Semiflows, Lyapunov functions and approach to equilibrium

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1 Introduction

Physical systems of PDE often have a *Lyapunov function*, that is a functional that decreases along solutions unless the system is in equilibrium. Typically this arises from the 2nd Law of Thermodynamics, with the Lyapunov function being the negative of the total entropy, or the total free energy.

Question: does the existence of such a functional enable one to prove that all solutions tend to equilibrium as time $t \to \infty$?

We first discuss this for a finite-dimensional example.

Example 1.1. Consider the ordinary differential equation

$$\ddot{u} + \dot{u} + u^3 - u = 0. \tag{1.1}$$

Note that $f(u) = u^3 - u = F'(u)$, where $F(u) = \frac{1}{4}(u^2 - 1)$ is a double-well potential (see Fig. 1.1). We write (1.1) as a first order system

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ -\dot{u} + u - u^3 \end{pmatrix}, \tag{1.2}$$

that is

$$\dot{w} = g(w), \tag{1.3}$$

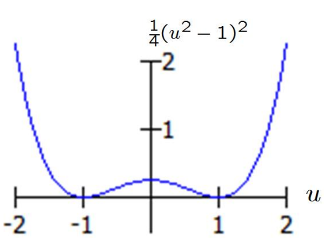


Figure 1.1: Double-well potential

where

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad g(w) = \begin{pmatrix} w_2 \\ -w_2 + w_1 - w_1^3 \end{pmatrix}. \tag{1.4}$$

The phase space for (1.3) is $X = \mathbb{R}^2$, and since g is smooth, for any $p \in \mathbb{R}^2$ there exists a unique solution $w(t)$ with initial data $w(0) = p$ for t in some maximal interval $[0, t_{\max}), t_{\max} > 0$. There are three rest points, namely $z_\pm = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, z_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The linearization of (1.3) about a rest point z is

$$\dot{y} = g'(z)y. \tag{1.5}$$

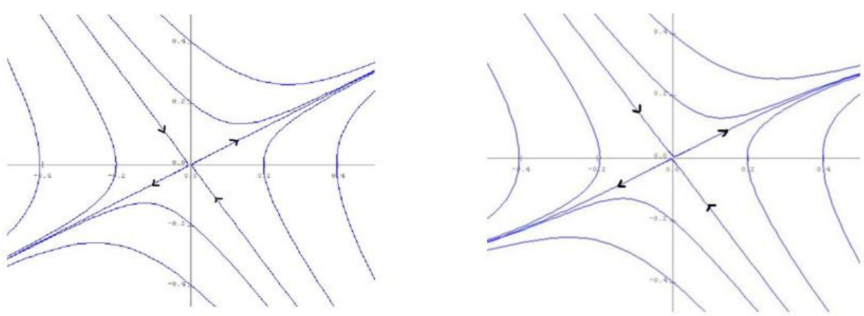
A short calculation shows that

$$g'(z_\pm) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \tag{1.6}$$

which has eigenvalues $\frac{-1 \pm i\sqrt{7}}{2}$, so that $z_\pm$ are spiral sinks, and that

$$g'(0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \tag{1.7}$$

which has eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$ and corresponding eigenvectors $\begin{pmatrix} \frac{-1 \pm \sqrt{5}}{2} \\ 1 \end{pmatrix}$, so that 0 is a saddle point.



(a) (b)

Figure 1.2: Phase portrait near zero (a) linearized (b) nonlinear.

(According to the theory of integral manifolds, the nonlinear equation (1.3) behaves like the linear one (1.5) in a sufficiently small neighbourhood of the critical point z. Thus, for example, near zero the linearized equation has the phase portrait in Fig. 1.2(a), while the nonlinear equation has the phase portrait in Fig. 1.2(b), with one-dimensional stable and unstable manifolds tangent at

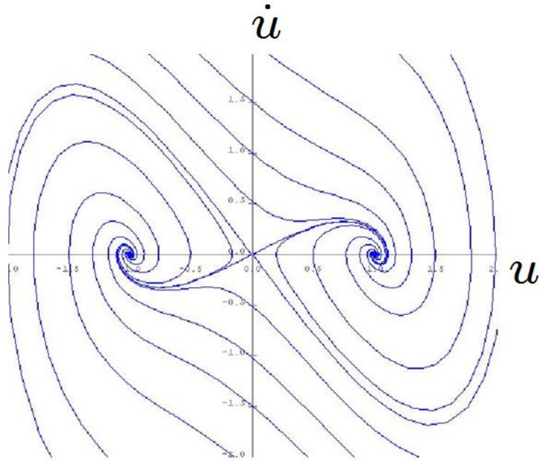


Figure 1.3: Phase-plane diagram for (1.1)

0 to the linearised ones.) The full phase portrait is shown in Fig. 1.3. We see from this that apparently every solution $w(t)$ converges to some rest point z as $t \to \infty$. The key to proving this will be the Lyapunov function

$$V(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \frac{1}{4}(u^2 - 1)^2, \tag{1.8}$$

which satisfies

$$\frac{d}{dt}V(u, \dot{u}) = -\dot{u}^2. \tag{1.9}$$

Note that ± 1 minimize F, so that the rest points $z_\pm$ are global minimizers of V. From (1.8), (1.9) we see that every solution is bounded for $t \geqslant 0$, so that in particular solutions exist for all time $t \geqslant 0$.

We make a first attempt at using (1.9) to prove convergence to a rest point as $t \to \infty$ by noting that it implies that $\int_0^\infty \dot{u}^2\, dt < 0$. Suppose $f : (0, \infty) \to [0, \infty)$ is C^1 and $\int_0^\infty f(t)\, dt < \infty$. This does not in itself prove that $f(t) \to 0$ as $t \to \infty$ (give an example). However if also $|\dot{f}(t)| \leqslant M < \infty$ then (exercise) $f(t) \to 0$ as $t \to \infty$. So, since $\frac{d}{dt}\dot{u}^2 = 2\dot{u}(-\dot{u} - u^3 + u)$ and $u(t), \dot{u}(t)$ are bounded for $t \geqslant 0$, in fact we do get that $\dot{u}(t) \to 0$ as $t \to \infty$. But this still doesn't prove that $u(t) \to z$ for some rest point z, for which a more subtle argument is required.

Now consider a large ball $B(0, R) \subset \mathbb{R}^2$ of initial data. How does it evolve under the flow? We prove later that it tends to the set A consisting of the three rest points and the two connecting orbits between them. The set A is the *global attractor* for (1.3).

2 Semiflows on a metric space.

Suppose we have an autonomous system with state space a metric space (X, d). We suppose that for each $p \in X$ there is a unique solution $w(t)$ with $w(0) = p$, defined for all $t \geqslant 0$ and depending continuously on p. Write $w(t) = T(t)p$. Then if $s \geqslant 0, t \geqslant 0$, the state of the system at time $s + t$ is $T(s+t)p$. But this is the same state as reached starting at $T(t)p$ sfter time s. Hence $T(s+t)p = T(s)T(t)p$.

Definition 2.1. A *semiflow* $\{T(t)\}_{t\geqslant 0}$ on a metric space (X, d) is a family of continuous maps $T(t) : X \to X$ satisfying

(i) $T(0) =$ identity,
(ii) $T(s + t) = T(s)T(t)$ for all $s \geqslant 0, t \geqslant 0$,
(iii) for each $p \in X$ the map $t \mapsto T(t)p$ is continuous from $[0, \infty) \to X$.

(In the literature a semiflow is sometimes called a (*nonlinear*) *semigroup* or *dynamical system.*)

It is possible to consider weaker versions of (iii), for example that for each p the map $t \mapsto T(t)p$ is strongly measurable from $[0, \infty) \to X$, and surprisingly this implies that $t \mapsto T(t)p$ is continuous from $(0, \infty) \to X$ (see [1]). Another similar example of the semigroup property (ii) strengthening continuity properties is:

Theorem 2.1 (Chernoff & Marsden [5])**.** *If $\{T(t)\}_{t\geqslant 0}$ is a semiflow on X, then the map $(t, p) \mapsto T(t)p$ is continuous from $(0, \infty) \times X \to X$.*

Proof. Let $p_j \to p$ in X. Let $0 < a < b < \infty$, and for $\varepsilon > 0, m = 1, 2, \ldots$, set

$$S_{m,\varepsilon} = \{t \in [a, b] : d(T(t)p_j, T(t)p) \leqslant \varepsilon \text{ for all } j \geqslant m\}.$$

By (iii) $S_{m,\varepsilon}$ is closed, and by the continuity of $T(t)$

$$\bigcup_{m=1}^{\infty} S_{m,\varepsilon} = [a, b].$$

By the Baire Category Theorem, some $S_{r,\varepsilon}$ contains an open interval. Since we may apply this argument to any $[a, b] \subset (0, \infty)$ there exists a dense open subset S_ε of $(0, \infty)$ such that if $t_0 \in S_\varepsilon$ there exists an open neighbourhood $N_\varepsilon(t_0)$ of t_0 and $r_\varepsilon(t_0)$ such that $d(T(t)p_j, T(t)p) \leqslant \varepsilon$ whenever $j \geqslant r_\varepsilon(t_0)$, $t \in N_\varepsilon(t_0)$. Let

$$K = \bigcap_{i=1}^{\infty} S_{1/i}.$$

Clearly $T(t_j)p_j \to T(t)p$ whenever $t_j \to t$ and $t \in K$. Again by the Baire Category Theorem, K is dense in $(0, \infty)$.

Now let $t > 0$ be arbitrary and $t_j \to t$. Let $t_1 \in K, 0 < t_1 < t$. Then $T(t_1 + t_j - t)p_j \to T(t_1)p$ and so

$$T(t_j)p_j = T(t - t_1)T(t_1 + t_j - t)p_j \to T(t - t_1)T(t_1)p = T(t)p.$$

$\square$

Corollary 2.2 (Dorroh [6], Chernoff [4])**.** *If X is locally compact then the map $(t,p) \mapsto T(t)p$ is continuous from $[0,\infty) \times X \to X$.*

(This is false in general if X is not locally compact; there are examples for $X =$ Hilbert space [4], [2].)

Proof of Corollary. We just need to show that if $p_j \to p, t_j \to 0+$, then $T(t_j)p_j \to p$. Suppose not. Then there is a subsequence (not relabelled) such that $d(T(t_j)p_j, p) \geqslant \varepsilon > 0$ for all j. We can also suppose that $d(p_j, p) < \varepsilon$. By (iii) there exists $s_j \in [0, t_j]$ with $d(T(s_j)p_j, p) = \varepsilon$. By local compactness we can assume that $T(s_j)p_j \to y$ with $d(y,p) = \varepsilon$. If $t > 0$ then $T(t+s_j)p_j \to T(t)p$ by the theorem. But $T(t+s_j)p_j = T(t)T(s_j)p_j \to T(t)y$. Hence $T(t)p = T(t)y$, and letting $t \to 0+$ we deduce that $p = y$, a contradiction. $\square$

Let $\{T(t)\}_{t\geqslant 0}$ be a semiflow on the metric space (X,d). The *positive orbit* of $p \in X$ is the set (see Fig. 2.1)

$$\gamma^+(p) = \{T(t)p : t \geqslant 0\}.$$

The $\omega-$*limit set* of p is the set

$$\begin{aligned} \omega(p) &= \{\chi \in X : T(t_j)p \to \chi \text{ for some sequence } t_j \to \infty\} \\ &= \bigcap_{t\geqslant 0} \overline{\bigcup_{\tau \geqslant t} T(\tau)p}. \end{aligned} \tag{2.1}$$

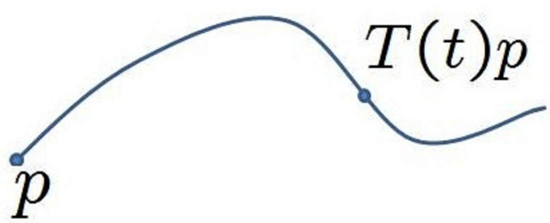


Figure 2.1: Positive orbit

A map $\psi : \mathbb{R} \to X$ is a *complete orbit* if

$$\psi(t+s) = T(t)\psi(s) \text{ for all } s \in \mathbb{R}, t \geqslant 0.$$

(Note that we do not assume backwards uniqueness, so there might be more than one complete orbit passing through a point $p \in X$ (see Fig. 2.2).)
If ψ is a complete orbit then the α-*limit set* of ψ is the set

$$\begin{aligned} \alpha(\psi) &= \{\chi \in X : \psi(t_j) \to \chi \text{ for some sequence } t_j \to -\infty\} \\ &= \bigcap_{t\leqslant 0} \overline{\bigcup_{\tau \leqslant t} \psi(\tau)}. \end{aligned}$$

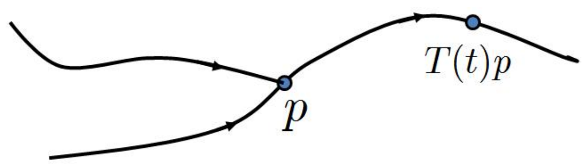


Figure 2.2: More than one complete orbit passing through a point.

If $E \subset X$, $t \geqslant 0$, we set

$$T(t)E = \{T(t)p : p \in E\}.$$

A subset $E \subset X$ is *positively invariant* if $T(t)E \subset E$ for all $t \geqslant 0$, and *invariant* if $T(t)E = E$ for all $t \geqslant 0$.

Note that if E invariant then there is a complete orbit contained in E passing through any point of E. Indeed if $p \in E$ then there exist $p_{-1} \in E$ with $T(1)p_{-1} = p$, $p_{-2} \in E$ with $T(1)p_{-2} = p_{-1}$, and so on, so that

$$\psi(t) = \begin{cases} T(t)p, & t \geqslant 0 \\ T(t+i)p_{-i}, & t \in [-i, -i+1), i = 1, 2, \ldots \end{cases}$$

defines a complete orbit passing through p.

Theorem 2.3. *(i) Let $\gamma^+(p)$ be relatively compact. Then $\omega(p)$ is nonempty, compact, invariant and connected. As $t \to \infty$,*

$$\operatorname{dist}(T(t)p, \omega(p)) \to 0,$$

where $\operatorname{dist}(q, E) := \inf_{\chi \in E} d(q, \chi)$.

(ii) Let ψ be a complete orbit with $\{\psi(t) : t \leqslant 0\}$ relatively compact. Then $\alpha(\psi)$ is nonempty, compact, invariant and connected, and as $t \to -\infty$

$$\operatorname{dist}(\psi(t), \alpha(\psi)) \to 0.$$

Proof. We prove (i). The proof of (ii) is similar and is left to Problem **??**. That $\omega(p)$ is nonempty is clear. Since $\omega(p)$ is by (2.1) the intersection of compact sets, it is compact. To prove the invariance, let $\chi \in \omega(p)$. Then $T(t_j)p \to \chi$ for some sequence $t_j \to \infty$. If $t \geqslant 0$ then, since $T(t)$ is continuous,

$$T(t + t_j)p = T(t)T(t_j)p \to T(t)\chi$$

and so $T(t)\omega(p) \subset \omega(p)$. Also $\{T(t_j - t)p\}$ is relatively compact, and so

$$T(t_{j_k} - t)p \to q \in X$$

for some subsequence $\{t_{j_k}\}$. Therefore

$$T(t_{j_k})p = T(t)T(t_{j_k} - t)p \to T(t)q = \chi.$$

Hence $T(t)\omega(p) \supset \omega(p)$ and so $\omega(p)$ is invariant.

If $\mathrm{dist}\,(T(t)p, \omega(p)) \not\to 0$ as $t \to \infty$, then there exist $\varepsilon > 0$ and a sequence $t_j \to \infty$ such that $d(T(t_j)p, z) \geq \varepsilon$ for all $z \in \omega(p)$. But a subsequence $T(t_{j_k})p \to \chi \in \omega(p)$, a contradiction.

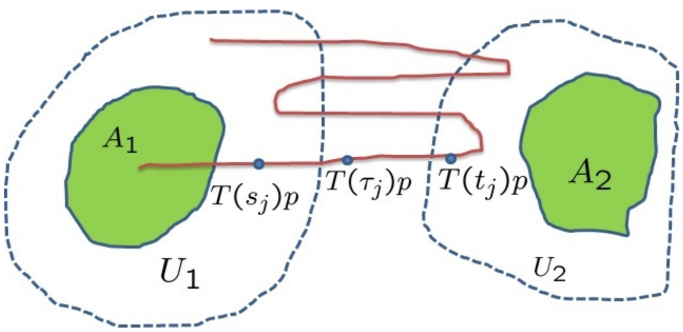


Figure 2.3: Proof of connectedness of $\omega(p)$

Suppose $\omega(p)$ is not connected. Then $\omega(p) = A_1 \cup A_2$ with A_1, A_2 nonempty disjoint compact sets. (Indeed, by the definition of connectedness we can write $\omega(p) = V_1 \cup V_2$ with $\bar{V_1} \cap V_2 = V_1 \cap \bar{V_2} = \emptyset$, and since $\omega(p)$ is closed we have $\omega(p) = \bar{V_1} \cup \bar{V_2}$. Thus $\bar{V_1} \cap \bar{V_2} = \emptyset$ and we can set $A_1 = \bar{V_1}, A_2 = \bar{V_2}$. Since A_1, A_2 are closed subsets of a compact set, they are themselves compact.) Let U_1, U_2 be disjoint open sets with $A_1 \subset U_1, A_2 \subset U_2$. We can take, for example, $U_i = \{q \in X : \mathrm{dist}\,(q, A_i) < \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. Then there exist sequences $s_j > t_j$ with $t_j \to \infty$ such that $T(s_j)p \in U_1, T(t_j)p \in U_2$ and hence, by Definition 2.1(iii), there exists $\tau_j \in (t_j, s_j)$ with $T(\tau_j)p \notin U_1 \cup U_2$ (see Fig. **??**). Hence by the relative compactness of $\gamma^+(p)$ there exists some $\chi \in \omega(p) \backslash (A_1 \cup A_2)$, a contradiction. □

3 Approach to equilibrium

A point $z \in X$ is a *rest point* if $T(t)z = z$ for all $t \geq 0$. The set Z of rest points is closed.
A function $V : X \to \mathbb{R}$ is a *Lyapunov function* if

(i) V is continuous,

(ii) $V(T(t)p) \leqslant V(p)$ for all $p \in X, t \geqslant 0$,

(iii) If $V(\psi(t)) = c$ for some complete orbit ψ, all $t \in \mathbb{R}$ and some constant c, then $\psi(t) = z$ for all $t \in \mathbb{R}$ for some rest point z.

(Note that (ii) implies that $V(T(t)p) \leqslant V(T(s)p)$ for all $t \geqslant s \geqslant 0$, since $V(T(t)p) = V(T(t-s)T(s)p) \leqslant V(T(s)p)$.)

Theorem 3.1 (LaSalle invariance principle)**.** *Let V be a Lyapunov function, and let $p \in X$ with $\gamma^+(p)$ relatively compact. Then $\omega(p)$ consists only of rest* *points. If the only nonempty connected subsets of Z are single points (for example, if there are only a finite number of rest points) then $\omega(p) = \{z\}$ for some rest point z, and $T(t)p \to z$ as $t \to \infty$.*

Proof. Since V is continuous and $\gamma^+(p)$ is relatively compact, $V(T(t)p)$ is bounded below for $t \geqslant 0$. But $t \mapsto V(T(t)p)$ is nonincreasing, and so

$$V(T(t)p) \to c \text{ as } t \to \infty$$

for some constant c. Let $z \in \omega(p)$. Then, since $\omega(p)$ is invariant, $z = \psi(0)$ for a complete orbit ψ contained in $\omega(p)$. Hence $V(\psi(t)) = c$ for all $t \in \mathbb{R}$, and so by (iii) z is a rest point.

If the only nonempty connected subsets of Z are single points then since $\omega(p)$ is connected, $\omega(p) = z$ for some rest point, so that $T(t)p \to z$ as $t \to \infty$ by Theorem 2.3. □

4 Lyapunov stability

Definitions 4.1. The rest point z is (*Lyapunov*) *stable* if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $p \in B(z, \delta)$ then $T(t)p \in B(z, \varepsilon)$ for all $t \geqslant 0$. The rest point z is *unstable* if it is not stable. The rest point z is *asymptotically stable* if z is stable and there exists $\rho > 0$ such that $p \in B(z, \rho)$ implies $T(t)p \to z$ as $t \to \infty$.

If the rest point z is asymptotically stable then clearly z is *isolated*, that is there is some $\varepsilon > 0$ such that z is the only rest point in $B(z, \varepsilon)$.

Theorem 4.1. *Let z be an isolated rest point, let V be a Lyapunov function, let $\gamma^+(p)$ be relatively compact for any p with $\gamma^+(p)$ bounded, and suppose that for all $\delta > 0$ sufficiently small*

$$\inf_{d(p,z)=\delta} V(p) > V(z) \qquad (\textit{Existence of a potential well}) \tag{4.1}$$

Then z is asymptotically stable.

Proof. Suppose z is not stable. Then there exist $\varepsilon > 0$, $p_j \to z$, $t_j \geqslant 0$ with $d(T(t_j)p_j, z) \geqslant \varepsilon$. We can suppose that ε is small enough such that

$$c_\varepsilon := \inf_{d(p,z)=\frac{\varepsilon}{2}} V(p) > V(z),$$

and such that z is the only rest point in $\overline{B(z, \varepsilon)}$. Let j be sufficiently large. Then since V is continuous, $V(p_j) < c_\varepsilon$. By the continuity of $t \mapsto T(t)p_j$ there exists $\tau_j \in (0, t_j)$ with $d(T(\tau_j)p_j, z) = \frac{\varepsilon}{2}$, and thus

$$c_\varepsilon \leqslant V(T(\tau_j)p_j) \leqslant V(p_j) < c_\varepsilon,$$

a contradiction.

By the stability, given $\varepsilon > 0$ there exists $\rho > 0$ such that if $d(p, z) < \rho$ then $T(t)p \in \overline{B(z, \varepsilon)}$ for all $t \geqslant 0$. Then $\gamma^+(p)$ is bounded, and so by the assumption of the theorem relatively compact. Thus, by Theorem 3.1, $\omega(p) \subset Z \cap \overline{B(z,\varepsilon)}$ and so $\omega(p) = \{z\}$ and $T(t)p \to z$ as $t \to \infty$. □

Remark 1. If $X = \mathbb{R}^n$ then the existence of a potential well (see (4.1)) is equivalent to the condition that z is a strict local minimizer of V, i.e. that there exists $\varepsilon > 0$ such that $V(p) > V(z)$ if $0 < d(p,z) \leqslant \varepsilon$. This follows easily from the fact that the sphere $S(z,\varepsilon)$ is compact, so that V attains a minimum on $S(z,\varepsilon) = \{p : d(p,z) = \varepsilon\}$. But if X is a metric space whose spheres $S(z,\varepsilon)$ are not compact (as is the case for infinite-dimensional normed vector spaces) then the existence of a potential well is a stronger condition than being a strict local minimizer. If we just assumed that z was a strict local minimizer then the danger would be that orbits could leak out of balls by going into higher and higher dimensions.

Theorem 4.2. *Let V be a Lyapunov function and suppose that $\gamma^+(p)$ is relatively compact for any p with $\gamma^+(p)$ bounded. Let z be an isolated rest point which is not a local minimizer of V (i.e. for any $\varepsilon > 0$ there is a point p with $d(p,z) < \varepsilon$ and $V(p) < V(z)$). Then z is unstable.*

Proof. Let $\varepsilon > 0$ be sufficiently small so that z is the only rest point in $\overline{B(z,\varepsilon)}$. Suppose for contradiction that z is stable. Then there exists $\delta > 0$ such that $d(p,z) < \delta$ implies $d(T(t)p,z) < \varepsilon$ for all $t \geqslant 0$. But since z is not a local minimizer there exists p with $d(p,z) < \delta$ and $V(p) < V(z)$. Since $\gamma^+(p) \subset \overline{B(z,\varepsilon)}$, $\gamma^+(p)$ is by assumption relatively compact. Hence by the invariance principle there exist a sequence $t_j \to \infty$ and a rest point $\tilde{z} = \lim_{j\to\infty} T(t_j)p$ in $\omega(p)$ with $\tilde{z} \in \overline{B(z,\varepsilon)}$. But $V(\tilde{z}) = \lim_{j\to\infty} V(T(t_j)p) < V(z)$. Hence $\tilde{z} \neq z$, a contradiction. □

The *region of attraction* of a rest point z is the set

$$A(z) = \{p \in X : T(t)p \to z \text{ as } t \to \infty\}.$$

Theorem 4.3. *$A(z)$ is connected.*

Proof. Suppose not, so that $A(z) = U \cup V$ with U, V nonempty and $U \cap \bar{V} = \bar{U} \cap V = \emptyset$. Let $p \in U, q \in V$. For any $t \geqslant 0$, $T(t)p \in A(z)$. Let $S = \{t \geqslant 0 : T(t)p \in U\}$. Let $t_j \in S$, $t_j \to t$. Then $T(t)p = \lim_{j\to\infty} T(t_j)p \in \bar{U}$ and so $T(t)p \in U$. Hence S is closed in $[0,\infty)$. Similarly S is open, and thus $\gamma^+(p) \subset U$. Similarly $\gamma^+(q) \subset V$. But $z \in \bar{U}$, hence $z \notin V$. Similarly $z \notin U$. But $z \in A(z) = U \cup V$, a contradiction. □

Theorem 4.4. *If z is an asymptotically stable rest point then $A(z)$ is open.*

Proof. Let $\rho > 0$ be as in Definition 4.1, and let $p \in A(z)$. Then there exists $s > 0$ such that $d(T(s)p,z) < \rho$. Hence by the continuity of $T(s)$ there exists $\sigma > 0$ such that $d(p,q) < \sigma$ implies $d(T(s)q,z) \leqslant d(T(s)q,T(s)p) + d(T(s)p,z) < \rho$, so that by asymptotic stability $T(t)q \to z$ as $t \to \infty$ and hence $q \in A(z)$. □

Applying these results to (1.1) we deduce that

$$\mathbb{R}^2 = A(z_-) \cup A(z_0) \cup A(z_+)$$

with $A(z_\pm)$ connected and open. $A(z_0)$ consists of $z_0 = 0$ together with the two orbits that approach 0 as $t \to \infty$.

5 Global attractors

Definitions 5.1. The semiflow $\{T(t)\}_{t \geqslant 0}$ is *asympttically compact* if for any bounded sequence $\{p_j\}$ in X and any sequence $t_j \to \infty$ $T(t_j)p_j$ has a convergent subsequence. It is *point dissipative* if there is a bounded set B_0 such that for any $p \in X$, $T(t)p \in B_0$ for all t sufficiently large.

A subset $A \subset X$ *attracts a set* $E \subset X$ if

$$\text{dist}\,(T(t)E, A) \to 0 \text{ as } t \to \infty,$$

where

$$\text{dist}\,(B, C) := \sup_{b \in B} \inf_{c \in C} d(b, c) = \sup_{b \in B} \text{dist}\,(b, C).$$

(If A is compact this is the same as saying that given any open neighbourhood $U \supset A$, $T(t)E \subset U$ for t sufficiently large.)

The subset A is a *global attractor* if A is compact, invariant, and attracts all bounded sets.

If $B \subset X$ is bounded, the ω-limit set of B is

$$\omega(B) = \{\chi \in X : T(t_j)p_j \to \chi \text{ for some sequences } p_j \in B, t_j \to \infty\}.$$

Theorem 5.1. *A semiflow $\{T(t)\}_{t \geqslant 0}$ has a global attractor if and only if it is point dissipative and asymptotically compact. The global attractor is unique and given by*

$$A = \bigcup \{\omega(B) : B \text{ a bounded subset of } X\}.$$

Furthermore A is the maximal compact invariant subset of X.

Proof. See [7, 8, 2]. □

We discuss now (following [3]) how asymptotic compactness and approach to equilibrium can be proved in an infinite-dimensional example for which (unlike for corresponding parabolic problems) $T(t)$ is not compact for $t > 0$ (so that compactness only occurs 'at infinity').

Let $\Omega \subset \mathbb{R}^3$ be bounded and open, and consider the semilinear hyperbolic PDE for $u = u(x, t), x \in \Omega, t \geqslant 0$

$$u_{tt} + \beta u_t - \Delta u + u^3 - u = 0, \tag{5.1}$$

where $\beta > 0$ is a constant, with boundary condition

$$u|_{\partial\Omega} = 0. \tag{5.2}$$

We can write (5.1) as the system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\beta v + \Delta u \end{pmatrix} + \begin{pmatrix} 0 \\ u - u^3 \end{pmatrix}, \tag{5.3}$$

or, setting $w = \begin{pmatrix} u \\ v \end{pmatrix}$, where $v = u_t$, as

$$\dot{w} = Aw + f(w), \tag{5.4}$$

where

$$A = \begin{pmatrix} 0 & I \\ \Delta & -\beta I \end{pmatrix}, \qquad f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ u - u^3 \end{pmatrix}.$$

We regard (5.4) as an equation in the Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega)$. Because of the embedding $H_0^1(\Omega) \subset L^6(\Omega)$, $f : X \to X$. We claim that f is locally Lipschitz (i.e. for any $K > 0$ there exists C_K such that $\|f(z) - f(\bar{z})\|_X \leqslant C_K\|z - \bar{z}\|_X$ if $\|z\|_X \leqslant K, \|\bar{z}\|_X \leqslant K$) because

$$\begin{aligned} \int_\Omega (u^3 - \bar{u}^3)^2 dx &= \int_\Omega (u - \bar{u})^2 P_4(u, \bar{u})\, dx \\ &\leqslant \left(\int_\Omega (u - \bar{u})^6 dx\right)^{\frac{1}{3}} \left(\int_\Omega P_4(u, \bar{u})^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \\ &\leqslant C_K \|u - \bar{u}\|_6^2, \end{aligned}$$

if $|u\|_6, \|\bar{u}\|_6 \leqslant K$, where $P(u, \bar{u})$ is a fourth order polynomial. Also $f : X \to X$ is *sequentially weakly continuous*. i.e. $w^{(j)} \rightharpoonup w$ in X implies $f(w^{(j)}) \rightharpoonup f(w)$ in X.

Formally (5.1), (5.2) have the Lyapunov function

$$V(u, u_t) = \int_\Omega \left(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2\right) dx, \tag{5.5}$$

for which

$$\dot{V} = -\beta \int_\Omega u_t^2\, dx. \tag{5.6}$$

These ingredients imply that weak solutions of (5.3) generate a semiflow on X with Lyapunov function V. The proof uses the equivalent *variation of constants* formula

$$w(t) = e^{At}w(0) + \int_0^t e^{A(t-s)} f(w(s))\, ds. \tag{5.7}$$

Furthermore $T(t) : X \to X$ is sequentially weakly continuous for each $t \geqslant 0$.

Theorem 5.2. $\{T(t)\}_{t \geqslant 0}$ *is asymptotically compact.*

Proof. We use the auxiliary functional

$$I(u,u_t) = V(u,u_t) + \frac{\beta}{2}(u,u_t), \tag{5.8}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. Then

$$\begin{aligned}
\frac{dI}{dt} &= -\beta\|u_t\|_2^2 + \frac{\beta}{2}\|u_t\|_2^2 + \frac{\beta}{2}\left(-\beta(u,u_t) - \|\nabla u\|_2^2 + \int_\Omega (u^2-u^4)\,dx\right)\\
&= -\beta I + \beta\int_\Omega \left(\frac{1}{4}(u^2-1)^2 + \frac{u^2-u^4}{2}\right)dx\\
&= -\beta I + H(u),
\end{aligned}$$

where $H(u) = \frac{\beta}{4}\int_\Omega (1-u^4)\,dx$.

Hence, given any $M>0$ and any $z\in X$,

$$I(T(M)z) = e^{-\beta M}I(z) + \int_0^M e^{\beta(t-M)}H(u(t))\,dt, \tag{5.9}$$

where $T(t)z = \begin{pmatrix} u(t)\\ \dot u(t)\end{pmatrix}$.

Let z_j be bounded, $t_j\to\infty$. Then $T(t_j)z_j$ is bounded and we may suppose that $T(t_j)z_j \rightharpoonup \chi, T(t_j-M)z_j \rightharpoonup \chi_{-M}$ for some $\chi,\chi_{-M}\in X$. Hence $T(t+t_j-M)z_j \rightharpoonup \chi_{-M}$ and thus $T(M)\chi_{-M}=\chi$. Apply (5.9) with $z=T(t_j-M)z_j$ to obtain

$$I(T(t_j)z_j) = e^{-\beta M}I(T(t_j-M)z_j) + \int_0^M e^{\beta(t-M)}H(u_j(t))\,dt, \tag{5.10}$$

where $T(t+t_j-M)z_j = \begin{pmatrix} u_j(t)\\ \dot u_j(t)\end{pmatrix}$. Passing to the limit and using again (5.9) with $z=\chi_{-M}$ we get

$$\limsup_{j\to\infty} I((t_j)z_j) \leqslant Ce^{-\beta M} + I(\chi) - e^{-\beta M}I(\chi_{-M}). \tag{5.11}$$

Letting $M\to\infty$ we obtain

$$\limsup_{j\to\infty} I(T(t_j)z_j) \leqslant I(\chi) \leqslant \liminf_{j\to\infty} I(T(t_j)z_j). \tag{5.12}$$

Hence $I(T(t_j)z_j)\to I(\chi)$ and from the form of I we deduce that $\|T(t_j)z_j\|_X \to \|\chi\|_X$ so that $T(t_j)z_j\to\chi$ strongly. □

Hence $\omega(p)$ consists only of rest points for every p. Also, the set Z of rest points is bounded, since for any rest point $z = \begin{pmatrix} u\\ 0\end{pmatrix}$,

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega (u^2-u^4)\,dx \leqslant \frac{1}{4}\operatorname{meas}\Omega.$$

Hence $\{T(t)\}_{t\geqslant 0}$ is point dissipative and there exists a global attractor.

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