Dynamical Systems and Energy Minimization

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Chapter 1

Approach to equilibrium of thermomechanical systems

1.1 Macroscopic examples.

We begin by considering some examples of thermomechanical systems which approach equilibrium as time $t \to \infty$ or as $t \to T$ for some finite $T > 0$.

Example 1.1. Stirred mug of hot coffee on table.

Figure 1.1: Stirred coffee

Initially the fluid (coffee) is rotating, and at a higher temperature than the surrounding air and table. As time $t \to \infty$ the fluid velocity $v \to 0$ and the fluid temperature $\theta \to \theta_0 =$ room temperature. Or does this happen as $t \to T-$ for some $T < \infty$? Since nothing ever really comes to rest, due to thermal vibrations etc, this is more a question about the mathematical model than reality.

Example 1.2. Bouncing rubber ball dropped from rest at height h.

Let us use a simplified model, in which the ball is represented by a point mass, and in which we neglect air resistance. The governing equations are then

$$\begin{aligned}
&\ddot{z}(t) = -g \quad \text{if } z(t) > 0, \\
&\dot{z}(t+) = -E\dot{z}(t-) \quad \text{if } z(t) = 0 \text{ with } \dot{z}(t-) < 0, \\
&z(0) = h,\ \dot{z}(0) = 0,
\end{aligned}$$

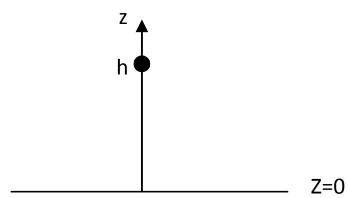


Figure 1.2: Elastic ball dropped from height h

where $0 < E < 1$ is the coefficient of restitution. Up to the first bounce we have

$$z(t) = h - g\frac{t^2}{2}.$$

So the first bounce is at $t_1 = \sqrt{\frac{2h}{g}}$ and $\dot{z}(t_1-) = -\sqrt{2hg}$. Hence $\dot{z}(t_1+) = E\sqrt{2hg} := v$. Now suppose that $z(0) = 0, \dot{z}(0+) = v > 0$. Then until the next bounce

$$z(t) = vt - g\frac{t^2}{2},$$

and so $z(\tau) = 0$ for $\tau = \frac{2v}{g} = 2E\sqrt{\frac{2h}{g}}$ and $\dot{z}(\tau-) = -v$. The second bounce is therefore at the time $t_2 = t_1 + \tau = \sqrt{\frac{2h}{g}}(1 + 2E)$, the third bounce at time $t_3 = \sqrt{\frac{2h}{g}}(1 + 2(E + E^2))$ and so on. Hence the ball comes to rest after time

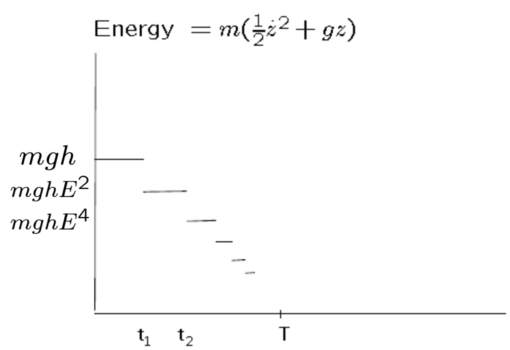


Figure 1.3: Decay of energy for bouncing ball

$$
\begin{aligned}
T &= \sqrt{\frac{2h}{g}}(1+2(E+E^2+\cdots)) \\
&= \sqrt{\frac{2h}{g}}\left(-1+\frac{2}{1-E}\right) \\
&= \sqrt{\frac{2h}{g}}\frac{1+E}{1-E} < \infty.
\end{aligned}
$$

Note that in between bounces the energy $m(\frac{1}{2}\dot{z}^2+gz)$ is constant, decreasing at each bounce (see Fig. 1.3).

See Problem 1.1 for the case when air resistance is added.

Example 1.3. Simple pendulum with air resistance proportional to velocity.

We assume that the pendulum consists of a thin massless rod with a bob of mass m on which air resistance exerts a force proportional to the velocity of the bob. The balance of forces in the tangential direction gives

$$ml\ddot{\theta} = -mg\sin\theta - cl\dot{\theta},$$

and so

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{l}\sin\theta = 0, \quad k = \frac{c}{m} > 0. \tag{1.1}$$

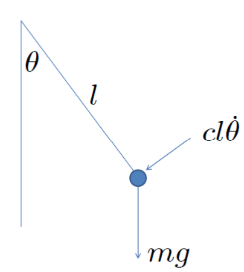


Figure 1.4: Forces on pendulum

The rest points (equilibria) are the time-independent solutions of (6.42), that is $\theta = n\pi, \dot{\theta} = 0$. Note that the rest points $\theta = n\pi, \dot{\theta} = 0$ are divided into two groups according to whether n is odd or even, all the rest points in each group representing the same physical state. In the case of even n the pendulum is in the (stable) state in which it hangs vertically downwards, while in the case of odd n the pendulum is in the (unstable) state when it is pointing vertically upwards. However this apparent redundancy contains useful information,

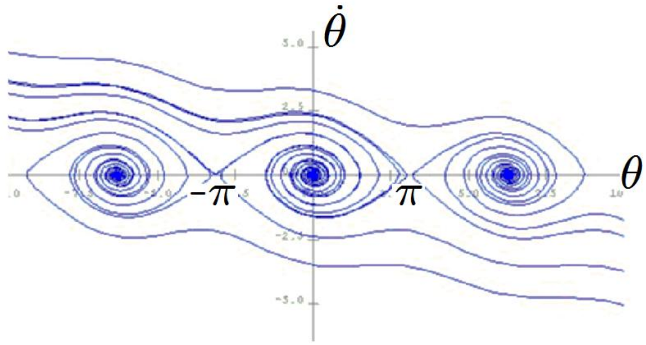


Figure 1.5: Phase-plane diagram for damped pendulum

for as the phase-plane diagram shows there are solutions which make arbitrary numbers of complete rotations in either direction before oscillating about and decaying to the stable rest point as $t \to \infty$, which would not be so easy to understand were we to identify all the odd and even rest points (that is work on the circle rather its 'lifting' to the real line).

Fact: every solution satisfies $\dot{\theta}(t) \to 0, \theta(t) \to n\pi$ as $t \to \infty$.

Why is this true? We will see that it is connected to the fact that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) = -k\dot{\theta}^2.$$

The energy

$$V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta$$

is a *Lyapunov function*, that is a function that is nonincreasing along solutions, and constant only for solutions that are rest points. A first attempt at a proof is to note that

$$V(\theta(T), \dot{\theta}(T)) + k\int_0^T \dot{\theta}^2(t)\,dt = V(\theta(0), \dot{\theta}(0))$$

implies that (since V is bounded below by a constant)

$$\int_0^\infty \dot{\theta}^2(t)\,dt < \infty,$$

suggesting that $\dot{\theta}(t) \to 0$ as $t \to \infty$. But
(a) f continuous with $f \geqslant 0$ and $\int_0^\infty f(t)\,dt < \infty$ does not imply $f(t) \to 0$ as $t \to \infty$ (see Problem 1.2).
(b) this doesn't show that $\theta(t) \to n\pi$ for some n.

In Examples 1.1-1.3 there are dissipative mechanisms that are apparently responsible for driving the system to equilibrium. But what is the origin of these dissipative mechanisms? This is a deep problem of science that is imperfectly understood. A first answer is that thermomechanical systems obey the *Second Law of Thermodynamics*, from which dissipative mechanisms result. Roughly the Second Law asserts that there is a quantity called *entropy* which increases in thermally isolated systems. Supplying a small quantity dQ of heat at temperature θ to a system increase the entropy by an amount $\frac{dQ}{\theta}$.

Example 1.4. The heat equation.

Consider a homogeneous rigid heat conductor occupying a bounded domain $\Omega \subset \mathbb{R}^3$. By rigid we mean that the body is assumed not to deform. We assume that the internal energy U per unit volume depends only on the temperature $\theta = \theta(x,t) > 0$, so that $U = U(\theta)$, and that the heat flux vector $q = q(\theta, \nabla\theta)$ depends only on the temperature θ and its spatial gradient $\nabla\theta$. (The internal energy can be thought of as consisting of the kinetic energy of constituent atoms plus the potential energy due to the interactions between them.) The balance law of energy asserts that the rate of change of the energy in any subvolume $E \subset \Omega$ equals the rate at which heat enters through the boundary ∂E, that is

$$\frac{d}{dt}\int_E U(\theta)\,dx = -\int_{\partial E} q\cdot n\,dS, \tag{1.2}$$

where $n = n(x)$ is the unit outward normal to ∂E. Using the divergence theorem we deduce that

$$\int_E [U(\theta)_t + \operatorname{div} q(\theta, \nabla\theta)]\,dx = 0 \tag{1.3}$$

for all subvolumes E, and so, assuming the integrand to be continuous in x, that

$$U(\theta)_t = -\operatorname{div} q(\theta, \nabla\theta) \qquad \text{(Heat equation)}. \tag{1.4}$$

Indeed, were the integrand in (2.3) nonzero at some point, it would also be nonzero in a sufficiently small neighbourhood E of that point, contradicting (2.3). (In the linear isotropic case we have $U(\theta) = c\theta, q(\theta, \nabla\theta) = -\kappa\nabla\theta$, where c, κ are the specific heat and thermal conductivity respectively, and we obtain from (1.4) the familiar linear heat equation

$$\theta_t = k\Delta\theta, \tag{1.5}$$

where $k = \kappa/c$.)

For the heat equation the entropy $\eta = \eta(\theta)$ is given by

$$\eta(\theta) = \int_{\bar\theta}^{\theta} \frac{U'(\tau)}{\tau}\,d\tau, \tag{1.6}$$

where $\bar\theta > 0$ is constant. Then the Second Law takes the form

$$\frac{d}{dt}\int_E \eta\,dx \geqslant -\int_{\partial E} \frac{q\cdot n}{\theta}\,dS \qquad \text{(Clausius-Duhem inequality)}, \tag{1.7}$$

for all $E \subset \Omega$, which is equivalent to

$$\int_E \left[\eta_t + \operatorname{div}\left(\frac{q}{\theta}\right)\right] dx \geqslant 0 \qquad \text{for all } E \subset \Omega, \tag{1.8}$$

or, using a similar argument to that used to derive (1.4),

$$\eta_t + \operatorname{div}\left(\frac{q}{\theta}\right) \geqslant 0. \tag{1.9}$$

Using the heat equation (1.4) this becomes

$$\frac{\partial \eta}{\partial \theta}\theta_t = \frac{1}{\theta}U(\theta)_t = -\frac{1}{\theta}\operatorname{div} q \geqslant -\operatorname{div}\left(\frac{q}{\theta}\right),$$

that is

$$-\frac{q \cdot \nabla\theta}{\theta^2} \geqslant 0,$$

or

$$q \cdot \nabla\theta \leqslant 0, \tag{1.10}$$

expressing the fact that heat flows from hot to cold. (In the linear case this becomes $\kappa \geqslant 0$.)

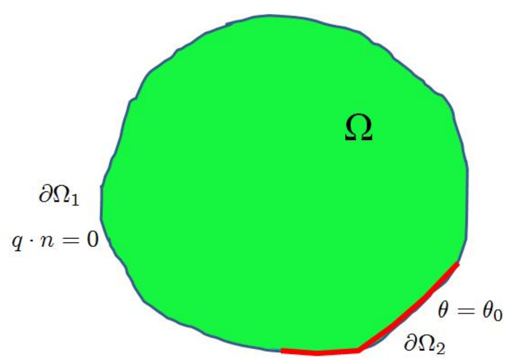


Figure 1.6: Boundary conditions for rigid heat conductor

Now we will see how the Second Law implies the existence of a Lyapunov function. Suppose (see Fig. 1.6) that $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_i$ disjoint, and that $\partial\Omega_1$ is insulated, so that $q \cdot n|_{\partial\Omega_1} = 0$, and $\partial\Omega_2$ is in contact with a heat bath at *constant* temperature θ_0 (independent of x and t), so that $\theta|_{\partial\Omega_2} = \theta_0$. Then, assuming that the various quantities are sufficiently smooth, we have that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (U-\theta_0\eta)\,dx &= \int_\Omega \left(\frac{\theta_0}{\theta}-1\right)\operatorname{div} q\,dx \\
&= \int_\Omega \left[\operatorname{div}\left(\left(\frac{\theta_0}{\theta}-1\right)q\right)+\theta_0\frac{q\cdot\nabla\theta}{\theta^2}\right]dx \\
&= \int_{\partial\Omega}\left(\frac{\theta_0}{\theta}-1\right)q\cdot n\,dS+\theta_0\int_\Omega\frac{q\cdot\nabla\theta}{\theta^2}dx \\
&= \theta_0\int_\Omega \frac{q\cdot\nabla\theta}{\theta^2}dx \leqslant 0, \qquad (1.11)
\end{aligned}$$

so that

$$V(\theta)=\int_\Omega (U-\theta_0\eta)\,dx \qquad (1.12)$$

is nonincreasing along solutions. If also (as in the linear case) $q\cdot\nabla\theta=0$ implies $\nabla\theta=0$, and the specific heat $U'(\theta)>0$ for all θ, then if V is constant along a solution we have $\nabla\theta=0$, hence $\theta=\theta(t)$. But $U'(\theta)\theta_t=\operatorname{div} q(\theta(t),0)=0$, so that $\theta_t=0$ and $\theta=$ constant ($=\theta_0$ if $\partial\Omega_2$ has positive area). Thus V is a Lyapunov function.

When θ_0 depends on x the situation is more complicated, though in some cases there is still a Lyapunov function (see Problem 1.3).

In Examples 1.1-1.3 the governing equations of a proper model are of course more complicated than the heat equation. For Example 1.1 they would consist of the equations of fluid mechanics coupled with heat conduction, and describe the transfer of heat from the fluid to the mug and surrounding air. In the case of a bouncing ball we would need to model the ball using elasticity theory and study the flow of air around the ball as it moves, together with making some assumptions about the floor and how it responds to impact. For the damped pendulum we would again need equations describing the flow of air around it, as well perhaps as those describing the pendulum arm as an elastic solid. But in all three cases there will be a Lyapunov function similar to (1.12).

1.2 The microscopic origins of dissipation.

(*Not for examination.*)

But what is entropy, what is the origin of the Second Law, and how exactly should it be formulated? In 1865, Ludwig Boltzmann, the founder of statistical physics, made a link between thermodynamics and the microscopic nature of matter. For example, the gas in a room consists of a huge number $N\sim 10^{28}$ of molecules obeying Newton's laws (or more generally the laws of quantum mechanics) and which collide with each other. The state of the system at time t can be described by a point in the $6N$-dimensional phase-space of positions and velocities. This system depends very sensitively on its initial data, so that

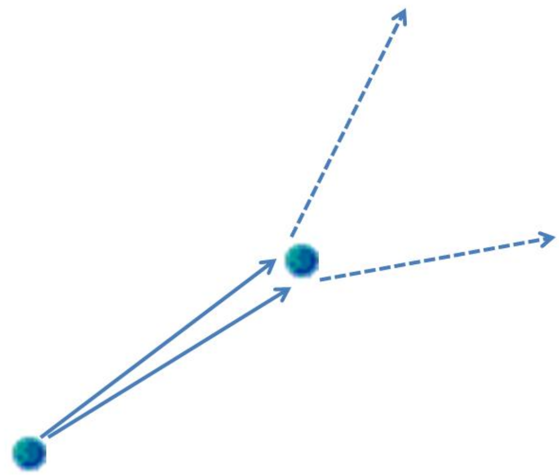


Figure 1.7: Small changes in initial data dramatically affect directions of motion after collisions

information about this initial data is rapidly lost, making any description other than a statistical one unrealistic.

A given 'macrostate' M of the system (specified by given values of macroscopic variables such as density and energy to within certain tolerances) corresponds to a phase-space volume $|\Gamma_M|$ of possible 'microstates' of the system giving these values. Boltzmann defined the entropy of M as

$$\eta(M) = K \log |\Gamma_M|, \tag{1.13}$$

where K = Boltzmann's constant, which can be interpreted as measuring the loss of information with respect to some initial configuration, or disorder, of the state M. He also derived, using a statistical hypothesis on the initial data, the *Boltzmann equation* of the kinetic theory of gases, and showed that it had a Lyapunov function, the Boltzmann H-function, related to entropy. (The same equation was proposed earlier by James Clerk Maxwell.) An objection was raised that this was inconsistent with the time-reversal invariance of Newton's laws (the 'reversibility-irreversibility' paradox). Thus, playing a film of Examples 1.1-1.3 backwards would produce unphysical behaviour, a cold cup of coffee beginning to heat up and start rotating, a ball at rest on a table starting spontaneously to bounce higher and higher, and a pendulum at rest starting to rock back and forth. Yet according to Newton's laws, if we instantaneously reversed the velocities of all the particles and let the system evolve in time, it would theoretically reproduce exactly these strange behaviours. However, in practice the incredibly sensitive dependence on initial data would mean that tiny disturbances from the external world would very soon reestablish thermodynamic behaviour. Another similar paradox is the Poincaré Recurrence Theorem, which states that any solution to the system of N particles obeying Newton's laws, will, after a sufficiently long time get arbitrarily close to its initial conditions: Boltzmann's reply to this was that one would have to wait a fantastically long time (perhaps longer than the age of the universe) for this to happen. However, there remains a large gap between the understanding of the Second Law, such as it is, for rarified gases, and the much more general use made of it in applied science and cosmology.

However, in this course we will not go further into these very interesting matters, but confine ourselves to the question of studying the approach to equilibrium of systems endowed with a Lyapunov function.

1.3 Problems

1.1. Consider a bouncing ball with air resistance, so that the height $z(t)$ satisfies

$$\begin{aligned} \ddot{z}(t) + k\dot{z}(t) &= -g \quad \text{if} \quad z(t) > 0 \\ \dot{z}(t+) &= -E\dot{z}(t-) \quad \text{if} \quad z(t) = 0 \text{ with } \dot{z}(t-) < 0, \end{aligned}$$

where $k > 0$ and $0 < E \leqslant 1$ are constants.

The ball is released from rest at height h. Show that
(i) if $E < 1$ then the ball comes to rest in finite time.
(ii) if $E = 1$ then the ball comes to rest in infinite time.

(*Hints.* (a) Make a linear change of variables in z, t so that the equation becomes:

$$\begin{aligned} \ddot{u}(t) + \dot{u}(t) + 1 = 0 \quad &\text{if} \quad u(t) > 0 \\ \dot{u}(t+) = -E\dot{u}(t-) \quad &\text{if} \quad u(t) = 0 \text{ with } \dot{u}(t-) < 0. \end{aligned} \tag{1.14}$$

(b) Estimate the time $\tau(v) > 0$ such that the solution of (1.14) with $u(0) = 0, \dot{u}(0) = v > 0$ satisfies $u(\tau) = 0$, by showing that $g(\tau) = v$, where

$$g(t) = \frac{t}{1 - e^{-t}} - 1,$$

and that

$$g(t) > t/2 \qquad \text{for } t > 0, \tag{1.15}$$

$$g(t) = \frac{t}{2}\left(1 + \frac{t}{6}\right) + o(t^2) \qquad \text{as } t \to 0+. \tag{1.16}$$

(c) Show that $\dot{u}(\tau+) = E(\tau - v)$, so that the times τ_n between successive bounces satisfy

$$v_{n+1} = E(\tau_n - v_n),$$

where $g(\tau_n) = v_n$.
(d) Show that v_n is decreasing in n, and hence using (1.15) that $\tau_n \to 0$ and $v_n \to 0$ as $n \to \infty$.
(e) Use (1.16) to show that

$$\frac{\tau_{n+1}}{\tau_n} \to E \text{ as } n \to \infty,$$

and hence deduce (i).
(f) If $E = 1$, use (1.16) to show that

$$\frac{1}{\tau_{n+1}} - \frac{1}{\tau_n} \to \frac{1}{3} \text{ as } n \to \infty$$

and deduce (ii).)

1.2. (i) Give an example of a nonnegative continuous function $f : [0, \infty) \to \mathbb{R}$ such that

$$\int_0^\infty f(t)\, dt < \infty \tag{1.17}$$

but $f(t_j) \to \infty$ for some sequence $t_j \to \infty$.

(ii) Show that if $f \geqslant 0$ is uniformly continuous on $[0, \infty)$, and in particular if f is C^1 with $|f'(t)| \leqslant C < \infty$ for all $t \geqslant 0$, and satisfies (1.17), then $f(t) \to 0$ as $t \to \infty$.

(iii) Deduce from (ii) that any solution of the damped pendulum equation

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{l} \sin\theta = 0,$$

where $k > 0$, satisfies $\dot{\theta}(t) \to 0$ as $t \to \infty$.

1.3. Let $\theta = \theta(x, t)$ be a solution to the heat equation

$$U(\theta)_t = -\operatorname{div} q(\theta, \nabla\theta), \quad x \in \Omega, \tag{1.18}$$

with boundary conditions

$$q(\theta(x, t), \nabla\theta(x, t)) \cdot n(x) = 0, \quad x \in \partial\Omega_1; \quad \theta(x, t) = \theta_0(x) > 0, \quad x \in \partial\Omega_2,$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ with $\partial\Omega_1, \partial\Omega_2$ disjoint, and where $n(x)$ is the unit outward normal to $\partial\Omega$. Let $\varphi = \varphi(x) > 0$ be a solution to the steady-state heat equation

$$\operatorname{div} q(\varphi, \nabla\varphi) = 0, \quad x \in \Omega,$$

with the same boundary conditions, that is

$$q(\varphi(x), \nabla\varphi(x)) \cdot n(x) = 0; \quad x \in \partial\Omega_1, \quad \varphi(x) = \theta_0(x), \quad x \in \partial\Omega_2.$$

Assume that all variables are sufficiently smooth, and let

$$V(\theta) = \int_\Omega [U(\theta) - \varphi(x)\eta(\theta)]\, dx,$$

where η is the entropy.

(i) Show that

$$\frac{dV(\theta)}{dt} = -I(\theta),$$

where

$$I(\theta) = \int_\Omega \nabla\left(\frac{\varphi}{\theta}\right) \cdot q(\theta, \nabla\theta)\, dx.$$

(ii) Suppose that $q(\theta, \nabla\theta) = -\kappa\nabla\theta$, where $\kappa > 0$ is a constant, and that the specific heat $U'(\theta) > 0$ for all θ. Show that

$$I(\theta) = \kappa \int_\Omega \varphi |\nabla \ln(\theta/\varphi)|^2 dx,$$

and thus that V is a Lyapunov function.

Chapter 2

Dynamics of ordinary differential equations

2.1 A simpler problem than the damped pendulum

Example 2.1. We consider a problem similar to the damped pendulum, but a little simpler (because, for example, it has only a finite number of rest points), namely the ordinary differential equation

$$\ddot{u} + \dot{u} + u^3 - u = 0. \tag{2.1}$$

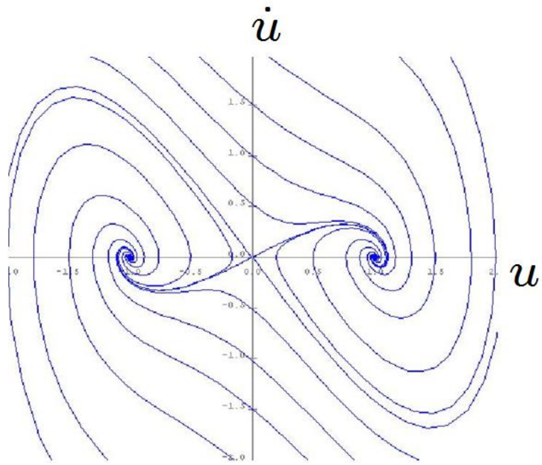


Figure 2.1: Phase-plane diagram for (2.1)

There are three rest points, namely $u = 0, \pm 1, \dot{u} = 0$, and the phase-plane

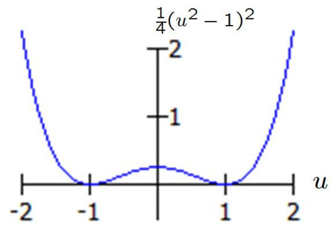


Figure 2.2: Double-well potential

diagram shows that each solution converges to one of these as $t \to \infty$, something we will prove. The key to the proof will be the Lyapunov function

$$V(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \frac{1}{4}(u^2-1)^2, \tag{2.2}$$

which satisfies

$$\frac{d}{dt}V(u,\dot{u}) = -\dot{u}^2. \tag{2.3}$$

We write (2.1) as a first order system

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ -\dot{u}+u-u^3 \end{pmatrix}, \tag{2.4}$$

that is

$$\dot{x} = f(x), \tag{2.5}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad f(x) = \begin{pmatrix} x_2 \\ -x_2+x_1-x_1^3 \end{pmatrix}. \tag{2.6}$$

Note that $u = \pm 1$ minimize the potential energy part of V, which is a double-well potential (see Fig. 2.2), so that the rest points $z_\pm = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$ are global minimizers of V. The linearization of (2.5) about a rest point z is

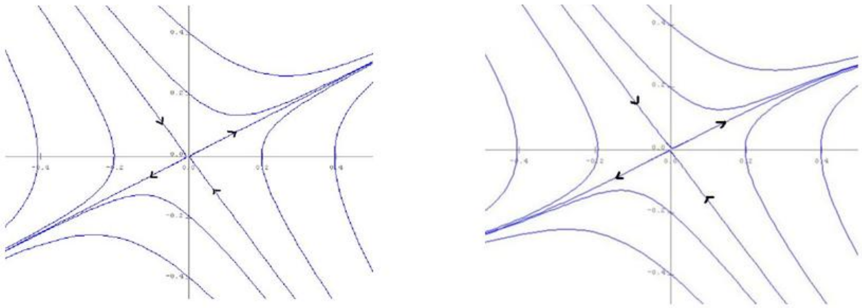
$$\dot{y} = f'(z)y. \tag{2.7}$$

A short calculation shows that

$$f'(z_\pm) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \tag{2.8}$$

which has eigenvalues $\frac{-1\pm i\sqrt{7}}{2}$, so that $z_\pm$ are spiral sinks, and that

$$f'(0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.9}$$



(a) (b)

Figure 2.3: Phase portrait near zero (a) linearized (b) nonlinear.

which has eigenvalues $\frac{-1\pm\sqrt{5}}{2}$ and corresponding eigenvectors $\begin{pmatrix} \frac{-1\pm\sqrt{5}}{2} \\ 1 \end{pmatrix}$, so that 0 is a saddle point.
(According to the theory of integral manifolds, the nonlinear equation (2.5) behaves like the linear one (2.7) in a sufficiently small neighbourhood of the critical point z. Thus, for example, near zero the linearized equation has the phase portrait in Fig. 2.3(a), while the nonlinear equation has the phase portrait in Fig. 2.3(b), with one-dimensional stable and unstable manifolds tangent at 0 to the linearised ones.)

We first need to prove that solutions exist for arbitrary initial data and all positive t.

2.2 Interlude on metric spaces

We recall that a *metric space* (X, d) is a set X equipped with a metric d, that is a function $d : X \times X \to [0, \infty)$ satisfying for $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

Given $x \in X$ and $r > 0$, the open ball of radius r is given by

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

A subset $U \subset X$ is *open* if for each $x \in U$ there exists an $r > 0$ with $B(x, r) \subset U$.
A sequence $x_j \to x$ in X as $j \to \infty$ if $d(x_j, x) \to 0$.
A subset $E \subset X$ is *closed* if $E^c = X \backslash E$ is open; equivalently, if $x_j \in E$ for all j and $x_j \to x$ then $x \in E$.

A subset $E \subset X$ is *compact* if it is closed and any sequence $\{x_j\} \subset E$ has a convergent subsequence.
A subset $E \subset X$ is *relatively compact*[1] if the closure $\bar{E} = \cap\{F \text{ closed}, E \subset F\}$ is compact.
A subset $E \subset X$ is *connected* if E cannot be written as $E = U \cup V$, where $U \subset X, V \subset X$ are nonempty with $U \cap \bar{V} = V \cap \bar{U} = \emptyset$. Thus a closed set $E \subset X$ is connected if E is not the union of two nonempty disjoint closed sets.
The sequence $\{x_j\} \subset X$ is a *Cauchy sequence* if $d(x_j, x_k) \to 0$ as $j, k \to \infty$.
(X, d) is *complete* if any Cauchy sequence $\{x_j\}$ is convergent, that is $x_j \to x$ for some $x \in X$.
A function $f : X \to X$ is *continuous* if given $x \in X, \varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$. Equivalently $x_r \to x$ implies $f(x_r) \to f(x)$.
A function $f : X \to X$ is a *contraction* if there exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \leqslant k\, d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 2.1 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point; that is there exists a unique $x^* \in X$ with $f(x^*) = x^*$.*

Proof. Pick any $x_1 \in X$ and define x_r iteratively by

$$x_{r+1} = f(x_r), \qquad r = 1, 2 \dots. \tag{2.10}$$

We first claim that

$$d(x_{r+1}, x_r) \leqslant k^{r-1} d(x_2, x_1) \quad \text{for all } r. \tag{2.11}$$

This is true for $r = 1$ and the claim follows in general by induction, noting that

$$d(x_{r+2}, x_{r+1}) = d(f(x_{r+1}), f(x_r)) \leqslant k\, d(x_{r+1}, x_r). \tag{2.12}$$

If $m > n$ we deduce from (2.11) that

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leqslant (k^{m-2} + \cdots + k^{n-1}) d(x_2, x_1) \\ &= k^{n-1}(1 + \cdots + k^{m-n-1}) d(x_2, x_1) \\ &\leqslant \frac{k^{n-1}}{1-k} d(x_2, x_1). \end{aligned}$$

Hence $\{x_r\}$ is a Cauchy sequence, and since X is complete $x_r \to x^*$ for some $x^* \in X$. Passing to the limit in (2.10) we get $x^* = f(x^*)$, so that x^* is a fixed point.

[1] Sometimes the term *precompact* is used, but more often the term precompact is reserved for sets that are *totally bounded*, i.e for any $\varepsilon > 0$ E can be covered by a finite number of open balls of radius ε, which is equivalent to relative compactness in a complete metric space.

If also $y^* = f(y^*)$ then

$$\begin{aligned} d(x^*, y^*) &= d(f(x^*), f(y^*)) \\ &\leqslant k\, d(x^*, y^*), \end{aligned}$$

and so $d(x^*, y^*) = 0$ and $x^* = y^*$. □

The following parametric version of Theorem 2.1 will be used to prove the continuous dependence of solutions with respect to parameters. The parameters are assumed to belong to a metric space Λ, for example $\Lambda = K \subset \mathbb{R}^m$.

Corollary 2.2. *Let Λ be a metric space of parameters, and suppose that $f : X \times \Lambda \to X$ is such that*
(i) f is a uniform contraction, i.e. there exists $k \in (0, 1)$ such that

$$d(f(x, \lambda), f(y, \lambda)) \leqslant k\, d(x, y) \text{ for all } x, y \in X, \lambda \in \Lambda. \tag{2.13}$$

(ii) for each $x \in X$, $f(x, \lambda)$ is continuous in λ.
Then for each $\lambda \in \Lambda$ there exists a unique fixed point $x^*(\lambda)$ of $f(\cdot, \lambda)$, i.e. $f(x^*(\lambda), \lambda) = x^*(\lambda)$, and $x^*(\lambda)$ is continuous in λ.

Proof. We just have to show that $\lambda_j \to \lambda$ in Λ implies $x^*(\lambda_j) \to x^*(\lambda)$ in X. But

$$\begin{aligned} d(x^*(\lambda_j), x^*(\lambda)) &= d(f(x^*(\lambda_j), \lambda_j), f(x^*(\lambda), \lambda)) \\ &\leqslant d(f(x^*(\lambda_j), \lambda_j), f(x^*(\lambda), \lambda_j)) + d(f(x^*(\lambda), \lambda_j), f(x^*(\lambda), \lambda)) \\ &\leqslant k\, d(x^*(\lambda_j), x^*(\lambda)) + o(1), \end{aligned}$$

and since $k < 1$, $d(x^*(\lambda_j), x^*(\lambda)) \to 0$. □

2.3 Existence of solutions

Consider the ODE in $\mathbb{R}^n$

$$\dot{x} = f(x), \tag{2.14}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally Lipschitz*, i.e. given any $M > 0$ there exists a constant $K_M > 0$ such that

$$|f(x) - f(y)| \leqslant K_M |x - y| \quad \text{if } |x|, |y| \leqslant M. \tag{2.15}$$

Note that any $f \in C^1(\mathbb{R}^n)$ is locally Lipschitz, since

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_0^1 \frac{d}{dt} f(ty + (1-t)x)\, dt \right| \\ &= \left| \int_0^1 f'(ty + (1-t)x) \cdot (y - x)\, dt \right| \\ &\leqslant K_M |x - y|, \end{aligned}$$

where $K_M = \max_{|w|\leqslant M} |f'(w)|$. Let $T > 0$. We denote by $C([0,T];\mathbb{R}^n)$ the space of continuous maps $x : [0,T] \to \mathbb{R}^n$. Given $x \in C([0,T];\mathbb{R}^n)$ define the norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|. \tag{2.16}$$

Then for $x, y, z \in C([0,T];\mathbb{R}^n)$ we have $\|x - z\| \leqslant \|x - y\| + \|y - z\|$, so that $C([0,T];\mathbb{R}^n)$ is a metric space with metric

$$d(x, y) = \|x - y\|. \tag{2.17}$$

Lemma 2.3. *$C([0,T];\mathbb{R}^n)$ is complete.*

Proof. Let x_j be a Cauchy sequence in $C([0,T];\mathbb{R}^n)$, and let $\varepsilon > 0$. Then there exists N so that for each $t \in [0,T]$

$$|x_j(t) - x_k(t)| < \varepsilon \text{ for } j, k \geqslant N. \tag{2.18}$$

Hence, for each $t \in [0,T]$, $x_j(t)$ is a Cauchy sequence in $\mathbb{R}^n$ and, since $\mathbb{R}^n$ is complete, $x_j(t)$ tends to a limit $x(t)$ in $\mathbb{R}^n$. Passing to the limit $k \to \infty$ in (2.18) we have that

$$|x_j(t) - x(t)| \leqslant \varepsilon \text{ for } j \geqslant N. \tag{2.19}$$

Since x_N is continuous there exists $\delta > 0$ such that

$$|x_N(s) - x_N(t)| < \varepsilon \text{ for } |s - t| < \delta, \tag{2.20}$$

and so by (2.19), (2.20) if $|s - t| < \delta$

$$|x(s) - x(t)| \leqslant |x(s) - x_N(s)| + |x_N(s) - x_N(t)| + |x_N(t) - x(t)| < 3\varepsilon. \tag{2.21}$$

Hence x is continuous, and by (2.19) $x_j \to x$ in $C([0,T];\mathbb{R}^n)$. □

Definition 2.1. A *solution* of (2.14) on the time interval $[0,T]$, with initial data $x_0 \in \mathbb{R}^n$, is a map $x \in C([0,T];\mathbb{R}^n)$ satisfying

$$x(t) = x_0 + \int_0^t f(x(s))\, ds \text{ for all } t \in [0,T]. \tag{2.22}$$

If x is a solution then clearly $x(0) = x_0$, $\dot{x}(t)$ exists for all $t \in [0,T]$ with $\dot{x} \in C([0,T];\mathbb{R}^n)$, and $\dot{x}(t) = f(x(t))$ for all $t \in [0,T]$. If $\tau \in (0,\infty]$ then by a solution of (2.14) on the half-open interval $[0,\tau)$ we mean a map $x : [0,\tau) \to \mathbb{R}^n$ which is a solution on each interval $[0,T]$ with $0 < T < \tau$.

Theorem 2.4. *Given $x_0 \in \mathbb{R}^n$ there exists a unique solution x of (2.14) with initial data x_0 defined on a maximal interval $[0, t_{\max})$ with $0 < t_{\max} \leqslant \infty$. If $t_{\max} < \infty$ then*

$$\lim_{t\to t_{\max}-} |x(t)| = \infty. \tag{2.23}$$

The solution $x = x(\cdot, x_0)$ depends continuously on x_0; more precisely, if the solution $x = x(\cdot, x_0)$ exists on the interval $[0,T]$ and if $x_{0j} \to x_0$ as $j \to \infty$ then for large enough j the solution $x_j = x(\cdot, x_{0j})$ exists on $[0,T]$ and $x_j \to x$ in $C([0,T];\mathbb{R}^n)$.

Proof. Step 1. We show that given $M > 0$ there exists $T_M > 0$ such that if $|x_0| \leqslant M$ then there is a unique solution $x = x(\cdot, x_0)$ on $[0, T_M]$ with initial data x_0, and if $x_{0j} \to x_0$, $|x_{0j}| \leqslant M$, and $x_j = x(\cdot, x_{0j})$ then $x_j \to x$ in $C([0,T_M];\mathbb{R}^n)$.

Let $T_M = \frac{M}{2MK_{2M}+|f(0)|}$. Let

$$X = \{x \in C([0,T_M];\mathbb{R}^n) : \|x\| \leqslant 2M\},$$

where $\|x\| = \max_{t\in[0,T_M]} |x(t)|$. Then X is a closed subset of $C([0,T_M];\mathbb{R}^n)$ and hence is complete with respect to the metric $d(x,y) = \max_{t\in[0,T_M]} |x(t) - y(t)|$.

For $|x_0| \leqslant M$ and $x \in X$ define

$$P(x, x_0)(t) = x_0 + \int_0^t f(x(s))\, ds. \tag{2.24}$$

We claim that $P(\cdot, x_0) : X \to X$ and is a uniform contraction for $|x_0| \leqslant M$. Indeed if $x \in X$ then $P(x,x_0)(t)$ is continuous in t and

$$\begin{aligned} |P(x,x_0)(t)| &\leqslant |x_0| + \int_0^t (|f(x(s)) - f(0)| + |f(0)|)\, ds \\ &\leqslant |x_0| + T_M(K_{2M}2M + |f(0)|) \\ &\leqslant M + M = 2M, \end{aligned}$$

and if $x, y \in X$

$$\begin{aligned} |P(x,x_0)(t) - P(y,x_0)(t)| &\leqslant \int_0^{T_M} |f(x(s)) - f(y(s))|\, ds \\ &\leqslant K_{2M}T_M d(x,y) \end{aligned}$$

and $K_{2M}T_M \leqslant \frac{1}{2}$. Since $d(P(x,x_0), P(x,y_0)) = |x_0 - y_0|$ it follows that $P(x,x_0)$ is continuous in x_0. Thus by Corollary 2.2 there exists a unique fixed point $x(\cdot, x_0)$, that is

$$x(t,x_0) = x_0 + \int_0^t f(x(s,x_0))\, ds \text{ for all } t \in [0,T_M],$$

and $x_0 \mapsto x(\cdot, x_0)$ is continuous for $|x_0| \leqslant M$.

Step 2. (Definition of $t_{\max}$ and uniqueness.) Given x_0, let $t_{\max} = \sup\{T > 0 :$ there is a solution x on $[0,T]$ with $x(0) = x_0\}$. Then, by Step 1, $t_{\max} > 0$. Suppose for contradiction that $y \neq x$ is another solution with $y(0) = x_0$, and set

$$\tau = \inf\{s > 0 : x, y \text{ defined on } [0,s] \text{ and } x(s) \neq y(s)\}.$$

Then, by Step 1, $\tau > 0$, and clearly $\tau < t_{\max}$. Since $x(t) = y(t)$ for all $t \in [0, \tau]$ we can apply Step 1 with initial data $x(\tau)$ to get that y is defined on $[\tau, \tau + \varepsilon]$ for some $\varepsilon > 0$ and equals x on that interval. This contradiction proves that x is unique.

Step 3. (Continuous dependence.) Let x be a solution on $[0, T]$ and choose $M > \max_{t\in[0,T]} |x(t)|$. Let $x_{0j} \to x_0$. We can suppose that $|x_{0j}| \leqslant M$ for all j. By Step 1, x_j exists on $[0, T_M]$ and $x_j \to x$ in $C([0, T_M]; \mathbb{R}^n)$. If $T \leqslant T_M$ we are done. If $T > T_M$ then $x_j(T_M) \to x(T_M)$ and repeating the argument with initial data $x_j(T_M)$ we have that $x_j \to x$ in $C([T_M, 2T_M]; \mathbb{R}^n)$, and hence in $C([0, 2T_M]; \mathbb{R}^n)$. After N such steps, where $(N-1)T_M < T \leqslant NT_M$, we obtain that $x_j \to x$ in $C([0, T]; \mathbb{R}^n)$ as required.

Step 4. (Blow-up if $t_{\max} < \infty$.) Suppose for contradiction that $t_{\max} < \infty$ but that (2.23) does not hold. Then there exists a sequence $t_j \to t_{\max}-$ with $|x(t_j)| \leqslant M$ for some $M < \infty$. Hence by Step 1 with initial data $x(t_j)$ the solution exists on the interval $[0, t_j + T_M]$, and $t_{\max} < t_j + T_M$ for large enough j, contradicting the definition of $t_{\max}$. □

Let $V \in C^1(\mathbb{R}^n)$. If $x(t)$ is a solution of (2.14) then on any interval of existence

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot f(x(t)).$$

So $\dot{V} \leqslant 0$ if and only if

$$\nabla V(x) \cdot f(x) \leqslant 0 \text{ for all } x \in \mathbb{R}^n. \tag{2.25}$$

Corollary 2.5. *Let $V \in C^1(\mathbb{R}^n)$ satisfy* (2.25) *and $V(x) \to \infty$ as $|x| \to \infty$. Then for any $x_0 \in \mathbb{R}^n$ the solution $x(t) = x(t, x_0)$ exists for all $t \geqslant 0$.*

Proof. For $t \in [0, t_{\max})$ we have $V(x(t)) \leqslant V(x_0)$. Hence $\sup_{t\in[0,t_{\max})} |x(t))| < \infty$, since otherwise there would exist $t_j \in [0, t_{\max})$ with $|x(t_j)| \to \infty$, implying that $V(x(t_j)) \to \infty$. So by (2.23) $t_{\max} = \infty$. □

For our equation (2.1), $\ddot{u} + \dot{u} + u^3 - u = 0$, we have that

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{4}(x_1^2 - 1)^2, \quad \nabla V(x) \cdot f(x) = -x_2^2,$$

and so solutions exist for all $t \geqslant 0$.

2.4 Problems

2.1. Let (X, d) be a metric space and $f : X \to X$. Give examples showing that f need not have a fixed point if

(i) f is a contraction but (X, d) is not complete,

(ii) (X, d) is complete and

$$d(f(x), f(y)) < d(x, y) \text{ whenever } x, y \in X \text{ with } x \neq y.$$

2.2. Let

$$\dot{x} = f(x), \tag{2.26}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, be an ODE in $\mathbb{R}^n$.

(i) Show that if $x_0 \in \mathbb{R}^n$ is not a rest point, then the solution $x(t)$ of (2.26) with $x(0) = x_0$ cannot tend to a rest point in finite time, i.e. there is no $T \in (0, \infty)$ such that $\lim_{t \to T-} x(t) = z$ for some z with $f(z) = 0$.

(ii) Give an example of an f such that for any $x_0 \in \mathbb{R}^n$ the solution of (2.26) with $x(0) = x_0$ satisfies $|x(t)| \leqslant 1$ for all $t \geqslant 1$.

2.3. (i) Prove that solutions to the equation

$$\ddot{u} + \dot{u} + u^3 - u = 0,$$

with arbitrary initial data $u(0) = u_0, \dot{u}(0) = u_1$, exist for all *negative* time.

(ii) Prove that solutions to the damped pendulum equation

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{l} \sin\theta = 0, \quad k = \frac{c}{m} > 0,$$

with arbitrary initial data $\theta(0) = \theta_0, \dot{\theta}(0) = \theta_1$, exist for all positive and negative time.

(*Hint.* In both cases get from the energy equation an estimate for the energy V of the form $\dot{V} \leqslant CV + D$ for constants C, D leading to a bound on the norm of the solution by an explicit function of t.)

Chapter 3

Semiflows and stability

3.1 The semiflow generated by an ODE

We consider again the ordinary differential equation

$$\dot{x} = f(x), \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally Lipschitz*. Suppose that for any $x_0 \in \mathbb{R}^n$ the solution $x(t, x_0)$ of (3.1) with initial data $x(0) = x_0$ exists for all $t \geqslant 0$. Conditions under which this is true were given in Corollary 2.5. Let us write for $t \geqslant 0$

$$T(t)x_0 = x(t, x_0). \tag{3.2}$$

Thus $T(t) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (by the continuous dependence of the

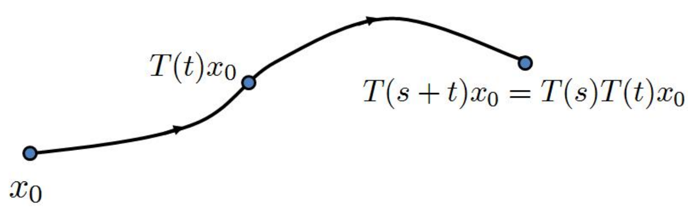


Figure 3.1: Uniqueness implies (ii).

solution on the initial data), and

(i) $T(0) =$ identity,
(ii) $T(s + t) = T(s)T(t)$ for all $s \geqslant 0, t \geqslant 0$,
(iii) for each $x_0 \in \mathbb{R}^n$ the map $t \mapsto T(t)x_0$ is continuous from $[0, \infty) \to \mathbb{R}^n$,

where (ii) holds because $y : [0, \infty) \to \mathbb{R}^n$ defined by

$$y(\tau) = \begin{cases} T(\tau)x_0, & \tau \in [0, t] \\ T(\tau - t)T(t)x_0, & \tau > t \end{cases}$$

is a solution of (2.14) with initial data $y(0) = x_0$, and hence by uniqueness $y(\tau) = T(\tau)x_0$ for all $\tau \geqslant 0$, so that $y(s+t) = T(s+t)x_0 = T(s)T(t)x_0$ (see Fig. 3.1).

3.2 Semiflows on a metric space.

Definition 3.1. A *semiflow* $\{T(t)\}_{t\geqslant 0}$ on a metric space (X,d) is a family of continuous maps $T(t) : X \to X$ satisfying

(i) $T(0) =$ identity,
(ii) $T(s+t) = T(s)T(t)$ for all $s \geqslant 0, t \geqslant 0$,
(iii) for each $p \in X$ the map $t \mapsto T(t)p$ is continuous from $[0,\infty) \to X$.

(In the literature a semiflow is sometimes called a (*nonlinear*) *semigroup* or *dynamical system*.)

Let $\{T(t)\}_{t\geqslant 0}$ be a semiflow on the metric space (X,d). The *positive orbit* of $p \in X$ is the set (see Fig. 3.2)

$$\gamma^+(p) = \{T(t)p : t \geqslant 0\}.$$

The $\omega-$*limit set* of p is the set

$$\begin{aligned} \omega(p) &= \{\chi \in X : T(t_j)p \to \chi \text{ for some sequence } t_j \to \infty\} \\ &= \bigcap_{t\geqslant 0} \overline{\bigcup_{\tau\geqslant t} T(\tau)p}. \end{aligned} \tag{3.3}$$

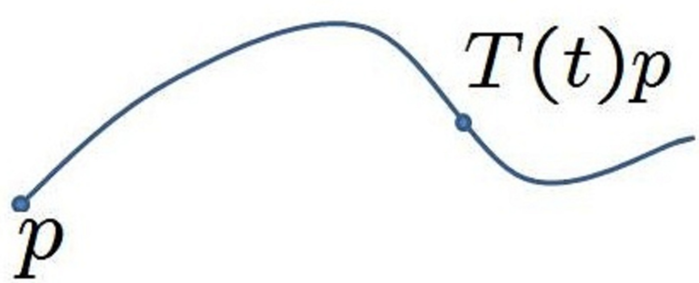


Figure 3.2: Positive orbit

A map $\psi : \mathbb{R} \to X$ is a *complete orbit* if

$$\psi(t+s) = T(t)\psi(s) \text{ for all } s \in \mathbb{R}, t \geqslant 0.$$

(Note that we do not assume backwards uniqueness, so there might be more than one complete orbit passing through a point $p \in X$ (see Fig. 3.3)).

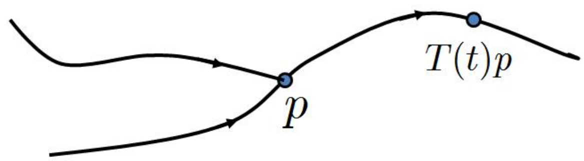


Figure 3.3: More than one complete orbit passing through a point.

If ψ is a complete orbit then the α-*limit set* of ψ is the set

$$\begin{aligned}\alpha(\psi) &= \{\chi \in X : \psi(t_j) \to \chi \text{ for some sequence } t_j \to -\infty\}\\ &= \bigcap_{t\leqslant 0} \overline{\bigcup_{\tau\leqslant t} \psi(\tau)}.\end{aligned}$$

If $E \subset X$, $t \geqslant 0$, we set

$$T(t)E = \{T(t)p : p \in E\}.$$

A subset $E \subset X$ is *positively invariant* if $T(t)E \subset E$ for all $t \geqslant 0$, and *invariant* if $T(t)E = E$ for all $t \geqslant 0$.

Note that if E invariant then there is a complete orbit contained in E passing through any point of E. Indeed if $p \in E$ then there exist $p_{-1} \in E$ with $T(1)p_{-1} = p$, $p_{-2} \in E$ with $T(1)p_{-2} = p_{-1}$, and so on, so that

$$\psi(t) = \begin{cases} T(t)p, & t \geqslant 0 \\ T(t+i)p_{-i}, & t \in [-i, -i+1), i = 1, 2, \ldots \end{cases}$$

defines a complete orbit passing through p.

Theorem 3.1. *(i) Let $\gamma^+(p)$ be relatively compact. Then $\omega(p)$ is nonempty, compact, invariant and connected. As $t \to \infty$,*

$$\operatorname{dist}(T(t)p, \omega(p)) \to 0,$$

where $\operatorname{dist}(q, E) := \inf_{\chi\in E} d(q, \chi)$.

(ii) Let ψ be a complete orbit with $\{\psi(t) : t \leqslant 0\}$ relatively compact. Then $\alpha(\psi)$ is nonempty, compact, invariant and connected, and as $t \to -\infty$

$$\operatorname{dist}(\psi(t), \alpha(\psi)) \to 0.$$

Proof. We prove (i). The proof of (ii) is similar and is left to Problem 3.2. That $\omega(p)$ is nonempty is clear. Since $\omega(p)$ is by (3.3) the intersection of compact sets, it is compact. To prove the invariance, let $\chi \in \omega(p)$. Then $T(t_j)p \to \chi$ for some sequence $t_j \to \infty$. If $t \geqslant 0$ then, since $T(t)$ is continuous,

$$T(t + t_j)p = T(t)T(t_j)p \to T(t)\chi$$

and so $T(t)\omega(p) \subset \omega(p)$. Also $\{T(t_j - t)p\}$ is relatively compact, and so

$$T(t_{j_k} - t)p \to q \in X$$

for some subsequence $\{t_{j_k}\}$. Therefore

$$T(t_{j_k})p = T(t)T(t_{j_k} - t)p \to T(t)q = \chi.$$

Hence $T(t)\omega(p) \supset \omega(p)$ and so $\omega(p)$ is invariant.

If $\operatorname{dist}(T(t)p, \omega(p)) \not\to 0$ as $t \to \infty$, then there exist $\varepsilon > 0$ and a sequence $t_j \to \infty$ such that $d(T(t_j)p, z) \geq \varepsilon$ for all $z \in \omega(p)$. But a subsequence $T(t_{j_k})p \to \chi \in \omega(p)$, a contradiction.

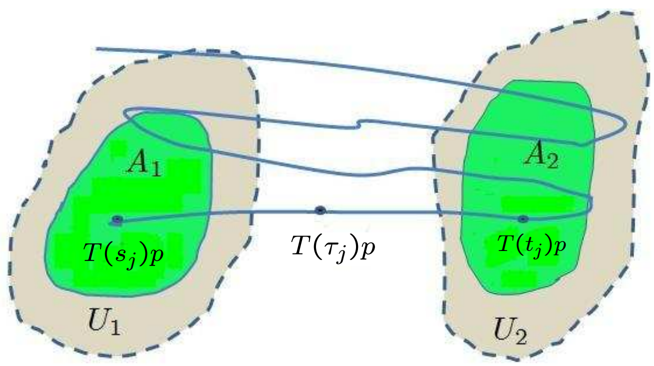


Figure 3.4: Proof of connectedness of $\omega(p)$

Suppose $\omega(p)$ is not connected. Then $\omega(p) = A_1 \cup A_2$ with A_1, A_2 nonempty disjoint compact sets. (Indeed, by the definition of connectedness we can write $\omega(p) = V_1 \cup V_2$ with $\bar{V_1} \cap V_2 = V_1 \cap \bar{V_2} = \emptyset$, and since $\omega(p)$ is closed we have $\omega(p) = \bar{V_1} \cup \bar{V_2}$. Thus $\bar{V_1} \cap \bar{V_2} = \emptyset$ and we can set $A_1 = \bar{V_1}, A_2 = \bar{V_2}$. Since A_1, A_2 are closed subsets of a compact set, they are themselves compact.) Let U_1, U_2 be disjoint open sets with $A_1 \subset U_1, A_2 \subset U_2$. We can take, for example, $U_i = \{q \in X : \operatorname{dist}(q, A_i) < \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. Then there exist sequences $s_j > t_j$ with $t_j \to \infty$ such that $T(s_j)p \in U_1, T(t_j)p \in U_2$ and hence, by Definition i(iii), there exists $\tau_j \in (t_j, s_j)$ with $T(\tau_j)p \notin U_1 \cup U_2$ (see Fig. 3.4). Hence by the relative compactness of $\gamma^+(p)$ there exists some $\chi \in \omega(p) \backslash (A_1 \cup A_2)$, a contradiction. □

3.3 Approach to equilibrium

A point $z \in X$ is a *rest point* if $T(t)z = z$ for all $t \geq 0$. The set Z of rest points is closed.

A function $V : X \to \mathbb{R}$ is a *Lyapunov function* if

(i) V is continuous,

(ii) $V(T(t)p) \leqslant V(p)$ for all $p \in X, t \geqslant 0$,

(iii) If $V(\psi(t)) = c$ for some complete orbit ψ, all $t \in \mathbb{R}$ and some constant c, then $\psi(t) = z$ for all $t \in \mathbb{R}$ for some rest point z.

(Note that (ii) implies that $V(T(t)p) \leqslant V(T(s)p)$ for all $t \geqslant s \geqslant 0$, since $V(T(t)p) = V(T(t-s)T(s)p) \leqslant V(T(s)p)$.)

Example 3.1. For the ordinary differential equation

$$\dot{x} = f(x), \tag{3.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, we have seen that if $V : \mathbb{R}^n \to \mathbb{R}$ is C^1 and satisfies

$$f(x) \cdot \nabla V(x) \leqslant 0, \text{ for all } x \in \mathbb{R}^n, \tag{3.5}$$

with $V(x) \to \infty$ as $|x| \to \infty$, then (3.4) generates a semiflow on $\mathbb{R}^n$ and (ii) holds. In order for V to be a Lyapunov function we need also (iii) to be satisfied, a sufficient (but not necessary) condition for which is that $f(x) \cdot \nabla V(x) = 0$ implies $f(x) = 0$.

Theorem 3.2 (LaSalle invariance principle)**.** *Let V be a Lyapunov function, and let $p \in X$ with $\gamma^+(p)$ relatively compact. Then $\omega(p)$ consists only of rest points. If the only nonempty connected subsets of Z are single points (for example, if there are only a finite number of rest points) then $\omega(p) = \{z\}$ for some rest point z, and $T(t)p \to z$ as $t \to \infty$.*

Proof. Since V is continuous and $\gamma^+(p)$ is relatively compact, $V(T(t)p)$ is bounded below for $t \geqslant 0$. But $t \mapsto V(T(t)p)$ is nonincreasing, and so

$$V(T(t)p) \to c \text{ as } t \to \infty$$

for some constant c. Let $z \in \omega(p)$. Then, since $\omega(p)$ is invariant, $z = \psi(0)$ for a complete orbit ψ contained in $\omega(p)$. Hence $V(\psi(t)) = c$ for all $t \in \mathbb{R}$, and so by (iii) z is a rest point.

If the only nonempty connected subsets of Z are single points then since $\omega(p)$ is connected, $\omega(p) = z$ for some rest point, so that $T(t)p \to z$ as $t \to \infty$ by Theorem 3.1. □

Corollary 3.3. *Every solution $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ of* (2.1) *converges to one of the three rest points* $0, z_\pm$ *as* $t \to \infty$.

Proof. We already showed that (2.1) generates a semiflow $\{T(t)\}_{t \geqslant 0}$ on $\mathbb{R}^2$. The function $V(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \frac{1}{4}(u^2 - 1)^2$ is a Lyapunov function; indeed properties (i) and (ii) are obvious, while if $V(u(t), \dot{u}(t)) = c$ for a complete orbit then $\dot{u}(t) = 0$ for all t and hence $\ddot{u}(t) = 0$, proving (iii). In particular, any positive orbit is bounded, hence relatively compact. □

3.4 Lyapunov stability

Let $\{T(t)\}_{t \geq 0}$ be a semiflow on a metric space (X, d).

Definitions 3.2. The rest point z is (*Lyapunov*) *stable* if given $\varepsilon > 0$, there exists $\delta > 0$ such that if $p \in B(z, \delta)$ then $T(t)p \in B(z, \varepsilon)$ for all $t \geqslant 0$. The rest point z is *unstable* if it is not stable. The rest point z is *asymptotically stable* if z is stable and there exists $\rho > 0$ such that $p \in B(z, \rho)$ implies $T(t)p \to z$ as $t \to \infty$.

If the rest point z is asymptotically stable then clearly z is *isolated*, that is there is some $\varepsilon > 0$ such that z is the only rest point in $B(z, \varepsilon)$. However one can have, for example, a line of stable rest points for an ODE in $\mathbb{R}^2$, such as $\dot{x}_1 = 0, \dot{x}_2 = -x_2$.

Example 3.2. (Vinograd) (*Not for examination.*) Consider the ODE in $\mathbb{R}^2$

$$\begin{aligned}\dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{r^2(1 + r^4)}\\ \dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{r^2(1 + r^4)},\end{aligned}$$

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ (see Fig. 3.5). Every solution $x(t) \to 0$ as $t \to \infty$, but 0 is unstable and thus not asymptotically stable.

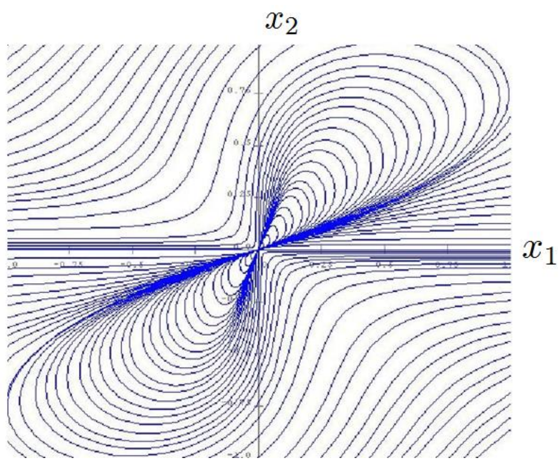


Figure 3.5: Vinograd example of attracting Lyapunov unstable rest point.

Theorem 3.4. *Let z be an isolated rest point, let V be a Lyapunov function, let $\gamma^+(p)$ be relatively compact for any p with $\gamma^+(p)$ bounded, and suppose that for all $\delta > 0$ sufficiently small*

$$\inf_{d(p,z)=\delta} V(p) > V(z) \qquad (\textit{Existence of a potential well}) \tag{3.6}$$

Then z is asymptotically stable.

Proof. Suppose z is not stable. Then there exist $\varepsilon > 0$, $p_j \to z$, $t_j \geqslant 0$ with $d(T(t_j)p_j, z) \geqslant \varepsilon$. We can suppose that ε is small enough such that

$$c_\varepsilon := \inf_{d(p,z)=\frac{\varepsilon}{2}} V(p) > V(z),$$

and such that z is the only rest point in $\overline{B(z,\varepsilon)}$. Let j be sufficiently large. Then since V is continuous, $V(p_j) < c_\varepsilon$. By the continuity of $t \mapsto T(t)p_j$ there exists $\tau_j \in (0, t_j)$ with $d(T(\tau_j)p_j, z) = \frac{\varepsilon}{2}$, and thus

$$c_\varepsilon \leqslant V(T(\tau_j)p_j) \leqslant V(p_j) < c_\varepsilon,$$

a contradiction.

By the stability there exists $\rho > 0$ such that if $d(p,z) < \rho$ then $T(t)p \in \overline{B(z,\varepsilon)}$ for all $t \geqslant 0$. Then $\gamma^+(p)$ is bounded, and so by the assumption of the theorem relatively compact. Thus, by Theorem 3.2, $\omega(p) \subset Z \cap \overline{B(z,\varepsilon)}$ and so $\omega(p) = \{z\}$ and $T(t)p \to z$ as $t \to \infty$. □

Remark 1. If $X = \mathbb{R}^n$ then the existence of a potential well (see (3.6)) is equivalent to the condition that z is a strict local minimizer of V, i.e. that there exists $\varepsilon > 0$ such that $V(p) > V(z)$ if $0 < d(p,z) \leqslant \varepsilon$. This follows easily from the fact that the sphere $S(z,\varepsilon)$ is compact, so that V attains a minimum on $S(z,\varepsilon) = \{p : d(p,z) = \varepsilon\}$. But if X is a metric space whose spheres $S(z,\varepsilon)$ are not compact (as is the case for infinite-dimensional normed vector spaces) then the existence of a potential well is a stronger condition than being a strict local minimizer. If we just assumed that z was a strict local minimizer then the danger would be that orbits could leak out of balls by going into higher and higher dimensions.

Theorem 3.5. *Let V be a Lyapunov function and suppose that $\gamma^+(p)$ is relatively compact for any p with $\gamma^+(p)$ bounded. Let z be an isolated rest point which is not a local minimizer of V (i.e. for any $\varepsilon > 0$ there is a point p with $d(p,z) < \varepsilon$ and $V(p) < V(z)$). Then z is unstable.*

Proof. Let $\varepsilon > 0$ be sufficiently small so that z is the only rest point in $\overline{B(z,\varepsilon)}$. Suppose for contradiction that z is stable. Then there exists $\delta > 0$ such that $d(p,z) < \delta$ implies $d(T(t)p,z) < \varepsilon$ for all $t \geqslant 0$. But since z is not a local minimizer there exists p with $d(p,z) < \delta$ and $V(p) < V(z)$. Since $\gamma^+(p) \subset \overline{B(z,\varepsilon)}$, $\gamma^+(p)$ is by assumption relatively compact. Hence by the invariance principle there exist a sequence $t_j \to \infty$ and a rest point $\tilde{z} = \lim_{j\to\infty} T(t_j)p$ in $\omega(p)$ with $\tilde{z} \in \overline{B(z,\varepsilon)}$. But $V(\tilde{z}) = \lim_{j\to\infty} V(T(t_j)p) < V(z)$. Hence $\tilde{z} \neq z$, a contradiction. □

Thus, for example, 0 is an unstable rest point for (2.1), since it is not a local minimizer of $V(x) = \frac{1}{2}x_2^2 + \frac{1}{4}(x_1^2 - 1)^2$.

The *region of attraction* of a rest point z is the set

$$A(z) = \{p \in X : T(t)p \to z \text{ as } t \to \infty\}.$$

The regions of attraction of the rest points $z_\pm, 0$ for (2.1) are shown in Fig. 3.6.

Theorem 3.6. *$A(z)$ is connected.*

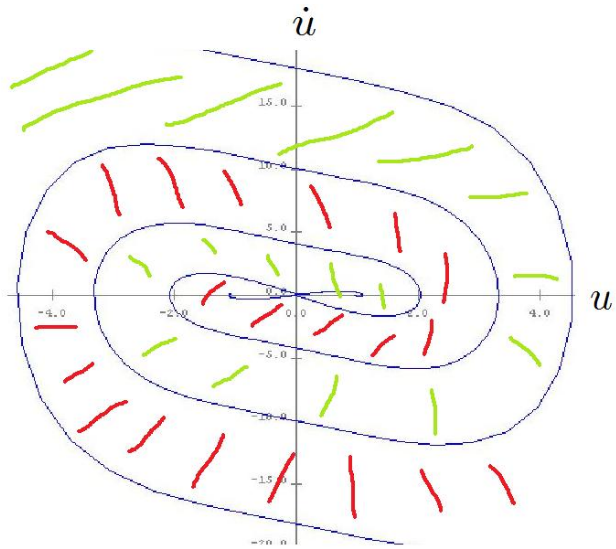


Figure 3.6: Regions of attraction of $z_{\pm}, 0$ for (2.1) (u axis rescaled)

Proof. Suppose not, so that $A(z) = U \cup V$ with U, V nonempty and $U \cap \bar{V} = \bar{U} \cap V = \emptyset$. Let $p \in U, q \in V$. For any $t \geqslant 0$, $T(t)p \in A(z)$. Let $S = \{t \geqslant 0 : T(t)p \in U\}$. Let $t_j \in S$, $t_j \to t$. Then $T(t)p = \lim_{j\to\infty} T(t_j)p \in \bar{U}$ and so $T(t)p \in U$. Hence S is closed in $[0, \infty)$. Similarly S is open, and thus $\gamma^+(p) \subset U$. Similarly $\gamma^+(q) \subset V$. But $z \in \bar{U}$, hence $z \notin V$. Similarly $z \notin U$. But $z \in A(z) = U \cup V$, a contradiction. $\square$

Theorem 3.7. *If z is an asymptotically stable rest point then $A(z)$ is open.*

Proof. Let $\rho > 0$ be as in Definition 3.2, and let $p \in A(z)$. Then there exists $s > 0$ such that $d(T(s)p, z) < \rho$. Hence by the continuity of $T(s)$ there exists $\sigma > 0$ such that $d(p, q) < \sigma$ implies $d(T(s)q, z) \leqslant d(T(s)q, T(s)p) + d(T(s)p, z) < \rho$, so that by asymptotic stability $T(t)q \to z$ as $t \to \infty$ and hence $q \in A(z)$. $\square$

It follows from Theorem 3.7 that if every semi-trajectory tends to a rest point as $t \to \infty$ and each rest point is isolated (so that z stable implies z asymptotically stable) then

$$\bigcup\{A(z) : z \text{ unstable}\} = \left(\bigcup\{A(z) : z \text{ stable}\}\right)^{\mathrm{c}}.$$

is closed.

We now consider again the damped pendulum equation:

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{l}\sin\theta = 0, \quad k > 0, \tag{3.7}$$

which has the Lyapunov function

$$V(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta. \tag{3.8}$$

Since $V(\theta,\dot{\theta}) \not\to \infty$ as $\sqrt{\dot{\theta}^2+\theta^2} \to \infty$, it is not immediately obvious that $\theta(t)$ is uniformly bounded for $t \geqslant 0$, which is needed to show relative compactness of positive orbits (though we showed in Problem 2.3 (i) that $|\theta(t)|$ is bounded on $[0,T]$ for any $T>0$, which was enough to prove that (3.8) generates a semiflow on $\mathbb{R}^2$).

Theorem 3.8. *Every solution of* (3.7) *tends to a rest point as* $t \to \infty$.

Proof. Each of the rest points $z_n = \begin{pmatrix} 2n\pi \\ 0 \end{pmatrix}$ is a strict local minimizer of V (hence lies is a potential well, either by Remark 1 or directly), and since they are all equivalent (because if θ is a solution so is $\theta + 2\pi$) there exists $\rho > 0$ independent of n such that if $p \in B(z_n,\rho)$ then $T(t)p \to z_n$ as $t \to \infty$. We proved in Problem 1.2 (iii) that $\dot{\theta}(t) \to 0$ as $t \to \infty$. Therefore for any solution there is a time $\tau > 0$ such that $|\dot{\theta}(t)| < \rho$ for all $t \geqslant \tau$. Let N be such that $2N\pi \leqslant \theta(\tau) \leqslant 2(N+1)\pi$. Then (see Fig. 3.7) either $\theta(t) \in (2N\pi, 2(N+1)\pi)$

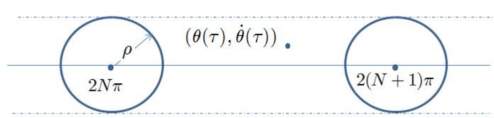


Figure 3.7: Proof of boundedness of $\theta(t)$.

for all $t \geqslant \tau$ or $\theta(\tau_1) = 2N\pi$ or $2(N+1)\pi$ for some $\tau_1 \geqslant \tau$, in which case $\theta(t) \to 2N\pi$ or $2(N+1)\pi$ as $t \to \infty$. Hence in all cases $\theta(t)$ is bounded for $t \geqslant 0$, and by Theorem 3.2 every solution tends to a rest point as $t \to \infty$. $\square$

3.5 Problems

3.1. Consider the ODE in $\mathbb{R}^2$ given in polar coordinates by

$$\begin{aligned} \dot{r} &= -r(r-1)^2 \\ \dot{\theta} &= r^2(r-1). \end{aligned}$$

(i) What are the rest points?
(ii) Show that $V = r^2$ is a Lyapunov function.
(iii) Is it true that every solution tends to a rest point as $t \to \infty$?
(iv) Determine the ω-limit set of every solution.

3.2. Prove Theorem 3.1 (ii), that if $\{T(t)\}_{t \geqslant 0}$ is a semiflow on a metric space (X, d), and if ψ is a complete orbit with $\{\psi(t) : t \leqslant 0\}$ precompact, then the α−limit set $\alpha(\psi)$ is nonempty, compact, invariant and connected, and, as $t \to -\infty$, $\operatorname{dist}(\psi(t), \alpha(\psi)) \to 0$.

3.3. Let $\{T(t)\}_{t \geqslant 0}$ be a semiflow on a metric space (X, d), and let $V : X \to \mathbb{R}$ be a Lyapunov function. For fixed $\tau > 0$ define

$$V_\tau(p) = V(T(\tau)p), \quad p \in X.$$

Show that V_τ is a Lyapunov function.

3.4. Show that every solution of the ordinary differential equation

$$\ddot{u} + u^2\dot{u} + u^3 = 0$$

satisfies $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \infty$.

3.5. Consider the ODE in $\mathbb{R}^2$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{aligned} \dot{x}_1 &= (x_2 - 1)(x_1 - x_2) \\ \dot{x}_2 &= (x_1 - 1)(x_1 + x_2 - 2x_1x_2). \end{aligned}$$

(i) Show that the rest points are 0 and $\bar{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
(ii) Show that $V(x) = |x|^2$ is Lyapunov function.
(iii) Prove that every solution $x(t) \to z$ as $t \to \infty$ where $z = 0$ or $z = \bar{z}$.
(iv) Prove that 0 is asymptotically stable and that $\bar{z}$ is unstable.

3.6. Consider the ordinary differential equation in $\mathbb{R}^2$

$$\begin{aligned} \dot{x}_1 &= x_2(x_1 - 1)^2 \\ \dot{x}_2 &= -x_1^2x_2 - x_1(x_1 - 1)^2. \end{aligned} \tag{3.9}$$

(a) Show that $V(x_1, x_2) = x_1^2 + x_2^2$ is a Lyapunov function.
(b) Show that (3.9) generates a semiflow on $\mathbb{R}^2$.
(c) What are the rest points of (3.9)?
(d) Prove that every solution $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ converges to a rest point as $t \to \infty$.

3.7. A system is governed by the ODE

$$\ddot{u}(t) + a(t)u(t) = 0,$$

where $a(t)$ is a smooth real-valued control. Show that given any initial data $u(0) = u_0, \dot{u}(0) = u_1$ there exists a control (depending on u_0, u_1) such that $u^2(t) + \dot{u}^2(t) \to 0$ as $t \to \infty$.

(*Hint.* Choose $a(t)$ as a function of $u(t), \dot{u}(t)$ (a *feedback control*) such that $\dot{u}^2 + u^2$ is a Lyapunov function.)

3.8. Consider the ordinary differential equation in $\mathbb{R}^n$

$$\dot{x} = f(x), \tag{3.10}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Assume that for some C^1 function $V : \mathbb{R}^n \to \mathbb{R}$

$$f(x) \cdot \nabla V(x) \leqslant 0 \text{ for all } x \in \mathbb{R}^n, \tag{3.11}$$

with equality if and only if $f(x) = 0$. Assume further that $V(x) \to \infty$ as $|x| \to \infty$ and that there are only finitely many rest points.

Let $z \in \mathbb{R}^n$ be a rest point that is not a local minimizer of V, so that there exists a sequence $z_j \to z$ such that $V(z_j) < V(z)$ for all j.
(i) Show that for each j the solution $T(t)z_j$ converges as $t \to \infty$ to a rest point different from z, and deduce that z is unstable.
(ii) For $\varepsilon > 0$ sufficiently small let $t_j(\varepsilon)$ denote the least value of $t > 0$ such that $|T(t)z_j - z| = \varepsilon$. Show that $t_j(\varepsilon)$ exists and is finite (for all sufficiently large j).
(iii) By using the continuous dependence on initial data for (3.10), or otherwise, prove that $t_j(\varepsilon) \to \infty$ as $j \to \infty$.
(iv) Show that there is a complete orbit ψ with

$$\psi(t) \to z \text{ as } t \to -\infty, \quad \psi(t) \to \bar{z} \text{ as } t \to \infty,$$

for some rest point $\bar{z} \neq z$.

(*Hint.* Consider the limit y of a subsequence of $T(t_j(\varepsilon))z_j$ and show that the solution $\psi(t)$ of (6.42) with initial data y satisfies $|\psi(t) - z| \leqslant \varepsilon$ for all $t \leqslant 0$.)

3.9. Let $V \in C^2(\mathbb{R}^n)$ satisfy $V(x) \to \infty$ as $|x| \to \infty$. Consider the 'gradient flow'

$$\dot{x} = -\nabla V(x). \tag{3.12}$$

(i) Prove that (3.12) generates a semiflow on $\mathbb{R}^n$ with Lyapunov function V.
(ii) Suppose that z_1, z_2 are distinct isolated strict local minimizers of V, so that in particular $\nabla V(z_1) = \nabla V(z_2) = 0$. Prove that there exists another critical point z_3, that is $\nabla V(z_3) = 0$, such that z_3 is not a strict local minimizer of V.

(*Hint.* Consider the asymptotic behaviour as $t \to \infty$ of $T(t)p$ for a point $p \in \partial(A(z_1) \cup A(z_2))$.)

3.10. Consider the coupled pair of ODEs

$$\begin{aligned} \ddot{u} + \dot{u} + uv^2 = 0 \\ \ddot{v} + \dot{v} + vu^2 = 0 \end{aligned} \tag{3.13}$$

regarded as an ODE in the vector $x = \begin{pmatrix} u \\ \dot{u} \\ v \\ \dot{v} \end{pmatrix} \in \mathbb{R}^4$.

(i) What are the rest points?
(ii) Letting

$$V(x) = \dot{u}^2 + \dot{v}^2 + u^2v^2$$

show that

$$\dot{V}(x) = -2(\dot{u}^2 + \dot{v}^2), \tag{3.14}$$

and hence that V is a Lyapunov function.
(iii) Deduce from (3.14) that $\dot{u}^2 + \dot{v}^2$ is bounded on the maximal interval of existence of any solution, and hence that (3.13) generates a semiflow on $\mathbb{R}^4$.
(iv) Letting $z = u^2 - v^2$, show that

$$\ddot{z} + \dot{z} = 2(\dot{u}^2 - \dot{v}^2),$$

and deduce from (3.14) that for $t \geqslant s \geqslant 0$

$$|\dot{z}(t) + z(t) - (\dot{z}(s) + z(s))| \leqslant V(s) - V(t).$$

Deduce that $\dot{z}(t)+z(t)$ tends to a limit as $t \to \infty$, and hence that $z(t)$ is bounded for all $t \geqslant 0$.
(v) Deduce that $x(t)$ is bounded for all $t \geqslant 0$, and that $\dot{u}(t) \to 0, \dot{v}(t) \to 0$ as $t \to \infty$. Hence show that every solution $x(t)$ of (3.13) tends to a rest point as $t \to \infty$.

Chapter 4

Elements of the one-dimensional calculus of variations I. Function spaces

4.1 Dynamics and the calculus of variations

Example 4.1. Consider the *reaction-diffusion equation* for $u = u(x,t)$

$$u_t = u_{xx} - f(x,u), \tag{4.1}$$

where $f = f(x,u)$ is a given sufficiently smooth function. This describes the concentration u of a chemical that diffuses and reacts. More generally one can consider systems of reaction-diffusion equations such as

$$\mathbf{u}_t = \Delta \mathbf{u} - \mathbf{f}(x, \mathbf{u}),$$

where $x \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ is a vector of concentrations of m different chemicals. However, we only consider the one-dimensional case (4.1). Special cases of f include

$$f = u^2 - u, \ \text{(Fisher's equation)}$$

from population dynamics,

$$f = u^3 - u, \ \text{(Newell-Whitehead-Segel equation)}$$

which is a model of convection, and

$$f = u(u-1)(u-\alpha), \ 0 < \alpha < 1 \ \text{(Zeldovich equation)}$$

which arises in combustion theory.

We seek to solve (4.1) on the interval $0 \leqslant x \leqslant 1$ subject to the initial condition

$$u(x,0) = u_0(x), \tag{4.2}$$

where u_0 is a given function, and boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{4.3}$$

(Note that the more general boundary conditions

$$u(0,t) = a, \; u(1,t) = b$$

can be reduced to this case by setting

$$v(x,t) = u(x,t) - [a + (b-a)x],$$

so that

$$\begin{aligned} v_t &= v_{xx} - \tilde{f}(x,v), \\ v(0,t) &= v(1,t) = 0, \end{aligned}$$

where $\tilde{f}(x,v) = f(x, v + a + (b-a)x)$.)

If u is a sufficiently smooth solution of (4.1), (4.3) then

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u_x^2 + F(x,u)\right) dx = -\int_0^1 u_t^2 \, dx \leqslant 0, \tag{4.4}$$

where

$$F(x,u) = \int_0^u f(x,s)\, ds, \tag{4.5}$$

so that

$$I(u) = \int_0^1 \left(\frac{1}{2}u_x^2 + F(x,u)\right) dx \tag{4.6}$$

is a Lyapunov function. We have seen that the local and global minimizers of Lyapunov functions play an important role in dynamics, and so we need to understand the properties of such minimizers. The functional (4.6) is a special case of the general one-dimensional functional of the calculus of variations:

$$I(u) = \int_0^1 F(x,u,u_x)\, dx. \tag{4.7}$$

In the following we will mostly only consider the special case (4.6) of interest for (4.1). The first issue is to decide on the appropriate space of functions over which to minimize I.

4.2 Review of the Lebesgue integral

We use the Lebesgue integral on $\mathbb{R}$. We recall that this integral is based on the concept of *measurable subsets* of the real line, those subsets $E \subset \mathbb{R}$ for which a 'length' $\operatorname{meas} E$ can be defined satisfying natural conditions. If $I = (a, b)$ is an open interval then $\operatorname{meas} I = b - a$. For an arbitrary $E \subset \mathbb{R}$ the *outer measure* $\lambda(E)$ of E is defined by

$$\lambda^*(E) = \inf \left\{ \sum_k \operatorname{meas}(I_k) : I_k \text{ open intervals with } E \subset \cup_k I_k \right\}.$$

The subset $E \subset \mathbb{R}$ is *measurable* if

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \text{ for all } A \subset \mathbb{R},$$

and then $\operatorname{meas} E \stackrel{\text{def}}{=} \lambda^*(E)$. Not every subset E is measurable. The set $\mathcal{Q}$ of rationals has measure zero, as is any countable set $\{q_i : i = 1, 2, \ldots\}$. In fact we can cover such a set by open intervals $(q_i - \varepsilon 2^{-i}, q_i + \varepsilon 2^{-i})$, where $\varepsilon > 0$ is arbitrarily small. Measurable sets satisfy natural properties. In particular $\operatorname{meas} E_1 \leqslant \operatorname{meas} E_2$ if $E_1 \subset E_2$, $\operatorname{meas}(E + a) = \operatorname{meas} E$ if $a \in \mathbb{R}$, and

$$\operatorname{meas} \bigcup_i E_i = \sum_i \operatorname{meas} E_i$$

for any countable sequence E_i of disjoint measurable sets.

A (*Lebesgue*) *measurable* function $f : \mathbb{R} \to \mathbb{R}$, is a function for which $f^{-1}([a, b])$ is measurable for any $a < b$. Two measurable functions f, g are regarded as being equivalent if $f = g$ 'almost everywhere' (a.e.), that is $f(x) = g(x)$ for all $x \notin E$, where $\operatorname{meas} E = 0$. If $E \subset \mathbb{R}$ is measurable then the characteristic function

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is measurable, and its Lebesgue integral is defined to be

$$\int_{\mathbb{R}} \chi_E \, dx = \operatorname{meas} E.$$

Similarly we can consider a *simple function* $s = \sum_{i=1}^N \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{R}$, and the E_i are measurable with $\operatorname{meas} E_i < \infty$ for all i, and define

$$\int_{\mathbb{R}} s \, dx = \sum_{i=1}^N \alpha_i \operatorname{meas} E_i.$$

(A given simple function can be represented in more than one way with different E_i and α_i, but it is easily checked that the formula gives the same result for any representation.) If $f \geqslant 0$ is measurable then its Lebesgue integral is defined by

$$\int_{\mathbb{R}} f \, dx = \sup \left\{ \int_{\mathbb{R}} s \, dx : 0 \leqslant s \leqslant f, s \text{ simple} \right\}.$$

If $f : \mathbb{R} \to \mathbb{R}$ is measurable, we can write $f = f^+ - f^-$, where $f^+(x) = \max\{0, f(x)\}$ and $f^-(x) = \max\{0, -f(x)\}$, and then the integral of f is defined as

$$\int_{\mathbb{R}} f\,dx = \int_{\mathbb{R}} f^+\,dx - \int_{\mathbb{R}} f^-\,dx,$$

provided the integrals on the right-hand side are not both $+\infty$. If $E \subset \mathbb{R}$ is measurable and $f : E \to \mathbb{R}$ is measurable (that is $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is measurable, where $\tilde{f}(x) = f(x)$ for $x \in E$, $\tilde{f}(x) = 0$ for $x \notin E$), then we define

$$\int_E f\,dx = \int_{\mathbb{R}} \chi_E f\,dx.$$

The integral so defined satisfies natural properties, such as

$$\begin{aligned} \int_E (\alpha f + \beta g)\,dx &= \alpha \int_E f\,dx + \beta \int_E g\,dx, \\ \left| \int_E f\,dx \right| &\leqslant \int_E |f|\,dx, \\ \int_E |f|\,dx &= 0 \text{ implies } f = 0 \text{ a.e.} \end{aligned}$$

There are important convergence theorems for the Lebesgue integral, for example

Theorem 4.1 (Monotone convergence theorem)**.** *Let $0 \leqslant u^{(1)} \leqslant u^{(2)} \leqslant \dots$ be a pointwise nondecreasing sequence of measurable functions on a measurable subset $E \subset \mathbb{R}$, and suppose that $\sup_j \int_E u^{(j)} dx < \infty$. Then $u(x) = \lim_{j\to\infty} u^{(j)}(x)$ is measurable and*

$$\lim_{j\to\infty} \int_E u^{(j)}\,dx = \int_E u\,dx.$$

Lemma 4.2 (Fatou)**.** *Let $u^{(j)} \geqslant 0$ be a sequence of Lebesgue measurable functions on E, $E \subset \mathbb{R}$ measurable, with $u^{(j)} \to u$ almost everywhere (a.e.) in E. Then*

$$\int_E u\,dx \leqslant \liminf_{j\to\infty} \int_E u^{(j)}\,dx.$$

and

Corollary 4.3 (Lebesgue dominated convergence theorem)**.** *Let $u^{(j)}$ be a sequence of Lebesgue measurable functions on E, $E \subset \mathbb{R}$ measurable, with $u^{(j)} \to u$ almost everywhere (a.e.) in E and $|u^{(j)}(x)| \leqslant \varphi(x)$ a.e., where $\int_E \varphi\,dx < \infty$. Then*

$$\lim_{j\to\infty} \int_E u^{(j)}\,dx = \int_E u\,dx.$$

Proof. Apply Lemma 4.2 to $\varphi \pm u^{(j)} \geqslant 0$. □

4.3 Inner-product spaces and Hilbert spaces

H is a (real) *inner-product space* if it is a real vector space endowed with an inner product (u, v) associating to each pair $u, v \in H$ the real number (u, v),and satisfying
(i) $(u, v) = (v, u)$ for all $u, v \in H$,
(ii) $(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v)$ for all $u_1, u_2, v \in H, \alpha, \beta \in \mathbb{R}$,
(iii) $(u, u) \geqslant 0$, with equality if and only if $u = 0$.

As an example we can take $H = \mathbb{R}^n$, with the inner product $(u, v) = u \cdot v$ being the dot product of vectors.

Given an inner-product space H we can define the *norm*

$$\|u\| = (u, u)^{\frac{1}{2}}$$

which satisfies
(a) $\|u\| \geqslant 0, \|u\| = 0$ if and only if $u = 0$,
(b) $\|\alpha u\| = |\alpha|\|u\|$ for all $u \in H, \alpha \in \mathbb{R}$,
(since by (i),(ii) $\|\alpha u\|^2 = (\alpha u, \alpha u) = \alpha(u, \alpha u) = \alpha(\alpha u, u) = \alpha^2\|u\|^2$.)
(c) $\|u + v\| \leqslant \|u\| + \|v\|$ for all $u, v \in H$.

To prove (c), note that $\|u + \alpha v\|^2 = \|u\|^2 + 2\alpha(u, v) + \alpha^2\|v\|^2 \geqslant 0$ for all α, so that

$$|(u, v)| \leqslant \|u\| \cdot \|v\|, \qquad \text{(Cauchy-Schwarz inequality)} \tag{4.8}$$

and then

$$\|u + v\|^2 = \|u\|^2 + 2(u, v) + \|v\|^2 \leqslant (\|u\| + \|v\|)^2.$$

Defining

$$d(u, v) = \|u - v\|$$

we see that (H, d) is a metric space, the triangle inequality following from (c). If (H, d) is complete, H is called a *Hilbert space*. Since $\mathbb{R}^n$ is complete it is a (finite-dimensional) Hilbert space. However In the following we will always assume that H is infinite-dimensional.

4.4 The space $L^2(0, 1)$

Definition 4.1. The space $L^2(0, 1)$ is the space of (equivalence classes of) measurable functions $u : (0, 1) \to \mathbb{R}$ with $\|u\|_2 < \infty$, where

$$\|u\|_2 = \left(\int_0^1 u^2\, dx\right)^{\frac{1}{2}}.$$

Writing

$$(u, v) = \int_0^1 uv\, dx \text{ for } u, v \in L^2(0, 1)$$

we see that $(\cdot, \cdot)$ is an inner product on $L^2(0, 1)$ (since $|uv| \leqslant \frac{1}{2}(u^2 + v^2)$ implies (u, v) is finite) with corresponding norm $\| \cdot \|_2$.

Theorem 4.4. *$L^2(0,1)$ is complete (and thus a Hilbert space).*

Proof. (*Not for examination.*) Let $u^{(j)}$ be a Cauchy sequence in $L^2(0,1)$, so that $\|u^{(j)} - u^{(k)}\|_2 \to \infty$ as $j, k \to \infty$. Pick j_1 such that $\|u^{(j_1)} - u^{(k)}\|_2 \leqslant \frac{1}{2}$ for all $k \geqslant j_1$, then j_2 such that $\|u^{(j_2)} - u^{(k)}\|_2 \leqslant \frac{1}{4}$ for all $k \geqslant j_2$, and so on. Hence we obtain j_k such that

$$\|u^{(j_k)} - u^{(j_{k+1})}\|_2 \leqslant 2^{-k} \quad \text{for } k = 1, 2, \dots .$$

Consider the sequence

$$v^{(l)}(x) = |u^{(j_1)}(x)| + \sum_{k=1}^{l} |u^{(j_{k+1})}(x) - u^{(j_k)}(x)| \geqslant 0.$$

Then

$$\|v^{(l)}\|_2 \leqslant \|u^{(j+1)}\|_2 + \sum_{k=1}^{l} 2^{-k} \leqslant \|u^{(j_1)}\|_2 + 1.$$

By the monotone convergence theorem

$$\int_0^1 \lim_{l\to\infty} |v^{(l)}|^2 \, dx = \lim_{l\to\infty} \int_0^1 |v^{(l)}|^2 \, dx < \infty$$

and so $v^{(l)} \to v$ a.e., where $v \in L^2(0,1)$. Hence

$$u^{(j_l)}(x) = u^{(j_1)}(x) + u^{(j_2)}(x) - u^{(j_1)}(x) + \cdots u^{(j_l)}(x) - u^{(j_{l-1})}(x)$$

converges a.e. to a limit $u(x)$ as $l \to \infty$. Thus, using Fatou's lemma

$$\int_0^1 |u^{(k)} - u|^2 \, dx \leqslant \liminf_{l\to\infty} \int_0^1 |u^{(k)} - u^{(j_l)}|^2 \, dx \to 0,$$

as $k \to 0$, since $u^{(k)}$ is Cauchy. Hence $u^{(k)} \to u$ in $L^2(0,1)$. □

Definition 4.2. Let H be a Hilbert space. An *orthonormal basis* of H is a countable family $\{\omega_j\}_{j=1}^{\infty}$ of elements of H such that

(i) $(\omega_j, \omega_k) = \delta_{jk}$ for all j, k,
(ii) every $u \in H$ can be represented as a convergent sum

$$u = \sum_{j=1}^{\infty} (u, \omega_j)\omega_j,$$

i.e. $\|u - \sum_{j=1}^{N} (u, \omega_j)\omega_j\| \to 0$ as $N \to \infty$.

If $\{\omega_j\}$ satisfies (i) then

$$\|u-\sum_{j=1}^{N}(u,\omega_j)\omega_j\|^2=\|u\|^2-\sum_{j=1}^{N}(u,\omega_j)^2,$$

so that

$$\sum_{j=1}^{\infty}(u,\omega_j)^2\leqslant\|u\|^2 \quad \text{(Bessel inequality)}.$$

Hence $\{\omega_j\}$ is an orthonormal basis provided for every $u\in H$

$$\|u\|^2=\sum_{j=1}^{\infty}(u,\omega_j)^2. \tag{4.9}$$

Theorem 4.5. *The following are orthonormal bases of $L^2(0,1)$:*

$$\{1,\sqrt{2}\sin 2j\pi x,\sqrt{2}\cos 2j\pi x,\ j=1,2,...\},$$
$$\{\sqrt{2}\sin j\pi x,\ j=1,2,...\}$$
$$\{1,\sqrt{2}\cos j\pi x,\ j=1,2,...\}.$$

Proof. (*Not for examination.*) See Problem 4.3. □

Definition 4.3. Let H be a Hilbert space. A sequence $u^{(j)}$ *converges weakly* to u in H (written $u^{(j)}\rightharpoonup u$) if and only if

$$(u^{(j)},v)\to(u,v) \text{ for all } v\in H.$$

Thus $u^{(j)}\rightharpoonup u$ in $L^2(0,1)$ if and only if

$$\int_0^1 u^{(j)}v\,dx\to\int_0^1 uv\,dx \quad \text{as } j\to\infty$$

for any $v\in L^2(0,1)$.

Weak convergence can be thought of as 'convergence of measurements'. That is, we can think of each v as representing a measuring device which delivers the value (u,v) for the state u of a system. For example, if u is temperature the value of the temperature measured by a probe near the point x_0 might be the average

$$\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}u\,dx=\int_0^1 uv_\varepsilon\,dx.$$

where $v_\varepsilon(x)=\frac{1}{2\varepsilon}\chi_{(x_0-\varepsilon,x_0+\varepsilon)}(x)$.

Note that weak limits are unique, since if $u^{(j)}\rightharpoonup u$ and $u^{(j)}\rightharpoonup\tilde{u}$ then $(u-\tilde{u},v)=0$ for all v, and so choosing $v=u-\tilde{u}$ we get $\|u-\tilde{u}\|^2=0$ and $u=\tilde{u}$.

If $u^{(j)}\to u$ strongly in H (i.e. $\|u^{(j)}-u\|\to 0$) then $u^{(j)}\rightharpoonup u$, since for any $v\in H$ we have $|(u^{(j)}-u,v)|\leqslant\|u^{(j)}-u\|\|v\|$. However weak convergence does not in general imply strong convergence.

Example 4.2. If $\{\omega_j\}_{j=1}^\infty$ is an orthonormal basis of H then $\omega_j \rightharpoonup 0$ as $j \to \infty$.

Proof. This follows immediately from (4.9). $\square$

Theorem 4.6. *Let H be a Hilbert space with an orthonormal basis (e.g. $H = L^2(0,1)$) and let $u^{(j)}$ be a bounded sequence in H. Then there exists a subsequence $u^{(r_j)}$ of $u^{(j)}$ converging weakly to some u in H.*

Proof. Let $M = \sup_j \|u^{(j)}\|_2$. Let $\{\omega_k\}_{k=1}^\infty$ be an orthonormal basis of H. Since $|(u^{(j)}, \omega_1)| \leqslant M\|\omega_1\|_2$ the sequence $(u^{(j)}, \omega_1)$ of real numbers is bounded. Hence there exists a subsequence $u^{(n_1(j))}$ such that $\lim_{j\to\infty}(u^{(n_1(j))}, \omega_1)$ exists. Similarly the sequence $(u^{(n_1(j))}, \omega_2)$ is bounded and so there exists a subsequence $u^{(n_2(j))}$ of $u^{(n_1(j))}$ such that $\lim_{j\to\infty}(u^{(n_2(j))}, \omega_2)$ exists. Proceeding in this way we obtain for each k a subsequence $u^{(n_k(j))}$ of $u^{(n_{k-1}(j))}$ such that $\lim_{j\to\infty}(u^{(n_k(j))}, \omega_k)$ exists. Consider the 'diagonal sequence' $u^{(n_j(j))}$ and write $r_j = n_j(j)$. Clearly $\lim_{j\to\infty}(u^{(r_j)}, \omega_k) = u_k$ exists for all k. But the fact that $\sum_{k=1}^\infty (u^{(r_j)}, \omega_k)^2 \leqslant M^2$ implies $\sum_{k=1}^\infty u_k^2 < \infty$. Thus $u = \sum_{k=1}^\infty u_k\omega_k \in H$. Let $v = \sum_{k=1}^\infty v_k\omega_k \in H$. Then

$$(u^{(r_j)} - u, v) = \sum_{k=1}^\infty v_k(u^{(r_j)} - u, \omega_k).$$

Given $\varepsilon > 0$ choose N sufficiently large for

$$\left|\sum_{k=N+1}^\infty v_k(u^{(r_j)} - u, \omega_k)\right| \leqslant \left(\sum_{k=N+1}^\infty v_k^2\right)^{\frac{1}{2}} \left(\sum_{k=N+1}^\infty (u^{(r_j)} - u, \omega_k)^2\right)^{\frac{1}{2}} \leqslant \varepsilon.$$

Since $\sum_{k=1}^N v_k(u^{(r_j)} - u, \omega_k) \to 0$ it follows that $|(u^{(r_j)} - u, v)| \leqslant 2\varepsilon$ for j sufficiently large, and so $u^{(r_j)} \rightharpoonup u$. $\square$

4.5 The Sobolev space $H^1(0,1)$

We denote by $C_0^1(0,1)$ the space of all C^1 functions $u : [0,1] \to \mathbb{R}$ satisfying $u(0) = u(1) = 0$.

Definition 4.4. Let $u \in L^2(0,1)$. A function $v \in L^2(0,1)$ is a *weak derivative* of u if

$$\int_0^1 u\varphi_x\,dx = -\int_0^1 v\varphi\,dx \quad \text{for all } \varphi \in C_0^1(0,1) \tag{4.10}$$

and we write $v = u_x$.

If $u \in C^1([0,1])$ then the formula (4.10) is clearly satisfied with $v = u_x$ the usual derivative since then

$$\int_0^1 (u\varphi)_x\,dx = \int_0^1 (u\varphi_x + u_x\varphi)\,dx = 0.$$

Note that weak derivatives are unique. Indeed if $u \in L^2(0,1)$ has weak derivatives $v, \tilde{v}$ then $\int_0^1 (v-\tilde{v})\varphi\, dx = 0$ for all $\varphi \in C_0^1(0,1)$ so that $v = \tilde{v}$ by the following lemma.

Lemma 4.7 (Fundamental lemma of the calculus of variations)**.** *Let $z \in L^2(0,1)$ satisfy*

$$\int_0^1 z\varphi\, dx = 0 \quad \textit{for all } \varphi \in C_0^1(0,1). \tag{4.11}$$

Then $z = 0$.

Proof. Let $\omega_j(x) = \sqrt{2}\sin j\pi x$. Then $\omega_j \in C_0^1(0,1)$ and so by (4.11) $(z, \omega_j) = 0$, Since by Theorem 4.5 $\{\omega_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(0,1)$ it follows that $z = 0$. □

Definition 4.5. The Sobolev space $H^1(0,1) = W^{1,2}(0,1)$ consists of those $u \in L^2(0,1)$ with weak derivative $u_x \in L^2(0,1)$.

Theorem 4.8. *$H^1(0,1)$ is a Hilbert space with inner product*

$$\langle u, v\rangle = \int_0^1 (uv + u_x v_x)\, dx, \quad u, v \in H^1(0,1) \tag{4.12}$$

and corresponding norm

$$\|u\|_{1,2} = \left(\int_0^1 (u^2 + u_x^2)\, dx\right)^{\frac{1}{2}}. \tag{4.13}$$

Proof. We just have to show that $H^1(0,1)$ is complete with respect to the norm (4.13). Let $u^{(j)}$ be a Cauchy sequence in $H^1(0,1)$. Then $u^{(j)}$ and $u_x^{(j)}$ are Cauchy sequences in $L^2(0,1)$, and since, by Theorem 4.4, $L^2(0,1)$ is complete we have that $u^{(j)} \to u, u_x^{(j)} \to v$ in $L^2(0,1)$ for some $u, v \in L^2(0,1)$. But by (4.10)

$$\int_0^1 u^{(j)}\varphi_x\, dx = -\int_0^1 u_x^{(j)}\varphi\, dx \quad \text{for all } \varphi \in C_0^1(0,1).$$

Passing to the limit $j \to \infty$ we obtain

$$\int_0^1 u\varphi_x\, dx = -\int_0^1 v\varphi\, dx \quad \text{for all } \varphi \in C_0^1(0,1),$$

so that $u \in H^1(0,1)$, $u_x = v$ and $u^{(j)} \to u$ in $H^1(0,1)$. □

Example 4.3. Let $u \in C([0,1])$ be piecewise C^1, i.e. there exist points $0 = x_0 < x_1 < \cdots < x_n = 1$ with $u \in C^1([x_{i-1}, x_i])$ for $i = 1, \ldots, n$. Then

$u \in H^1(0,1)$. In fact let $\varphi \in C_0^1(0,1)$ and denote by u_x the usual derivative in each interval (x_{i-1}, x_i). Then

$$\begin{aligned}\int_0^1 (u\varphi_x + u_x\varphi)\,dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (u\varphi)_x\,dx \\ &= \sum_{i=1}^n (u(x_i)\varphi(x_i) - u(x_{i-1})\varphi(x_{i-1})) = 0,\end{aligned}$$

since $\varphi(0) = \varphi(1) = 0$. Thus u_x is the weak derivative of u and since $u_x \in L^2(0,1)$ it follows that $u \in H^1(0,1)$ as required.

Lemma 4.9 (du Bois Reymond lemma)**.** *Let $z \in L^2(0,1)$ satisfy*

$$\int_0^1 z\varphi_x\,dx = 0 \quad \textit{for all } \varphi \in C_0^1(0,1). \tag{4.14}$$

Then $z = c$ a.e. for some constant c.

Proof. Write $z(x) = c + 2\sum_{j=1}^\infty (z, \cos j\pi x)\cos j\pi x$ as a convergent series in $L^2(0,1)$ using the basis $\{1, \sqrt{2}\cos j\pi x\}$ (see Theorem 4.2). Choosing $\varphi(x) = \sqrt{2}\sin j\pi x$ we deduce from (4.14) that $(z, \cos j\pi x) = 0$ for all j, giving the result. □

Theorem 4.10 (A characterization of $H^1(0,1)$)**.** *Let $u, v \in L^2(0,1)$. Then the following are equivalent:*
(i) $u \in H^1(0,1)$ and $u_x = v$.
(ii) $u(x) - \int_0^x v(y)\,dy = c$ for some constant c and a.e. $x \in (0,1)$.

Proof.

$$\begin{aligned}(i) \text{ holds} \quad &\Leftrightarrow \quad \int_0^1 (u\varphi_x + v\varphi)\,dx = 0 \text{ for all } \varphi \in C_0^1(0,1) \\ &\Leftrightarrow \quad \int_0^1 \left(u - \int_0^x v\,dy\right)\varphi_x\,dx = 0 \text{ for all } \varphi \in C_0^1(0,1) \\ &\qquad\ \left(\text{since } \int_0^1 \frac{d}{dx}\left(\int_0^x v\,dy\,\varphi\right)dx = 0\right) \\ &\Leftrightarrow \quad (ii) \text{ by Lemma 4.9.}\end{aligned}$$

□

Corollary 4.11. *Any $u \in H^1(0,1)$ has a representative τu belonging to $C([0,1])$, and*

$$\|\tau u\|_{C([0,1])} = \max_{x\in[0,1]} |(\tau u)(x)| \leqslant \sqrt{2}\|u\|_{1,2}, \tag{4.15}$$

so that the embedding $\tau : H^1(0,1) \to C([0,1])$ is a continuous linear map.

Proof. By Theorem 4.10

$$(\tau u)(x) = c + \int_0^x v(y)\,dy$$

is a continuous representative of u. Note that

$$(\tau u)(x) - (\tau u)(z) = \int_z^x u_y(y)\,dy \quad \text{for all } x, z \in [0,1]. \tag{4.16}$$

Integrating (4.16) with respect to z we obtain

$$|(\tau u)(x)| \leqslant \int_0^1 |u(z)|\,dz + \int_0^1 |u_y(y)|\,dy, \tag{4.17}$$

and so

$$\|\tau u\|_{C([0,1])} \leqslant \left(\int_0^1 |u|^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 |u_x|^2\,dx\right)^{\frac{1}{2}},$$

where we have used

$$\int_0^1 |u|\,dx = \int_0^1 1\cdot|u|\,dx \leqslant \left(\int_0^1 1^2\,dx\right)^{\frac{1}{2}} \left(\int_0^1 |u|^2\,dx\right)^{\frac{1}{2}}$$

etc., and the result follows from the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$. □

From now on we always choose the continuous representative of u and simply write $\tau u = u$.

Theorem 4.12.
(i) A sequence $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$ if and only if $u^{(j)} \rightharpoonup u$ in $L^2(0,1)$ and $u_x^{(j)} \rightharpoonup u_x$ in $L^2(0,1)$.
(ii) Any bounded sequence $u^{(j)}$ in $H^1(0,1)$ has a weakly convergent subsequence.

Proof. (i) Suppose $u^{(j)} \rightharpoonup u$ in $L^2(0,1)$ and $u_x^{(j)} \rightharpoonup u_x$ in $L^2(0,1)$. Then

$$\int_0^1 u^{(j)} v\,dx \to \int_0^1 uv\,dx, \int_0^1 u_x^{(j)} v_x\,dx \to \int_0^1 u_x v_x\,dx \text{ for all } v \in H^1(0,1),$$

so that

$$\int_0^1 (u^{(j)} v + u_x^{(j)} v_x)\,dx \to \int_0^1 (uv + u_x v_x)\,dx \text{ for all } v \in H^1(0,1). \tag{4.18}$$

Hence $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$.

Conversely suppose that $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$, so that (4.18) holds, and suppose for contradiction that $u^{(j)} \not\rightharpoonup u$ in $L^2(0,1)$ or $u_x^{(j)} \not\rightharpoonup u_x$ in $L^2(0,1)$. Then there exist $\varepsilon > 0$ and $\tilde{v} \in L^2(0,1)$ and a subsequence $u^{(j_k)}$ such that for all k

$$\text{either } \left|\int_0^1 (u^{(j_k)} - u)\tilde{v}\,dx\right| > \varepsilon \text{ or } \left|\int_0^1 (u_x^{(j_k)} - u_x)\tilde{v}\,dx\right| > \varepsilon. \tag{4.19}$$

Since $u^{(j_k)} \rightharpoonup u$ in $H^1(0,1)$ it follows by the same argument as in Problem Sheet 5 1(ii) that $u^{(j)}$ is bounded in $H^1(0,1)$, so that $u^{(j_k)}$ and $u_x^{(j_k)}$ are bounded in $L^2(0,1)$. Hence by Theorem 4.6 we can assume that $u^{(j_k)} \rightharpoonup w, u_x^{(j_k)} \rightharpoonup z$ in $L^2(0,1)$ for some $w, z \in L^2(0,1)$. Thus from (4.18) we have that

$$\int_0^1 (wv + zv_x)\,dx = \int_0^1 (uv + u_x v_x)\,dx$$

for all $v \in H^1(0,1)$, Choosing first $v \in C_0^1(0,1)$ we deduce from (4.10) that $(w-u)_x = z - u_x$, from which it follows using (4.10) that $z = w_x$. Then setting $v = w - u$ we get that $w = u, z = u_x$. But this contradicts (4.19).
(ii) If $u^{(j)}$ is bounded in $H^1(0,1)$ then there is a subsequence $u^{(j_k)}$ such that $u^{(j_k)} \rightharpoonup u, u_x^{(j_k)} \rightharpoonup z$ in $L^2(0,1)$ for some $u, z \in L^2(0,1)$, and by passing to the limit in (4.10) we have that $z = u_x$. Hence by part (i) (the easy direction) $u^{(j_k)} \rightharpoonup u$ in $H^1(0,1)$.

□

Theorem 4.13. *The embedding of $H^1(0,1)$ in $C([0,1])$ is compact, i.e. if $u^{(j)}$ is a bounded sequence in $H^1(0,1)$ then there is a subsequence $u^{(r_j)} \to u$ in $C([0,1])$ for some $u \in C([0,1])$.*

Proof. By Theorem 4.12(ii) we may assume that $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$ for some u. By Corollary 4.11, $u^{(j)}(0)$ is bounded, so that we may also assume that $u^{(j)}(0) \to a$ for some constant a. But by Theorem 4.10

$$u^{(j)}(x) - u^{(j)}(0) = \int_0^x u_y^{(j)}(y)\,dy,$$

and since $u_x^{(j)} \rightharpoonup u_x$ in $L^2(0,1)$ it follows that $u^{(j)}(x) \to \tilde{u}(x)$ for all $x \in [0,1]$, where

$$\tilde{u}(x) = a + \int_0^x u_y(y)\,dy.$$

But since, by Corollary 4.11, $|u^{(j)}(x)|$ is bounded independent of x and j we have by the dominated convergence theorem that for any $v \in L^2(0,1)$

$$\lim_{j\to\infty} \int_0^1 u^{(j)} v\,dx = \int_0^1 \tilde{u} v\,dx = \int_0^1 uv\,dx.$$

Hence $\tilde{u} = u$. To prove that $u^{(j)} \to u$ in $C([0,1])$ it is enough to show that if $x_j \to x$ in $[0,1]$ then $u^{(j)}(x_j) \to u(x)$. But

$$\begin{aligned} |u^{(j)}(x_j) - u^{(j)}(x)| &\leqslant \left| \int_x^{x_j} u_y^{(j)}(y)\,dy \right| \\ &\leqslant |x_j - x|^{\frac{1}{2}} \left(\int_0^1 |u_y^{(j)}|^2\,dy \right)^{\frac{1}{2}}, \end{aligned}$$

and by the boundedness of $\|u_x^{(j)}\|_2$ we deduce that $u^{(j)}(x_j) \to u(x)$ as required.

□

Definition 4.6.

$$H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}.$$

We will use the fact that $H_0^1(0,1)$ is *weakly closed*, i.e. if $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$ and $u^{(j)} \in H_0^1(0,1)$ for all j then $u \in H_0^1(0,1)$. This follows by passing to the limit in the equation (see (4.17))

$$u^{(j)}(x) = \int_0^1 u^{(j)}(y)\, dy + \int_0^1 \int_y^x u_z^{(j)}(z)\, dz\, dy$$

for $x = 0, 1$ and using the corresponding equation for u. Alternatively one can follow the proof of Theorem 4.13. In particular $H_0^1(0,1)$ is a closed linear subspace of $H^1(0,1)$ and hence itself a Hilbert space.

Theorem 4.14 (Poincaré inequality)**.** *There is a constant $C > 0$ such that*

$$\int_0^1 u^2\, dx \leqslant C \int_0^1 u_x^2\, dx \tag{4.20}$$

for all $u \in H^1(0,1)$ satisfying $u(0) = 0$ (in fact we can take $C = \frac{1}{2}$). In particular $||u_x\|_2$ is an equivalent norm for $H_0^1(0,1)$.

Proof. For any such u we have that

$$u(x) = \int_0^x u_y\, dy$$

and so

$$|u(x)| \leqslant \left(\int_0^x 1^2\, dy\right)^{\frac{1}{2}} \left(\int_0^x u_y^2\, dy\right)^{\frac{1}{2}},$$

and thus

$$\int_0^1 u^2\, dx \leqslant \int_0^1 x\, dx \int_0^1 u_x^2\, dx = \frac{1}{2} \int_0^1 u_x^2\, dx.$$

□

Theorem 4.15. *$\tilde{\omega}_j(x) = c_j \sin j\pi x$, $c_j = \sqrt{\frac{2}{1+j^2\pi^2}}$, is an orthonormal basis of $H_0^1(0,1)$ with respect to the inner product $\langle \cdot, \cdot \rangle$.*

Proof. We have that

$$\langle \tilde{\omega}_j, \tilde{\omega}_k \rangle = c_j c_k \int_0^1 (\sin j\pi x\, \sin k\pi x + jk\pi^2 \cos j\pi x\, \cos k\pi x)\, dx = \delta_{jk}$$

for all j, k. Let $u \in H_0^1(0,1)$. Then since $\{\sqrt{2} \sin j\pi x, j = 1, 2, \dots\}$ is an orthonormal basis of $L^2(0,1)$ we have that

$$u(x) = \sum_{j=1}^{\infty} 2(u, \sin j\pi x) \sin j\pi x, \tag{4.21}$$

the series being convergent in $L^2(0,1)$. But also $u_x \in L^2(0,1)$, and since $\{1, \sqrt{2}\cos j\pi x, j = 1,2,\dots\}$ is an orthonormal basis of $L^2(0,1)$ we have that

$$u_x(x) = \sum_{j=1}^{\infty} 2(u_x, \cos j\pi x)\cos j\pi x + a, \tag{4.22}$$

where a is a constant and the series in convergent in $L^2(0,1)$. Since $\int_0^1 u_x dx = 0$ it follows that

$$a = -\lim_{N\to\infty}\sum_{j=1}^{N}\int_0^1 2(u_x, \cos j\pi x)\cos j\pi x\, dx = 0.$$

Thus from (4.21), (4.22) we get

$$u = \sum_{j=1}^{\infty}\langle u, \tilde{\omega}_j\rangle\tilde{\omega}_j,$$

the series being convergent in $H_0^1(0,1)$ as required. $\square$

4.6 Problems

4.1. (i) Prove that if $u^{(j)} \rightharpoonup u$ in $L^2(0,1)$ and $\|u^{(j)}\|_2 \to \|u\|_2$ then $u^{(j)} \to u$ strongly in $L^2(0,1)$.

(ii) The *Baire Category Theorem* states that if X is a complete metric space and $X = \cup_{j=1}^{\infty} E_j$, where E_j is closed for each j, then some E_{j_0} contains an open ball. Deduce from this that if $u^{(j)}$ is a weakly convergent sequence in $L^2(0,1)$ then $\|u^{(j)}\|_2$ is bounded.
(*Hint.* Suppose without loss of generality that $u^{(j)} \rightharpoonup 0$, and define $E_j = \{v \in L^2(0,1) : |(u^{(k)}, v)| \leqslant 1 \text{ for all } k \geqslant j\}$.)

(iii) Let

$$u^{(j)}(x) = \begin{cases} j^s & \text{if } 0 < x < j^{-1} \\ 0 & \text{otherwise} \end{cases}$$

where $0 < s < \infty$. For what s does $u^{(j)} \to 0$ in $L^2(0,1)$? For what s does $u^{(j)} \rightharpoonup u$ in $L^2(0,1)$?

4.2. Let $\alpha, \beta \in \mathbb{R}$, $0 < \lambda < 1$, and define $w : \mathbb{R} \to \mathbb{R}$ by

$$w(x) = \begin{cases} \alpha, & 0 < x \leqslant \lambda \\ \beta, & \lambda < x \leqslant 1 \end{cases}$$

extended to the whole of $\mathbb{R}$ as a function of period 1. Define $w^{(j)}(x) = w(jx)$.

(i) Prove that $w^{(j)} \rightharpoonup \lambda\alpha + (1-\lambda)\beta$ in $L^2(0,1)$.

(*Hint.* Show first that for the characteristic function χ of the interval $(r,s) \subset (0,1)$

$$\lim_{j\to\infty} \int_0^1 w^{(j)} \chi \, dx = (\lambda\alpha + (1-\lambda)\beta)(s-r),$$

and then use the fact that step functions are dense in $L^2(0,1)$.)

(ii) By considering the functions $f(w^{(j)})$, where f is continuous, show that if f satisfying $\sup_u \frac{|f(u)|}{1+|u|} < \infty$ has the property that

$$u^{(j)} \rightharpoonup u \text{ in } L^2(0,1) \text{ implies } f(u^{(j)}) \rightharpoonup f(u) \text{ in } L^2(0,1)$$

then f is *affine*, i.e. $f(v) = rv + s$ for $r, s \in \mathbb{R}$.

4.3. (i) Let $\{\omega_j\}_{j=1}^{\infty}$ be an orthonormal subset of $L^2(0,1)$, i.e. $(\omega_j, \omega_k) = \delta_{jk}$ for all j, k. Show that if there is no nonzero $u \in L^2(0,1)$ that is orthogonal to every ω_j, i.e. $(u, \omega_j) = 0$ for all j, then $\{\omega_j\}_{j=1}^{\infty}$ is an orthonormal basis.

(ii) Suppose that u is continuous on $[0,1]$ and orthogonal in $L^2(0,1)$ to every element of the orthonormal set

$$S = \{1, \sqrt{2}\sin 2j\pi x, \sqrt{2}\cos 2j\pi x, \ j = 1, 2, ...\}.$$

Prove that $u = 0$.
(*Hint.* Suppose for contradiction that $u \neq 0$ and without loss of generality that $|u|$ attains a maximum $u(x_0) > 0$ at some $x_0 \in [0,1]$. Choose $\delta > 0$ sufficiently small so that $u(x) > u(x_0)/2$ for $|x - x_0| < \delta$ and consider the function

$$\theta(x) = 1 + \cos 2\pi(x - x_0) - \cos 2\pi\delta.$$

Show that $(u, \theta^n) = 0$ for each nonnegative integer n. Then write

$$\int_0^1 u\theta^n\,dx = \int_{|x-x_0|>\delta} u\theta^n\,dx + \int_{|x-x_)|<\delta} u\theta^n\,dx$$

and note that $|\theta(x)| \leqslant 1$ for $|x - x_0| > \delta$ and $\theta(x) > 1$ for $|x - x_0| < \delta$. Let $n \to \infty$ to get a contradiction.)

(iii) Suppose that $u \in L^2(0,1)$ is orthogonal to every element of S. Prove that $u = 0$.
(*Hint.* Consider the continuous function $\tilde{u}(x) = \int_0^x u(y)\,dy + c$, where c is a suitable constant, and apply part (ii).)

(iv). Deduce from (i) that

$$\{1, \sqrt{2}\sin 2j\pi x, \sqrt{2}\cos 2j\pi x,\ j = 1,2,...\},$$
$$\{\sqrt{2}\sin j\pi x,\ j = 1,2,...\}$$
$$\{1, \sqrt{2}\cos j\pi x,\ j = 1,2,...\}$$

are orthonormal bases of $L^2(0,1)$.

4.4. Let $u(x) = x^\alpha$, where $\alpha \in \mathbb{R}$. For what α does $u \in H^1(0,1)$? Show that for these values of α the weak derivative of u is $u_x(x) = \alpha x^{\alpha-1}$.

4.5. Prove that $C^\infty([0,1])$ is dense in $H^1(0,1)$.
(*Hint.* Use the fact that $\{c_j \sin j\pi x,\ j = 1,2,\ldots\}$ is an orthonormal basis of $H^1_0(0,1)$ for suitable c_j.)

4.6. Prove that if $u, v \in H^1(0,1)$ then $uv \in H^1(0,1)$ with weak derivative

$$(uv)_x = uv_x + u_x v.$$

(*Hint.* Approximate by smooth functions.)

4.7. Prove that there is a constant $C > 0$ such that

$$\int_0^1 u^2\,dx \leqslant C\left(\int_0^1 u_x^2\,dx + \left(\int_0^1 u\,dx\right)^2\right)$$

for all $u \in H^1(0,1)$.
(*Hint.* Suppose not, that for $j = 1,2,\ldots$ there exist $u^{(j)} \in H^1(0,1)$ such that

$$\int_0^1 u^{(j)2}\,dx > j\left(\int_0^1 u_x^{(j)2}\,dx + \left(\int_0^1 u^{(j)}\,dx\right)^2\right)$$

and note that we can assume the LHS$\,= 1$ for all j. Then use weak convergence.)

Chapter 5

Elements of the one-dimensional calculus of variations II. Global and local minimizers.

5.1 Existence of minimizers

Theorem 5.1. *Let $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded below. Then the integral*

$$I(u) = \int_0^1 \left(\frac{1}{2} u_x^2 + F(x,u) \right) dx \tag{5.1}$$

attains an absolute minimum on $H_0^1(0,1)$.

Proof. The proof uses the *direct method of the calculus of variations*. By hypothesis there exists M with $F(x,u) \geqslant M$ for all $x \in [0,1], u \in \mathbb{R}$. Let

$$l = \inf_{v \in H_0^1(0,1)} I(v).$$

Then, since $0 \in H_0^1(0,1)$, $-\infty < M \leqslant l \leqslant I(0) < \infty$. Let $u^{(j)}$ be a minimizing sequence, so that $u^{(j)} \in H_0^1(0,1)$ for all j and $I(u^{(j)}) \to l$. Then $I(u^{(j)}) \leqslant l + 1$ for sufficiently large j, and so for such j

$$M + \frac{1}{2} \int_0^1 u_x^{(j)2} \, dx \leqslant l + 1.$$

Hence by the Poincaré inequality (Theorem 4.14) $u^{(j)}$ is bounded in $H^1(0,1)$ and we may assume that $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$ for some u. Since $H_0^1(0,1)$

is weakly closed, $u \in H_0^1(0,1)$. We show that I is *sequentially weakly lower semicontinuous* in $H^1(0,1)$, i.e. $u^{(j)} \rightharpoonup u$ in $H^1(0,1)$ implies

$$I(u) \leqslant \liminf_{j\to\infty} I(u^{(j)}). \tag{5.2}$$

From this it follows that $I(u) \leqslant l$, and since $u \in H_0^1(0,1)$, $I(u) \geqslant l$. Thus $I(u) = l$ and u is a minimizer.

To prove (5.2) note that

$$\begin{aligned} I(u^{(j)}) &= \int_0^1 \left(\frac{1}{2}u_x^{(j)2} + F(x,u^{(j)})\right) dx \\ &= \int_0^1 \left(\frac{1}{2}(u_x^{(j)} - u_x)^2 + u_x^{(j)}u_x - \frac{1}{2}u_x^2 + F(x,u^{(j)})\right) dx \\ &\geqslant \int_0^1 \left(u_x^{(j)}u_x - \frac{1}{2}u_x^2 + F(x,u^{(j)})\right) dx. \end{aligned}$$

But $\lim_{j\to\infty}\int_0^1 u_x^{(j)}u_x\,dx = \int_0^1 u_x^2\,dx$ since $u_x^{(j)} \rightharpoonup u_x$ in $L^2(0,1)$. Further $u^{(j)} \to u$ in $C([0,1])$. Hence by dominated convergence (or Fatou's lemma)

$$\liminf_{j\to\infty} I(u^{(j)}) \geqslant \int_0^1 \left(\frac{1}{2}u_x^2 + F(x,u)\right) dx = I(u)$$

as required. □

5.2 Local minimizers

Definitions 5.1. Let $u \in H_0^1(0,1)$. We say that u is a *weak local minimizer* of I if there exists $\varepsilon > 0$ such that

$$I(v) \geqslant I(u) \quad \text{for all } v \in H_0^1(0,1) \text{ with } \|v-u\|_{1,\infty} < \varepsilon,$$

where $\|w\|_{1,\infty} = \operatorname{ess\,sup}_{x\in[0,1]}[|w(x)| + |w_x(x)|]$, an H^1 *local minimizer* of I if there exists $\varepsilon > 0$ such that

$$I(v) \geqslant I(u) \quad \text{for all } v \in H_0^1(0,1) \quad \text{with } \|v-u\|_{1,2} < \varepsilon,$$

and a *strong local minimizer* of I if there exists $\varepsilon > 0$ such that

$$I(v) \geqslant I(u) \quad \text{for all } v \in H_0^1(0,1) \quad \text{with } \|v-u\|_{C([0,1])} < \varepsilon.$$

Because of the embedding of $H^1(0,1)$ in $C([0,1])$ (Corollary 4.11) we have that $\|v-u\|_{C([0,1])} \leqslant C\|v-u\|_{1,2}$ for $u,v \in H_0^1(0,1)$, where $C > 0$ is a constant. From this we see that

$$\begin{aligned} u \text{ a strong local minimizer} \quad &\Rightarrow \quad u \text{ an } H^1 \text{ local minimizer} \\ &\Rightarrow \quad u \text{ a weak local minimizer.} \end{aligned}$$

We shall see later that for the functional (5.1) the definitions are equivalent, which is not true for general one-dimensional problems of the calculus of variations of the form (4.1).

If there exists $\varepsilon > 0$ such that

$$I(v) > I(u) \text{ for all } v \in H_0^1(0,1) \text{ with } 0 < \|v-u\|_{1,2} < \varepsilon$$

then we say that u is a *strict* H^1 local minimizer.

Theorem 5.2. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^1 and let u be a weak local minimizer of I in $H_0^1(0,1)$. Then $u \in C^2([0,1])$ and satisfies the Euler-Lagrange equation*

$$u_{xx} = F_u(x,u) \tag{5.3}$$

for all $x \in [0,1]$.

Proof. Let $\varphi \in C_0^1(0,1)$. Then $I(u+t\varphi)$ has a local minimum at $t=0$, so that

$$\frac{d}{dt}I(u+t\varphi)|_{t=0} = 0,$$

provided the derivative exists. But by dominated convergence we see that

$$\frac{d}{dt}I(u+t\varphi)|_{t=0} = \int_0^1 [u_x\varphi_x + F_u(x,u)\varphi]\,dx = 0 \tag{5.4}$$

for all $\varphi \in C_0^1(0,1)$. Therefore

$$\int_0^1 \left(u_x - \int_0^x F_u(y,u)\,dy\right)\varphi_x\,dx = 0 \text{ for all } \varphi \in C_0^1(0,1),$$

so that by the du Bois Reymond lemma (Lemma 4.9) we have that

$$u_x(x) = \int_0^x F_u(y,u)\,dy + c$$

for a.e. $x \in (0,1)$ and some constant c. Thus u_x has a continuous representative on $[0,1]$, so that, by Theorem 4.10, $u \in C^1([0,1])$, and $u_x \in C^1([0,1])$ with continuous derivative $F_u(x,u)$. □

Remark 2. This is an example of a *regularity theorem.* A weak local minimizer u of I in $H_0^1(0,1)$ is a *weak solution* of the Euler-Lagrange equation, i.e. u satisfies (5.4), and any weak solution has better regularity, i.e. $u \in C^2([0,1])$. If $F \in C^\infty(\mathbb{R}^2)$ then $u \in C^\infty([0,1])$ (see Problem 5.3).

Theorem 5.3 (Existence of a potential well in $H_0^1(0,1)$)**.** *Let u be a strict H^1 local minimizer of I. Then for $\delta > 0$ sufficiently small*

$$\inf_{v \in H_0^1(0,1), \|v-u\|_{1,2}=\delta} I(v) > I(u). \tag{5.5}$$

Proof. Let $0 < \delta < \varepsilon$, where $\varepsilon > 0$ is as in the definition of strict H^1 local minimizer, and suppose that (5.5) is false. Then there exist $v^{(j)} \in H_0^1(0,1)$ with $\|v^{(j)} - u\|_{1,2} = \delta$ and $I(v^{(j)}) \to I(u)$. Since $\|v^{(j)}\|_{1,2} \leqslant \|u\|_{1,2} + \delta$ it follows that $v^{(j)}$ is bounded in $H^1(0,1)$, and so we may assume that $v^{(j)} \rightharpoonup v$ in $H_0^1(0,1)$ for some $v \in H_0^1(0,1)$. But then by the lower semicontinuity argument in the proof of Theorem 5.1 we have that

$$\|v-u\|_{1,2}^2 \leqslant \liminf_{j\to\infty} \|v^{(j)} - u\|_{1,2}^2 = \delta^2$$

and $I(v) \leqslant I(u)$. Since u is a strict H^1 local minimizer it follows that $v = u$. But by the compactness of the embedding of $H^1(0,1)$ in $C([0,1])$ and dominated convergence it follows that $\lim_{j\to\infty} \int_0^1 F(x, v^{(j)})\,dx = \int_0^1 F(x,u)\,dx$ and so $\lim_{j\to\infty} \int_0^1 v_x^{(j)2}\,dx = \int_0^1 u_x^2\,dx$. Hence by Problem 4.1(i) $v^{(j)} \to u$ in $H_0^1(0,1)$, contradicting $\|v^{(j)} - u\|_{1,2} = \delta$. □

Theorem 5.4. *Let u be a weak local minimizer of I in $H_0^1(0,1)$. Then u is a strong local minimizer.*

Proof. (*Not for examination.*) We can suppose without loss of generality that $u = 0$. Let $v \in H_0^1(0,1)$ with $\|v\|_{C([0,1])} \leqslant h^2$, where $h = \frac{1}{N}$ and N is a sufficiently large integer. Define v_h in each interval $[jh,(j+1)h]$, $j = 0, 1, \ldots, N$ to be a minimizer of

$$I_j(z) = \int_{jh}^{(j+1)h} \left(\frac{1}{2} z_x^2 + F(x,z) \right) dx$$

in $H^1(0,1)$ subject to $z(jh) = v(jh), z((j+1)h) = v((j+1)h)$, which exists from Theorem 5.1. Then by Theorem 5.2 v_h is continuous and piecewise C^1, and thus by Example 4.3 belongs to $H_0^1(0,1)$. We claim that there is a constant C independent of h such that $\|v_h\|_{1,\infty} \leqslant Ch$. Since 0 is a weak local minimizer it follows that $I(v_h) \geqslant I(0)$ for h sufficiently small, and by construction $I(v) \geqslant I(v_h)$.

To prove the claim note that

$$\left| \frac{z(j+h) - z(jh)}{h} \right| \leqslant \frac{2h^2}{h} = 2h. \tag{5.6}$$

In particular, setting $l_j(x) = z(jh) + h^{-1}(x - jh)(z((j+1)h) - z(jh))$ we have

$$I_j(v_h) \leqslant I_j(l_j) \leqslant \int_{jh}^{(j+1)h} [2h^2 + F(x,l_j)]\,dx,$$

and since F is bounded below this yields

$$\int_{jh}^{(j+1)h} (v_h)_x^2\,dx \leqslant C_0 h \tag{5.7}$$

for some constant C_0. But making a linear change of variables in the Poincaré inequality (4.20) we deduce that

$$\|v_h - l_j\|^2_{C([jh,(j+1)h])} \leqslant C_1 h \int_{jh}^{(j+1)h} (v_h - l_j)_x^2 \, dx$$

for some constant C_1 (in fact we can take $C_1 = 1$ but this is not important). Since $|l_j(x)| \leqslant h^2$ and $|v_h - l_j)_x|^2 \leqslant 2((v_h)_x^2 + (l_j)_x^2)$ it follows from (5.6), (5.7) that

$$|v_h(x)| \leqslant C_2 h \tag{5.8}$$

for some constant C_2. Now by the mean value theorem there is a point $\bar{x} \in [jh,(j+1)h]$ with $(v_h)_x(\bar{x}) = h^{-1}(z((j+1)h) - z(jh))$, and this is by (5.6) of order h. But the Euler-Lagrange equation $(v_h)_{xx} = F_u(x, v_h)$ and the boundedness of v_h then imply that $|(v_h)_x(x)| \leqslant C_3 h$ for all $x \in [jh,(j+1)h]$, and since the constants are independent of j we have that $\|v_h\|_{1,\infty} \leqslant Ch$ as required. □

5.3 Problems

5.1. Let

$$I(u) = \int_0^1 \left[\frac{(u_x^2 - 1)^2}{u_x^2 + 1} + u^2 \right] dx.$$

Show that

$$\inf_{u \in H_0^1(0,1)} I(u) = 0,$$

but that the infimum is not attained.

(*Hint.* Consider piecewise affine functions with slopes ± 1.)

5.2. Consider the integral

$$I(u) = \int_0^1 [g(u_x) + F(x,u)]\, dx,$$

where F is continuous and bounded below, and where $g = g(p)$ is C^2 and satisfies

(i) g is convex, i.e. $g''(p) \geqslant 0$ for all p,
(ii) $g(p) \geqslant C_1 p^2 + C_2$, for all p, where $C_1 > 0$ and C_2 are constants,
(iii) $|g'(p)| \leqslant C_3(|p| + 1)$ for all p, where $C_3 > 0$ is a constant.

Prove that I attains an absolute minimum on $H_0^1(0,1)$.

(*Hint.* Show using (i) that g satisfies the inequality

$$g(q) \geqslant g(p) + g'(p)(q-p)$$

for all p, q.)
Show that the example in Qn. 1 satisfies (ii) and (iii) but not (i).

5.3. Show that if $F : \mathbb{R}^2 \to \mathbb{R}$ is C^∞ then if u is a weak local minimizer of

$$I(u) = \int_0^1 \left[\frac{1}{2} u_x^2 + F(x,u) \right] dx$$

in $H_0^1(0,1)$ then $u \in C^\infty([0,1])$.

5.4. Consider the integral

$$I(u) = \int_0^1 (u_x^4 - 3u_x^3 + u_x^2)\, dx. \tag{5.9}$$

Show that $u = 0$ is a weak local minimizer of I in $H_0^1(0,1)$ but is not an H^1 or strong local minimizer.

(*Hint.* Consider the function

$$v(x) = \begin{cases} px, & x \in [0, \lambda] \\ \frac{p\lambda}{1-\lambda}(1-x) & x \in [\lambda, 1] \end{cases}$$

for suitable λ, p.)

5.5. Consider the integral

$$I(u) = \int_0^1 \left(\frac{1}{2} u_x^2 - 60 \sin^6 u \right) dx.$$

Show that I attains an absolute minimum in $H_0^1(0,1)$. Show that there is a potential well in $H_0^1(0,1)$ at $u = 0$. By choosing a suitable piecewise affine function, or otherwise, show that $u = 0$ is not an absolute minimizer of I in $H_0^1(0,1)$ and that there are at least two distinct absolute minimizers.

Chapter 6

Approach to equilibrium for a parabolic PDE.

We consider the semilinear parabolic PDE for $u = u(x,t)$

$$u_t = u_{xx} - f(x,u), \quad x \in [0,1], \tag{6.1}$$

where f is a sufficiently smooth function, with boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{6.2}$$

and initial condition

$$u(x,0) = u_0(x), \tag{6.3}$$

where u_0 is a given function. We have seen that formally

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u_x^2 + F(x,u)\right) dx = -\int_0^1 u_t^2\, dx \leqslant 0, \tag{6.4}$$

where

$$F(x,u) = \int_0^u f(x,s)\, ds, \tag{6.5}$$

so that

$$I(u) = \int_0^1 \left(\frac{1}{2}u_x^2 + F(x,u)\right) dx \tag{6.6}$$

is a Lyapunov function.

We will study the approach to equilibrium of solutions to this problem by applying the invariance principle Theorem 3.2 using as the metric space the Sobolev space $X = H_0^1(0,1)$, which incorporates the boundary conditions (6.2) automatically. In the following we shall use the norm $\|v\|_X = \left(\int_0^1 v_x^2\, dx\right)^{\frac{1}{2}}$ on X, which we have seen in Theorem 4.14 is an equivalent norm to $\|\cdot\|_{1,2}$ on $H_0^1(0,1)$. In order to verify the hypotheses of Theorem 3.2 we need to

(i) Prove that (6.1) generates a semiflow on X (see Theorem 6.4),

(ii) Show that I is a Lyapunov function, so that the above formal calculation is correct (see Theorem 6.5),

(iii) Prove that positive orbits are relatively compact in X (see Theorem 6.7),

(iv) Give conditions under which the rest points are isolated in X (see Example 6.1).

To carry out (i) we will regard (6.1) as a perturbation of the linear heat equation, which we therefore first study in some detail.

6.1 The heat equation

Consider the linear heat equation

$$u_t = u_{xx}, \quad x \in [0,1], \tag{6.7}$$

with the boundary conditions (6.2), which are incorporated in the space $H_0^1(0,1)$. By Theorem 4.5 we can write any $v \in L^2(0,1)$ as the convergent series in $L^2(0,1)$

$$v(x) = \sum_{j=1}^{\infty} v_j \sin j\pi x, \tag{6.8}$$

where $v_j = 2\int_0^1 v(x)\sin j\pi x\,dx$, and if $v \in H_0^1(0,1)$ then by Theorem 4.15 the series is convergent in $H_0^1(0,1)$.

In the following we will also use the space

$$H^2(0,1) = \{v \in H^1(0,1) : v_x \in H^1(0,1)\},$$

and if $v \in H^2(0,1)$ we write $v_{xx} = (v_v)_x$. $H^2(0,1)$ is a Hilbert space with norm

$$\|v\|_{H^2} = \left(\int_0^1 (v^2 + v_x^2 + v_{xx}^2)\,dx\right)^{\frac{1}{2}}.$$

Let $T > 0$. By a *weak solution* $u = u(x,t)$ of the heat equation (6.7) on $[0,T]$ we mean a continuous function $u : [0,T] \to L^2(0,1)$ (so that $u(t)(x) = u(x,t)$) with $u : (0,T] \to X$ continuous, such that for any $\varphi \in X$ the function $t \mapsto (u(t),\varphi)$ belongs to $C^1((0,T])$ and

$$\frac{d}{dt}(u(t),\varphi) = -(u(t)_x, \varphi_x), \quad t \in (0,T]. \tag{6.9}$$

This is what is obtained formally by multiplying (6.7) by φ, and integrating by parts, so that

$$\frac{d}{dt}\int_0^1 u\varphi\,dx = \int_0^1 u_{xx}\varphi\,dx = -\int_0^1 u_x\varphi_x\,dx.$$

The definition of a weak solution gives a meaning to a solution for which the second derivative u_{xx} might not exist.

Theorem 6.1. *Given any $u_0 \in L^2(0,1)$ there exists a unique weak solution u of the heat equation (6.7) satisfying $u(0) = u_0$, defined for all $t \geqslant 0$ and given by*

$$u(x,t) = \sum_{j=1}^{\infty} e^{-j^2\pi^2 t} u_{0j} \sin j\pi x. \tag{6.10}$$

The solution $u(x,t) = (e^{\Delta t}u_0)(x)$ generates a semiflow $\{e^{\Delta t}\}_{t \geqslant 0}$ on $L^2(0,1)$ satisfying

$$\|e^{\Delta t}u_0\|_2 \leqslant e^{-\pi^2 t}\|u_0\|_2, \quad t \geqslant 0, \tag{6.11}$$

$$\|e^{\Delta t}u_0\|_X \leqslant \frac{1}{(et)^{\frac{1}{2}}} e^{-\frac{1}{2}\pi^2 t}\|u_0\|_2, \quad t > 0. \tag{6.12}$$

Furthermore, $e^{\Delta t}u_0 \in H^2(0,1)$ for all $t > 0$, $t \mapsto e^{\Delta t}u_0$ is continuous from $(0,\infty) \to H^2(0,1)$ and

$$\|u_{xx}(\cdot,t)\|_2 \leqslant \frac{2}{et} e^{-\frac{1}{2}\pi^2 t}\|u_0\|_2, \quad t > 0. \tag{6.13}$$

$\{e^{\Delta t}\}_{t \geqslant 0}$ is also a semiflow on X, satisfying for $u_0 \in X$

$$\|e^{\Delta t}u_0\|_X \leqslant e^{-\pi^2 t}\|u_0\|_X, \tag{6.14}$$

$$\|u_{xx}(\cdot,t)\|_2 \leqslant \frac{1}{(et)^{\frac{1}{2}}} e^{-\frac{1}{2}\pi^2 t}\|u_0\|_X, \quad t > 0. \tag{6.15}$$

Proof. If u is a weak solution of (6.7) on $[0,T]$ satisfying $u(0) = u_0 \in L^2(0,1)$ then writing $u(x,t) = \sum_{j=1}^{\infty} u_j(t) \sin j\pi x$ and choosing $\varphi(x) = \sin j\pi x$ in (6.9) we see that

$$\begin{aligned} \dot{u}_j(t) &= -j^2\pi^2 u_j(t) \\ u_j(0) &= u_{0j}, \end{aligned}$$

so that u is given by (6.10). In particular u is unique.

Conversely, suppose u is given by (6.10). Note first that $u(t) \in L^2(0,1)$ for all $t \geqslant 0$ and that $u : [0,\infty) \to L^2(0,1)$ is continuous at zero. In fact

$$\|u(t) - u_0\|_2^2 \quad = \quad \frac{1}{2}\sum_{j=1}^{\infty}(1 - e^{-j^2\pi^2 t})^2 u_{0j}^2,$$

and $\|u_0\|_2^2 = \frac{1}{2}\sum_{j=1}^{\infty} u_{0j}^2 < \infty$. Given $\varepsilon > 0$ choose J sufficiently large for $\sum_{j=J}^{\infty} u_{0j}^2 < \varepsilon^2$, and then t sufficiently small for $\frac{1}{2}\sum_{j=1}^{J-1}(1 - e^{-j^2\pi^2 t})^2 u_{0j}^2 < \frac{1}{2}\varepsilon^2$, so that $\|u(t) - u_0\|_2 < \varepsilon$, proving the continuity at zero.

Next define $u^{(m)}(t)(x) = u^{(m)}(x,t)$ by

$$u^{(m)}(x,t) = \sum_{j=1}^{m} e^{-j^2\pi^2 t} u_{0j} \sin j\pi x.$$

Then

$$u^{(m)}(t)_x = \sum_{j=1}^{m} j\pi e^{-j^2\pi^2 t} u_{0j} \cos j\pi x$$

so that $u^{(m)}(t) \in X$ and if $n \geqslant m$

$$\|u^{(n)}(t) - u^{(m)}(t)\|_X^2 = \frac{1}{2} \sum_{j=m+1}^{n} j^2\pi^2 e^{-2j^2\pi^2 t} u_{0j}^2.$$

Hence for $t \geqslant \delta > 0$

$$\|u^{(n)}(t) - u^{(m)}(t)\|_X^2 \leqslant \frac{1}{2} \sum_{j=m+1}^{n} j^2\pi^2 e^{-2j^2\pi^2 \delta} u_{0j}^2,$$

and thus $u^{(m)}$ is a Cauchy sequence in $C([\delta, T]; X)$ for any $T > 0$, and so its limit $u \in C([\delta, T]; X)$. By the Poincaré inequality we thus have $u \in C([\delta, T]; L^2(0, 1))$ and hence $u : [0, \infty) \to L^2(0, 1)$ and $u : (0, \infty) \to X$ are continuous.

If $\varphi \in X$ then for any $\delta > 0, t \geqslant \delta$

$$(u^{(m)}(t), \varphi) = (u^{(m)}(\delta), \varphi) - \int_\delta^t (u^{(m)}(s)_x, \varphi_x)\, ds,$$

and we may pass to the limit $m \to \infty$ to obtain

$$(u(t), \varphi) = (u(\delta), \varphi) - \int_\delta^t (u(s)_x, \varphi_x)\, ds,$$

so that u is a weak solution.

The semiflow properties

(i) $e^{\Delta 0} =$ identity,

(ii) $e^{\Delta(s+t)} = e^{\Delta s} e^{\Delta t}, \quad s \geqslant 0, t \geqslant 0,$

are obvious. Also

$$\|e^{\Delta t} u_0\|_2^2 = \frac{1}{2} \sum_{j=1}^{\infty} e^{-2j^2\pi^2 t} u_{0j}^2 \leqslant e^{-2\pi^2 t} \|u_0\|_2^2,$$

so that (6.11) holds. Since $e^{\Delta t} : L^2(0, 1) \to L^2(0, 1)$ is linear, it follows from (6.11) that $e^{\Delta t}$ is continuous. Hence $\{e^{\Delta t}\}_{t \geqslant 0}$ is a semiflow on $L^2(0, 1)$.

To prove (6.12) note that for $t > 0$

$$
\begin{aligned}
\|e^{\Delta t}u_0\|_X^2 &= \frac{1}{2}\sum_{j=1}^{\infty} j^2\pi^2 e^{-2j^2\pi^2 t}u_{0j}^2 \\
&\leqslant \frac{1}{2}e^{-\pi^2 t}\sum_{j=1}^{\infty} j^2\pi^2 e^{-j^2\pi^2 t}u_{0j}^2 \\
&\leqslant \frac{1}{2}e^{-\pi^2 t}\max_{\tau\geqslant 0}\tau e^{-\tau t}\sum_{j=1}^{\infty} u_{0j}^2 \\
&= \frac{1}{et}e^{-\pi^2 t}\|u_0\|_2^2.
\end{aligned}
$$

Using $u^{(m)}$ as above we obtain that $u_{xx} : (0,\infty) \to L^2(0,1)$ is continuous, and for $t > 0$

$$
\begin{aligned}
\|u_{xx}(\cdot,t)\|_2^2 &= \frac{1}{2}\sum_{j=1}^{\infty}(j\pi)^4 e^{-2j^2\pi^2 t}u_{0j}^2 \\
&\leqslant e^{-\pi^2 t}\max_{\tau\geqslant 0}\tau^2 e^{-\tau t}\|u_0\|_2^2 \\
&= \left(\frac{2}{et}\right)^2 e^{-\pi^2 t}\|u_0\|_2^2,
\end{aligned}
$$

giving (6.12).

If $u_0 \in X$ then $\|u_0\|_X^2 = \frac{1}{2}\sum_{j=1}^{\infty} j^2\pi^2 u_{0j}^2$, from which the continuity of $u : [0,\infty) \to X$ at zero, and the estimates (6.14), (6.15) follow by similar arguments. In particular (6.14) implies that $\{e^{\Delta t}\}_{t\geqslant 0}$ is a semiflow on X $\square$

Remark 3.
1. Using similar arguments one can prove that $u(x,t)$ is a smooth function of $x \in [0,1], t > 0$.
2. Note that we can have, for example, $u_0 = 1$, which does not satisfy the boundary conditions, but that the solution $u(t) = e^{\Delta t}u_0$ satisfies the boundary conditions for arbitrarily small $t > 0$.

6.2 The inhomogeneous equation

We now consider the inhomogeneous equation

$$
u_t = u_{xx} + g(x,t), \quad x \in [0,1], t \in (0,T], \tag{6.16}
$$

with the same boundary conditions (6.2), where $g(x,t) = g(t)(x)$ and $g : [0,T] \to L^2(0,1)$ is continuous. For each t we can write $g(t)$ as the convergent series in $L^2(0,1)$

$$
g(t)(x) = \sum_{j=1}^{\infty} g_j(t)\sin j\pi x
$$

with each $g_j : [0.T] \to \mathbb{R}$ continuous.

By a *weak solution* of (6.16) on $[0,T]$ we mean a continuous function $u : [0,T] \to L^2(0,1)$ with $u : (0,T] \to X$ continuous, such that for any $\varphi \in X$ the function $t \mapsto (u(t),\varphi)$ belongs to $C^1((0,T])$ and

$$\frac{d}{dt}(u(t),\varphi) = -(u(t)_x,\varphi_x) + (g(t),\varphi), \quad t \in (0,T]. \tag{6.17}$$

Let $u_0 \in L^2(0,1)$. Choosing $\varphi(x) = \sin j\pi x$ in (6.17) we find that if $u(x,t) = \sum_{j=1}^{\infty} u_j(t) \sin j\pi x$ is a weak solution with $u(0) = u_0$ then

$$\begin{aligned} \dot{u}_j(t) &= -j^2\pi^2 u_j(t) + g_j(t) \\ u_j(0) &= u_{0j}, \end{aligned}$$

that is

$$u_j(t) = e^{-j^2\pi^2 t}u_{0j} + \int_0^t e^{-j^2\pi^2(t-s)} g_j(s)\,ds. \tag{6.18}$$

Multiplying (6.18) by $\sin j\pi x$ and summing we obtain the *variation of constants formula*

$$u(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)} g(s)\,ds. \tag{6.19}$$

In (6.19) the integral is an integral in the Hilbert space $L^2(0,1)$ (*details not for examination*). If H is a Hilbert space with norm $\|\cdot\|$, inner product $\langle\cdot,\cdot\rangle$ and orthonormal basis $\{\omega_j\}_{j=1}^{\infty}$, and if $h : [a,b] \to H$ is continuous, where $a < b$, then we can write $h(t) = \sum_{j=1}^{\infty} \langle h(t),\omega_j\rangle\omega_j$ and define

$$\int_a^b h(t)\,dt = \sum_{j=1}^{\infty} \left(\int_a^b \langle h(t),\omega_j\rangle\,dt \right) \omega_j. \tag{6.20}$$

This is well defined as an element of H since

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\int_a^b \langle h(t),\omega_j\rangle dt \right)^2 &\leqslant \sum_{j=1}^{\infty} (b-a) \int_a^b \langle h(t),\omega_j\rangle^2 dt \\ &\leqslant (b-a) \int_a^b \|h(t)\|^2 dt < \infty. \end{aligned}$$

With the choice $\omega_j(x) = \sqrt{2}\sin j\pi x$ this definition corresponds to the integral in (6.19).

Lemma 6.2. *The definition is independent of the choice of orthonormal basis,*

$$\langle \int_a^b h(t)\,dt, v\rangle = \int_a^b \langle h(t),v\rangle\,dt, \quad \textit{for all } v \in H \tag{6.21}$$

and

$$\| \int_a^b h(t)\,dt \| \leqslant \int_a^b \|h(t)\|\,dt. \tag{6.22}$$

Proof. Let $v \in H$. Then $v = \sum_{j=1}^{\infty} \langle v, \omega_j \rangle \omega_j$ and

$$\begin{aligned}\langle \int_a^b h(t)\,dt, v\rangle &= \sum_{j=1}^{\infty} \int_a^b \langle h(t), \omega_j\rangle\,dt \langle v, \omega_j\rangle \\ &= \int_a^b \langle h(t), v\rangle\,dt,\end{aligned}$$

where we can take the infinite sum inside the integral by dominated convergence. Since v is arbitrary the definition is independent of the orthonormal basis (since if two different orthonormal bases gave different values to the integral, the difference would be orthogonal to any v). Furthermore

$$\left|\langle \int_a^b h(t)\,dt, v\rangle\right| \leqslant \int_a^b \|h(t)\|\,dt\,\|v\|,$$

giving (6.21), and (6.22) follows by choosing $v = \int_a^b h(t)\,dt$. □

Remark 4. If $h : (a,b) \to H$ is continuous with $\int_a^b \|h(t)\|_X\,dt < \infty$ then we can define

$$\int_a^b h(t)\,dt = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b-\varepsilon} h(t)\,dt,$$

noting that the integral on the right-hand side is a Cauchy sequence in H as $\varepsilon \to 0$, and (6.21), (6.22) still hold.

Lemma 6.3. *u is a weak solution of* (6.16) *with $u(0) = u_0$ if and only if u is given by the variation of constants formula*

$$u(t) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} g(s)\,ds, \quad t \in [0,T]. \tag{6.23}$$

If, further, $g(s) = z + \gamma(s)$, where $z \in L^2(0,1)$ and $\gamma : [0,T] \to X$ is continuous, then $u : (0,T] \to H^2(0,1)$ and

$$\|u(t)_{xx}\|_2 \leqslant \frac{2}{et} e^{-\frac{1}{2}\pi^2 t}\|u_0\|_2 + \|e^{\Delta t} z - z\|_2 + m(t) \max_{s \in [0,t]} \|\gamma(s)\|_X, \;\; t \in [0,T] \tag{6.24}$$

where $m(t) = \int_0^t \frac{1}{(e\tau)^{\frac{1}{2}}} e^{-\frac{1}{2}\pi^2 \tau} d\tau$.

Proof. (*Not for examination.*) We have already shown that if u is a weak solution with $u(0) = u_0$ then u is given by (6.23). Conversely let u be given by (6.23). Clearly $u(0) = u_0$. Let

$$u^{(m)}(t) = e^{\Delta t} u_0^{(m)} + \int_0^t e^{\Delta(t-s)} g^{(m)}(s)\,ds, \tag{6.25}$$

where $u_0^{(m)}(x) = \sum_{j=1}^m u_{0j}\sin j\pi x$, $g^{(m)}(s)(x) = \sum_{j=1}^m g_j(s)\sin j\pi x$, so that $u_0^{(m)} = u^{(m)}(0) \to u_0$ in $L^2(0,1)$, and we have[1] that $g^{(m)} \to g$ in $C([0,T];L^2(0,1))$. Then by (6.11) for $n \geqslant m$

$$\|u^{(n)}(t) - u^{(m)}(t)\|_2 \leqslant \|u_0^{(n)} - u_0^{(m)}\|_2 + \int_0^t \|g^{(n)}(s) - g^{(m)}(s)\|ds.$$

Hence $u^{(m)}$ is a Cauchy sequence in $C([0,T];L^2(0,1))$, and so its limit $u : [0,T] \to L^2(0,1)$ is continuous.

Similarly, using (6.12), for $t > 0$

$$\begin{aligned}\|u^{(n)}(t) - u^{(m)}(t)\|_X \leqslant{}& \frac{1}{(et)^{\frac12}} e^{-\frac12\pi^2 t}\|u_0^{(n)} - u_0^{(m)}\|_2 \\ &+ \int_0^t \frac{1}{(e(t-s))^{\frac12}} e^{-\frac12\pi^2(t-s)}\|g^{(n)}(s) - g^{(m)}(s)\|_2 ds,\end{aligned}$$

and so $u^{(m)}$ is a Cauchy sequence in $C([\delta,T];X)$ for any $\delta > 0$. Thus $u : (0,T] \to X$ is continuous.

If $\varphi \in X$ then we have that

$$(u^{(m)}(t),\varphi) = (u^{(m)}(\delta),\varphi) - \int_\delta^t (u^{(m)}(s)_x, \varphi_x)ds + \int_\delta^t (g^{(m)}(s),\varphi)ds,$$

and passing to the limit $m \to \infty$ we deduce that u is a weak solution.

Now suppose that $g(s) = z + \gamma(s)$, where $z \in L^2(0,1)$ and $\gamma \in C([0,T];X)$. The integral term in (6.23) has two parts, the first being

$$\begin{aligned}\left(\int_0^t e^{\Delta(t-s)} z\, ds\right)(x) &= \int_0^t \sum_{j=1}^\infty e^{-j^2\pi^2(t-s)} z_j \sin j\pi x\, ds \\ &= \sum_{j=1}^\infty \frac{1}{j^2\pi^2}(1 - e^{-j^2\pi^2 t}) z_j \sin j\pi x. \end{aligned} \tag{6.26}$$

Thus

$$\begin{aligned}\left(\int_0^t e^{\Delta(t-s)} z\, ds\right)_{xx} &= -\sum_{j=1}^\infty (1 - e^{-j^2\pi^2 t}) z_j \sin j\pi x \\ &= e^{\Delta t} z - z,\end{aligned}$$

so that the left-hand side is a continuous map from $[0,T] \to L^2(0,1)$.

The second part of the integral term is

$$\int_0^t e^{\Delta(t-s)}\gamma(s)\, ds,$$

[1] If $f \in C([0,T];H)$, where H is a Hilbert space with orthonormal basis $\{\omega_j\}$, and if $f^{(m)}(t) = \sum_{j=1}^m f_j(t)\omega_j$, where $f_j(t) = \langle f(t),\omega_j\rangle$, then $\|f^{(m)}(t) - f(t)\|^2 = \sum_{j=m+1}^\infty f_j(t)^2 \stackrel{\text{def}}{=} h_m(t)$ and h_m is continuous with $h_{m+1} \leqslant h_m$, and tends to zero as $m \to \infty$ for each t. Hence by Dini's theorem $h_m \to 0$ uniformly and hence $f^{(m)} \to f$ in $C([0,T];H)$.

and by (6.15) we have that

$$\begin{aligned}
\left\| \left(\int_0^t e^{\Delta(t-s)} \gamma(s)\, ds \right)_{xx} \right\|_2 &= \left\| \int_0^t (e^{\Delta(t-s)}\gamma(s))_{xx} ds \right\|_2 \\
&\leqslant \int_0^t \|(e^{\Delta(t-s)}\gamma(s))_{xx}\|_2 ds \\
&\leqslant \int_0^t \frac{1}{(e(t-s))^{\frac{1}{2}}} e^{-\frac{1}{2}\pi^2(t-s)} \|\gamma(s)\|_X ds \\
&\leqslant m(t) \max_{s\in[0,t]} \|\gamma(s)\|_X, \qquad (6.27)
\end{aligned}$$

which can be justified using partial sums as above. Indeed, setting $\gamma^{(m)}(t) = \sum_{j=1}^m \gamma_j(t) \sin j\pi x$, this is true with $\gamma(t)$ replaced by $\gamma^{(m)}(t)$ and also when $\gamma(t)$ is replaced by $\gamma^{(m)}(t) - \gamma^{(n)}(t)$. Hence

$$\left(\int_0^t e^{\Delta(t-s)} \gamma^{(m)}(s)\, ds \right)_{xx} = \int_0^t (e^{\Delta(t-s)}\gamma^{(m)}(s))_{xx} ds$$

is a Cauchy sequence in $L^2(0,1)$ and thus converges to some $v \in L^2(0,1)$ as $m \to \infty$. Passing to the limit in the definition of weak derivative, noting that $w = \int_0^t e^{\Delta(t-s)}\gamma(s)\, ds \in X$, we see that $w \in H^2(0,1)$ with $w_{xx} = v$. Then we can pass to the limit $m \to \infty$ to get (6.27). □

6.3 Existence of a semiflow

We suppose that $f = f(x,u)$ is C^2 in (x,u). For $u \in X$ define

$$\begin{aligned}
\gamma(u)(x) &= f(x,0) - f(x,u(x)), \qquad (6.28)\\
z(x) &= -f(x,0). \qquad (6.29)
\end{aligned}$$

Then $\gamma(u) \in C([0,1])$ and $\gamma(u)(0) = \gamma(u)(1) = 0$. Also the weak derivative $\gamma(u)_x$ exists and is given by

$$\gamma(u)_x(x) = f_x(x,0) - f_x(x,u(x)) - f_u(x,u(x))u_x(x), \qquad (6.30)$$

and so $\gamma : X \to X$ Furthermore

$$\|\gamma(u) - \gamma(v)\|_X \leqslant K_M \|u - v\|_X, \quad \text{if } \|u\|_X \leqslant M, \|v\|_X \leqslant M, \qquad (6.31)$$

since, for example,

$$\begin{aligned}
\|f_u(\cdot,u)u_x - f_u(\cdot,v)v_x\|_2 &\leqslant \|f_u(\cdot,u)(u_x - v_x)\|_2 + \|(f_u(\cdot,v) - f_u(\cdot,u))v_x\|_2 \\
&\leqslant c_M\|u_x - v_x\|_2 + d_M\|u - v\|_{C([0,1])} \\
&\leqslant K_M\|u_x - v_x\|_2,
\end{aligned}$$

where $c_M = \max_{x\in[0,1],|z|\leqslant M} |f_u(x,z)|$, $d_M = M \max_{x\in[0,1],|z|\leqslant M} |f_{uu}(x,z)|$, and where we have used the inequality $\|u\|_{C([0,1])} \leqslant \|u\|_X$. Thus $\gamma : X \to X$ is locally Lipschitz.

Theorem 6.4. *Let $G : X \to L^2(0,1)$ have the form $G(u) = z + \gamma(u)$, where $z \in L^2(0,1)$ and $\gamma : X \to X$ is locally Lipschitz. Given $u_0 \in X$ there exists a unique continuous solution $u : [0, t_{\max}) \to X$ of the equation*

$$u(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}G(u(s))\,ds \tag{6.32}$$

defined on a maximal interval of existence $[0, t_{\max})$ with $0 < t_{\max} \leqslant \infty$. If $t_{\max} < \infty$ then

$$\lim_{t\to t_{\max}-} \|u(t)\|_X = \infty. \tag{6.33}$$

The solution $u = u(\cdot, u_0)$ depends continuously on u_0; more precisely, if the solution $u = u(\cdot, u_0)$ exists on the interval $[0,T]$ and if $u_0^{(j)} \to u_0$ in X then for large enough j the solution $u^{(j)} = u(\cdot, u_0^{(j)})$ exists on $[0,T]$ and $u^{(j)} \to u$ in $C([0,T];X)$.

Proof. (*Not for examination.*) The proof is very similar to that for Theorem 2.4. We suppose that γ satisfies (6.31) and for $T > 0$ let $X_M = C([0,T];X)$ with norm $\|u\|_{X_M} = \max_{t\in[0,T]} \|u(t)\|_X$. Since X is complete, the same proof as for Lemma 2.3 shows that X_M is complete. Let

$$Z_M = \{u \in X_M : \|u\|_{X_M} \leqslant 2M\},$$

which is a closed subset of X_M and thus also a complete metric space with metric $d(u,v) = \|u - v\|_{X_M}$. For $u_0 \in X$ with $\|u_0\|_X \leqslant M$, and $u \in Z_M$, define

$$P(u,u_0)(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}G(u(s))\,ds, \quad t \in [0.T].$$

Then

$$P(u,u_0)(t) = e^{\Delta t}u_0 + \zeta(t) + \int_0^t e^{\Delta(t-s)}\gamma(u(s))\,ds,$$

where

$$\zeta(t) = \int_0^t e^{\Delta(t-s)}z\,ds.$$

Arguing as in Lemma 6.3, we have that $\zeta : [0,\infty) \to X$ is continuous with $\zeta(0) = 0$. In particular $\|\zeta(t)\|_X \leqslant \frac{M}{2}$ for t sufficiently small. By Theorem 6.1 and similar arguments to those in the proof of Lemma 6.3 we also have that $P(u,u_0) : [0,T] \to X$ is continuous. Now

$$\begin{aligned}
\|P(u,u_0)(t)\|_X &\leqslant \|u_0\|_X + \|\zeta(t)\|_X + \int_0^t \|\gamma(u(s))\|_X ds \\
&\leqslant \|u_0|_X + \|\zeta(t)\|_X + \int_0^t (K_{2M}\|u(s)\|_X + \|\gamma(0)\|_X) ds \\
&\leqslant M + \frac{M}{2} + (MK_{2M} + \|\gamma(0)\|_X)T \\
&\leqslant 2M
\end{aligned}$$

for all $t \in [0,T]$, provided T is sufficiently small, and thus $P(\cdot, u_0) : Z_M \to Z_M$. Also if $u, v \in Z_M$

$$\|P(u,u_0)(t) - P(v,u_0)(t)\|_X \leqslant TMK_{2M}\|u - v\|_{X_M}, \quad t \in [0,T]$$

and so $P(\cdot, u_0)$ is a uniform contraction for T sufficiently small. Since $\|P(u,u_0)(t) - P(u,v_0)(t)\|_X \leqslant \|u_0 - v_0\|_X$ it follows that $P(u,u_0)$ is continuous in u_0. Hence by Corollary 2.2 there exists a unique fixed point of $P(\cdot, u_0)$ depending continuously on u_0, so that there is a unique solution on the interval $[0,T]$. The remaining assertions are proved in exactly the same way as for Theorem 2.4. $\square$

With $G(u)(x) = -f(x, u(x))$, let

$$I(u) = \int_0^1 \left[\frac{1}{2}u_x^2 + F(x,u)\right] dx, \tag{6.34}$$

where $F(x,u) = \int_0^u f(x,s)\, ds$ and we assume F is bounded below.

Theorem 6.5. *For $0 \leqslant t < t_{\max}$ the solution satisfies*

$$I(u(t)) = I(u_0) - \int_0^t \|u_{xx} - f(x,u)\|_2^2\, ds. \tag{6.35}$$

Proof. (*Not for examination.*) This is a little tricky, since the formal computation of $\frac{d}{dt}I(u(t))$ does not obviously make sense. Therefore we need to approximate the solution in such a way that we can carry out the computation. Let $0 < T < t_{\max}$, and for $t \in [0,T]$ set $\gamma(t) = \gamma(u(t))$, so that $\gamma : [0,T] \to X$ is continuous. For $m = 1, 2, \ldots$ let

$$\begin{aligned} u^{(m)}(t) &= \sum_{j=1}^m u_j(t) \sin j\pi x, \\ u_0^{(m)} &= \sum_{j=1}^m u_{0j} \sin j\pi x, \\ z^{(m)} &= \sum_{j=1}^m z_j \sin j\pi x, \\ \gamma^{(m)}(t)(x) &= \sum_{j=1}^m \gamma_j(t) \sin j\pi x, \end{aligned}$$

so that

$$u^{(m)}(t) = e^{\Delta t}u_0^{(m)} + \int_0^t e^{\Delta(t-s)}(z^{(m)} + \gamma^{(m)}(s))\, ds.$$

Then

$$\|u(t) - u^{(m)}(t)\|_X \leqslant \|u_0 - u_0^{(m)}\|_X + \|z - z^{(m)}\|_2 + \int_0^t \|\gamma(s) - \gamma^{(m)}(s)\|_X\, ds,$$

where we have used (6.26), so that $u^{(m)} \to u$ in $C([0,T];X)$. Since

$$\dot{u}_j^m(t) = -j^2\pi^2 u_j^m(t) + z_j + \gamma_j(t).$$

it follows that

$$\begin{aligned}
\frac{d}{dt}I(u^{(m)}(t)) &= -(u_{xx}^{(m)} - f(\cdot, u^{(m)}), \dot{u}^{(m)}) \\
&= -(u_{xx}^{(m)} - f(\cdot, u^{(m)}), u_{xx}^{(m)} + z^{(m)} + \gamma^{(m)}(t)) \\
&= -\|u_{xx}^{(m)} - f(\cdot, u^{(m)})\|_2^2 \\
&\qquad -(u_{xx}^{(m)} - f(\cdot, u^{(m)}), f(\cdot, u^{(m)}) + z^{(m)} + \gamma^{(m)}(t)).
\end{aligned}$$

The last term can be written in the form

$$-(u_{xx}^{(m)} - f(\cdot, u^{(m)}), f(\cdot, u^{(m)}) - f(\cdot, u) + z^{(m)} - z + \gamma^{(m)}(t) - \gamma(t))$$

and hence for $\delta > 0$ converges to zero in $C([\delta, T])$ as $m \to \infty$ (since $u^{(m)} \to u$, and $\gamma^{(m)} \to \gamma$ in $C([0,T];X)$ and since by (6.24) $u^{(m)}(t)_{xx}$ is bounded on $[\delta, T]$. Hence if $0 < t_1 \leqslant t \leqslant T$

$$I(u(t)) - I(u(t_1)) = -\lim_{m\to\infty} \int_{t_1}^{t} \|u_{xx}^{(m)} - f(\cdot, u^{(m)})\|_2^2 \, ds. \tag{6.36}$$

But from Lemma 6.3 applied to the initial data $u_0 - u_0^{(m)}$ etc we have that $u^{(m)} \to u$ in $C([\delta, T]; H^2(0,1))$ and from this it follows that $\|u^{(m)}(t)_{xx} - f(\cdot, u^{(m)}(t))\|_2^2 \to \|u(t)_{xx} - f(\cdot, u(t))\|_2^2$ in $C([\delta, T])$. Thus

$$I(u(t)) - I(u(t_1)) = -\int_{t_1}^{t} \|u(s)_{xx} - f(\cdot, u(s))\|_2^2 \, ds,$$

and passing to the limit $t_1 \to 0+$ we obtain (6.35). □

Corollary 6.6. *For any $u_0 \in X$ the solution $u(\cdot, u_0)$ exists for all $t \geqslant 0$ and*

$$T(t)u_0 = u(t, u_0)$$

defines a semiflow $\{T(t)\}_{t\geqslant 0}$ on X.

Proof. From (6.35) we deduce that $\|u(t, u_0)\|_X \leqslant \|u_0\|_X + C$ for all $t \in [0, t_{\max})$ and some constant C. Thus $t_{\max} = \infty$ and $\{T(t)\}_{t\geqslant 0}$ defines a semiflow by Theorem 6.4. □

6.4 Asymptotic behaviour

Theorem 6.7. *Suppose that there are only finitely many solutions $w \in C^2([0,1])$ to the equilibrium equation*

$$w_{xx} = f(x, w), \quad x \in [0,1] \tag{6.37}$$

with zero boundary conditions $w(0) = w(1) = 0$. *Then given any* $u_0 \in H_0^1(0,1)$ *the solution* $u = u(x,t)$ *to* (6.1) *satisfies*

$$u(\cdot,t) \to w \quad \text{in } H_0^1(0,1) \text{ as } t \to \infty$$

for some equilibrium solution w. *If* w *is a strict local minimizer of* I *in* $H_0^1(0,1)$ *then* w *is asympotically stable.*

Proof. (*Not for examination.*) By Theorem 6.5 the functional $I : X \to \mathbb{R}$ is a Lyapunov function. Indeed I is continuous and nonincreasing along solutions, and if $I(u(t))$ is constant for a complete orbit then $u_{xx}(x,t) = f(x,u(x,t))$ for all t and so for any $\varphi \in X$ we have $\frac{d}{dt}(u(t),\varphi) = 0$, implying that $(u(t)-u(0),\varphi) = 0$ and hence that $u(t)$ is a rest point. By the invariance principle, Theorem 3.2, to prove that each solution u tends to a rest point we just need to show that the positive orbit of any solution is relatively compact in X. But by Lemma 6.3, the boundedness of $m(t)$ and the boundedness of $u(t)$ in X for $t \geqslant 0$ we have that $u(t)_{xx}$ is bounded in $L^2(0,1)$ for all $t \geqslant \varepsilon > 0$ for any $\varepsilon > 0$. Given any sequence $t_j \to \infty$ there is thus a subsequence t_{j_k} and $v \in X$ with $u(t_{j_k}) \rightharpoonup v$ in X and $u(t_{j_k})_{xx}$ bounded in $L^2(0,1)$, that is $u(t_{j_k})_x$ bounded in $H^1(0,1)$. By the compactness of the embedding of $H^1(0,1)$ in $C([0,1])$ we can thus suppose that $u(t_{j_k}) \to v$ in $C([0,1])$ (hence in $L^2(0,1)$) and $u(t_{j_k})_x \to \chi$ in $C([0,1])$ for some $\chi \in C([0,1])$. Since $u(t_{j_k})_x \rightharpoonup v_x$ in $L^2(0,1)$, by the uniqueness of weak limits $\chi = v_x$. Thus $u(t_{j_k}) \to v$ strongly in X, proving the relative compactness.

The asymptotic stability of strict local minimizers follows immediately from Theorems 3.4 and 5.3. $\square$

Example 6.1. Suppose that $f(x,u) = \lambda(u^3 - u)$, where $\lambda > 0$ is a constant. Then there are just finitely many solutions of

$$w_{xx} = \lambda(w^3 - w) \tag{6.38}$$

satisfying the boundary conditions

$$w(0) = w(1) = 0, \tag{6.39}$$

so that the hypotheses of Theorem 6.7 are satisfied. To prove this (*not for examination*) we first show that given $a < b$ there is at most one solution to (6.38) on $[a,b]$ satisfying $w(a) = w(b) = 0$ and such that $w(x) > 0$ for $x \in (a,b)$. Indeed suppose there are two distinct such solutions w^1, w^2, and let (c,d) be a maximal subinterval of (a,b) on which they differ, so that without loss of generality $w^1(x) > w^2(x)$ for all $x \in (c,d)$ and $w^1(c) = w^2(c), w^1(d) = w^2(d)$. Letting $z(x) = e^{\sqrt{\lambda}x}w(x)$ the equation (6.38) becomes

$$z_{xx} - 2\sqrt{\lambda}z_x = \lambda e^{-2\sqrt{\lambda}x}z^3.$$

Letting $z^i(x) = e^{\sqrt{\lambda}x}w(x)$ for $i = 1,2$ we have that at a point $x_0 \in (c,d)$ at which $z^1(x) - z^2(x)$ is maximized $z^1_x(x_0) = z^2_x(x_0)$ and $z^1(x_0) > z^2(x_0)$, so that $z^1_{xx}(x_0) - z^2_{xx}(x_0) > 0$, a contradiction.

Now suppose that w is any nonzero solution of (6.38), and that $w(x_1) > 0$ for some $x_1 \in (0,1)$, so that there exists $x_2 \in (0,1)$ with $w(x_2) > 0, w_x(x_2) = 0$. Let (a,b) be the maximal subinterval of $(0,1)$ containing x_2 on which $w > 0$. Then the function

$$v(x) = \begin{cases} w(x - 2j\delta), & x \in [a + 2j\delta, b + 2j\delta] \\ -w(a + b - x + (2j-1)\delta), & x \in [a + (2j-1)\delta, b + (2j-1)\delta], \end{cases}$$

where j is any integer and $\delta = b - a$, defines a solution of (6.38) on $\mathbb{R}$ with $v(x_2) = w(x_2), v_x(x_2) = w(x_2)$, and hence by uniqueness $w = v$ in $[0,1]$ and $b - a = k^{-1}$ for some positive integer k.

Multiplying (6.38) by w and integrating by parts we find that

$$\int_a^b w_x^2 \, dx = \lambda \int_a^b (w^2 - w^4) \, dx. \tag{6.40}$$

We now use the Poincaré inequality (4.20), the best constant for which is $C = \pi^2$ (obtained by minimizing $\int_0^1 w_x^2 \, dx$ in $H_0^1(0,1)$ subject to the constraint $\int_0^1 w^2 \, dx = 1$), which implies using a linear change of variables that

$$\int_a^b w_x^2 \, dx \geqslant \frac{\pi^2}{(b-a)^2} \int_a^b w^2 \, dx. \tag{6.41}$$

From (6.40), (6.41) we deduce that $k^2 \leqslant \frac{\lambda}{\pi^2}$. Hence by the uniqueness of positive solutions and the fact that $-w$ is a solution for any solution w we deduce that there are at most $2k + 1$ solutions (including the zero solution) where k is the integer part of $\sqrt{\lambda}/\pi$.

6.5 Problems

6.1. Prove that the solution $u = u(x,t)$ of the linear heat equation

$$u_t = u_{xx}, \quad x \in [0,1],$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = u_0(x)$, where $u_0 \in L^2(0,1)$, is a smooth function of $x \in [0,1]$ and $t > 0$. If $u_0(x) = 1$ for a.e. $x \in (0,1)$ deduce that for all sufficiently small $t > 0$ there exist at least two points $x_i(t) \in (0,1)$, $i = 1,2$, at which $u(x_i(t),t) = \frac{1}{2}$.

6.2. Prove that if u is the solution of the linear heat equation

$$u_t = u_{xx}, \quad x \in [0,1],$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = u_0(x)$, where $u_0 \in X = H_0^1(0,1)$, then

$$\|u(\cdot,t)_{xxx}\|_2 \leqslant \frac{2}{et} e^{-\frac{1}{2}\pi^2 t} \|u_0\|_X$$

6.3. Prove that every solution $u = u(x,t)$ of the equation

$$u_t = u_{xx} - u^5, \quad x \in [0,1]$$

with boundary conditions $u(x,t) = u(1,t) = 0$ and initial condition $u_0 \in H_0^1(0,1)$ converges to zero in $C^1([0,1])$ (i.e. $u(\cdot,t) \to 0$ and $u_x(\cdot,t) \to 0$ in $C([0,1])$) as $t \to \infty$.

6.4. Let $F = F(x,u)$ be C^1, bounded below, and satisfy $F_u(0,0) = F_u(1,0) = 0$. Suppose that w is an asymptotically stable rest point for the semiflow generated on $H_0^1(0,1)$ by

$$u_t = u_{xx} - F_u(x,u).$$

Show that if u_0 belongs to the boundary of the region of attraction $A(w)$ of w then there is a rest point $\bar{w}$ in the $\omega-$limit set of u_0, that $\bar{w}$ lies in the boundary of $A(w)$, and that $\bar{w}$ is unstable. Deduce that if there is a rest point w that is isolated in $H_0^1(0,1)$ and a strict H^1 local minimizer of

$$I(u) = \int_0^1 \left[\frac{1}{2}u_x^2 + F(x,u)\right] dx$$

in $H_0^1(0,1)$ and another solution of

$$v_{xx} = F_v(x,v) \tag{6.42}$$

in $H_0^1(0,1)$ then there is a solution $v \in H_0^1(0,1)$ of (6.42) which is not a strict H^1 local minimizer of I.