Advanced Fluid Dynamics lecture 1

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# The Stokes flow regime

Stokes flow, or "slow viscous flow" describes fluids at small Mach number (incompressible) and small Reynolds number.

In Kinetic Theory we derived the compressible Navier–Stokes equations:

$$\partial_t \rho + \nabla\cdot(\rho \underline{u}) = 0,$$

$$\partial_t(\rho\underline{u}) + \nabla\cdot(\rho\underline{u}\,\underline{u} + p\underline{\underline{I}} - \underline{\underline{\tau}}) = \underline{f}$$

viscous stress

$$\tfrac{3}{2}\rho(\partial_t\theta + \underline{u}\cdot\nabla\theta) + \rho\theta\,\nabla\cdot\underline{u} = \underline{\underline{\tau}} : \nabla\underline{u} - \nabla\cdot\underline{q}$$

heat flux

The viscous stress

$$\tau = 2\mu\,\underline{\underline{e}}$$

where $\underline{\underline{e}} = \frac{1}{2}\left((\nabla\underline{u}) + (\nabla\underline{u})^T - \frac{2}{3}\underline{\underline{I}}\,\nabla\cdot\underline{u}\right)$

is the symmetric, traceless part of $\nabla\underline{u}$.

Taking $Ma = u/\sqrt{\theta} \to 0$ gives (sound speed is $\sqrt{\gamma\theta}$, $\theta$ is temperature in "energy units") gives incompressible Navier–Stokes

$$\rho_0(\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} + \underline{f}$$

$$\nabla\cdot\underline{u} = 0.$$

Another simplification when

$$Re = \frac{UL}{\mu/\rho_0} = \frac{UL}{\nu} \ll 1 \qquad \nu = \mu/\rho_0$$

This estimates $\dfrac{\underline{u}\cdot\nabla\underline{u}}{\nu\nabla^2\underline{u}}$

The most general version is the unsteady Stokes equations with a body force:

$$\rho_0\partial_t\underline{u} = \underline{f} + \nabla\cdot\underline{\underline{\sigma}}, \quad \nabla\cdot\underline{u} = 0$$

where $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\underline{\underline{e}}$ is the stress tensor. We've dropped the nonlinear $\underline{u}\cdot\nabla\underline{u}$ but kept $\partial_t\underline{u}$. This is relevant for high frequency regimes (small timescale $T$) so $\frac{U}{T} \gg \frac{U^2}{L}$

$$\partial_t\underline{u} \gg \underline{u}\cdot\nabla\underline{u}$$

The (quasi)steady Stokes equations with a body force $\underline{f}$ are

$$\nabla\cdot\underline{\underline{\sigma}} + \underline{f} = 0, \quad \nabla\cdot\underline{u} = 0.$$

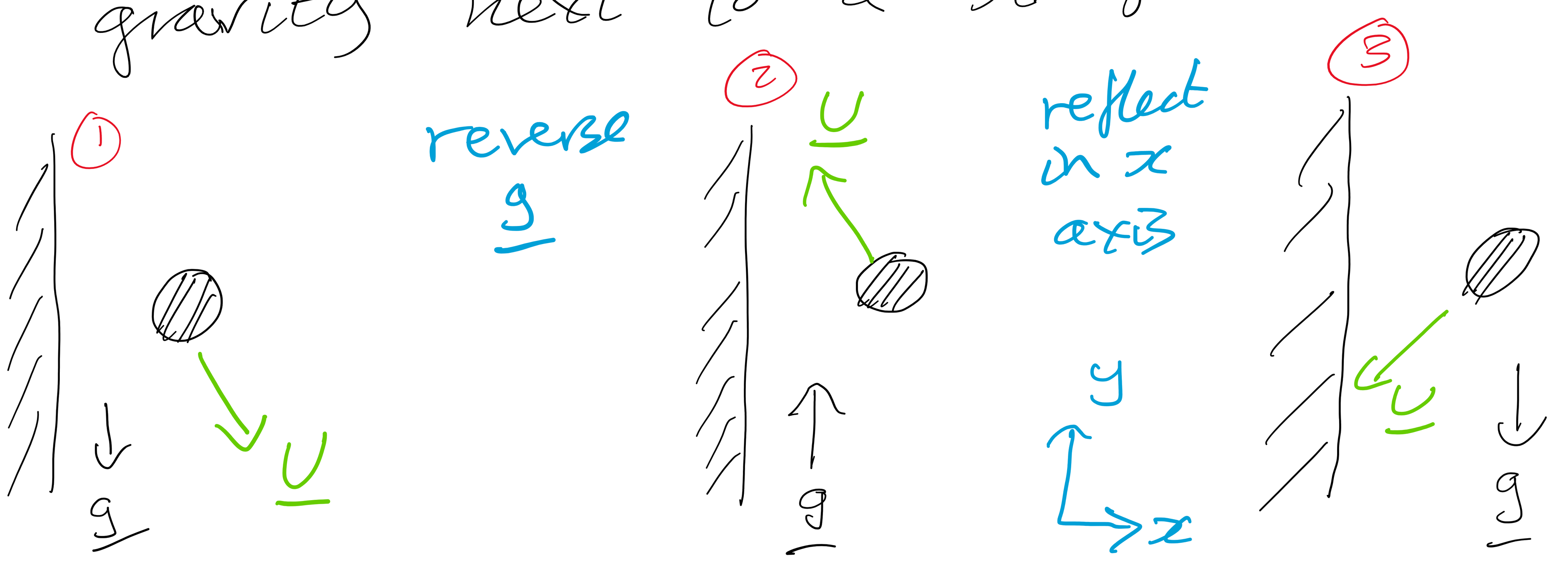
The flow responds instantaneously to body forces & boundary conditions.

The (homogeneous) "Stokes equations" are $\nabla\cdot\underline{\underline{\sigma}} = 0$, $\nabla\cdot\underline{u} = 0$

or $\nabla p = \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u} = 0$.

Taking the divergence $\Rightarrow \nabla^2 p = 0$, so $p$ is a harmonic fn of $\underline{x}$.

Consider a sphere falling under gravity next to a straight wall.




Stokes flow solutions are unique so 1 and 3 must be the same flows. $\underline{U}$ must point straight down parallel to $\underline{g}$.

1 → 2 is $\underline{g} \to -\underline{g}$, $\underline{u} \to -\underline{u}$

$\underline{U} \to \underline{U}$, $p \to -p$

Boundary conditions $\underline{u} = \underline{0}$ on $x = 0$ (wall) and $\underline{u} = \underline{U}$ on sphere.

2 → 3 is $y \to -y$, $x \to x$

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# Three Stokes flow theorems

## A. Dissipation of kinetic energy

Consider an unsteady Stokes flow

$$\rho_0 \partial_t \underline{u} = \underline{f} + \nabla \cdot \underline{\underline{\sigma}}, \quad \nabla \cdot \underline{u} = 0,$$

in a space-fixed volume $V$ with boundary $\partial V$ on which $\underline{u} = \underline{U}$.

The kinetic energy inside $V$ is

$$K = \tfrac{1}{2} \rho_0 \int_V |\underline{u}|^2 \, dV.$$

It's rate of change is

$$\frac{dK}{dt} = \int_V \rho_0 \frac{\partial \underline{u}}{\partial t} \cdot \underline{u} \, dV$$

$\underline{\underline{\sigma}}$ is symmetric

$$= \int_V f_i u_i + u_i \partial_j \sigma_{ij} \, dV$$

$$= \underbrace{\int_V f_i u_i \, dV}_{\text{rate of working of } \underline{f} \text{ in } V} + \underbrace{\int_{\partial V} U_i \sigma_{ij} n_j \, dS}_{\text{rate of working of } \underline{\underline{\sigma}} \text{ on } \partial V}$$

$$- \underbrace{\int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} \, dV}_{\text{viscous dissipation } \Phi}$$

Using incompressibility,

$$\Phi = \int_V e_{ij} \sigma_{ij} \, dV$$

$$= 2\mu \int_V e_{ij} e_{ij} \, dV \geq 0$$

The (quasi) steady Stokes equations omit $\partial_t \underline{u}$, so $\partial_t K = 0$, or

$$\Phi = \int_V \underline{f} \cdot \underline{u} \, dV + \int_{\partial V} \underline{U} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \, dS$$

## B. Minimum dissipation Theorem

A (homogeneous) Stokes flow minimizes $\Phi$ in a domain $V$ among incompressible vector fields with prescribed values on $\partial V$.

Suppose $\underline{u}^s$ solves $\nabla \cdot \underline{\underline{\sigma}} = 0$ in $V$, and $\nabla \cdot \underline{u}^s = 0$ in $V$, $\underline{u}^s = \underline{U}$ on $\partial V$.

Suppose $\nabla \cdot \underline{u} = 0$ in $V$, $\underline{u} = \underline{U}$ on $\partial V$.

Then
$$\int_V \underline{\underline{e}}^s : \underline{\underline{e}}^s \, dV \leq \int_V \underline{\underline{e}} : \underline{\underline{e}} \, dV$$

with equality iff $\underline{\underline{e}} = \underline{\underline{e}}^s$.

Let $\delta \underline{u} = \underline{u} - \underline{u}^s$ with $\delta \underline{u} = 0$ on $\partial V$.

$$2\mu \int_V \underline{\underline{e}} : \underline{\underline{e}} - \underline{\underline{e}}^s : \underline{\underline{e}}^s \, dV$$

$$= 2\mu \int_V \delta e_{ij} \left( e_{ij} + e^s_{ij} \right) dV$$

$$= 2\mu \int_V \delta e_{ij} \left( \delta e_{ij} + 2 e^s_{ij} \right) dV$$

$$= \underbrace{2\mu \int_V \delta e_{ij} \delta e_{ij} \, dV}_{\geq 0}$$

$$+ \underbrace{4\mu \int_V \delta e_{ij} e^s_{ij} \, dV}_{\text{This vanishes}}$$

$$= 2 \int_V \delta e_{ij} \sigma^s_{ij} \, dV$$

$$= 2 \int_V (\partial_j \delta u_i) \sigma^s_{ij} \, dV$$

$$= 2 \int_V \partial_j (\delta u_i \sigma^s_{ij}) \, dV$$

$$- 2 \int_V \delta u_i \partial_j \sigma^s_{ij} \, dV$$

$$= 2 \int_{\partial V} \delta u_i \sigma^s_{ij} n_j \, dV$$

$= 0$ as $\nabla \cdot \underline{\underline{\sigma}}^s = 0$

$= 0$ as $\delta \underline{u} = 0$ on $\partial V$

In particular, changing the flow, say by adding small particles, can only increase the dissipation. We'll calculate by how much later for a dilute suspension of rigid spheres.

## C Uniqueness theorem

Suppose $(\underline{u}^{(1)}, p^{(1)})$ and $(\underline{u}^{(2)}, p^{(2)})$ are two homogeneous Stokes flows in a volume $V$ with $\underline{u}^{(1)} = \underline{u}^{(2)} = \underline{U}$ on $\partial V$, with dissipations $\Phi^{(1)}$ and $\Phi^{(2)}$.

Minimum dissipation $\Rightarrow$
$$\Phi^{(1)} \leq \Phi^{(2)} \quad \text{and} \quad \Phi^{(2)} \leq \Phi^{(1)}$$

$$\Rightarrow \quad \Phi^{(1)} = \Phi^{(2)}$$

$$\Rightarrow \quad e^{(1)}_{ij} = e^{(2)}_{ij}$$

by proof of minimum dissipation theorem.

Hence $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ can only differ by a rigid body motion (a rotation plus a translation).

The boundary condition requires $\underline{u}^{(1)} = \underline{u}^{(2)} = \underline{U}$ on $\partial V$.

$\therefore \quad \underline{u}^{(1)} = \underline{u}^{(2)}$ in $V$.

The Stokes equations then imply $\nabla p^{(1)} = \nabla p^{(2)}$ in $V$.

Hence $p^{(1)} = p^{(2)} + \text{constant}$,

best one can hope for as pressures are arbitrary up to a constant in incompressible flows where only $\nabla p$ appears.