Non-positive curvature and complexity for finitely presented groups

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Outline

Finitely Presented Groups
- From presentations to geometry
- The 2 Strands of Geometric Group Theory
- The universe of finitely presented groups

Non-Positive Curvature
- local curvature, CAT(0)
- NPC groups

Isoperimetric functions
- Dehn functions
- Isoperimetric spectra

Drawing in and Reaching out
- Subdirect products of hyperbolic groups
- Solution of Grothendieck's Problems

Finitely presented groups

$$\Gamma \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle \equiv \mathcal{P}$$

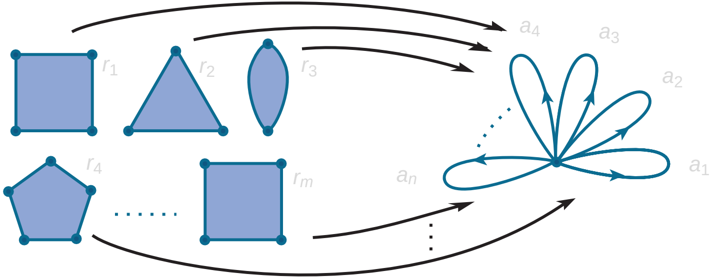

Figure: The standard 2-complex $K(\mathcal{P})$

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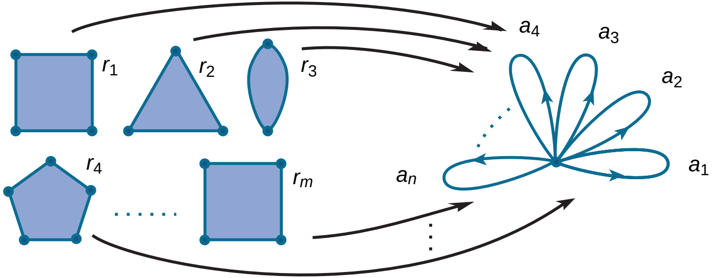

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The group springing into action

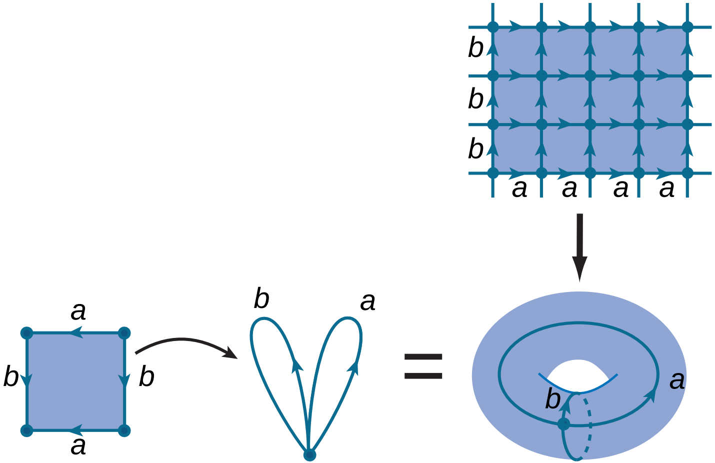

Figure: The 2-complex and Cayley graph for $\langle a, b \mid [a, b] \rangle$

The first strand in Geometric Group Theory

Strand 1: Study (and manufacture) group actions on spaces in order to elucidate the structure of both the groups and the spaces.

One may prefer discrete cocompact actions by isometries but . . .

sometimes weaken admission criteria to obtain a more diverse class of groups,

sometimes demand more structure to narrow the focus on groups and spaces of exceptional character.

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Groups as geometric objects (Gromov)

Study finitely generated groups as geometric objects in their own right, via their intrinsic geometry.

$$\Gamma = \langle a_1, \ldots, a_n \mid r_1, r_2, \ldots \rangle$$

Word Metric:

$$d(\gamma_1, \gamma_2) = \min\{|w| : w \in F(\mathcal{A}),\ w \stackrel{\Gamma}{=} \gamma_1^{-1}\gamma_2\}.$$

Cayley Graph (1878) = $\widetilde{K(\mathcal{P})}^{(1)}$

- Word metric and Cayley graph are independent of generating set, up to quasi-isometry.
- Thus one is particularly interested in properties of groups and spaces invariant under quasi-isometry.
- Large-scale (coarse) geometry and topology

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Combinatorial Group Theory (Dehn 1912)

$$\Gamma \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$$

"The general discontinuous group is given [as above]. There are above all three fundamental problems.

- The identity [word] problem
- The transformation [conjugacy] problem
- The isomorphism problem

[. . .] One is already led to them by necessity with work in topology. Each knotted space curve, in order to be completely understood, demands the solution of the three"

Higman Embedding (1961): Every recursively presented group is a subgroup of a finitely presented group.

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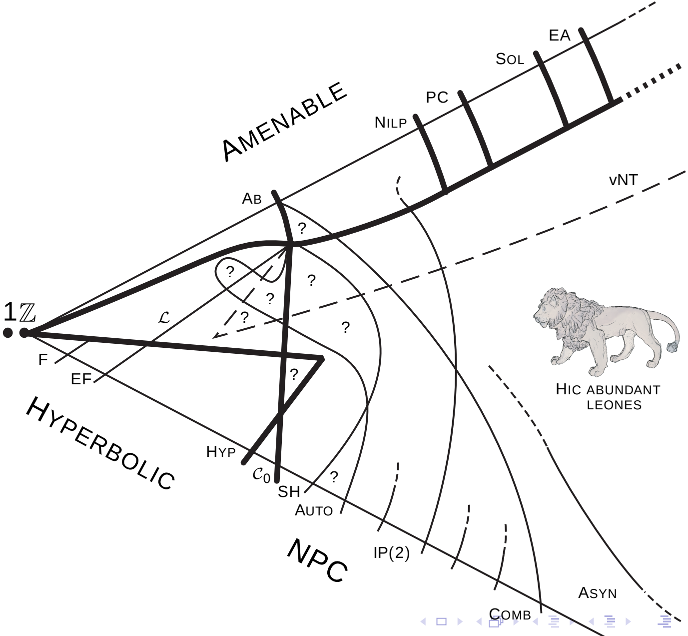
The universe of finitely presented groups



Gromov's hyperbolic groups

If Γ is hyperbolic then it

• acts properly, cocompactly on a contractible complex

• has only finitely many conjugacy classes of finite subgroups and its abelian subgroups are virtually cyclic

• striking algorithmic properties: the set of geodesic words for Γ (*wrt* any finite gen set) is a regular language: ∃ finite state automaton recognising words labelling geodesics in Cayley graph; hyperbolic groups are automatic.

• Rapidly-solvable word and conjugacy problems.

• [Sela] The isomorphism problem is solvable among torsion-free hyperbolic groups.

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CAT(0) and CAT(-1) conditions

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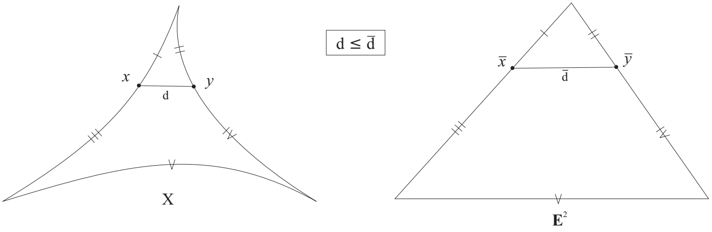

Figure: The CAT(0) inequality

Local-to-global: If X is complete and every point has a neighbourhood in which triangles satisfy this inequality, then in $\tilde{X}$ all triangles satisfy this inequality.

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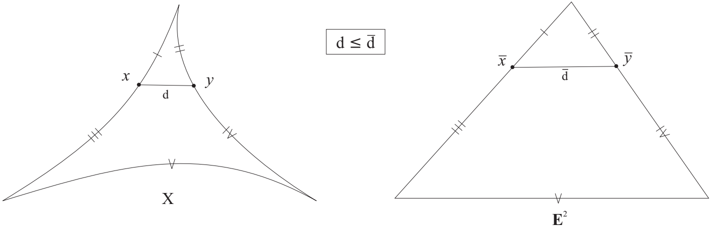

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CAT(0) spaces

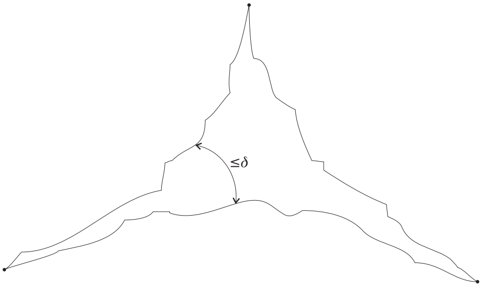
Metric spaces of non-positive curvature, *Bridson-Haefliger* Grund. Math. Wiss. **319**

- connections with many branches of mathematics
- local-to-global phenomena
- rigidity
- complexes of groups (à la Haefliger)
- connections with 3-dimensional geometry
- combination theorems; verifiability
- A great deal one can say about the structure of groups that act by isometries on CAT(0) and CAT(-1) spaces

Negative curvature and hyperbolic groups

Identify key *robust feature* of $\mathrm{CAT}(-1)$ spaces X

If Γ acts geometrically on X (basepoint p), articulate what remains of the feature when it is pulled-back via the Γ-equivariant quasi-isometry $\gamma \mapsto \gamma.p$ (fixed $p \in X$).



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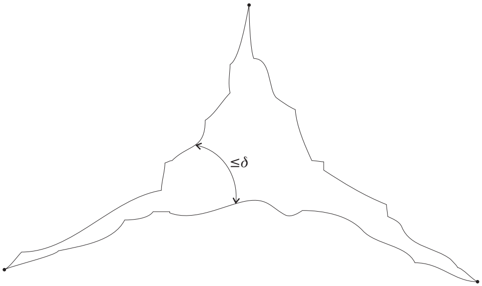

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Coarse convexity

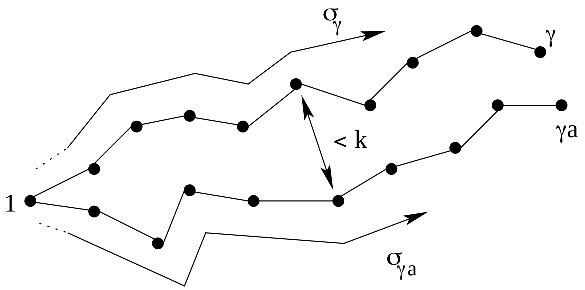

Figure: The fellow-traveller property

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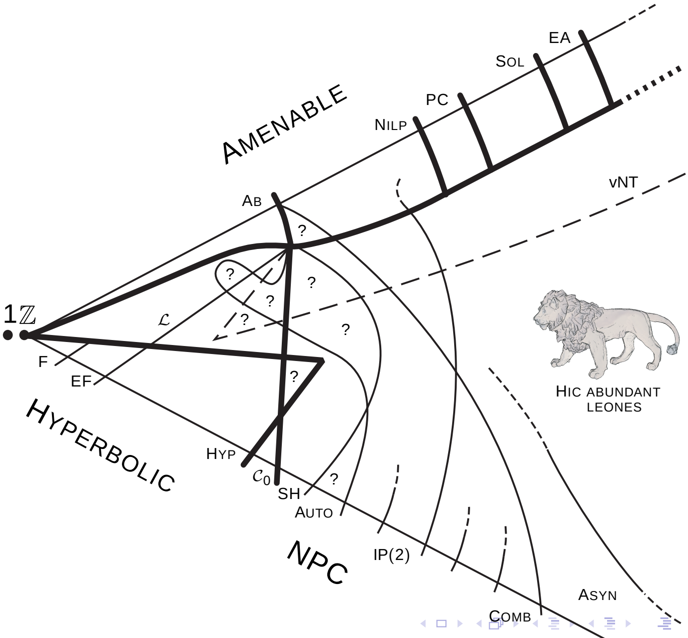

Discriminating among NPC Groups [B, c.'00]

Theorem

∃ combable groups that are not bicombable or automatic.

Hierarchy of formal languages, $\text{Reg} \subset \text{CF} \subset \text{Ind}.$

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∃ Ind*-combable groups that are not* Reg*-combable (automatic); some have quadratic Dehn functions, some cubic.*

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∃ combable groups with unsolvable conjugacy problem.

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Dehn functions of groups

Finitely presented $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$, word $w \in F(\mathcal{A})$ with $w = 1$ in Γ

$$\mathrm{Area}(w) := \min \left\{ N \ : \ w = \prod_{j=1}^{N} u_j r_j u_j^{-1} \text{ freely, } r_j \in \mathcal{R}^{\pm 1} \right\}$$

The Dehn function $\delta(n)$ of the presentation is

$$\delta(n) = \max\{\mathrm{Area}(w) \mid w \in \ker(F(\mathcal{A}) \to \Gamma), \ |w| \le n\}$$

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The Filling Theorem

[Gromov]
W smooth, complete, Riemannian manifold.

- $c : S^1 \to W$ null-homotopic, rectifiable loop

$$\mathrm{FArea}(c) = \inf\{\mathrm{Area}(F) \mid F : \mathbb{D}^2 \to W,\ F|_{\partial \mathbb{D}^2} = c\}.$$

- The 2-diml isoperimetric function $[0,\infty) \to [0,\infty)$ is

$$\mathrm{Fill}_0^{\mathrm{M}}(r) := \sup\{\mathrm{FArea}(c) \mid c : S^1 \to \widetilde{M},\ l(c) \leq r\}.$$

- **Filling Theorem:** $\forall M$ closed, $\mathrm{Fill}_0^{\mathrm{M}}(x) \simeq \delta_{\pi_1 M}(x)$.

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The isoperimetric spectrum

$$\mathbf{IP} = \{\alpha \mid n^\alpha \simeq \text{ Dehn function}\} \subseteq [1, \infty)$$

- Gromov (Bowditch, Papasoglu, Olshanskii,...): $(1, 2) \cap \mathbf{IP} = \emptyset$ and $\alpha = 1$ are the *hyperbolic groups*.
- At $\alpha = 2$ one has a rich and diverse class of groups.
- $\mathbb{N} \subseteq \mathbf{IP}$ [Baumslag-Miller-Short, B-Pittet, Gromov]
- Bridson ('94): $\exists$ infinitely many non-integers in **IP**.
- Brady-B ('98): $\forall p \geq q$ integers, $2\log_2(2p/q) \in \mathbf{IP}$.
- Sapir-Birget-Rips ('98): Deeper analysis $\delta(x) \succeq x^4$.
- B,B,Forester,Shankar ('04) $\mathbb{Q} \cap [2, \infty) \subseteq \mathbf{IP}$.

Theorem (B-Brady '98)

$$\overline{\mathbf{IP}} = \{1\} \cup [2, \infty).$$

Basic snowflake groups [Brady-B]

$$G = \langle a_1, a_2, c, s_1, s_2 \mid a_1 a_2 = c = a_2 a_1, s_i^{-1} a_i^r s_i = c \rangle$$

Figure: Half of snowflake diagram and dual tree

If k is the radius of the dual tree,

- Growth of the base of central Δ is r^k
- Area of central Δ is roughly square of the base.
- Length of boundary is roughly 2^k.
- Area $\geq (r^k)^2 = (2^k)^{2\log_2(r)} = |\partial|^{2\log_2(r)}$

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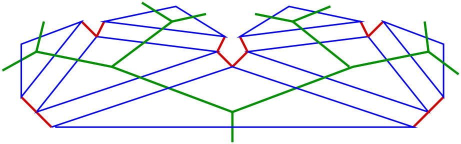

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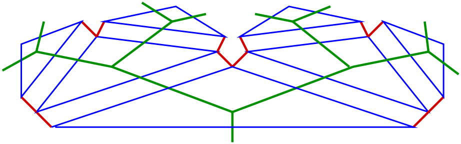

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General BBFS construction based on Perron-Frobenius theory

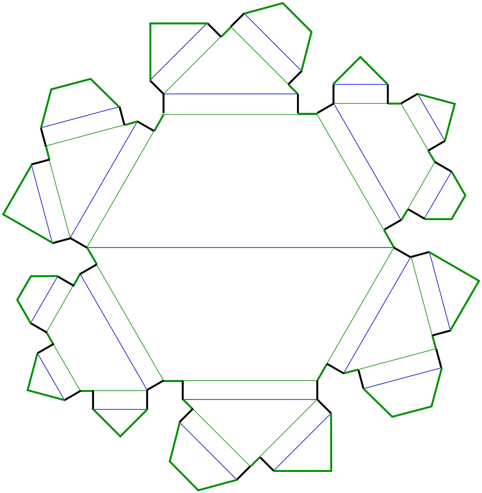

Figure: A snowflake disk based on the matrix $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Take iterated mapping tori by homotheties

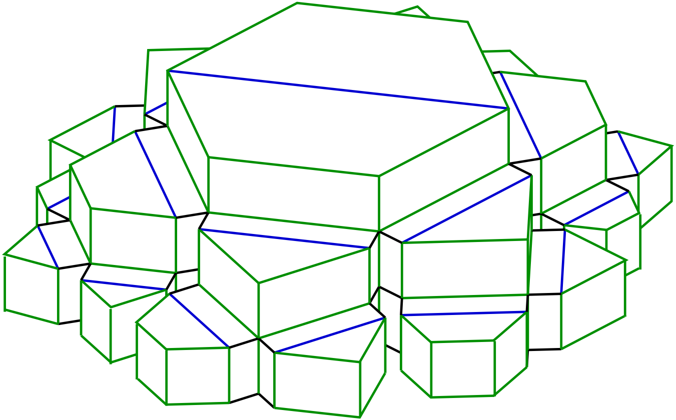

Figure: Snowflake spheres

Isoperimetric spectra

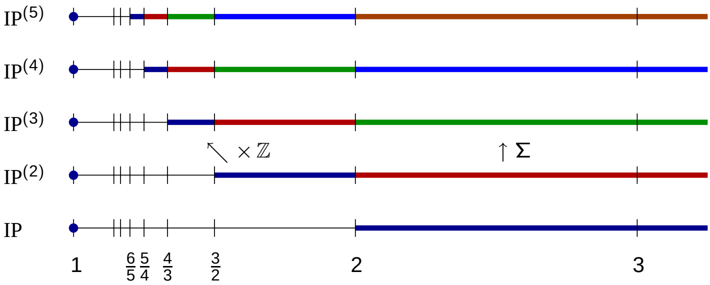

Figure: Complete picture of IP, partial for $\mathrm{IP}^{(k)}$.

$$\mathbf{IP}^{(k)} = \{\alpha \mid n^\alpha \simeq k\text{-diml Dehn function}\}$$

Subdirect products of hyperbolic groups

Tame: Finitely presented subdirect products of free groups, surface groups and limit groups are remarkably tame, for example:

Theorem (B,Howie,Miller,Short)

If the Γ_i are limit groups and $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is of type $\mathrm{FP_n}$, then S is virtually a direct product of limit groups.

Wild: Finitely presented subgroups of more general hyperbolic groups can be utterly wild.

An embracing algorithm (Rips & 1-2-3 Thm)

$\exists$ algorithm with input a finite aspherical presentation $\mathcal{Q}$ and output a FINITE presentation of $P \subset H \times H$, with H hyperbolic.

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subset H \times H$$

is the fibre-product associated to s.e.s.

$$1 \to N \to H \xrightarrow{p} Q \to 1$$

with N fin gen, H 2-diml hyperbolic, $Q = |\mathcal{Q}|$.

“1-2-3 Thm" refers to fact that N, H and Q are of type $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ respectively. [Baumslag, B, Miller, Short]

Refinements (B-Haefliger, Wise) place more stringent conditions on H, e.g. locally CAT(-1) or residually finite.

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Grothendieck's Question (1970)

$A \neq 0$ a commutative ring, Γ a finitely generated group, $\mathrm{Rep}_A(\Gamma)$ the category of Γ-actions on fin. pres. A-modules. Any homomorphism $u : \Gamma_1 \to \Gamma_2$ of groups induces

$$u_A^* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1).$$

Theorem (G, 1970)

If $u : \Gamma_1 \to \Gamma_2$ is a homomorphism of finitely generated groups, u_A^* is an equivalence of categories if and only if $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism of profinite groups.

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$\exists$ residually finite, hyperbolic groups H and finitely presented subgroups $P \hookrightarrow \Gamma := H \times H$ of infinite index, such that P is not abstractly isomorphic to Γ, but the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$.

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Outline of the proof

Suppose Q has no finite quotients.

$$1 \to N \to H \to Q \to 1$$

Lemma. $H_2(Q,\mathbb{Z}) = 0 \implies \hat{N} \to \hat{H}$ is iso.

- Build Q with aspherical presentation, no finite quotients and $H_2(Q,\mathbb{Z}) = 0$
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