Variational methods - lectures 1-8

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January 3, 2019

1 A user's guide to Sobolev spaces

In order to give an unambiguous definition of what is meant by a solution of a system of partial differential equations appropriate function spaces must be defined. By far the most important of these spaces for variational methods are the *Sobolev spaces* based on the classical L^p spaces of functions whose pth powers are integrable.

The reader not familiar with Banach spaces, L^p spaces and weak convergence will need to supplement the material given here by reference to standard texts on Lebesgue integration and functional analysis (see, for example, Brezis [6], Dunford & Schwartz [12], Rudin [21]).

For general references on Sobolev spaces see Adams & Fournier [1], Brezis [6], Evans [15], Maz'ya [19].

1.1 Review of L^p spaces

If $x \in \mathbb{R}^n$ we write $x = (x_1, ..., x_n)$, where the x_i are the coordinates of x with respect to a fixed orthonormal basis e_i of $\mathbb{R}^n$. Let $\mathcal{L}^n$ denote n-dimensional Lebesgue measure; if $E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable we denote its measure by $\mathcal{L}^n(E)$, writing $d\mathcal{L}^n = dx$. If $E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable and $1 \leq p \leq \infty$ then $L^p(E)$ is the space of (equivalence classes of) $\mathcal{L}^n$-measurable functions $u : E \to \mathbb{R}$ with $\|u\|_p < \infty$, where

$$\|u\|_p = \left(\int_E |u(x)|^p \, dx\right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty, \tag{1.1}$$

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x \in E} |u(x)|. \tag{1.2}$$

Here two functions u, v are equivalent if $u(x) = v(x)$ $\mathcal{L}^n$ almost everywhere (that is, for all $x \in E \backslash N$ where $\mathcal{L}^n(N) = 0$). In (1.2),

$$\operatorname*{ess\,sup}_{x \in E} |u(x)| \stackrel{\text{def}}{=} \inf\{\alpha \geq 0 : |u(x)| \leq \alpha \text{ for a.e. } x \in E\}.$$

Most of the time we will consider $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. Endowed with the norm $\|\cdot\|_p$, $L^p(E)$ is a *Banach space* (i.e. a complete normed linear space;

complete means that each Cauchy sequence converges). The triangle inequality

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p$$

is *Minkowski's inequality*. We also have *Hölder's inequality*

$$\|uv\|_1 \leq \|u\|_p \|v\|_{p'} \quad \text{ for all } u \in L^p(\Omega), v \in L^{p'}(\Omega), \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. In particular, since

$$\| \, |u|^q \| \leq \| \, |u|^q \|_{p/q} \|1\|_{(p/q)'}$$

we have that $L^p(E) \subset L^q(E)$ whenever $1 \leq q \leq p$ and $\mathcal{L}^n(E) < \infty$.

If $1 \leq p < \infty$ then the dual space $L^p(E)^*$ of $L^p(E)$ (that is the Banach space of all continuous linear mappings from $L^p(E)$ to $\mathbb{R}$) can be identified with $L^{p'}(E)$. More precisely, if $T \in L^p(E)^*$ there exists a unique $\varphi = \varphi_T$ in $L^{p'}(E)$ such that

$$\langle T, u \rangle = \int_E u\varphi \, dx \quad \text{ for all } u \in L^p(E), \tag{1.4}$$

and the mapping $T \mapsto \varphi_T$ is an isometric isomorphism of $L^p(E)^*$ onto $L^{p'}(E)$ (i.e. it is 1-1, onto and $\|T\|_{L^p(E)^*} = \|\phi_T\|_{L^{p'}(E)}$). From this it follows easily that if $1 < p < \infty$ then $L^p(E)$ is *reflexive*. (Recall that a Banach space X is reflexive if the natural embedding $\tau : X \to X^{**}$ defined by

$$< \tau u, T > = < T, u > \quad \text{for all } u \in X, T \in X^*$$

is onto, so that in particular we can identify X^{**} with X.)

If $1 \leq p < \infty$ then $L^p(\Omega)$ is *separable* (that is, contains a countable dense subset); a suitable dense subset is given by finite linear combinations with rational coefficients of the characteristic functions $\{\chi_{E \cap Q}\}$, where Q runs through all n-cubes of the form $Q = q + (0, 1/j)^n$, the coordinates q_i of $q = (q_1, \ldots, q_n)$ are rational, and $j = 1, 2, \ldots$. But if $\mathcal{L}^n(E) > 0$ then $L^\infty(E)$ is not separable;for example if E is open and $x \in E$ the uncountable family of functions χ_Q, where Q runs through all n-cubes of the form $Q = x + (0, a)^n$, $a > 0$ sufficiently small, are all distance 1 apart in $L^\infty(E)$.

Assume $1 \leq p < \infty$ and et $u^{(j)} \to u$ in $L^p(\Omega)$. Then there exists a subsequence $u^{(j_k)}$ of $u^{(j)}$ which converges to u a.e. in Ω (i.e. $u^{(j_k)}(x) \to u^{(j)}(x)$ for all $x \in E \backslash N$, where $\mathcal{L}^n(N) = 0$). More generally, this holds if $u^{(j)} \to u$ in *measure* i.e. given any $\varepsilon > 0$

$$\lim_{j \to \infty} \mathcal{L}^n(\{x \in \Omega : |u^{(j)}(x) - u(x)| > \varepsilon\}) = 0.$$

1.2 Approximation by smooth functions

Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < \infty$ and $u \in L^p(\Omega)$. How can we approximate u by smooth functions?

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}$ and denote by $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$ consisting of those $\varphi : \Omega \to \mathbb{R}$ with compact support in Ω (i.e. such that $\varphi(x) = 0$ for $x \in \Omega \backslash K$, where $K \subset \Omega$ is compact; the smallest such K is called the *support* $\operatorname{supp} \varphi$ of φ. Note that a nonzero $\varphi \in C_0^\infty(\Omega)$ cannot be analytic (i.e. representable as the sum of a convergent power series), since all the Taylor coefficients are zero for $x \notin \operatorname{supp} \varphi$; an example of a nonzero $\varphi \in C_0^\infty(\mathbb{R}^n)$ is given by (see Example 1.4)

$$\varphi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{1.5}$$

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$(i) \qquad \rho \geq 0, \qquad \rho(x) = 0 \ \text{ if } \ |x| \ \geq 1, \tag{1.6}$$

$$(ii) \qquad \int_{\mathbb{R}^n} \rho \, dx = 1. \tag{1.7}$$

For $\varepsilon > 0$ define

$$\rho_\varepsilon(x) = \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right). \tag{1.8}$$

ρ_ε is called a *mollifier*. Clearly

$$(i) \qquad \rho_\varepsilon \geq 0, \qquad \rho_\varepsilon(x) = 0 \ \text{ if } \ |x| \geq \varepsilon, \tag{1.9}$$

$$(ii) \qquad \int_{\mathbb{R}^n} \rho_\varepsilon(x) \, dx = \int_{\mathbb{R}^n} \rho(y) \, dy = 1, \tag{1.10}$$

so that ρ_ε approximates the delta function (see Figure 1). We therefore expect

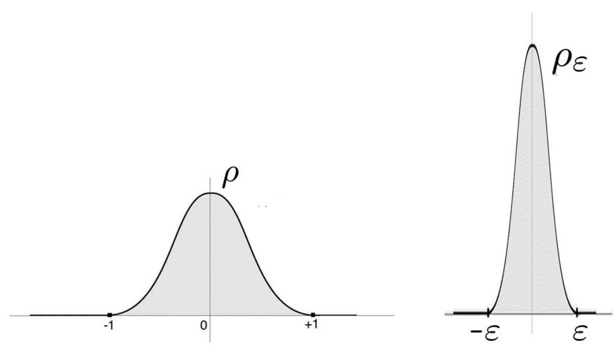

Figure 1: Approximating the δ function; the functions ρ and ρ_ε.

the convolution

$$(\rho_\varepsilon * u)(x) := \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) u(y) \, dy \tag{1.11}$$

to approximate u.

Theorem 1. *Let $1 \le p < \infty$ and $u \in L^p(\Omega)$. Define u to be zero outside Ω. Then*
(i) $\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n)$,
(ii) $\|\rho_\varepsilon * u\|_p \le \|u\|_p$,
(iii) $\lim_{\varepsilon\to 0} \|\rho_\varepsilon * u - u\|_p = 0$.
In particular $C^\infty(\Omega)$ is dense in $L^p(\Omega)$.

We make use of the following lemma.

Lemma 2. *Let $1 \le p < \infty$, $h \in C_0^\infty(\mathbb{R}^n)$ and $u \in L^p(\mathbb{R}^n)$. Then $h * u$ is continuously differentiable on $\mathbb{R}^n$ and for $i = 1, ...n$*

$$\frac{\partial(h*u)}{\partial x_i}(x) = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x_i}(x-y)u(y)\,dy. \tag{1.12}$$

Proof. Let $x_j \to x$. By definition

$$(h*u)(x_j) = \int_{\mathbb{R}^n} h(x_j - y)u(y)\,dy. \tag{1.13}$$

The integrand vanishes for all j for y outside some bounded set, and is bounded in absolute value by $const.|u(y)|$. Hence by the dominated convergence theorem $(h*u)(x_j) \to (h*u)(x)$ and so $h*u$ is continuous.

For $x \in \Omega$, and $|t| \le 1$ we have

$$\frac{(h*v)(x+te_i) - (h*v)(x)}{t} = \int_{\mathbb{R}^n} \left(\frac{h(x+te_i-y) - h(x-y)}{t}\right) v(y)\,dy. \tag{1.14}$$

Since $h \in C_0^\infty(\mathbb{R}^n)$ the integrand is bounded by $const.|v(y)|$ and is zero for y outside some bounded set. Hence by the dominated convergence theorem $\partial(h*v)/\partial x_i$ exists and is given by (1.12).

By the first part of the argument applied to the kernel $\partial h/\partial x_i$ we see that each $\partial(h*v)/\partial x_i$ is continuous and so by a standard result $h*v$ is continuously differentiable. □

Proof of Theorem 1. (i) This follows by applying Lemma 2 inductively to u and its partial derivatives.
(ii) We write

$$\rho_\varepsilon(x-y)u(y) = \rho_\varepsilon(x-y)^{\frac{1}{p'}}\rho_\varepsilon(x-y)^{\frac{1}{p}}u(y).$$

Thus

$$\left|\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)u(y)\,dy\right| \le \left(\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\,dy\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)|u(y)|^p dy\right)^{\frac{1}{p}}, \tag{1.15}$$

and hence, using Fubini's theorem and $\int_{\mathbb{R}^n}\rho_\varepsilon(z)\,dz = 1$,

$$\begin{aligned}\int_\Omega |\rho_\varepsilon * u|^p dx &\le \int_{\mathbb{R}^n}|u(y)|^p\left(\int_\Omega \rho_\varepsilon(x-y)dx\right)dy\\ &\le \int_\Omega |u(y)|^p dy. \qquad (1.16)\end{aligned}$$

(*iii*) Given $\tau > 0$ there exists a continuous function w of compact support in Ω with $\|u-w\|_p < \tau$. Since

$$\begin{aligned}\int_\Omega |(\rho_\varepsilon * w)(x) - w(x)|^p dx &= \int_\Omega \left|\int_{\mathbb{R}^n}\rho_\varepsilon(x-y)(w(y)-w(x))dy\right|^p dx,\\ &\le \kappa(\varepsilon)^p\mathcal{L}^n(N_\varepsilon), \qquad (1.17)\end{aligned}$$

where $\kappa(\varepsilon) := \sup_{|x-y|<\varepsilon}|w(x)-w(y)|$, and $N_\varepsilon = \{x\in\mathbb{R}^n : \mathrm{dist}\,(x, \mathrm{supp}\, w) \le \varepsilon\}$, it follows that

$$\lim_{\varepsilon\to 0}\|\rho_\varepsilon * w - w\|_p = 0. \qquad (1.18)$$

Since

$$\|\rho_\varepsilon * u - u\|_p \le \|\rho_\varepsilon * w - w\|_p + \|\rho_\varepsilon * (u-w) - (u-w)\|_p, \qquad (1.19)$$

it follows from (*ii*) that $\lim_{\varepsilon\to 0}\|\rho_\varepsilon * u - u\|_p \le 2\tau$. Since τ is arbitrary this completes the proof. □

1.3 Weak and weak* convergence

Let X be a Banach space with dual space X^*.

Definitions 1. *A sequence $u^{(j)}$ converges weakly to u in X (written $u^{(j)} \rightharpoonup u$ in X) if*

$$\langle T, u^{(j)}\rangle \to \langle T, u\rangle \qquad \textit{for all } T\in X^*.$$

A sequence $T^{(j)}$ converges weak to T in X^* (written $T^{(j)} \overset{*}{\rightharpoonup} T$) if*

$$\langle T^{(j)}, u\rangle \to \langle T, u\rangle \qquad \textit{for all } u\in X.$$

Applying these definitions to $X = L^p(E)$, and using the characterization of $L^p(E)^*$ in Section 1.1, we find that if $1\le p<\infty$ then $u^{(j)} \rightharpoonup u$ in $X = L^p(E)$ if and only if

$$\int_E u^{(j)}\varphi\,dx \to \int_E \varphi\,dx \qquad \text{for all } \varphi\in L^{p'}(E), \qquad (1.20)$$

and $u^{(j)} \overset{*}{\rightharpoonup} u$ in $L^\infty(E)$ if and only if

$$\int_E u^{(j)}\varphi\,dx \to \int_\Omega u\varphi\,dx \qquad \text{for all } \varphi\in L^1(E). \qquad (1.21)$$

Example 1.1. (Rademacher functions) Let $\Omega = (0,1)$, $0 < \lambda < 1$, $a, b \in \mathbb{R}$ and define $\theta : \mathbb{R} \to \mathbb{R}$ by

$$\theta(x) = \begin{cases} a, & 0 < x \leq \lambda \\ b, & \lambda < x \leq 1 \end{cases} \tag{1.22}$$

extended to the whole of $\mathbb{R}$ as a function of period 1. (See Figure 2(i).) Now

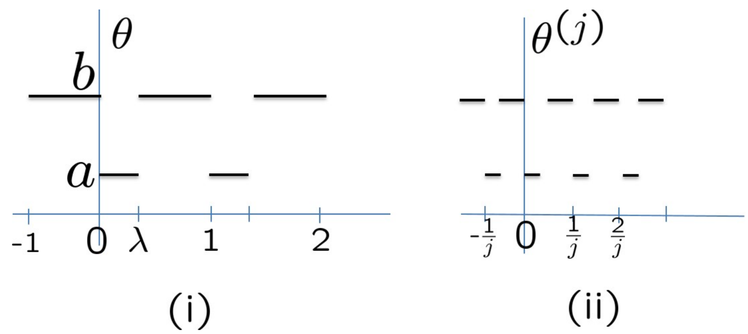

Figure 2: (i) The 1-periodic function θ, (ii) The function $\theta^{(j)}(x) = \theta(jx)$ for large j.

define $\theta^{(j)}(x) = \theta(jx)$, $j = 1, 2, \ldots$. For large j, $\theta^{(j)}$ oscillates fast between the values a and b (see Figure 2 (ii)), taking these values with relative frequency λ to $1 - \lambda$. Let $c = \lambda a + (1-\lambda)b$. Thus we guess that

Proposition 3. *$\theta^{(j)} \overset{*}{\rightharpoonup} c$ in $L^\infty(0,1)$ as $j \to \infty$.*

Proof. We first calculate $\lim_{j\to\infty} \int_r^s \theta^{(j)}\,dx$ for $0 \leq r < s \leq 1$. We have that

$$\begin{aligned} \int_r^s \theta^{(j)}(x)\,dx &= \int_r^s \theta(jx)\,dx \\ &= \frac{1}{j}\int_{jr}^{js} \theta(\tau)\,d\tau. \end{aligned} \tag{1.23}$$

The interval (jr, js) contains N_j integers, where $|N_j - (js - jr)| \leq 1$. Since θ is 1-periodic and $\int_0^1 \theta(\tau)\,d\tau = c$ it follows that

$$\int_{jr}^{js} \theta(\tau)\,d\tau = (js - jr)c + \epsilon_j, \tag{1.24}$$

where $|\epsilon_j| \leq constant$. Combining (1.23), (1.24) we deduce that

$$\lim_{j\to\infty} \int_r^s \theta^{(j)}(x)\,dx = \int_r^s c\,dx. \tag{1.25}$$

It follows from (1.25) that

$$\lim_{j\to\infty} \int_r^s \theta^{(j)}\varphi\,dx = \int_r^s c\varphi\,dx \tag{1.26}$$

for any step function φ (i.e. for any function φ with finitely many values, each taken on an interval). But step functions are dense in $L^1(0,1)$; given any $\varphi \in L^1(0,1)$ there exists a sequence $\varphi^{(k)}$ of step functions converging strongly to φ in $L^1(0,1)$. Hence

$$\begin{aligned} &| \int_0^1 \theta^{(j)} \varphi \, dx - \int_0^1 c\varphi \, dx| \\ \leq \; & | \int_0^1 (\theta^{(j)} - c)\varphi^{(k)} \, dx| + | \int_0^1 (\theta^{(j)} - c)(\varphi - \varphi^{(k)}) \, dx| \\ \leq \; & | \int_0^1 (\theta^{(j)} - c)\varphi^{(k)} \, dx| + K \parallel \varphi^{(k)} - \varphi \parallel_1, \end{aligned} \tag{1.27}$$

where K is a constant. Letting $j \to \infty$ and then $k \to \infty$ we deduce that

$$\lim_{j\to\infty} \int_0^1 \theta^{(j)} \varphi \, dx = \int_0^1 c\varphi \, dx \tag{1.28}$$

for all $\varphi \in L^1(0,1)$, and thus $\theta^{(j)} \stackrel{*}{\rightharpoonup} c$ in $L^\infty(0,1)$.

□

A key reason why weak convergence is important for variational methods is that suitably bounded sequences have weakly (or weak*) convergent subsequences.

Theorem 4. *Let X be a separable Banach space, and let $T^{(j)}$ be a bounded sequence in X^*, i.e. $sup_j \parallel T^{(j)} \parallel_{X^*} = M < \infty$. Then there exists a subsequence $T^{(j_k)}$ of $T^{(j)}$ converging weak* to some T in X^*.*

Proof. Let $\{\psi_i\}_{i=1}^\infty$ be a countable dense subset of X. Since

$$| \langle T^{(j)}, \psi_1 \rangle | \leq M \parallel \psi_1 \parallel \tag{1.29}$$

the sequence $\langle T^{(j)}, \psi_1 \rangle$ of real numbers is bounded. Hence there exists a subsequence $T^{(n_1(j))}$ of $T^{(j)}$ such that $\lim_{j\to\infty} \langle T^{(n_1(j))}, \psi_1 \rangle$ exists. Similarly, the sequence $\langle T^{(n_1(j))}, \psi_2 \rangle$ is bounded, and so there exists a subsequence $T^{(n_2(j))}$ of $T^{(n_1(j))}$ such that $\lim_{j\to\infty} \langle T^{(n_2(j))}, \psi_2 \rangle$ exists. Proceeding in this way we obtain for each i a subsequence $T^{(n_i(j))}$ of $T^{(n_{i-1}(j))}$ such that $\lim_{j\to\infty} \langle T^{(n_i(j))}, \psi_i \rangle$ exists. Consider the 'diagonal sequence' $T^{(n_j(j))}$. Since $\{T^{(n_j(j))}\}_{j=i}^\infty$ is a subsequence of $\{T^{(n_i(j))}\}_{j=i}^\infty$ it follows that $\lim_{j\to\infty} \langle T^{(n_j(j))}, \psi_i \rangle$ exists for each i.

Now let $\psi \in X$ be arbitrary. Given $\varepsilon > 0$ there exists I with

$$\parallel \psi - \psi_I \parallel \leq \frac{\varepsilon}{2M}. \tag{1.30}$$

Then

$$| \langle T^{(n_j(j))}, \psi \rangle - \langle T^{(n_k(k))}, \psi \rangle | \leq | \langle T^{(n_j(j))}, \psi_I \rangle - \langle T^{(n_k(k))}, \psi_I \rangle | + \varepsilon \tag{1.31}$$

and hence $\langle T^{(n_k(k))}, \psi \rangle$ is a Cauchy sequence, so that

$$T(\psi) \stackrel{\text{def}}{=} \lim_{k\to\infty} \langle T^{(n_k(k))}, \psi \rangle \tag{1.32}$$

exists. Clearly T is linear in ψ, and since $| T(\psi) | \leq M \parallel \psi \parallel$ it follows that $T \in X^*$. Thus $T^{(j_k)} \stackrel{*}{\rightharpoonup} T$ in X^* with $j_k = n_k(k)$. □

A related result is

Theorem 5 ([12, p68])**.** *A bounded sequence in a reflexive Banach space X has a weakly convergent subsequence.*

Thus a bounded sequence in $L^p(E), 1 < p < \infty$, has a weakly convergent subsequence, and a bounded sequence in $L^\infty(E)$ has a weak* convergent subsequence. A bounded sequence in $L^1(E)$ need not have a weakly convergent subsequence (consider, for example, the case $E = (0,1), u^{(j)} = j\chi_{(0,\frac{1}{j})}$), and an extra condition is needed to ensure this.

Theorem 6 (de la Vallée Poussin, see [11, p24])**.** *A sequence $u^{(j)}$ in $L^1(E)$ has a weakly convergent sequence if*

$$\sup_j \int_E \Phi(|u^{(j)}|)\, dx < \infty$$

for some continuous $\Phi : [0,\infty) \to [0,\infty)$ with

$$\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Exercises

1.1. Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$. For $\alpha \in \mathbb{R}$ define

$$u_\alpha(x) = |x|^\alpha.$$

For which p, $1 \leq p \leq \infty$, does $u_\alpha \in L^p(B)$?

1.2. Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Are the following statements true or false?

$$\text{(i)} \quad L^1(\Omega) = \bigcup_{1<p<\infty} L^p(\Omega),$$

$$\text{(ii)} \quad L^\infty(\Omega) = \bigcap_{1<p<\infty} L^p(\Omega).$$

1.3. For $j = 1, 2, ...$ let $a_j = \sum_{i=1}^{j} \frac{1}{i}$, and define E_j to be the interval (a_j, a_{j+1}) (mod 1) (i.e. $x \in E_j$ if and only if $x \in (0,1)$ and $x + m \in (a_j, a_{j+1})$ for some integer m). Show that $u^{(j)} = \chi_{E_j}$ converges to zero in $L^p(0,1)$ as $j \to \infty$, but that $u^{(j)} \not\to 0$ a.e..

1.4. Show that the function φ given by (1.5) belongs to $C_0^\infty(\mathbb{R}^n)$.
Hint. Prove by induction that for $|t| < 1$ the n^{th} derivative $f^{(n)}$ of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} \exp\left(\frac{1}{t^2-1}\right) & |t| < 1 \\ 0 & |t| \geq 1 \end{cases} \tag{1.33}$$

has the form

$$f^{(n)}(t) = \frac{P_n(t)}{(t^2-1)^{2n}} \exp\left(\frac{1}{t^2-1}\right), \qquad |t| < 1, \tag{1.34}$$

where P_n is a polynomial.

1.5. Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p < \infty$.
(i) Prove that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.
(ii) Is $C_0^\infty(\Omega)$ dense in $L^\infty(\Omega)$?

1.6. Let $\theta : \mathbb{R} \to \mathbb{R}$ be continuous with $\theta(t) = 0$ for $|t| \geq 1$, and define $\theta^{(j)}(x) = \theta(x + j)$.
(i) Prove that $\theta^{(j)} \rightharpoonup 0$ in $L^p(\mathbb{R})$ for $1 < p < \infty$, and that $\theta^{(j)} \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R})$ as $j \to \infty$.
(ii) Does $\theta^{(j)} \rightharpoonup 0$ in $L^1(\mathbb{R})$?

1.7. Prove the following generalization of Proposition 3. If $\theta \in L^\infty(\mathbb{R})$ is 1-periodic and if $\theta^{(j)}(x) := \theta(jx)$, then

$$\theta^{(j)} \overset{*}{\rightharpoonup} \bar{\theta} := \int_0^1 \theta(t)\, dt$$

in $L^\infty(\mathbb{R})$ as $j \to \infty$.

1.8. Let

$$u^{(j)}(x) = \begin{cases} j & \text{for} \quad 0 < x < j^{-1}, \\ 0 & \text{otherwise} \end{cases}$$

.
(i) If $1 < p < \infty$ prove that $(u^{(j)})^{\frac{1}{p}} \rightharpoonup 0$ in $L^p(0,1)$ as $j \to \infty$.
(ii) Is $u^{(j)}$ weakly convergent in $L^1(0,1)$?

1.9. Let $\Omega \subset \mathbb{R}^n$ be open, and let $f^{(j)} \rightharpoonup f$ in $L^1(\Omega)$, $f^{(j)} \to g$ a.e. in Ω. Prove that $f = g$ a.e..
Hint. Use Mazur's theorem, that if $f^{(j)} \rightharpoonup f$ in a Banach space X then there exists a sequence $\{\theta^{(k)}\}$ of finite convex combinations of the $f^{(j)}$ converging strongly to f in X.

1.4 The multi-index notation for derivatives

It is convenient to have a compact notation for expressing mixed partial derivatives of functions. A *multi-index* α is an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers α_i, and we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be smooth. Then we define

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{1.35}$$

For example, if $n = 3$ and $\beta = (2, 1, 0)$, then

$$D^\beta u = \frac{\partial^3 u}{\partial x_1^2 \partial x_2}. \tag{1.36}$$

Note that if α, β are multi-indices then so is $\alpha + \beta = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)$, and

$$D^{\alpha+\beta} u = D^\alpha D^\beta u = D^\beta D^\alpha u. \tag{1.37}$$

We will use the multi-index notation also for weak derivatives as defined in the next section.

1.5 Weak derivatives

Let $\Omega \subset \mathbb{R}^n$ be open with boundary $\partial\Omega$, and let $v \in C^1(\Omega)$, $\varphi \in C_0^\infty(\Omega)$. Then for any $j = 1, ..., n$

$$\frac{\partial}{\partial x_j}(v\varphi) = v\frac{\partial \varphi}{\partial x_j} + \frac{\partial v}{\partial x_j}\varphi, \tag{1.38}$$

so that integrating over Ω and using the divergence theorem [1] we have that

$$\int_\Omega v \frac{\partial \varphi}{\partial x_j}\, dx = -\int_\Omega \frac{\partial v}{\partial x_j} \varphi\, dx. \tag{1.39}$$

This can be thought of as the formula for integration by parts in n dimensions.

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[1] The divergence theorem states that if $f : R^n \to R^n$ is C^1, and if $E \subset R^n$ is open and has sufficiently smooth boundary, then

$$\int_E \operatorname{div} f\, dx = \int_{\partial E} f \cdot n\, dS,$$

where n denotes the unit outward normal to ∂E. To obtain (1.39) we cannot apply the theorem directly because $\partial\Omega$ may not be smooth. Instead, we extend $v\varphi$ by zero to the whole of R^n and apply the theorem with E a large ball containing Ω and $f = v\varphi e_j$. Then

$$\int_\Omega \operatorname{div} f\, dx = \int_E \operatorname{div} f\, dx = 0,$$

and since

$$\operatorname{div} f = \frac{\partial}{\partial x_j}(v\varphi)$$

we obtain (1.39).

Now let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multi-index and $u \in C^{|\alpha|}(\Omega)$. Applying (1.39) α_j times for each j we deduce that

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \cdot \varphi \, dx, \tag{1.40}$$

there being $|\alpha| = \alpha_1 + \cdots + \alpha_n$ changes of sign all together.

Define

$$L^1_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} : u|_E \in L^1(E) \text{ for all bounded open } E \text{ with } \bar{E} \subset \Omega\}.$$

Definition 1. *Let $u \in L^1_{loc}(\Omega)$ and α be a multi-index. A function $v \in L^1_{loc}(\Omega)$ is said to be an α^{th} weak derivative of u if*

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega), \tag{1.41}$$

and we write $v = D^\alpha u$.

If v_1 and v_2 are two α^{th} weak derivatives, their difference $w = v_1 - v_2$ satisfies

$$\int_\Omega w \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

and so by the following lemma $v_1 = v_2$. Hence weak derivatives are unique.

Lemma 7. (*The fundamental lemma of the calculus of variations.*) *Let $w \in L^1_{loc}(\Omega)$ satisfy*

$$\int_\Omega w \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega) \tag{1.42}$$

Then $w = 0$.

Proof. Let ρ_ε be a mollifier. Let E be bounded and open with $\overline{E} \subset \Omega$. If $\varepsilon < \text{dist}(E, \partial\Omega)$ then for each $x \in E$ the function $\varphi_{\varepsilon,x}$ defined by $\varphi_{\varepsilon,x}(y) = \rho_\varepsilon(x - y)$ belongs to $C_0^\infty(\Omega)$. Hence by (1.42)

$$(\rho_\varepsilon * w)(x) = \int_\Omega \rho_\varepsilon(x - y) w(y) \, dy = 0 \tag{1.43}$$

for all $x \in E$. But $\rho_\varepsilon * w \to w$ in $L^1(E)$ as $\epsilon \to 0$, and so $w = 0$ a.e. in E. Since E is arbitrary the result follows. □

1.6 The Sobolev space $W^{m,p}(\Omega)$

Definition 2. *Let m be a non-negative integer and let $1 \le p \le \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the linear space of functions $u \in L^p(\Omega)$ such that for each α, $0 \le |\alpha| \le m$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$. We norm $W^{m,p}(\Omega)$ by*

$$\|u\|_{m,p} = \begin{cases} \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_\infty & \text{if } p = \infty. \end{cases}$$

If $p = 2$ an alternative notation is often used, namely

$$H^m(\Omega) = W^{m,2}(\Omega).$$

Note that $W^{0,p}(\Omega) = L^p(\Omega)$, while

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \frac{\partial u}{\partial x_j} \in L^p(\Omega) \quad \text{for } i = 1, ..., n\}$$

with norm

$$\|u\|_{1,p} = \left(\int_\Omega |u|^p dx + \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^p dx \right)^{\frac{1}{p}}, \tag{1.44}$$

if $1 \leq p < \infty$ and

$$\|u\|_{1,\infty} = \max \left(\|u\|_\infty, \|\frac{\partial u}{\partial x_1}\|_\infty, ..., \|\frac{\partial u}{\partial x_n}\|_\infty \right), \tag{1.45}$$

where the $\partial u / \partial x_i$ are weak derivatives.

If $(a, b) \subset \mathbb{R}$ is an interval we will write $W^{m,p}(a, b)$ instead of $W^{m,p}((a, b))$.

Theorem 8. *$W^{m,p}(\Omega)$ is a Banach space.*

Proof. $W^{m,p}(\Omega)$ is clearly a normed linear space, and we have to show that it is complete. Let $u^{(j)}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $u^{(j)}$ is a Cauchy sequence in $L^p(\Omega)$, and since $L^p(\Omega)$ is complete $u^{(j)} \to u$ in $L^p(\Omega)$ as $j \to \infty$ for some u. Similarly, if $0 < |\alpha| \leq m$ then $D^\alpha u^{(j)}$ is a Cauchy sequence in $L^p(\Omega)$ and so $D^\alpha u^{(j)} \to u_\alpha$ in $L^p(\Omega)$. But by (1.41)

$$\int_\Omega u^{(j)} D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u^{(j)} \cdot \varphi \, dx \tag{1.46}$$

for all $\varphi \in C_0^\infty(\Omega)$. Passing to the limit $j \to \infty$ using Hölder's inequality we obtain

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi \, dx, \tag{1.47}$$

for all $\varphi \in C_0^\infty(\Omega)$ so that $u_\alpha = D^\alpha u$. Hence $u^{(j)} \to u$ in $W^{m,p}(\Omega)$, so that $W^{m,p}(\Omega)$ is complete. □

Let $\kappa = \kappa(m, n)$ denote the number of multi-indices α with $0 \leq |\alpha| \leq m$, and consider the product space $L^p(\Omega)^\kappa$ with the norm of $v = (v_1, ..., v_\kappa)$ given by

$$\|v\|_{p;\kappa} = \begin{cases} \left(\sum_{i=1}^\kappa \|v_i\|_p^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq i \leq \kappa} \|v_i\|_\infty & \text{if } p = \infty. \end{cases}$$

Then, since $L^p(\Omega)$ is a Banach space which is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$, by well-known results of functional analysis the space

$L^p(\Omega)^\kappa$ has the same properties. Choose a definite ordering of the multi-indices α with $0 \le |\alpha| \le m$. Given $u \in W^{m,p}(\Omega)$ define $Pu \in L^p(\Omega)^\kappa$ by

$$Pu = (D^\alpha u)_{0 \le |\alpha| \le m}. \tag{1.48}$$

Then P is an isometric isomorphism of $W^{m,p}(\Omega)$ onto a linear subspace Z of $L^p(\Omega)^\kappa$, and by a similar argument to that in the proof of Theorem 8 it is easily seen that Z is closed. Recalling that a closed subspace of a separable (resp. reflexive) Banach space is separable (resp. reflexive) we have thus proved

Theorem 9. *$W^{m,p}(\Omega)$ is separable if $1 \le p < \infty$ and is reflexive if $1 < p < \infty$.*

1.7 Examples

In this section we give examples of various functions that do or do not belong to Sobolev spaces, giving proofs from first principles.

1.7.1 Smooth functions

Let $u \in C^m(\Omega)$ with $\|u\|_{m,p} < \infty$. Then by (1.40) the weak derivatives $D^\alpha u$ for $0 \le |\alpha| \le m$ equal the usual ones, and hence $u \in W^{m,p}(\Omega)$. In particular, if Ω is bounded and $u \in C^\infty(\mathbb{R}^n)$ then $u|_\Omega \in W^{m,p}(\Omega)$ for all m, p.

1.7.2 Piecewise affine functions

Let $n = 1$, $\Omega = (0,1)$, and let u be defined by

$$u(x) = \begin{cases} x & \text{if } 0 < x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} < x < 1 \end{cases}. \tag{1.49}$$

Let us show that $u \in W^{1,\infty}(0,1)$ (and hence, since (0,1) is bounded, $u \in W^{1,p}(0,1)$ for $1 \le p \le \infty$). This looks obvious, since

$$\frac{du}{dx}(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x < 1 \end{cases} \tag{1.50}$$

and so $\|u\|_\infty = \frac{1}{2}, \|du/dx\|_\infty = 1$. However, there is a crucial detail to check, namely that du/dx given by (1.50) is indeed the weak derivative of u. To prove this we must show that

$$\int_0^1 u\frac{d\varphi}{dx}\,dx = -\int_0^1 \frac{du}{dx}\varphi\,dx \quad \text{for all } \varphi \in C_0^\infty(0,1), \tag{1.51}$$

where du/dx is given by (1.50). But, integrating by parts on the intervals $(0,\frac{1}{2}), (\frac{1}{2},1)$ we have that

$$\begin{aligned} \int_0^1 u\frac{d\varphi}{dx}\,dx &= \int_0^{\frac{1}{2}} x\frac{d\varphi}{dx}dx + \int_{\frac{1}{2}}^1 (1-x)\frac{d\varphi}{dx}\,dx \\ &= \frac{1}{2}\varphi(\frac{1}{2}) - \int_0^{\frac{1}{2}} \frac{d\varphi}{dx}\,dx - \frac{1}{2}\varphi(\frac{1}{2}) - \int_{\frac{1}{2}}^1 \frac{d\varphi}{dx}\,dx \\ &= -\int_0^1 \frac{du}{dx}\varphi\,dx \end{aligned}$$

as required. Hence $u \in W^{1,\infty}(0,1)$.

A similar proof shows that if u is a piecewise affine function on (0,1) (i.e. u is continuous on (0,1) and affine on each interval (a_i, a_{i+1}), where $0 = a_1 < a_2 < ... < a_n = 1$) then $u \in W^{1,\infty}(0,1)$.

1.7.3 The Heaviside function

The Heaviside function H is defined by

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} . \tag{1.52}$$

Clearly $H \in L^\infty(-1,1)$. We ask whether $H \in W^{1,p}(-1,1)$. Since the derivative

$$\frac{dH}{dx}(x) = 0 \text{ for } x \in (-1,0) \cup (0,1)$$

it is tempting to conclude that $dH/dx \in L^\infty(-1,1)$, so that $H \in W^{1,\infty}(-1,1)$. But this is *false*. In fact, we have

Proposition 10. *$H \notin W^{1,p}(-1,1)$ for any $p, 1 \leq p \leq \infty$.*

Proof. Suppose for contradiction that $H \in W^{1,1}(-1,1)$. Let $dH/dx \in L^1(-1,1)$ denote the weak derivative of H. Then, since H is smooth in $(-1,0) \cup (0,1)$, $dH/dx = 0$ a.e. in $(-1,0) \cup (0,1)$ and so $dH/dx = 0$ a.e. in (-1,1). But by (1.41)

$$\int_{-1}^{1} H \frac{d\varphi}{dx}\, dx = -\int_{-1}^{1} \frac{dH}{dx} \varphi\, dx, \tag{1.53}$$

so that

$$\int_{-1}^{1} H \frac{d\varphi}{dx}\, dx = \int_{0}^{1} \frac{d\varphi}{dx}\, dx = -\varphi(0) = 0 \tag{1.54}$$

for all $\varphi \in C_0^\infty(-1,1)$, a contradiction. □

1.7.4 The function $\ln|x|$ on $\mathbb{R}^n$

Let $n > 1$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$. For $x \neq 0$ define

$$u(x) = \ln r, \qquad r = |x|. \tag{1.55}$$

We show that $u \in W^{1,p}(B)$ if and only if $1 \leq p < n$.

Step 1. *Formal calculation.* For $r > 0$, u is smooth and

$$\frac{\partial u}{\partial x_i} = \frac{1}{r}\frac{\partial r}{\partial x_i} = \frac{x_i}{r^2}. \tag{1.56}$$

Hence $|\nabla u|^2 = \frac{1}{r^2}$ and so

$$\int_B (|u|^p + |\nabla u|^p)\, dx = \omega_{n-1} \int_0^1 r^{n-1}(|\log r|^p + r^{-p})\, dr, \tag{1.57}$$

where $\omega_{n-1} = \mathcal{H}^{n-1}(S^{n-1})$, and this is finite if and only if $1 \le p < n$.

Step 2. Proof that u has weak derivatives given by $\frac{\partial u}{\partial x_i} = \frac{x_i}{r^2}$.

We must show that

$$\int_B u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_B \frac{x_i}{r^2} \varphi \, dx \tag{1.58}$$

for all $\varphi \in C_0^\infty(B)$.Let $\varepsilon > 0$, $B_\varepsilon = B(0, \varepsilon)$. Then

$$\begin{aligned} \int_{B \setminus B_\varepsilon} u \frac{\partial \varphi}{\partial x_i}\, dx &= \int_{B \setminus B_\varepsilon} \left(\frac{\partial (\varphi u)}{\partial x_i} - \varphi \frac{\partial u}{\partial x_i} \right) dx \\ &= -\int_{\partial B_\varepsilon} \varphi u n_i \, dS - \int_{B \setminus B_\varepsilon} \frac{x_i}{r^2} \varphi \, dx. \end{aligned} \tag{1.59}$$

We need to pass to the limit $\varepsilon \to 0$. The volume integrals converge to the obvious limits by dominated convergence; for example, the first integral can be written as

$$\int_{B_R} (1 - \chi_\varepsilon(x)) u \frac{\partial \varphi}{\partial x_i}\, dx, \tag{1.60}$$

where χ_ε denotes the characteristic function of B_ε, and the integrand in (1.60) is bounded in absolute value by $const.|\log r|$, which belongs to $L^1(B_R)$. For the surface integral we have

$$\left| \int_{\partial B_\varepsilon} \varphi u n_i \, dS \right| \le \int_{\partial B_\varepsilon} |\varphi| \cdot |\log \varepsilon|, dS \le const.|\log \varepsilon| \varepsilon^{n-1}, \tag{1.61}$$

which tends to zero as $\varepsilon \to 0$. This proves (1.58).

1.8 Approximation by smooth functions

Let $u \in W^{m,p}(\Omega)$. Let $E \subset \Omega$ be open with $\varepsilon_0 := \operatorname{dist}(E, \partial\Omega) > 0$. Let ρ_ε be a mollifier. Then if $0 < \varepsilon \le \varepsilon_0$ the mollified function

$$\begin{aligned} (\rho_\varepsilon * u)(x) &= \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) u(y)\, dy \\ &= \int_\Omega \rho_\varepsilon(x-y) u(y)\, dy \end{aligned} \tag{1.62}$$

is well-defined for all $x \in E$. If $|\alpha| \le m$ then for $x \in E$

$$\begin{aligned} D^\alpha(\rho_\varepsilon * u)(x) &= \int_\Omega D_x^\alpha \rho_\varepsilon(x-y) u(y)\, dy \\ &= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \rho_\varepsilon(x-y) u(y)\, dy \end{aligned} \tag{1.63}$$

where D_x^α, D_y^α denote derivatives with respect to x, y respectively. Let $\varphi_\varepsilon(y) = \rho_\varepsilon(x-y)$. Since $\varphi_\varepsilon \in C_0^\infty(\Omega)$ it follows from the definition of weak derivatives that for $x \in E$

$$\begin{aligned} D^\alpha(\rho_\varepsilon * u)(x) &= \int_\Omega \rho_\varepsilon(x-y) D^\alpha u(y)\,dy \\ &= (\rho_\varepsilon * D^\alpha u)(x), \end{aligned} \tag{1.64}$$

i.e. *the derivatives of the mollified function are the mollified derivatives.* Applying Proposition 1 we deduce that if $1 \le p < \infty$ then $\rho_\varepsilon * u \to u$ in $W^{m,p}(E)$ as $\varepsilon \to 0$.

Because of the restriction that $\operatorname{dist}(E, \partial\Omega) > 0$ this does not provide an approximation of u in $W^{m,p}(\Omega)$ by functions in $C^\infty(\Omega)$. However, by a more careful argument using a partition of unity one can prove

Theorem 11 (Meyers & Serrin)**.** *Let $1 \le p < \infty$. Then $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

For $\Omega \subset \mathbb{R}^n$ open and $m = 1, 2, \ldots$ or $m = \infty$ define

$$C^m(\bar\Omega) = \{v : \Omega \to \mathbb{R} : \text{ there exists } w \in C^m(\mathbb{R}^n) \text{ with } w|_\Omega = v\}.$$

Can any $u \in W^{m,p}(\Omega)$ be approximated by functions in $C^\infty(\bar\Omega)$? In general the answer is no.

Example 1.2. Let $\Omega = (-1,0)\cup(0,1)$, $u(x) = H(x)$. Then $u \in C^\infty(\Omega)$, so that $u \in W^{m,p}(\Omega)$ for any m, p. Suppose that there were a sequence $u^{(j)} \in C^1(\mathbb{R})$ with $u^{(j)} \to u$ in $W^{1,p}(\Omega)$. Then we may assume by Proposition **??** that $u^{(j)} \to u$ a.e. in Ω. Choosing $x_- \in (-1,0), x_+ \in (0,1)$ with $u^{(j)}(x_-) \to 0, u^{(j)}(x_+) \to 1$ we have that $u^{(j)}(x_+) - u^{(j)}(x_-) \to 1$. But

$$\lim_{j\to\infty}(u^{(j)}(x_+) - u^{(j)}(x_-)) = \lim_{j\to\infty}\int_{x_-}^{x_+} \frac{du^{(j)}}{dx}\,dx = 0, \tag{1.65}$$

a contradiction.

In the example, Ω lies on both sides of the boundary point 0. To prevent this kind of situation and to deal with boundary values we make the following definition.

Definition 3. *An open set $\Omega \subset \mathbb{R}^n$ has a C^m (respectively Lipschitz) boundary if given any $\bar x \in \partial\Omega$ there exist $r > 0$ and a C^m (respectively Lipschitz) function $a : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, in a suitable Cartesian coordinate system,*

$$\Omega \cap B(\bar x, r) = \{x \in \mathbb{R}^n : x_n > a(x_1, ..., x_{n-1})\} \cap B(\bar x, r). \tag{1.66}$$

For brevity we write $x' = (x_1, ..., x_{n-1})$, so that $x = (x', x_n)$. Notice that each of the definitions implies that

$$\partial\Omega \cap B(\bar x, r) = \{x \in \mathbb{R}^n : x_n = a(x')\} \cap B(\bar x, r), \tag{1.67}$$

so that the boundary is locally the graph of a C^m (resp. Lipschitz) function.

Theorem 12. *Let Ω have C^0 boundary, and let $1 \le p < \infty$. Then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$. In particular, $C^\infty(\bar\Omega)$ is dense in $W^{m,p}(\Omega)$.*

1.9 Boundary values

Let $\Omega \subset \mathbb{R}^n$ have Lipschitz boundary. How can we define the boundary values of a function $u \in W^{1,p}(\Omega)$? this is not a trivial matter even if $\partial\Omega$ is smooth, since (a) u is in principle defined only in Ω, (b) even if u could be extended to a function $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ the values of $\tilde{u}$ on $\partial\Omega$ appear to have no meaning since $\mathcal{L}^n(\partial\Omega) = 0$ and $\tilde{u}$ may be altered at will on sets of $\mathcal{L}^n$ measure zero.

If Ω has Lipschitz boundary we can define $L^p(\partial\Omega)$ as the space of (equivalence classes of) $\mathcal{H}^{n-1}$ measurable functions $u : \partial\Omega \to \mathbb{R}$ such that $\|u\|_{L^p(\partial\Omega)} < \infty$, where

$$\|u\|_{L^p(\partial\Omega)} = \begin{cases} \left(\int_{\partial\Omega} |u(x)|^p d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}} & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |u(x)| & p = \infty. \end{cases}$$

$L^p(\partial\Omega)$ is a Banach space, and we can use the usual formulae to calculate integrals, e.g. in a neighbourhood of $\bar{x} \in \partial\Omega$

$$d\mathcal{H}^{n-1}(x) = \left(1 + \sum_{i=1}^{n-1}\left(\frac{\partial a}{\partial x_i}\right)^2\right)^{\frac{1}{2}} dx_1 \ldots dx_{n-1}.$$

The key idea for defining boundary values is contained in the following theorem.

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with Lipschitz boundary, and let $1 \le p < \infty$. Then there exists a constant $c > 0$ such that*

$$\int_{\partial\Omega} |u|^p d\mathcal{H}^{n-1} \le c\|u\|_{1,p}^p \tag{1.68}$$

for all $u \in C^1(\bar{\Omega})$.

Proof for $\Omega = (0,1)^n$.

$$u(x',1) - u(x',x_n) = \int_{x_n}^1 \frac{\partial u}{\partial x_n}(x',s)\, ds.$$

Hence

$$|u(x',1)|^p \le c\left(|u(x',x_n)|^p + \int_0^1 \left|\frac{\partial u}{\partial x_n}(x',s)\right|^p ds\right). \tag{1.69}$$

Integrate (1.69) with respect to $x_n \in (0,1)$ to obtain

$$|u(x',1)|^p \le c\int_0^1 \left(|u(x',x_n)|^p + \left|\frac{\partial u}{\partial x_n}(x',x_n)\right|^p\right) dx_n. \tag{1.70}$$

Then, integrating (1.70) with respect to $x' \in (0,1)^{n-1}$ we obtain

$$\int_{(0,1)^{n-1}} |u(x',1)|^p d\mathcal{H}^{n-1} \le c\|u\|_{1,p}^p.$$

Adding up the corresponding estimates for each face of the cube gives the result. □

If $u \in W^{1,p}(\Omega)$ there exists a sequence $u^{(j)} \in C^1(\bar{\Omega})$ with $u^{(j)} \to u$ in $W^{1,p}(\Omega)$. Hence $u^{(j)}$ is a Cauchy sequence in $W^{1,p}(\Omega)$, and by the theorem is also a Cauchy sequence in $L^p(\partial\Omega)$. Hence

$$u^{(j)}|_{\partial\Omega} \to \operatorname{tr} u \text{ in } L^p(\partial\Omega)$$

for some function $\operatorname{tr} u$, the *trace* of u on $\partial\Omega$. Since we can interlace any two different approximating sequences $u^{(j)}, \widetilde{u^{(j)}}$ it easily follows that $\operatorname{tr} u$ is independent of the approximating sequence. The mapping $\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is a bounded linear operator.

There is an alternative way of describing zero boundary values independent of the regularity of the boundary. For $1 \le p < \infty$ denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. If $p = \infty$ we define $W_0^{m,\infty}(\Omega)$ to be the set of $v \in W^{m,\infty}(\Omega)$ that are the a.e. limit of a sequence $\varphi^{(j)} \in C_0^\infty(\Omega)$ that is bounded in $W^{m,\infty}(\Omega)$. $W_0^{m,p}(\Omega)$ is a closed linear subspace of $W^{m,p}(\Omega)$, and hence is a Banach space with the same norm. We write $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. Then we have

Theorem 14. *Let $\Omega \subset \mathbb{R}^n$ be open with Lipschitz boundary. Then if $1 \le p \le \infty$*

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) : \operatorname{tr} D^\alpha u = 0 \text{ if } |\alpha| < m\}.$$

Theorem 15. *If $1 \le p < \infty$ then $W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$.*

1.10 Lipschitz mappings and $W^{1,\infty}$.

Theorem 16. *A mapping $u \in W_{loc}^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if u has a representative that is locally Lipschitz.*

Theorem 17. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with Lipschitz boundary. Then $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if u has a representative that is Lipschitz on Ω.*

1.11 Embedding theorems

Example 1.3. Let $n = 1$, $-\infty < a < b < \infty$. then $W^{1,1}(a, b)$ is continuously embedded in $C([a, b])$ i.e. each equivalence class v of functions in $W^{1,1}(a, b)$ has a representative $\tau v \in C([a, b])$ and there is a constant $K > 0$ such that

$$\|\tau v\|_{C([a,b])} \le K\|v\|_{1,1}.$$

Proof. Suppose v is smooth. Then

$$v(y) - v(x) + \int_x^y v'(t)\, dt,$$

and so

$$|v(y)) \le |v(x)| + \int_a^b |v'(t)|)\, dt.$$

Integrating with respect to x we find

$$(b-a)|v(y)| \leq \int_a^b (|v(t)| + (b-a)|v'(t)|)\,dt$$

and so

$$\|v\|_{C([a,b])} \leq K\|v\|_{1,1}. \tag{1.71}$$

Now let $v \in W^{1,1}(a,b)$. There exists a sequence of smooth functions $v^{(j)}$ with $v^{(j)} \to v$ in $W^{1,1}(a,b)$. Then $v^{(j)}$ is a Cauchy sequence in $W^{1,1}(a,b)$ and thus by (1.71) is a Cauchy sequence in $C([a,b])$. Hence $v^{(j)} \to \tau v$ in $C([a,b])$ and $\tau v = v$ a.e. with

$$\|\tau v\|_{C([a,b])} \leq K\|v\|_{1,1}.$$

□

Note that the argument also shows that the continuous representative of v satisfies the fundamental theorem of calculus

$$v(y) = v(x) + \int_x^y v'(t)\,dt \text{ for all } x, y \in [a,b],$$

so that v is absolutely continuous.

Now let $p > 1$, and suppose $\|u^{(j)}\|_{1,p} \leq M < \infty$. Then by (1.71) $\|u^{(j)}\|_{C([a,b])}$ is bounded, and if $x \leq y$

$$\begin{aligned} |u^{(j)}(x) - u^{(j)}(y)| &\leq \int_x^y |u^{(j)\prime}(t)|\,dt \\ &\leq \left(\int_x^y 1^{p'}dt\right)^{\frac{1}{p'}} \left(\int_x^y |u^{(j)\prime}(t)|^p dt\right)^{\frac{1}{p}} \\ &\leq m|y-x|^{\frac{1}{p'}}. \end{aligned}$$

Hence $u^{(j)}$ is bounded and equicontinuous, so that by the Arzela-Ascoli theorem $u^{(j)}$ has a convergent subsequence in $C([a,b])$. So for $p > 1$ the embedding $W^{1,p}(a,b) \to C([a,b])$ is *compact* (bounded sequences in $W^{1,1}(a,b)$ are relatively compact in $C([a,b])$).

In general we have

Theorem 18 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, open with Lipschitz boundary, and let $1 \leq p \leq \infty$.*
If $mp < n$ then $W^{m,p}(\Omega) \subset L^q(\Omega), \frac{1}{q} \geq \frac{1}{p} - \frac{m}{n}$,
if $mp = n$ then $W^{m,p}(\Omega) \subset L^q(\Omega), 1 \leq q < \infty$,
(if $p = 1$ and $m = n$ then in addition $W^{n,1}(\Omega) \subset L^\infty(\Omega)$),
if $mp > n$ then $W^{m,p}(\Omega) \subset C^0(\bar{\Omega})$.

Theorem 19 (Rellich-Kondrachoff)**.** *The embedding $W^{m,p}(\Omega) \subset L^q(\Omega)$ is compact if $mp < n, \frac{1}{q} > \frac{1}{p} - \frac{m}{n}$ or if $mp = n, 1 \leq q < \infty$.*
The embedding $W^{m,p}(\Omega) \subset C^0(\bar{\Omega})$ is compact if $mp > n$.

Example 1.4. Let $n = 3, m = 1$. Then

$$H^1(\Omega) = W^{1,2}(\Omega) \subset L^6(\Omega)$$

and the embedding $W^{1,2}(\Omega) \subset L^{6-\varepsilon}(\Omega)$ is compact.
$W^{1,3}(\Omega) \subset L^q(\Omega)$ for $1 \leq q < \infty$ but $W^{1,3}(\Omega) \not\subset L^\infty(\Omega)$.
$W^{1,p}(\Omega) \subset C^0(\bar{\Omega})$ compact if $p > 3$.

As an example of use of the embedding theorems we prove

Theorem 20 (Generalized Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (i.e. open and connected) with Lipschitz boundary, and let $1 < p < \infty$. Then there exists a constant $C + C(\Omega, p)$ such that*

$$\int_\Omega |u|^p dx \leq C \left(\left| \int_\Omega u\, dx \right|^p + \int_\Omega |\nabla u|^p dx \right)$$

for all $u \in W^{1,p}(\Omega)$.

Proof. Suppose not. Then there exist $u^{(j)} \in W^{1,p}(\Omega)$ with

$$1 = \int_\Omega |u^{(j)}|^p dx > j \left(\left| \int_\Omega u^{(j)}\, dx \right|^p + \int_\Omega |\nabla u^{(j)}|^p dx \right).$$

Hence $u^{(j)}$ is bounded in $W^{1,p}(\Omega)$ and we can suppose that $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega)$. By the compactness of the embedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ we have $\int_\Omega |u|^p dx = 1$. We now use the inequality

$$|\mathbf{a}|^p \geq |\mathbf{b}|^p + p|\mathbf{b}|^{p-2}\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \text{ for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n.$$

Thus

$$\int_\Omega |\nabla u^{(j)}|^p dx \geq \int_\Omega |\nabla u|^p dx + p \int_\Omega |\nabla u|^{p-2} \nabla u \cdot (\nabla u^{(j)} - \nabla u)\, dx.$$

Thus

$$\begin{aligned} 0 &= \lim_{j\to\infty} \left(\left| \int_\Omega u^{(j)}\, dx \right|^p + \int_\Omega |\nabla u^{(j)}|^p dx \right) \\ &\geq \left| \int_\Omega u\, dx \right|^p + \int_\Omega |\nabla u|^p dx. \end{aligned}$$

(since $\nabla u^{(j)} \rightharpoonup \nabla u$ in $(L^p)^n$ and $|\nabla u|^{p-2}\nabla u \in (L^{p'})^n$). Hence $\nabla u = 0$, so u is constant and thus $u = 0$. Contradiction. □

Exercises

1.10. Let $n > 1$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$.

(a) For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, define

$$u_\alpha(x) = |x|^\alpha, \quad x \neq 0.$$

Prove that if $1 \leq p < \infty$ then $u_\alpha \in W^{1,p}(B)$ if and only if $n > p(1-\alpha)$. For what α does $u_\alpha \in W^{1,\infty}(B)$? For what p does $u_\alpha \in W^{1,p}(\mathbb{R}^n)$?

(b) Prove that the function u defined for $x \neq 0$ by

$$u(x) = \log\log(2|x|^{-1})$$

belongs to $W^{1,n}(B)$ but not to $W^{1,p}(B)$ for any $p > n$.

(c) Let $u : B \to \mathbb{R}^n$ be defined for $x \neq 0$ by

$$u(x) = \frac{x}{|x|}.$$

Show that $u \in W^{1,p}(B)^n$ if and only if $1 \leq p < n$. Interpret u geometrically.

1.11. Let $R > \rho > 0$. Show that there exists $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subset B(0,R)$, $\varphi|_{B(0,\rho)} = 1$, $0 \leq \varphi \leq 1$ and $|D\varphi| \leq \frac{2}{R-\rho}$.
Hint. Reduce the problem to the case $n = 1$ by considering a radial function $\varphi = \varphi(r)$, $r = |x|$. Then mollify a suitable piecewise affine function.

1.12. Prove that the ellipsoid $\Omega = \{x \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i^2}{a_i^2} < 1\}$, where $a_i > 0$, $i = 1, ..., n$, has C^∞ boundary.

2 The one-dimensional calculus of variations

For the one-dimensional calculus of variations see Buttazzo, Giaquinta & Hildebrandt [7]. As a general reference for the calculus of variations there is a new book of Rindler [20].

Consider for $-\infty < a < b < \infty$ the integral functional

$$I(u) = \int_a^b f(x, u(x), u_x(x))\, dx \tag{2.1}$$

for f continuous and bounded below. Here $u \in W^{1,1}(a,b) = AC[a,b]$, and satisfies the boundary conditions:

$$\text{either} \quad u(a) = \alpha,\ u(b) = \beta, \tag{2.2}$$

$$\text{or} \quad u(a) = \alpha. \tag{2.3}$$

(Note that for such u we may have $I(u) = +\infty$.)

2.1 Existence of minimizers

We begin with some counterexamples.

Example 2.1 (Bolza)**.**

$$I(u) = \int_0^1 [(u_x^2 - 1)^2 + u^2]\, dx, \quad u(0) = u(1) = 0.$$

Theorem 21. *I does not attain an absolute minimum in* $W_0^{1,1}(0,1)$.

Proof. Let $u^{(j)}$ be as shown (Fig. 3), so that $u_x^{(j)}(x) = \pm 1$ a.e. and $|u^{(j)}(x)| \leq \frac{1}{2j}$. Then

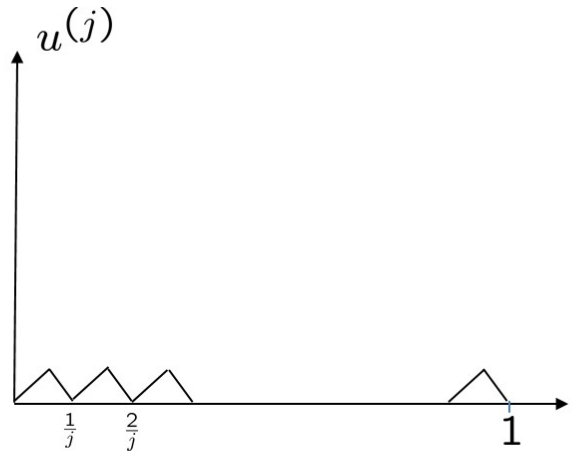

Figure 3: Minimizing sequence for Bolza problem.

$$I(u^{(j)}) = \int_0^1 u^{(j)2}\, dx \leq \frac{1}{4j^2} \to 0 \text{ as } j \to \infty.$$

Hence $\inf_{W_0^{1,1}} I = 0$. But $I(u) = 0$ implies $u = 0$, hence $u_x = 0$ and $I(u) = 1$. Contradiction. □

Remarks 1.
1. The same argument works for the boundary conditions $u(0) = 0, u(1)$ free.
2. We can think of there being a minimizer which is a 'generalized curve' in the sense of L.C. Young [22], with track $u = 0$ and derivative given by the probability measure $\nu = \frac{1}{2}(\delta_{-1} + \delta_1)$.

Example 2.2.

$$I(u) = \int_0^1 x^2 u_x^2\, dx, \quad u(0) = 0, u(1) = 1.$$

To show that the minimum is not attained we can take as a minimizing sequence $u^{(j)}$ as shown in Fig. 4 for which

$$I(u^{(j)}) = \int_0^{\frac{1}{j}} x^2 j^2\, dx = \frac{1}{3j} \to 0,$$

and note that there is no $u \in W^{1,1}(0,1$ with $u(0) = 0, u(1) = 1$ and $I(u) = 0$.

Example 2.3.

$$I(u) = \int_0^1 (|u_x| + (u-1)^2)\, dx, \quad u(0) = 0, u(1) = 1.$$

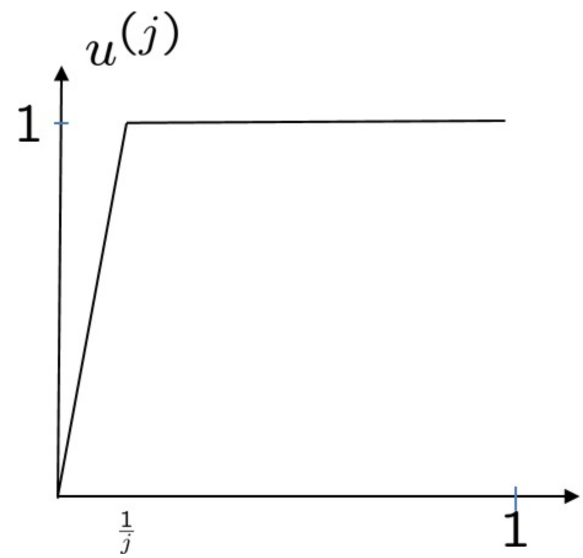

Figure 4: Minimizing sequence for Examples 2.2, 2.3.

Then

$$I(u) \geq \left| \int_0^1 u_x \, dx \right| + \int_0^1 (u-1)^2 \, dx = 1 + \int_0^1 (u-1)^2 dx.$$

But if $u^{(j)}$ is as in Fig. 4,

$$I(u^{(j)}) = \int_0^{\frac{1}{j}} [j + (jx-1)^2] \, dx \to 1 \text{ as } j \to \infty.$$

Thus $\inf I = 1$ and is not attained.

In Example 2.1 $f(x,u,\cdot)$ is not convex (recall that a function $g : X \to \mathbb{R} \cup \{+\infty\}$, X a vector space, is *convex* if

$$g(\lambda p + (1-\lambda)q) \leq \lambda g(p) + (1-\lambda)g(q)$$

for all $p, q \in X$ and $\lambda \in [0,1]$), while in Examples 2.2, 2.3 $f(x,u,p)$ does not have superlinear growth in p.

In order to prove the existence of minimizers we need an appropriate lower semicontinuity theorem.

Theorem 22 (Berkowitz [4], Cesari [8], Ekeland & Temam [14], Ioffe [18, 17], Eisen [13], [2] ...)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded open, and let $f : \Omega \times \mathbb{R}^s \times \mathbb{R}^\sigma \to [0,\infty]$ satisfy:*
(i) $f(\cdot, z, v) : \Omega \to [0,\infty]$ is measurable for every $z \in \mathbb{R}^s, v \in \mathbb{R}^\sigma$,
(ii) $f(x,\cdot,\cdot) : \mathbb{R}^s \times \mathbb{R}^\sigma \to [0,\infty]$ is continuous for a.e. $x \in \Omega$,
(iii) $f(x,z,\cdot) : \mathbb{R}^\sigma \to [0,\infty]$ is convex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^s$.

Let $z^{(j)}, z : \Omega \to \mathbb{R}^s$ be measurable mappings such that $z^{(j)} \to z$ a.e., and let $v^{(j)} \rightharpoonup v$ in $L^1(\Omega;\mathbb{R}^\sigma)$. Then

$$\int_\Omega f(x, z(x), v(x))\,dx \le \liminf_{j\to\infty} \int_\Omega f(x, z^{(j)}(x), v^{(j)}(x))\,dx.$$

Proof. We may assume that

$$\liminf_{j\to\infty} \int_\Omega f(x, z^{(j)}(x), v^{(j)}(x))\,dx = a < \infty. \tag{2.4}$$

We first claim that

$$h^{(j)}(x) = f(x, z^{(j)}(x), v^{(j)}(x)) - f(x, z(x), v^{(j)}(x))$$

converges to zero in measure as $j \to \infty$. If this were false there would exist $\varepsilon > 0, \delta > 0$ and subsequences $z^{(j_k)}, v^{(j_k)}$ such that $\mathcal{L}^n(M_k) \ge \delta$ for all k, where

$$\begin{aligned} M_k \quad = \quad & \{x \in \Omega : |f(x, z^{(j_k)}(x), v^{(j_k)}(x)) - f(x, z(x), v^{(j_k)}(x))| \ge \varepsilon \\ & \qquad z^{(j_k)}(x) \to z(x), f(x,\cdot,\cdot) \text{ continuous } \}. \end{aligned}$$

Since $v^{(j_k)} \rightharpoonup v$ in $L^1(\Omega;\mathbb{R}^\sigma)$, and by (2.4), there exists $K > 0$ such that

$$\int_\Omega |v^{(j_k)}(x)|\,dx \le K, \ \int_\Omega f(x, z^{(j_k)}(x), v^{(j_k)}(x))\,dx \le K$$

for all k, and thus $\mathcal{L}^n(N_k) \le \frac{\delta}{2}$, where

$$N_k = \left\{ x \in \Omega : |v^{(j_k)}(x)| > \frac{4K}{\varepsilon} \text{ or } f(x, z^{(j_k)}(x), v^{(j_k)}(x)) > \frac{4K}{\varepsilon} \right\}.$$

Let $M_k' = M_k \setminus N_k$. Then $\mathcal{L}^n(M_k') \ge \frac{\delta}{2}$ for all k. Therefore

$$\mathcal{L}^n\left(\limsup_{k\to\infty} M_k'\right) \ge \frac{\delta}{2},$$

where

$$\limsup_{k\to\infty} M_k' := \bigcap_{l=1}^\infty \bigcup_{k=l}^\infty M_k'.$$

For $x \in \limsup_{k\to\infty} M_k'$ we have, for a further subsequence not relabelled,

$$|v^{(k)}(x)| \le \frac{4K}{\delta}, |f(x, z^{(j_k)}(x), v^{(j_k)}(x))| \le \frac{4K}{\delta},$$

$$|f(x, z^{(j_k)}(x), v^{(j_k)}(x)) - f(x, z(x), v^{(j_k)}(x))| \ge \varepsilon,$$

$$z^{(j_k)}(x) \to z(x), f(x,\cdot,\cdot) \text{ continuous},$$

which is impossible (choosing a convergent subsequence of $v^{(j_k)}(x)$), proving the claim.

Extracting a subsequence from $h^{(j)}$, we may suppose that $h^{(j)}(x) \to 0$ a.e. in Ω. By Mazur's theorem there exist convex combinations $\xi^{(k)} = \sum_{j=k}^{\infty} \lambda_j^k v^{(j)}$, where only finitely many λ_j^k are nonzero for each k, such that $\xi^{(k)} \to v(x)$ a.e. as $k \to \infty$. Since $f(x, z(x), \cdot)$ is convex,

$$f(x, z(x), \xi^{(k)}(x)) + \sum_{j=k}^{\infty} \lambda_j^k h^{(j)}(x) \leq \sum_{j=k}^{\infty} \lambda_j^k f(x, z^{(j)}(x), v^{(j)}(x))$$

for a.e. x and large enough k.

Integrating over Ω, taking the lim inf as $k \to \infty$, and applying Fatou's Lemma, we obtain the result. □

Theorem 23 (Tonelli)**.** *Let $f = f(x, u, p)$ be convex in p for each x, u and suppose that*

$$f(x, u, p) \geq \Phi(p) \text{ for all } x, u$$

for some continuous Φ with $\frac{\Phi(p)}{|p|} \to \infty$ as $|p| \to \infty$. Let

$$\mathcal{A} = \{v \in W^{1,1}(a, b) : v(a) = \alpha, v(b) = \beta\} \tag{2.5}$$

or

$$\mathcal{A} = \{v \in W^{1,1}(a, b) : v(a) = \alpha\}. \tag{2.6}$$

Then I attains an absolute minimum on $\mathcal{A}$.

Proof. Let $l = \inf_{\mathcal{A}} I$. Then $\infty > l > -\infty$. Let $u^{(j)} \in \mathcal{A}$ be a minimizing sequence, so that $I(u^{(j)}) \to l$. Then

$$\sup_j \int_a^b \Phi(u_x^{(j)})\, dx < \infty$$

and so by Theorem 6 there exists a subsequence, still denoted $u^{(j)}$, such that $v^{(j)} := u_x^{(j)} \rightharpoonup v$ in $L^1(a, b)$ for some v. Therefore

$$u^{(j)}(x) = \alpha + \int_a^x v^{(j)}(s)\, ds \to u(x) := \alpha + \int_a^x v(s)\, ds \text{ for all } x \in [a, b].$$

In particular for the boundary conditions (2.5) we have $u(b) = \beta$. By the lower semicontinuity Theorem 22 below,

$$\begin{aligned} l = \liminf_{j\to\infty} I(u^{(j)}) &= \lim_{j\to\infty} \int_a^b f(x, u^{(j)}(x), v^{(j)}(x))\, dx \\ &\geq \int_a^b f(x, u(x), v(x))\, dx = I(u) \geq l, \end{aligned}$$

and hence u is a minimizer. □

2.2 Local minimizers

Consider again the integral functional

$$I(u) = \int_a^b f(x, u(x), u_x(x))\, dx \tag{2.7}$$

with f continuous and bounded below, with set of admissible functions

$$\mathcal{A} = \{u \in W^{1,1}(a, b) : u(a) = \alpha, u(b) = \beta\}. \tag{2.8}$$

Definitions 2. $u \in \mathcal{A}$ is a *weak local minimizer* of I if $I(u) < \infty$ and there exists $\varepsilon > 0$ such that $I(v) \geq I(u)$ for all $v \in \mathcal{A}$ with

$$\operatorname*{ess\,sup}_{x\in[a,b]} [|v(x) - u(x)| + |v_x(x) - u_x(x)|] < \varepsilon.$$

$u \in \mathcal{A}$ is a *strong local minimizer* of I if $I(u) < \infty$ and there exists $\varepsilon > 0$ such that $I(v) \geq I(u)$ for all $v \in \mathcal{A}$ with

$$\max_{x\in[a,b]} |v(x) - u(x)| < \varepsilon.$$

Thus u is a weak (resp. strong) local minimizer if it is a local minimizer with respect to the $W^{1,\infty}$ (resp. L^∞) norm (see Fig. 5). A strong local minimizer

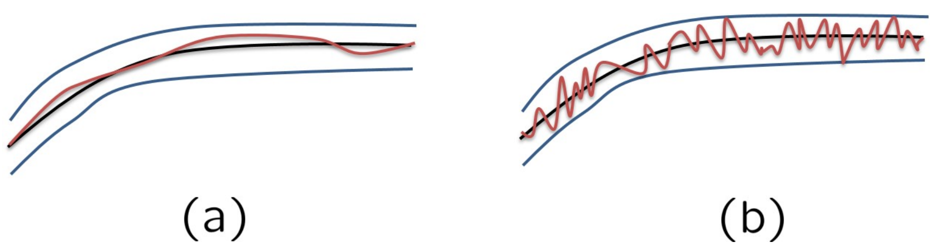

Figure 5: Schematic of typical function v (in red) in (a) a $W^{1,\infty}$ neighbourhood of a smooth function u (in black) (b) an L^∞ neighbourhood of u. In the second case the derivative v_x can be arbitrarily large, whereas in the first it must be close to u_x.

is a weak local minimizer, but in general a weak local minimizer need not be a strong local minimizer.

2.3 Necessary conditions for local minimizers

We now assume for simplicity that $f = f(x, u, p)$ is C^3 in its arguments x, u, p. Let $u \in \mathcal{A} \cap W^{1,\infty}(a, b)$ be a weak local minimizer. If $\varphi \in C_0^\infty(a, b)$ then $I(u+\tau\varphi)$ has a local minimum at $\tau = 0$, so that $\frac{d}{d\tau}I(u+\tau\varphi)|_{\tau=0} = 0$, provided this derivative exists. In fact by the mean-value theorem

$$\begin{aligned}\frac{I(u+\tau\varphi) - I(u)}{\tau} \quad &= \quad \int_a^b [f_u(x, u(x) + \tau(x)\varphi(x), u_x(x) + \tau(x)\varphi_x(x))\varphi(x) \\ &\quad + f_p(x, u(x) + \tau(x)\varphi(x), u_x(x) + \tau(x)\varphi_x(x))\varphi_x(x)]\, dx\end{aligned}$$

where $|\tau(x)| \leq |\tau|$, so that by the bounded convergence theorem

$$\int_a^b [f_u\varphi + f_p\varphi_x]\,dx = 0 \text{ for all } \varphi \in C_0^\infty(a,b), \tag{WEL}$$

i.e. u satisfies the Euler-Lagrange equation

$$\frac{d}{dx}f_p = f_u \tag{EL}$$

in the sense of distributions. Note that since

$$f_u\varphi = \frac{d}{dx}\left(\int_a^x f_u\,ds\cdot\varphi\right) - \left(\int_a^x f_u\,ds\right)\varphi_x,$$

(WEL) is equivalent to

$$\int_a^b \left(f_p - \int_a^x f_u\,ds\right)\varphi_x\,dx = 0 \text{ for all } \varphi \in C_0^\infty(a,b),$$

and hence to the integrated Euler-Lagrange equation

$$f_p = \int_a^x f_u\,ds + c, \quad x \in [a,b], \tag{IEL}$$

where c is a constant.

Similarly we have that the second variation

$$\delta^2 I(u)(\varphi,\varphi) := \frac{d^2}{d\tau^2}I(u+\tau\varphi) \geq 0,$$

that is

$$\int_a^b [f_{uu}\varphi^2 + 2f_{up}\varphi\varphi_x + f_{pp}\varphi_x^2]\,dx \geq 0 \text{ for all } \varphi \in C_0^\infty(a,b),$$

which we abbreviate to

$$\delta^2 I(u) \geq 0. \tag{2.9}$$

Now let $u \in \mathcal{A}\cap W^{1,\infty}(a,b)$ be a *strong* local minimizer. For $\varphi \in C_0^\infty(a,b)$ and $|\tau|$ small enough there is a unique smooth increasing solution $z_\tau(x)$ to $z+\tau\varphi(z) = x$ for $x \in [a,b]$. Define the *inner variation*

$$u_\tau(x) = u(z_\tau(x)),$$

which rearranges the values of u. Then $\lim_{\tau\to 0}\max_{x\in[a,b]}|u_\tau(x) - u(x)| = 0$, and so

$$\frac{d}{d\tau}I(u_\tau)|_{\tau=0} = \frac{d}{d\tau}\int_a^b f(z+\tau\varphi(z), u(z), u_z(z)\cdot\frac{1}{1+\tau\varphi_z(z)})(1+\tau\varphi_z(z))\,dz|_{\tau=0} = 0,$$

giving

$$\int_a^b [f_x\varphi + (f - u_x f_p)\varphi_x]\,dx = 0 \text{ for all } \varphi \in C_0^\infty(a,b). \tag{WDBR}$$

That is u satisfies the Du Bois-Reymond equation

$$\frac{d}{dx}(f - u_x f_p) = f_x \tag{DBR}$$

in the sense of distributions. Equivalently, u satisfies the integrated form

$$f - u_x f_p = \int_a^x f_x\,ds + c, \quad x \in [a,b], \tag{IDBR}$$

for some constant c.

Note that (WDBR) does not follow from (WEL). In the special case $f = f(p)$ the 'broken extremal'

$$u(x) = \begin{cases} qx & x \in [-1,0] \\ rx & x \in [0,1] \end{cases}$$

satisfies (WEL) on $[-1,1]$ if and only if $f_p(q) = f_p(r)$, i.e. the tangents to f at q, r have the same slope. If also (WDBR) holds then

$$f(q) - qf_p(q) = f(r) - rf_p(r),$$

i.e. the tangents at q, r are a common tangent (see Fig. 6).

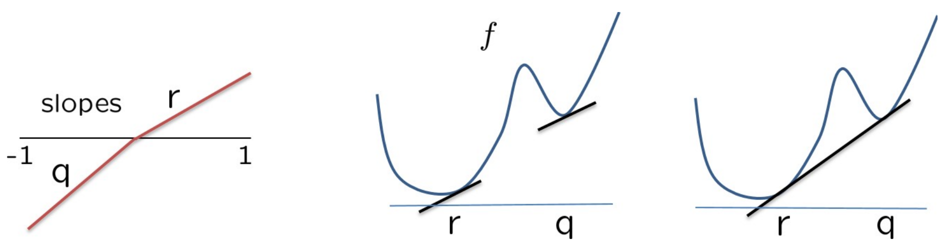

Figure 6: The broken extremal with slopes q, r satisfies (WEL) if the slopes of f at q, r are the same, and satisfies also (WDBR) if there is a common tangent at q, r.

Suppose again that $u \in \mathcal{A} \cap W^{1,\infty}(a,b)$ is a strong local minimizer. Let $[c,d] \subset (a,b)$, $\psi \in W_0^{1,\infty}(-1,1)$ and consider for $\varepsilon > 0$ and $x_0 \in [c,d]$ the localized variation

$$u_\varepsilon(x_0, x) = u(x) + \varepsilon\psi\Big(\frac{x - x_0}{\varepsilon}\Big).$$

For $\varepsilon > 0$ sufficiently small (independent of x_0) we have that $I(u_\varepsilon(x_0,\cdot)) \geq I(u)$, and so

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \Bigg(f\Big(x, u(x) + \varepsilon\psi\Big(\frac{x-x_0}{\varepsilon}\Big), u_x(x) + \psi_y\Big(\frac{x-x_0}{\varepsilon}\Big)\Big) - f(x, u(x), u_x(x))\Bigg)\,dx \geq 0. \tag{2.10}$$

Let $\varphi \in C_0^\infty(c,d)$, $\varphi \geq 0$. Multiplying (2.10) by $\varphi(x_0)$, integrating with respect to x_0 over (c,d), and making the change of variables $y = \frac{x-x_0}{\varepsilon}$ we obtain

$$\varepsilon \int_a^b \varphi(x-\varepsilon y)\Bigg(\int_{-1}^1 (f(x, u(x)+\varepsilon\psi(y), u_x(x)+\psi_y(y)) \\ - f(x,u(x),u_x(x))\, dy\Bigg)\, dx \geq 0.$$

Dividing by ε and passing to the limit $\varepsilon \to 0$ we deduce that

$$\int_c^d \varphi(x)\int_{-1}^1 \big(f(x,u(x),u_x(x)+\psi_y(y)) - f(x,u(x),u_x(x))\big)\, dy\, dx \geq 0,$$

and since $\varphi \geq 0$ is arbitrary it follows that for a.e. $x \in [c,d]$, and hence for a.e. $x \in (a,b)$,

$$\int_{-1}^1 f(x,u(x),u_x(x)+\psi_y(y))\, dy \geq \int_{-1}^1 f(x,u(x),u_x(x)\, dy. \tag{2.11}$$

(This is quasiconvexity in 1D.)

Define $F(p) = f(x,u(x),u_x(x)+p)$, so that (2.11) becomes

$$\int_{-1}^1 F(\psi_y(y))\, dy \geq \int_{-1}^1 F(0)\, dy.$$

Choosing ψ as shown in Fig 7 we deduce that

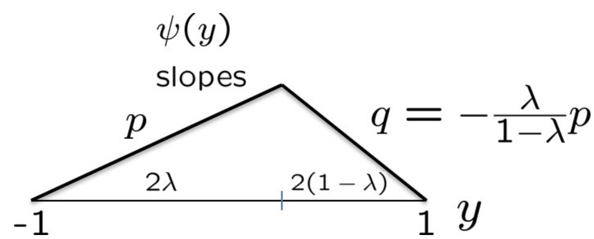

Figure 7: Function $\psi(y)$ with slopes p and $q = -\frac{\lambda}{1-\lambda}p$, where $0 < \lambda < 1$.

$$\lambda F(p) + (1-\lambda)F(-\frac{\lambda}{1-\lambda}p) \geq F(0).$$

Hence $\frac{d}{d\lambda}(LHS)|_{\lambda=0} \geq 0$, and hence $F(p) \geq F(0) + pF_p(0)$, giving the *Weierstrass necessary condition*, that for a.e. $x \in (a,b)$,

$$f(x,u(x),u_x(x)+p) \geq f(x,u(x),u_x(x)) + pf_p(x,u(x),u_x(x)) \text{ for all } p.$$

Thus the possible values of $u_x(x)$ in a strong local minimizer are those for which the tangent at $u_x(x)$ to the graph of $f(x,u(x),\cdot)$ does not lie above the graph (see Fig. 8).

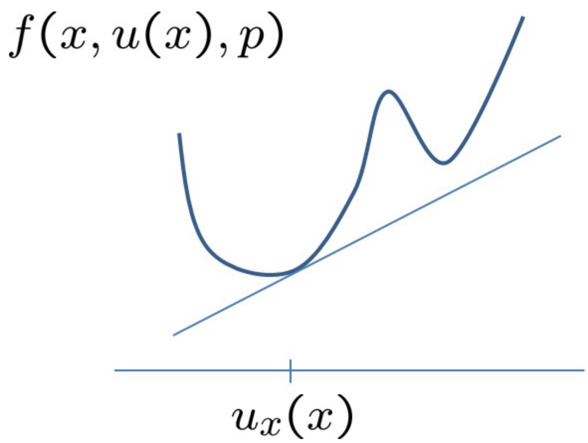

Figure 8: The Weierstrass condition is that the tangent at $u_x(x)$ to the graph of $f(x, u(x), \cdot)$ does not lie above the graph.

2.4 Sufficient conditions for local minimizers

By slightly strengthening the necessary conditions we can obtain sufficient conditions for a sufficiently regular $u \in \mathcal{A}$ to be a weak or strong local minimizer.

For $u \in \mathcal{A} \cap W^{1,\infty}(a, b)$ write

$$\delta^2 I(u) > 0 \tag{2.12}$$

if

$$\int_a^b (f_{uu}\varphi^2 + 2f_{up}\varphi\varphi_x + f_{pp}\varphi_x^2)\, dx \geq \mu \int_a^b (\varphi^2 + \varphi_x^2)\, dx \tag{2.13}$$

for all $\varphi \in C_0^\infty(a, b)$ and some constant $\mu > 0$. Note that (2.13) then holds for all $\varphi \in W_0^{1,2}(a, b)$ by density. Note also that (2.12) implies the *strong Legendre condition* that for a.e. $x \in (a, b)$

$$f_{pp}(x, u(x), u_x(x)) \geq \mu. \tag{2.14}$$

Indeed, (2.12) implies that $\varphi = 0$ is a global minimizer for the functional

$$\delta^2 I(u)(\varphi, \varphi) - \mu \int_a^b (\varphi^2 + \varphi_x^2)\, dx,$$

so that by the (proof of the) Weierstrass condition $\tau = 0$ is a point of convexity of the function $(f_{pp}(x, u(x), u_x(x)) - \mu)\tau^2$, giving (2.14).

Theorem 24. *If $u \in \mathcal{A} \cap W^{1,\infty}(a, b)$ satisfies* (WEL) *and $\delta^2 I(u) > 0$ then u is a strict weak local minimizer (i.e. there exists $\varepsilon > 0$ such that $I(v) > I(u)$ for all $v \in \mathcal{A}$ with $0 < \|v - u\|_{1,\infty} < \varepsilon$).*

Proof. Let $\varphi \in W_0^{1,\infty}(a,b)$. Then setting $\theta(t) = f(x, u+t\varphi, u_x+t\varphi_x)$ and using

$$\theta(1) - \theta(0) = \theta'(0) + \int_0^1 (1-t)\theta''(t)dt$$

we obtain

$$I(u+\varphi) - I(u) \quad = \quad \cancel{\int_a^b (f_u\varphi + f_p\varphi_x)\,dx} + \frac{1}{2}\delta^2 I(u)(\varphi,\varphi) + R(u,\varphi)$$

where

$$R(u,\varphi) = \int_a^b \int_0^1 (1-t)[(f_{uu}(x, u+t\varphi, u_x+t\varphi_x) - f_{uu}(x,u,u_x))\varphi^2 + \cdots]\,dtdx.$$

For $\varepsilon > 0$ sufficiently small and $\|\varphi\|_{1,\infty} < \varepsilon$, we have that

$$R(u,\varphi) \geq -\frac{\mu}{4}\int_a^b (\varphi^2 + \varphi_x^2)\,dx,$$

and hence

$$I(u+\varphi) - I(u) \geq \frac{\mu}{4}\int_a^b (\varphi^2 + \varphi_x^2)\,dx,$$

as required. □

We say that $u \in \mathcal{A} \cap C^1([a,b])$ satisfies the *strengthened Weierstrass condition* if there exists $\delta > 0$ such that for all $x \in [a,b]$ and $p \in \mathbb{R}$

$$f(x,v,p) \quad \geq \quad f(x,v,q) + (p-q)f_p(x,v,q) \qquad \text{whenever } |v-u(x)| < \delta, |q-u_x(x)| < \delta. \tag{2.15}$$

Theorem 25 (Weierstrass)**.** *Let $u \in \mathcal{A} \cap C^1([a,b])$ satisfy* (WEL)*, $\delta^2 I(u) > 0$ and the strengthened Weierstrass condition. Then u is a strong local minimizer. If strict inequality holds in* (2.15) *for $p \neq q$ then u is a strict strong local minimizer.*

Proof. We sketch a version of Hilbert's amazing proof of this theorem. The part we do not do concerns the analysis of the second variation in terms of the *Jacobi equation* (the Euler-Lagrange equation of $\delta^2 I(u)(\varphi,\varphi)$) and conjugate points (see, for example, [5, 7, 9]). Using $\delta^2 I(u) > 0$ leads to the conclusion that u is embedded in a *field of extremals*, that is there is a one-parameter family

$$U(x,\gamma), \gamma \in [-\tau,\tau], \tau > 0,$$

of solutions to the Euler-Lagrange equation (EL) for f on $[a,b]$ such that

(i) $u(x) = U(x,0)$ for all $x \in [a,b]$,
(ii) the field simply covers a neighbourhood of the graph of u, i.e. there exists $\varepsilon > 0$ such that for each $x \in [a,b], v \in \mathbb{R}$, with $|v-u(x)| < \varepsilon$, there is a unique $\gamma = \gamma(x,v) \in [-\tau,\tau]$ with $U(x,\gamma) = v$ (see Fig. 9). We assume that $U(\cdot,\cdot)$ is

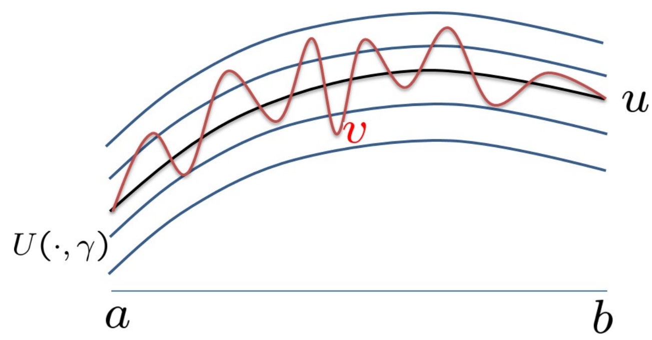

Figure 9: A field of extremals simply covering an L^∞ neighbourhood of the graph of u and a typical $v \in \mathcal{A}$ lying in this neighbourhood.

C^2 in (x, γ). We write $p(x, v) = U_x(x, \gamma(x, v))$ and call $p(\cdot, \cdot)$ the *slope function* of the field.

Now let $v \in \mathcal{A}$ with $\|v - u\|_\infty$ sufficiently small. Then we claim that

$$\begin{aligned} I(v) - I(u) \quad = \quad & \int_a^b [f(x, v, v_x) - f(x, v, p(x, v)) \\ & \qquad - f_p(x, v, p(x, v))(v_x - p(x, v))] \, dx, \end{aligned} \tag{2.16}$$

where $p(x, v)$ is the slope function of the field. Thus $I(v) \geq I(u)$ by the strengthened Weierstrass condition.
To prove the claim, we compute

$$\begin{aligned} & \frac{d}{dx} \int_0^{\gamma(x, v(x))} f_p(x, U(x, \gamma), U_x(x, \gamma)) U_\gamma(x, \gamma) \, d\gamma \\ & \quad = \int_0^{\gamma(x, v(x))} [f_u(x, U(x, \gamma), U_x(x, \gamma)) U_\gamma(x, \gamma) + \\ & \qquad f_p(x, U(x, \gamma), U_x(x, \gamma)) U_{x\gamma}(x, \gamma)] \, d\gamma \\ & \qquad + f_p(x, v(x), p(x, v(x))) U_\gamma(x, \gamma(x, v)) \frac{d}{dx} \gamma(x, v) \\ & \quad = f(x, U(x, \gamma), U_x(x, \gamma))|_0^{\gamma(x, v)} + f_p(x, v, p(x, v))(v_x - p(x, v)) \\ & \quad = f(x, v, p(x, v)) - f(x, u, u_x) + f_p(x, v, p(x, v))(v_x - p(x, v)), \end{aligned}$$

where we used that $\frac{d}{dx} U(x, \gamma(x, v)) = v_x$, and integrating with respect to x we are done. □

Remarks 2.
1. Note that the key computation can be interpreted as showing that

$$L(x, v, v_x) = f(x, v, p(x, v)) + f_p(x, v, p(x, v))(v_x - p(x, v))$$

is a *null Lagrangian*, i.e. the corresponding Euler-Lagrange equation reduces to $0 = 0$.
2. Another completely different method is due to Hestenes [16].

2.5 Regularity and the Lavrentiev phenomenon

We assumed above that $u \in C^1([a,b])$. But when is this true? A first regularity result is:

Theorem 26. *Suppose that $f \in C^2$ and that $f_{pp}(x,v,p) > 0$ for all x, v, p. If $u \in \mathcal{A} \cap W^{1,\infty}(a,b)$ solves* (WEL) *then $u \in C^2([a,b])$ and*

$$u_{xx} = F(x,u,u_x) \text{ for all } x \in [a,b],$$

where

$$F = \frac{f_u - f_{xp} - f_{up}p}{f_{pp}}.$$

Proof. Step 1. We prove that $u \in C^1([a,b])$. Choose the continuous representative of u. We have that $|u_x(x)| \leq M < \infty$ and

$$f_p(x,u(x),u_x(x)) = c + \int_a^x f_u\, dy \qquad \text{(IEL)}$$

for all $x \in E$, where $\operatorname{meas} E = b - a$. Suppose $x \in [a,b]$. We claim that

$$p(x) := \lim_{z \to x, z \in E} u_x(z) \text{ exists.}$$

Suppose not, Then $u_x(x_j) \to p_1, u_x(y_j) \to p_2$ for sequences $x_j \to x, y_j \to x$, with $x_j, y_j \in E, p_1 \neq p_2$. But from (IEL) we deduce that

$$f_p(x,u(x),p_1) = f_p(x,u(x),p_2).$$

Since $f_{pp} > 0$ this is a contradiction.

Step 2. We prove that $u \in C^2([a,b])$. For each $x \in [a,b]$ we have that

$$\lim_{h \to 0} \frac{f_p(x+h,u(x+h),u_x(x+h)) - f_p(x,u(x),u_x(x))}{h} = f_u(x,u(x),u_x(x)).$$

But the LHS equals

$$\begin{aligned}
&\lim_{h\to 0}\Bigg[\frac{f_p(x+h,u(x+h),u_x(x+h))-f_p(x,u(x+h),u_x(x+h))}{h}\\
&+\frac{f_p(x,u(x+h),u_x(x+h))-f_p(x,u(x),u_x(x+h))}{h}\\
&+\frac{f_p(x,u(x),u_x(x+h))-f_p(x,u(x),u_x(x))}{h}\Bigg].\\
=\ & f_{xp}(x,u(x),u_x(x))+f_{up}(x,u(x),u_x(x))u_x(x)\\
&+\lim_{h\to 0}\frac{1}{h}\int_{u_x(x)}^{u_x(x+h)} f_pp(x,u(x),\tau)\,d\tau\\
=\ & f_xp+f_{up}+\lim_{h\to 0}\frac{u_x(x+h)-u_x(x)}{h}f_{pp},
\end{aligned}$$

and since $f_{pp}>0$ we get that u_x is differentiable with $u_{xx}=F(x,u,u_x)$. □

Remark 1. Another way to do Step 2 is to note that $p(x)=u_x(x)$ solves $G(x,p)=0$, where

$$G(x,p)=f_p(x,u(x),p)-\int_a^x f_u\,dy-c,$$

and use the implicit function theorem.

But does the global minimizer u given by Theorem 23 belong to $W^{1,\infty}(a,b)$ or satisfy (WEL)?

Example 2.4 (adapted from [3])**.** Let

$$I(u)=\int_{-1}^{1}[(u^5-x^3)^2u_x^{20}+\varepsilon u_x^2]\,dx,$$

where $\varepsilon>0$ is sufficiently small, and

$$\mathcal{A}=\{v\in W^{1,1}(-1,1):v(-1)=-1,v(1)=1\}.$$

Note that $f(x,u,p)=(u^5-x^3)^2p^{20}+\varepsilon p^2$ is a polynomial with $f_{pp}\geq 2\varepsilon>0$, and that f has superlinear growth in p, so that f satisfies the hypotheses of Theorem 23. Hence there exists an absolute minimizer u^*.

We claim that if $u\in\mathcal{A}\cap W^{1,\infty}(-1,1)$ then

$$I(u)\geq\frac{2^{14}}{3^{20}}. \tag{2.17}$$

To prove the claim, suppose that $u(0)\leq 0$. If $u(0)=0$ then $|u(x)|\leq Cx$ for $x\in[-1,1]$ and a constant $C>0$. Hence there exist $0\leq x_0<x_1<1$ with $0<u(x)<\left(\frac{x^3}{2}\right)^{\frac{1}{5}}$ for $x\in(x_0,x_1)$, $u(x_0)=0,u(x_1)=\left(\frac{x_1^3}{2}\right)^{\frac{1}{5}}$ (see Fig. 10). Hence

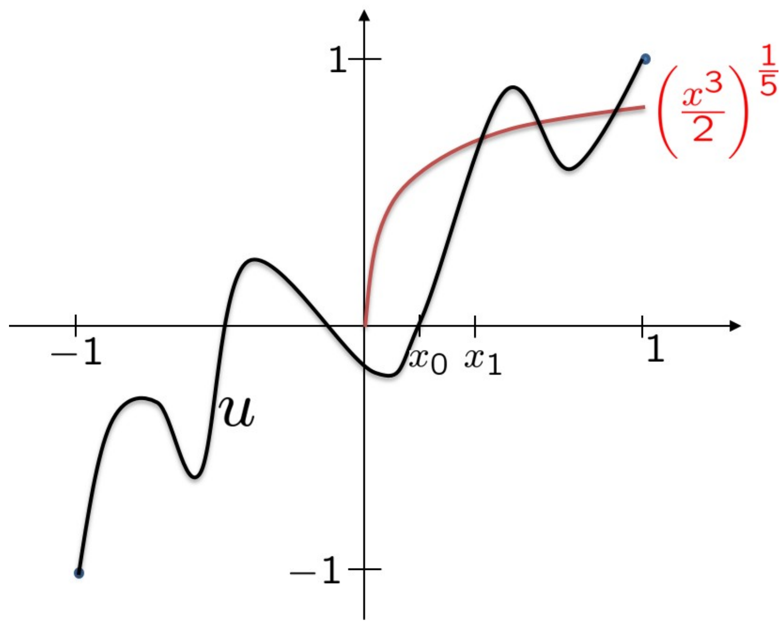

Figure 10: Argument for establishing the Lavrentiev phenomenon.

$$\begin{aligned} I(u) &\geq \int_{x_0}^{x_1} (u^5 - x^3)^2 u_x^{20}\, dx \\ &\geq \int_{x_0}^{x_1} u^{10} u_x^{20}\, dx \\ &= \int_{x_0}^{x_1} (u^{\frac{1}{2}} u_x)^{20}\, dx. \end{aligned}$$

Since t^{20} is convex in t by Jensen's inequality

$$\begin{aligned} I(u) &\geq (x_1 - x_0)\left(\frac{1}{x_1 - x_0}\int_{x_0}^{x_1} u^{\frac{1}{2}} u_x\, dx\right)^{20} \\ &= \frac{1}{(x_1 - x_0)^{19}}\left[\frac{2}{3}\left(u(x_1)^{\frac{3}{2}} - u(x_0)^{\frac{3}{2}}\right)\right]^{20} \\ &= \frac{\left(\frac{2}{3}\right)^{20}\left(\frac{x_1^3}{2}\right)^{6}}{(x_1 - x_0)^{19}} \\ &\geq \frac{2^{14}}{3^{20}} \cdot \frac{1}{x_1} \geq \frac{2^{14}}{3^{20}}. \end{aligned}$$

If $u(0) \geq 0$ we argue similarly. Hence

$$\inf_{\mathcal{A} \cap W^{1,\infty}(-1,1)} I \geq \frac{2^{14}}{3^{20}}. \tag{2.18}$$

But choosing $u = |x|^{\frac{3}{5}} \operatorname{sign} x$ we have that

$$\inf_{\mathcal{A}} I \leq 2\varepsilon \int_0^1 \left(\frac{3}{5x^{-\frac{2}{5}}}\right)^2 dx = 2\varepsilon \cdot \frac{9}{5}.$$

Hence if $\varepsilon < \varepsilon_0 := \frac{5}{18} \cdot \frac{2^{14}}{3^{20}}$ we have that

$$\inf_{\mathcal{A} \cap W^{1,\infty}} I > \inf_{\mathcal{A}} I \quad !!! \tag{2.19}$$

This is the *Lavrentiev phenomenon*, that the infimum can be different in different function spaces.

Now let u^* be a global minimizer of I in $\mathcal{A}$. We claim that if $0 < \varepsilon < \varepsilon_0$ then $u^*(0) = 0$ and $f_p(x, u^*, u^*_x)$ is unbounded in the neighbourhood of $x = 0$. In particular (IEL) *does not hold*. Indeed if $u^*(0) \neq 0$ we get $I(u^*) \geq \frac{2^{14}}{3^{20}} > I(|x|^{\frac{3}{5}} \operatorname{sign} x) \geq I(u^*)$, a contradiction. If $|u^*_x| \leq C$ in a neighbourhood of 0, and $u^*(0) = 0$ we get the same contradiction. Hence u^*_x is unbounded near 0 and hence so is $|f_p| = |20(u^5 - x^3)^2 u_x^{*19} + 2\varepsilon u^*_x| \geq 2\varepsilon |u^*_x|$.

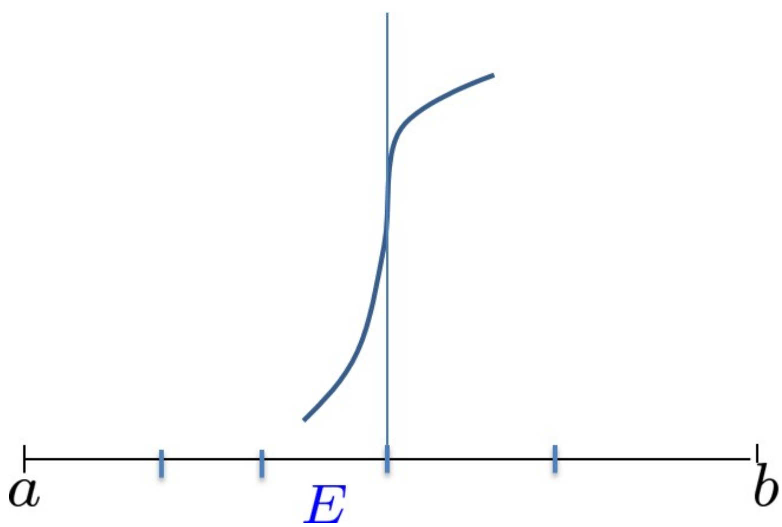

Figure 11: A strong local minimizer has $|u'(x)| = \infty$ on its Tonelli set E.

Remarks 3.
1. The example shows that an elliptic regularization (adding εu_x^2 to a degenerate elliptic problem) may not smooth minimizers.
2. If $\varphi \in C_0^\infty(-1, 1), \varphi(0) \neq 0$, then $I(u^* + t\varphi) = \infty$ for all $t \neq 0$, since $I(u^* + t\varphi) \geq \delta \int_{-r}^{r} u_x^{*20}\, dx = \infty$.
3. The Lavrentiev phenomenon shows that typical finite element schemes for minimizing I among piecewise affine functions may not converge to a minimizer.

Theorem 27 (Tonelli's Partial Regularity Theorem)**.** *Let f be C^3 with $f_{pp} > 0$. If $u \in \mathcal{A}$ is a strong local minimizer of I in $\mathcal{A}$, then there is a closed set $E \subset [a,b]$ with* $\operatorname{meas} E = 0$ *such that u is a C^3 solution of EL on $[a,b] \setminus E$. Furthermore the derivative*

$$u'(x) := \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}$$

exists for all $x \in [a,b]$ as an element of $\bar{\mathbb{R}}$ (one-sided limits if $x = a$ or $x = b$), and $u' : [a,b] \to \bar{\mathbb{R}}$ is continuous with $E = \{x \in [a,b] : |u'(x)| = \infty\}$.

See Fig. 11. The theorem is optimal [10].

Exercises

2.1. Consider the integral

$$I(u) = \int_a^b f(u_x)\, dx,$$

where f is continuous and bounded below, defined for the set of admissible functions

$$\mathcal{A} = \{u \in W^{1,1}(a,b) : u(a) = \alpha, u(b) = \beta\},$$

where α, β are given.

(i) Show that if

$$\frac{f(p)}{|p|} \to \infty \text{ as } |p| \to \infty \qquad (\dagger)$$

then I attains a minimum on $\mathcal{A}$.

(*Hint.* Consider the convex envelope of f, i.e. the sup of all linear functions $rp + s \leq f(p)$ for all p.)

(ii) Is the minimum in general attained if (†) does not hold?

2.2. (i) Let

$$I(u) = \int_0^1 [u_x^4 - 4u_x^2 + x^2 u_x + u^2]\, dx,$$

$$\mathcal{A} = \{u \in W^{1,1}(0,1) : u(0) = 0, u(1) = 1\}.$$

Show that $\bar{u}(x) = x$ is a weak local minimizer of I in $\mathcal{A}$. Is $\bar{u}$ a strong local minimizer?

(ii) Let

$$I(u) = \int_0^1 [(u_x^2 - 1)^2 + u^2]\, dx.$$

Show that there is no strong local minimizer of I in

$$\mathcal{A} = \{u \in W^{1,1}(0,1) : u(0) = u(1) = 0\}.$$

(*Hint.* Consider the maximum and minimum of a possible strong local minimizer.)

2.3. Let

$$I(u) = \int_a^b f(x, u(x), u_x(x))\,dx,$$

$$\mathcal{A} = \{u \in W^{1,1}(a,b) : u(a) = \alpha\},$$

where $-\infty < a < b < \infty$, $\alpha \in \mathbb{R}$, and f is C^1 and bounded below.
(i) Show that if $u \in \mathcal{A} \cap W^{1,\infty}(a,b)$ is a weak local minimizer of I in $\mathcal{A}$ (i.e. a local minimizer in $\mathcal{A} \cap W^{1,\infty}(a,b)$) then

$$f_p(x, u(x), u_x(x)) = \int_b^x f_u(y, u(y), u_y(y))\,dy \text{ for a.e. } x \in [a,b].$$

(ii) Show that if $u \in \mathcal{A} \cap W^{1,\infty}(a,b)$ is a strong local minimizer of I in $\mathcal{A}$ (i.e. a local minimizer in $\mathcal{A} \cap L^\infty(a,b)$), and if u is C^1 in a neighbourhood of b, then $f(b, u(b), p)$ is minimized at $p = u_x(b)$.
(iii) Is the minimum of

$$I(u) = \int_0^1 (u_x^2 + u^2)\,dx$$

among $u \in C^1([0,1])$ satisfying $u(0) = 0, u_x(1) = 1$ attained?

2.4. Let

$$I(u) = \int_0^1 (u^5 - x)^2 u_x^4\,dx,$$

$$\mathcal{A} = \{u \in W^{1,1}(0,1) : u(0) = 0, u(1) = 1\}.$$

(i) Prove that the unique minimizer of I in $\mathcal{A}$ is $\bar{u}(x) = x^{\frac{1}{5}}$.
(ii) Prove that if $p \geq \frac{5}{4}$ then

$$\inf_{u \in \mathcal{A} \cap W^{1,p}(0,1)} I(u) > 0 = I(\bar{u}).$$

(iii) Prove the *repulsion property*, that if $u^{(j)} \in W^{1,\frac{5}{4}}(0,1)$ and $\lim_{j\to\infty} u^{(j)}(\xi_k) = \bar{u}(\xi_k)$ for some sequence $\xi_k > 0$ with $\xi_k \to 0$, then $\lim_{j\to\infty} I(u^{(j)}) = \infty$.

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