ in X and any sequence $t_j \to \infty$ $T(t_j)p_j$ has a convergent subsequence. It is *point dissipative* if there is a bounded set B_0 such that for any $p \in X$, $T(t)p \in B_0$ for all t sufficiently large.

A subset $A \subset X$ *attracts a set* $E \subset X$ if

$$\text{dist}\,(T(t)E, A) \to 0 \text{ as } t \to \infty,$$

where

$$\text{dist}\,(B, C) := \sup_{b\in B} \inf_{c\in C} d(b,c) = \sup_{b\in B} \text{dist}\,(b, C).$$

(If A is compact this is the same as saying that given any open neighbourhood $U \supset A$, $T(t)E \subset U$ for t sufficiently large.)

Definitions. The subset A is a *global attractor* if A is compact, invariant, and attracts all bounded sets. If $B \subset X$ is bounded, the ω-limit set of B is

$$\begin{aligned}\omega(B) &= \{\chi \in X : T(t_j)p_j \to \chi \\ &\qquad \text{for some sequences } p_j \in B, t_j \to \infty\}.\end{aligned}$$

Theorem 39. A semiflow $\{T(t)\}_{t\geq 0}$ has a global attractor if and only if it is point dissipative and asymptotically compact. The global attractor is unique and given by

$$A = \bigcup \{\omega(B) : B \text{ a bounded subset of } X\}.$$

Furthermore A is the maximal compact invariant subset of X.

Proof. See Hale, Ladyzhenskaya, JB (*Nonlinear Science* **7** 1997, erratum 1998).

We apply this theory to the semilinear damped hyperbolic equation

$$u_{tt} + \beta u_t - \Delta u + u^3 - u = 0,$$

where $\Omega \subset \mathbb{R}^3$ is open with $\mathcal{L}^3(\Omega) < \infty$ and $\beta > 0$ is a constant, with boundary condition $u|_{\partial\Omega} = 0$.

In order to apply our previous theory we let as before $X = H_0^1(\Omega) \times L^2(\Omega)$, and write the equation as

$$\dot{w} = Aw + f(w), \qquad (15)$$

where $w = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ and $f(w) = \begin{pmatrix} 0 \\ -\beta v - u^3 + u \end{pmatrix}$.

Then $f : X \to X$ is Lipschitz.

Theorem 39 Given $p = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$ there exists a unique weak solution $w \in C([0,\infty); X)$ to (15) with $w(0) = p$, and $T(t)p = w(t) = \begin{pmatrix} u \\ u_t \end{pmatrix}$ defines a semi-flow $\{T(t)\}_{t\geq 0}$ on X with $(t,p) \mapsto T(t)p$ continuous from $[0,\infty) \times X \to X$. The solution satisfies the energy equation

$$V(T(t)p) + \beta \int_0^t \int_\Omega u_t(x,s)^2 \, dx \, ds = V(p), \text{ for all } t \geq 0,$$

where

$$V(w) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |v|^2) \, dx + \int_\Omega \frac{1}{4}(u^2 - 1)^2 \, dx.$$

Proof. Local existence and continuity in p follows from Theorem 26. The energy equation follows from Theorem 27 using the same calculations as for the undamped case, which in turn implies global existence. □

In order to understand the asymptotic behaviour of solutions we will need to know that $T(t)$ is not only continuous, but (sequentially) *weakly continuous*.

Lemma 40 If $T : X \to X$ is a continuous linear map, then it is sequentially weakly continuous, i.e.

$$p_j \rightharpoonup p \text{ in } X \text{ implies } Tp_j \rightharpoonup Tp.$$

Proof. If $v \in X^*$ then $v \circ T \in X^*$. Hence $\langle Tp_j, v\rangle \to \langle Tp, v\rangle$. □

Lemma 41 Let $p_j \rightharpoonup p$ in X, $t_j \to t$ in $[0,\infty)$. Then $e^{At_j}p_j \rightharpoonup e^{At}p$.

Proof. We have that for any $v \in D(A^*)$

$$\langle e^{At_j}p_j, v\rangle = \langle p_j, v\rangle + \int_0^{t_j} \langle e^{As}p_j, A^*v\rangle\, ds.$$

By Lemma 40 we therefore have that

$$\lim_{j\to\infty} \langle e^{At_j}p_j, v\rangle = \langle e^{At}p, v\rangle,$$

giving the result since $D(A^*)$ is dense. □

Lemma 42 $f : X \to X$ is sequentially weakly continuous, i.e. $p_j \rightharpoonup p$ in X implies $f(p_j) \rightharpoonup f(p)$ in X.

Proof. We just need to show that $u_j \rightharpoonup u$ in $H_0^1(\Omega)$ implies $u_j^3 \rightharpoonup u^3$ in $L^2(\Omega)$. But since $H_0^1(\Omega)$ is continuously embedded in $L^6(\Omega)$, we can extract a subsequence such that $u_{j_k}^3 \rightharpoonup \chi$ in $L^2(\Omega)$, and by the compactness of the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ we may suppose that $u_{j_k} \to u$ a.e..

Thus $\chi = u^3$ a.e. (e.g. by Lusin's or Mazur's Theorem), and since the limit is unique the whole sequence converges weakly to u^3 as required. □

Lemma 43. Let $\tau > 0$ and let z_j be a bounded sequence in $C([0,\tau];X)$ such that for any $v \in X^*$ the functions $\langle z_j(\cdot), v\rangle$ are equicontinuous on $[0,\tau]$. Then there exist a subsequence z_{j_k} and a weakly continuous map $z : [0,\tau] \to X$ such that $\langle z_{j_k}(\cdot), v\rangle \to \langle z(\cdot), v\rangle$ uniformly on $[0,\tau]$ for all $v \in X^*$.

Proof. Since $z_j(t)$ is bounded in the reflexive space X for each $t \in [0,\tau]$ there exists a diagonal subsequence z_{j_k} such that for each rational $q \in [0,\tau]$ there exists $z(q) \in X$ with $z_{j_k}(q) \rightharpoonup z(q)$.

We claim that $\langle z_{j_k}(r), v\rangle$ is convergent for every $r \in [0,\tau]$ and $v \in X^*$. Indeed let $q_j \to r$ with q_j rational.

Then

$$
\begin{aligned}
|\langle z_{j_k}(r), v\rangle - \langle z_{j_l}(r), v\rangle| \;\le\; & |\langle z_{j_k}(r), v\rangle - \langle z_{j_k}(q_m), v\rangle| \\
& +|\langle z_{j_k}(q_m), v\rangle - \langle z_{j_l}(q_m), v\rangle| \\
& +|\langle z_{j_l}(q_m), v\rangle - \langle z_{j_l}(r), v\rangle|,
\end{aligned}
$$

and the claim follows from the equicontinuity.

It then follows easily that $z_{j_k}(r) \rightharpoonup z(r)$ for some $z(r)$ and that $z : [0, \tau] \to X$ is weakly continuous. $\square$

Theorem 43 The semiflow $\{T(t)\}_{t\ge 0}$ is jointly sequentially weakly continuous, i.e. if $p_j \rightharpoonup p$ in X and $t_j \to t$ in $[0, \infty)$ then $T(t_j)p_j \rightharpoonup T(t)p$ in X.

Proof. Let $w_j(t) = T(t)p_j$, and let $0 \leq t \leq t+h \leq \tau$. Then if $v \in X^*$

$$\begin{aligned}
\langle w_j(t+h) - w_j(t), v\rangle &= \langle (e^{A(t+h)} - e^{At})p_j, v\rangle \\
&\quad + \langle \int_0^{t+h} e^{A(t+h-s)} f(w_j(s))\, ds, v\rangle \\
&\quad - \langle \int_0^{t} e^{A(t-s)} f(w_j(s))\, ds, v\rangle \\
&= \langle (e^{A(t+h)} - e^{At})p_j, v\rangle + \int_t^{t+h} \langle e^{A(t+h-s)} f(w_j(s)), v\rangle\, ds \\
&\quad + \int_0^{t} \langle (e^{A(t+h-s)} - e^{A(t-s)}) f(w_j(s)), v\rangle\, ds,
\end{aligned}$$

and it follows from Lemmas 40, 41 and the uniform boundedness of $\|w_j(t)\|_X$, $\|f(w_j(t))\|_X$ that $\langle w_j(\cdot), v\rangle$ is equicontinuous on $[0, \tau]$.

Hence by Lemma 43 there is a subsequence w_{j_k} and a weakly continuous $w : [0, \tau] \to X$ such that $\langle w_{j_k}(t), v\rangle \to \langle w(t), v\rangle$ uniformly on $[0, \tau]$ for each $v \in X^*$.

Then we can pass to the limit in

$$\langle w_{j_k}(t), v\rangle = \langle e^{At}p_{j_k}, v\rangle + \int_0^t \langle e^{A(t-s)} f(w_{j_k}(s)), v\rangle\, ds$$

to deduce that $w(t) = T(t)p$. The uniqueness of the limit then shows that $\langle T(t)p_j, v\rangle$ converges to $\langle T(t)p, v\rangle$ uniformly on $[0, \tau]$ as required. □

Theorem 44 $\{T(t)\}_{t\geq 0}$ is asymptotically compact.

Proof. Let

$$I(u, u_t) = V(u, u_t) + \frac{\beta}{2}(u, u_t).$$

Then

$$
\begin{aligned}
\frac{dI}{dt} &= -\beta\|u_t\|_2^2 + \frac{\beta}{2}\|u_t\|_2^2 \\
&\quad + \frac{\beta}{2}\left(-\beta(u, u_t) - \|\nabla u\|_2^2 + \int_\Omega (u^2 - u^4)\,dx\right) \\
&= -\beta I + \beta \int_\Omega \left(\frac{1}{4}(u^2-1)^2 + \frac{u^2 - u^4}{2}\right)\,dx \\
&= -\beta I + H(u),
\end{aligned}
$$

where $H(u) = \frac{\beta}{4}\int_\Omega (1 - u^4)\,dx$.

Hence, given any $M > 0$ and any $z \in X$,

$$
I(T(M)z) = e^{-\beta M} I(z) + \int_0^M e^{\beta(t-M)} H(u(t))\,dt, \quad (16)
$$

where $T(t)z = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$.

Let p_j be bounded, $t_j \to \infty$. Then $T(t_j)p_j$ is bounded and we may suppose that $T(t_j)p_j \rightharpoonup \chi, T(t_j - M)p_j \rightharpoonup \chi_{-M}$ for some $\chi, \chi_{-M} \in X$. Hence $T(t+t_j-M)p_j \rightharpoonup T(t)\chi_{-M}$ and thus $T(M)\chi_{-M} = \chi$.

Apply (16) with $z = T(t_j - M)p_j$ to obtain

$$I(T(t_j)p_j) = e^{-\beta M} I(T(t_j - M)p_j) + \int_0^M e^{\beta(t-M)} H(u_j(t))\, dt,$$

where $T(t + t_j - M)p_j = \begin{pmatrix} u_j(t) \\ \dot{u}_j(t) \end{pmatrix}$.

Passing to the limit and using again (16) with $z = \chi_{-M}$ we get

$$\limsup_{j\to\infty} I(T(t_j)p_j) \leq Ce^{-\beta M} + I(\chi) - e^{-\beta M} I(\chi_{-M}).$$

Letting $M \to \infty$ we obtain

$$\limsup_{j\to\infty} I(T(t_j)p_j) \le I(\chi) \le \liminf_{j\to\infty} I(T(t_j)p_j).$$

Hence $I(T(t_j)p_j) \to I(\chi)$ and from the form of I we deduce that $\|T(t_j)p_j\|_X \to \|\chi\|_X$ so that $T(t_j)p_j \to \chi$ strongly. $\square$

Hence $\omega(p)$ consists only of rest points for every p. Also, the set Z of rest points is bounded, since for any rest point $z = \begin{pmatrix} u \\ 0 \end{pmatrix}$,

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega (u^2 - u^4)\, dx \le \frac{1}{4}\mathcal{L}^3(\Omega).$$

Hence $\{T(t)\}_{t\geq 0}$ is point dissipative and so we have proved

Theorem 45 There exists a global attractor for (15). For each complete orbit ψ in A the α and ω limit sets of ψ are connected subsets of the set Z of rest points on which V is constant. If Z is totally disconnected then the limits

$$z_- = \lim_{t\to-\infty}\psi(t),\ z_+ = \lim_{t\to\infty}\psi(t)$$

exist and z_-, z_+ are rest points; furthermore, $T(t)p$ tends to a rest point in X as $t\to\infty$ for every solution $p\in X$.

Remarks. 1. It is not true in general that Z is totally disconnected. To construct an example, let $\Omega = B_R := B(0,R)$ for some $R>0$, and $B_R^{\pm} = \{x\in B_R : \pm x_3 > 0\}$.

Let u^+ minimize

$$E(u) = \int_{B_R^+} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 \right) dx$$

in $H_0^1(B_R^+)$. It is easily checked that for $R > 0$ sufficiently large $u^+ \neq 0$, since we can take as a test function $u_\varepsilon(x) = 1$ for $\mathrm{dist}\,(x, \partial B_R^+)$ with a suitable interpolation to zero in the $\varepsilon-$neighbourhood of ∂B_R^+. Also u^+ is a smooth solution of $-\Delta u + u^3 - u = 0$ in B_R^+ which is smooth up to $\{x_3 = 0\}$.

Now define $\tilde{u}$ as the odd extension of u^+ to B_R, i.e.

$$\tilde{u}(x) = \begin{cases} u^+(x), & x_3 > 0, \\ -u^+(x_1, x_2, -x_3), & x_3 < 0. \end{cases}$$

Then $\tilde{u}$ is also a solution to $-\Delta u + u^3 - u = 0$ in B_R^- which is smooth up to $\{x_3 = 0\}$. Let $v \in C_0^\infty(B_R)$. Then

$$\begin{aligned}\int_{B_R} (\nabla \tilde{u} \cdot \nabla v + (\tilde{u}^3 - \tilde{u})v)\, dx &= \int_{B_R^-} (-\Delta \tilde{u} + \tilde{u}^3 - \tilde{u}) v\, dx \\ &\quad + \int_{B_R^+} (-\Delta u^+ + u^{+3} - u^+) v\, dx \\ &\quad + \int_{x_1^2 + x_2^2 < R^2} \left(\frac{\partial \tilde{u}}{\partial x_3} - \frac{\partial u^+}{\partial x_3} \right) v\, dS \\ &= 0,\end{aligned}$$

so that $\tilde{u}$ is a rest point.

Clearly $\tilde{u}$ is not radial, since there exists $x = (x_1, x_2, x_3) \in B_R$ with $\tilde{u}(x_1, x_2, x_3) = -\tilde{u}(x_1, x_2, -x_3) \neq 0$, and $Q_0 \in SO(3)$ with $Q_0 x = (x_1, x_2, -x_3)$, so that $\tilde{u}(Q_0 \cdot) \neq \tilde{u}(\cdot)$.

Note that $\tilde{u}(Q \cdot)$ is a rest point for any $Q \in SO(3)$. Let $E = \{Q \in SO(3) : \tilde{u}(Q \cdot) = \tilde{u}(\cdot)\}$. Then $E \subset SO(3)$ is closed, and $\mathbf{1} \in E$. Let $Q^* \in \partial E$.

Then in any neighbourhood of Q^* there is a rotation $Q' \notin E$, so that $\tilde{u}$ is not isolated.

However Saut & Temam, Comm. PDE 1979 show that for generic Ω diffeomorphic to a ball there are only finitely many rest points.

2. Kalantarov, Savostianov & Zelik, Ann. Henri Poincaré 2016, prove a version of Theorem 45 in $\mathbb{R}^3$ with a quintic nonlinear term.

Remark added after course. An alternative way to prove Lemma 21 is to use the fact that Δ with domain $D(\Delta) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$ is self-adjoint on $L^2(\Omega)$. This also simplifies the proof of Theorem 20, in which part of the argument effectively proves the self-adjointness.