Hyperbolic Equations
Assignments

(To be sent to jb101@hw.ac.uk by 8 April either in Latex or handwritten and scanned.)

1. Show that Kirchhoff's formula for a C^2 solution to the linear wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3$ with initial data $u(\cdot, 0) = u_0(\cdot)$, $u_t(\cdot, 0) = u_1(\cdot)$ can be written in the form

$$u(x,t) = \frac{1}{4\pi t^2} \int_{S(x,t)} (tu_1(y) + u_0(y) + \nabla u_0(y) \cdot (y - x))\, dS_y.$$

Deduce that for $t \geqslant 1$

$$|u(x,t)| \leqslant \frac{1}{4\pi t} \int_{\mathbb{R}^3} (3|u_1| + |\nabla u_1| + 3|u_0| + |\nabla u_0| + |\Delta u_0|)\, dx.$$

Hence show that if u_0, u_1 are smooth and have compact support and if $\alpha > 3$ then

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} |u(x,t)|^\alpha dx = 0.$$

2. Let $\{T(t)\}_{t\geqslant 0}$ be a C^0-semigroup on a real Banach space X with infinitesimal generator A.
(i) Show that $D(A)$ is a Banach space under the norm $\|w\|_{D(A)} := \|w\|_X + \|Aw\|_X$.
(ii) Show that $T(t)$ restricted to $D(A)$ is a C^0-semigroup on $D(A)$, with infinitesimal generator A with domain $D(A^2) = \{p \in D(A) : Ap \in D(A)\}$.
(iii) Defining inductively $D(A^m) = \{p \in D(A^{m-1}) : A^{m-1}p \in D(A)\}$, with norm $\|w\|_{D(A^m)} = \sum_{j=1}^m \|A^j w\|_X$, deduce that $T(t)$ restricted to $D(A^m)$ is a C^0-semigroup.
(iv) Deduce that the linear wave equation $u_{tt} = \Delta u$ generates a C^0- semigroup on $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ for any positive integer s.

3. Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and consider the biharmonic wave equation for $u = u(x,t)$

$$u_{tt} + \Delta^2 u = 0,$$

with boundary conditions $u|_{\partial\Omega} = 0, \nabla u|_{\partial\Omega} = 0$ and initial conditions $u(\cdot, 0) = u_0(\cdot), u_t(\cdot, 0) = u_1(\cdot)$. Use the Hille-Yosida theorem to show that weak solutions of this equation generate a contraction semigroup $\{T(t)\}|_{t\geqslant 0}$ on $X = H_0^2(\Omega) \times L^2(\Omega)$ with respect to the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_X = \left(\int_\Omega (|D^2 u|^2 + v^2)\, dx \right)^{\frac{1}{2}},$$

and that for $p = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$ the solution $T(t)p = \begin{pmatrix} u \\ u_t \end{pmatrix}$ satisfies the energy equation

$$\|T(t)p\|_X^2 = \|p\|_X^2, \quad t \geqslant 0.$$

(*Hint. You may find it helpful to show that*

$$\int_\Omega ((\Delta u)^2 - |D^2 u|^2)\, dx = 0$$

for all $u \in H_0^2(\Omega)$.)

4. Let $\Omega \subset \mathbb{R}^3$ be open with $\mathcal{L}^3(\Omega) < \infty$, and consider for a constant $\beta > 0$ the damped hyperbolic equation

$$u_{tt} + \beta u_t - \Delta u + u^3 - u = 0,$$

and the corresponding semiflow $\{T(t)\}_{t \geqslant 0}$ on $X = H_0^1(\Omega) \times L^2(\Omega)$.
(i) Show that (although in general rest points need not be isolated) that $\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an isolated rest point in X.
(*Hint. Suppose not, that there is a sequence* $\begin{pmatrix} u_j \\ 0 \end{pmatrix}$ *of other rest points converging to* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *in* X *and consider* $w_j = \frac{u_j}{\|u_j\|_{H_0^1}}$. *Extract a weakly convergent subsequence and prove it converges strongly.*)
(ii) Hence or otherwise show that either $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable or it is unstable.

JMB 24/02/20