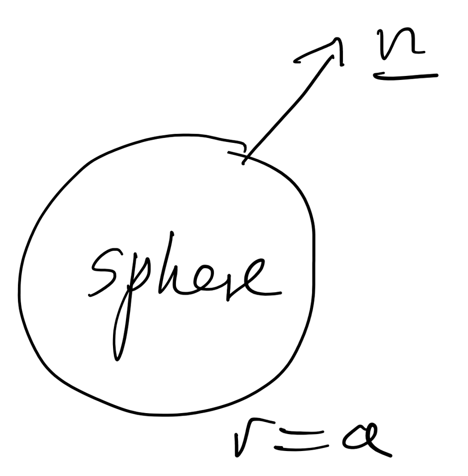
Advanced Fluid Dynamics lecture 3

Monday, 25 January 2021 00:00

Force, torque & stresslet

The normal $\underline{n}$ points into the fluid.



$$\underline{F} = \int_{r=a} \underline{\underline{\sigma}}\cdot\underline{n}\, dS$$

$$F_i = \int_{r=a} (-p\delta_{ij} + 2\mu e_{ij})\, n_j\, dS$$

From the solution in §III.B

$$-p\,\delta_{ij} n_j\Big|_{r=a} = -\frac{3\mu}{2a} U_k n_k n_i$$

$$2\mu e_{ij} n_j\Big|_{r=a} = -\frac{3\mu}{2a} U_k (\delta_{ik} - n_k n_i)$$

$$\underline{\underline{\sigma}}\cdot\underline{n} = -\frac{3\mu}{2a}\underline{U}$$ is uniform over the sphere's surface

Multiplying by the area $4\pi a^2$ gives $\underline{F} = -6\pi\mu a\,\underline{U}$

Stokes drag law

Similarly the torque on a rotating sphere (angular velocity $\underline{\Omega}$) is

$$\underline{T} = \int_{r=a} (\underline{x}\times\underline{\underline{\sigma}})\cdot\underline{n}\, dS$$

$$= -8\pi\mu a^3\,\underline{\Omega}$$

The torque is the antisymmetric part of the complete first moment

$$M_{ij} = \int_{r=a} \sigma_{ik} n_k x_j\, dS.$$

$$M_{ij} = \tfrac{1}{2}(M_{ij} + M_{ji}) + \tfrac{1}{2}(M_{ij} - M_{ji})$$

$$= S_{ij} - \tfrac{1}{2}\epsilon_{ijk} T_k$$

The symmetric part is called the stresslet:

$$S_{ij} = \tfrac{1}{2}\int_{r=a} (\sigma_{ik} x_j + \sigma_{jk} x_i)\, n_k\, dS.$$

This characterises the response of a rigid object in a pure straining flow.

For a sphere in strain flow

$$S_{ij} = -\frac{20}{3}\pi\mu a^3 E_{ij}.$$

Faxen relations

These use the above solutions to find the force, torque & stresslet on a sphere moving with velocity $\underline{U}$, rotating with angular velocity $\underline{\Omega}$, in a Stokes flow with velocity $\underline{u}^\infty$ outside the sphere, vorticity $\underline{\omega}^\infty$, strain rate $\underline{\underline{E}}^\infty$.

$$\underline{F} = 6\pi\mu a\left[\left(1 + \frac{a^2}{6}\nabla^2\right)\underline{u}^\infty(\underline{x}=0) - \underline{U}\right]$$

The flow without the sphere, evaluated at the centre of the sphere

$$\underline{T} = 8\pi\mu a^3\left[\tfrac{1}{2}\underline{\omega}^\infty(\underline{x}=0) - \underline{\Omega}\right]$$

$$\underline{\underline{S}} = \frac{20}{3}\pi\mu a^3\left(1 + \frac{a^2}{10}\nabla^2\right)\underline{\underline{E}}^\infty(\underline{x}=0)$$

$\tfrac{1}{2}\underline{\omega}^\infty$ is the angular velocity of the flow without the sphere ($\tfrac{1}{2}$ because $\nabla\times(\underline{\Omega}\times\underline{x}) = 2\underline{\Omega}$)

No subtracted term as the sphere is rigid

The $1 + \frac{a^2}{6}\nabla^2$ and $1 + \frac{a^2}{10}\nabla^2$ operators are the same operators that give the finite $O(a^2/r^2)$ corrections for flows around a sphere.

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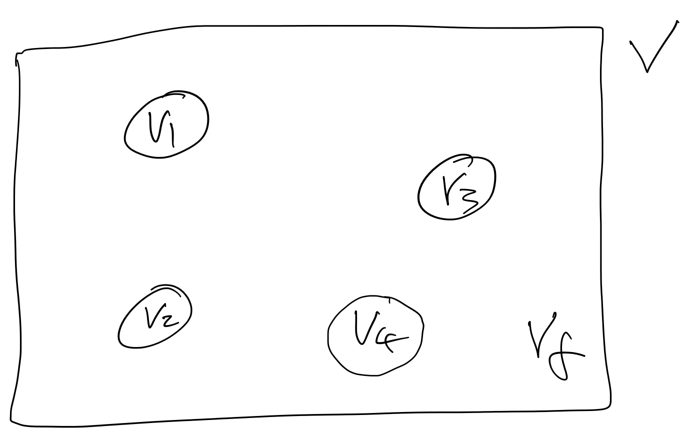
Einstein viscosity of a dilute suspension of rigid spheres



Consider lots of rigid spheres occupying volumes $V_1 \dots V_N$, a box of volume V, and a remaining volume V_f of fluid.

Consider the volume-averaged stress $\bar{\underline{\underline{\sigma}}} = \langle \underline{\underline{\sigma}} \rangle = \frac{1}{|V|} \int_V \underline{\underline{\sigma}} \, dV$

$$\bar{\underline{\underline{\sigma}}} = \frac{1}{|V|} \left(\int_{V_f} \underline{\underline{\sigma}} \, dV + \int_{V_p} \underline{\underline{\sigma}} \, dV \right)$$

V_p = $V_1 \cup V_2 \cup \dots \cup V_p$ volume occupied by particles.

$$= \frac{1}{|V|} \left(\int_{V_f} (-p \underline{\underline{I}} + 2\mu \underline{\underline{e}}) \, dV + \int_{V_p} \underline{\underline{\sigma}} \, dV \right)$$

$$= \frac{1}{|V|} \left(\int_V -p \underline{\underline{I}} \, dV + 2\mu \int_V \underline{\underline{e}} \, dV + \int_{V_p} \underline{\underline{\sigma}} + p \underline{\underline{I}} \, dV \right)$$

This holds because $\underline{\underline{e}} = 0$ inside rigid particles.

In a Newtonian fluid,

$$\underline{\underline{\sigma}} = -p \underline{\underline{I}} + 2\mu \underline{\underline{e}}$$

and $\underline{\underline{e}}$ is traceless (as $\nabla \cdot \underline{u} = 0$) so $p = -\frac{1}{3} \mathrm{Trace}(\underline{\underline{\sigma}})$. We can use this to define a "mechanical pressure" p in the whole domain.

We then get

$$\bar{\underline{\underline{\sigma}}} = -\langle p \rangle \underline{\underline{I}} + 2\mu \langle \underline{\underline{e}} \rangle + \frac{1}{|V|} \int_{V_p} \underline{\underline{\sigma}} - \frac{1}{3} (\mathrm{Trace}\, \underline{\underline{\sigma}}) \underline{\underline{I}} \, dV$$

($\underline{\underline{\sigma}} - \frac{1}{3}(\mathrm{Trace}\,\underline{\underline{\sigma}})\underline{\underline{I}}$ = $\underline{\underline{\sigma}} + p \underline{\underline{I}}$; the last term $= \underline{\underline{\sigma}}^{p}$)

The problem now is that we don't know $\underline{\underline{\sigma}}$ inside rigid particles, only that $\underline{\underline{e}} = 0$.

We do know that $\nabla \cdot \underline{\underline{\sigma}} = 0$ inside rigid particles on scales where the Stokes flow regime is valid.

For particle m occupying volume V_m,

$$\int_{V_m} \sigma_{ij} \, dV = \int_{V_m} \frac{\partial}{\partial x_k} (\sigma_{ik} x_j) - x_j \frac{\partial \sigma_{ik}}{\partial x_k} \, dV$$

($x_j \frac{\partial \sigma_{ik}}{\partial x_k} = 0$ as $\nabla \cdot \underline{\underline{\sigma}} = 0$)

$$= \int_{\partial V_m} \sigma_{ik} x_j n_k \, dS$$

$$= M_{ij}^{(m)}$$

the first moment of the stress on the surface

We can decompose $M_{ij}^{(m)}$ into antisymmetric, symmetric-traceless, and isotropic pressure parts:

$$\int_{V_m} \sigma_{ij} \, dV = \underbrace{\frac{1}{2} \int_{\partial V_m} (\sigma_{ik} x_j - \sigma_{jk} x_i) n_k \, dS}_{A_{ij}^{(m)}}$$

$$+ \underbrace{\frac{1}{2} \int_{\partial V_m} \left(\sigma_{ik} x_j + \sigma_{jk} x_i - \frac{2}{3} \delta_{ij} \sigma_{kl} x_l \right) n_k \, dS}_{S_{ij}^{(m)}}$$

$$+ \delta_{ij} \underbrace{\frac{1}{3} \int_{\partial V_m} \sigma_{kl} x_l n_k \, dS}_{-p^{(m)}}$$

The last term is included in the average pressure.

$$A_{ij}^{(m)} = -\frac{1}{2} \varepsilon_{ijk} T_k^{(m)}$$

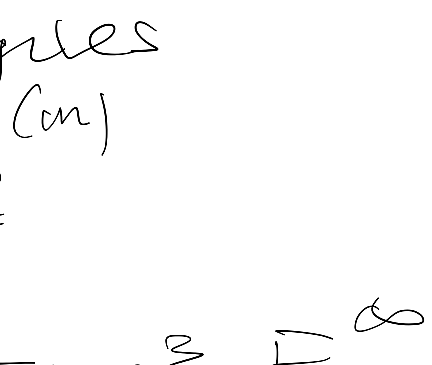
is proportional to the external torque on particle m. It vanishes when there are no external torques.

This leaves $\bar{\underline{\underline{\sigma}}}^{(p)}$ as a sum of contributions from symmetric traceless stresslets:

$$\sigma_{ij}^{(p)} = \frac{1}{|V|} \sum_{m=1}^{N} S_{ij}^{(m)}$$

So far this is exact, for any collection of non-overlapping rigid particles.

If we specialize to a dilute suspension of spheres, each of radius a, and assume the flow around each particle is described by §III.C with $\underline{\underline{E}} = \langle \underline{\underline{e}} \rangle$, e.g.



Then (for particle m)

$$S_{ij}^{(m)} = \frac{20}{3} \pi \mu a^3 E_{ij}^{\infty}$$

Adding them up gives

$$\bar{\underline{\underline{\sigma}}}^{(p)} = \frac{1}{|V|} \sum_{m=1}^{N} \underline{\underline{S}}^{(m)}$$

$$= \frac{N}{|V|} \frac{20}{3} \pi \mu a^3 \underline{\underline{E}}^{\infty}$$

$$= 5 \mu \phi \underline{\underline{E}}^{\infty}$$

where $n = N/|V|$ is the number density of spheres, and $\phi = \frac{4}{3} \pi \mu a^3 n$ is their volume fraction.

$$\therefore \quad \bar{\underline{\underline{\sigma}}} = -\langle p \rangle \underline{\underline{I}} + 2\mu \left(1 + \frac{5}{2}\phi\right) \underline{\underline{E}}^{\infty}$$

($\mu\left(1 + \frac{5}{2}\phi\right) = \mu_E$, i.e. $2\mu\left(1 + \frac{5}{2}\phi\right) = 2\mu_E$)

The suspension behaves like a Newtonian fluid, but with an enhanced Einstein viscosity

$$\mu_E = \left(1 + \frac{5}{2}\phi\right) \mu .$$