Advanced Fluid Dynamics lecture 5

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Suspensions of non-spherical particles

Consider an arbitrarily-shaped body in a linear flow: $\underline{u}(\underline{x}) = \underline{u}(0) + \underline{x}\cdot\nabla\underline{u}$ (cst)

(No $\frac{a^2}{6}\nabla^2$-type finite size corrections.)

Generalising the Faxen relations gives

$$\begin{pmatrix} \underline{F} \\ \underline{T} \\ \underline{\underline{S}} \end{pmatrix} = \mu \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} & \underline{\underline{\tilde{G}}} \\ \underline{\underline{B}}^T & \underline{\underline{C}} & \underline{\underline{\tilde{H}}} \\ \underline{\underline{G}} & \underline{\underline{H}} & \underline{\underline{M}} \end{pmatrix} \begin{pmatrix} \underline{U}^\infty - \underline{U} \\ \underline{\Omega}^\infty - \underline{\Omega} \\ \underline{\underline{E}}^\infty \end{pmatrix}$$

$\underline{U}^\infty$, $\underline{\Omega}^\infty$, $\underline{\underline{E}}^\infty$ are velocity, angular velocity $\underline{\Omega}^\infty = \frac{1}{2}\underline{\omega}^\infty$, and strain rate in the linear flow away from the body.

The particle translates with velocity $\underline{U}$ and rotates with angular velocity $\underline{\Omega}$.

This big 3×3 block matrix is the resistance matrix. It is block diagonal for spheres, but not in general (extra blue terms).

The reciprocal theorem imposes some symmetries:

$$\underline{\underline{A}} = \underline{\underline{A}}^T, \quad \underline{\underline{C}} = \underline{\underline{C}}^T, \quad M_{ijkl} = M_{klij}$$

$$G_{ijk} = \tilde{G}_{kij}$$

$$H_{ijk} = \tilde{H}_{kij}$$

For example, the torque is

$$T_i = \mu\left(B_{ij}(U_j^\infty - U_j) + C_{ij}(\Omega_j^\infty - \Omega_j) + \tilde{H}_{ijk}E_{jk}\right).$$

It depends on translation & strain as well as angular velocity.

Torque about where? We can define a "hydrodynamic centre" $\underline{x}_0$ so that $\underline{\underline{B}}$ is symmetric when $\underline{T}$ is the torque about $\underline{x}_0$.

How are torques $\underline{T}^{(0)}$ and $\underline{T}^{(1)}$ about centres $\underline{x}^{(0)}$ and $\underline{x}^{(1)}$ related?

$$\underline{T}^{(1)} - \underline{T}^{(0)} = \int_S (\underline{x} - \underline{x}^{(1)}) \times (\underline{\underline{\sigma}}\cdot\underline{n})\,dS - \int_S (\underline{x} - \underline{x}^{(0)}) \times (\underline{\underline{\sigma}}\cdot\underline{n})\,dS$$

$$= -\int_S (\underline{x}^{(1)} - \underline{x}^{(0)}) \times (\underline{\underline{\sigma}}\cdot\underline{n})\,dS$$

$$= -(\underline{x}^{(1)} - \underline{x}^{(0)}) \times \underline{F} \quad \circledast$$

For pure translation, no rotation or strain, $\underline{F} = \mu\underline{\underline{A}}(\underline{U}^\infty - \underline{U})$, and

$$\underline{T}^{(i)} = \mu\underline{\underline{B}}^{(i)}(\underline{U}^\infty - \underline{U}) \text{ for } i = 1, 2$$

$\circledast$ must hold for arbitrary vectors $\underline{U}^\infty - \underline{U}$

$$B_{ij}^{(1)} - B_{ij}^{(0)} = -\varepsilon_{ikl}(x_k^{(1)} - x_k^{(0)})A_{lj} \quad \odot$$

Suppose we can choose $\underline{x}^{(0)}$ to make $\underline{\underline{B}}^{(0)}$ symmetric. Taking the antisymmetric part of $\odot$ then gives

$$\varepsilon_{ijk}B_{ij}^{(1)} = (A_{kj} - A_{ll}\delta_{kj})(x_j^{(0)} - x_j^{(1)}).$$

We can now solve for

$$x_j^{(0)} = x_j^{(1)} + \left[\left(\underline{\underline{A}} - (\mathrm{Tr}\,\underline{\underline{A}})\underline{\underline{I}}\right)^{-1}\right]_{jk}\varepsilon_{kpq}B_{pq}^{(1)}$$

because $\underline{\underline{A}}$ is symmetric, hence diagonalizable, with positive eigenvalues because the drag must oppose motion, i.e. $\underline{F}\cdot(\underline{U}^\infty - \underline{U}) \geq 0$ (rate of working)

When diagonalized

$$\underline{\underline{A}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

and $\mathrm{Tr}\,\underline{\underline{A}} = \lambda_1 + \lambda_2 + \lambda_3$,

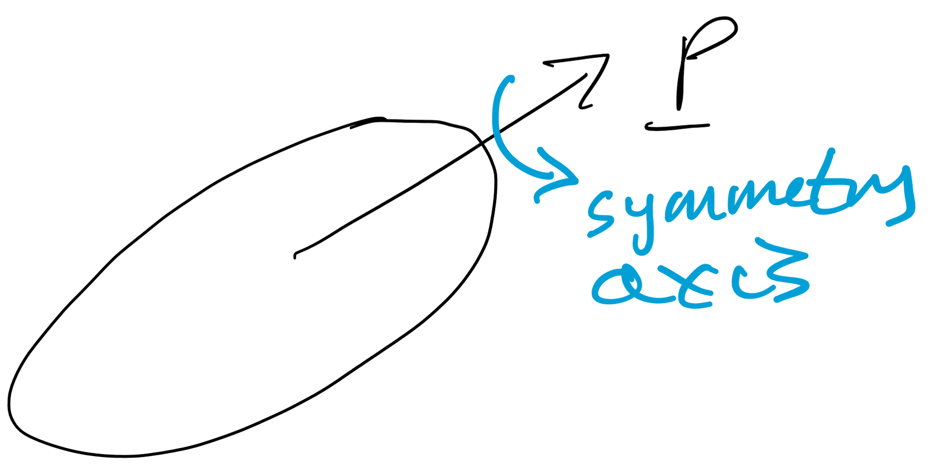
so
$$\underline{\underline{A}} - (\mathrm{Tr}\,\underline{\underline{A}})\underline{\underline{I}} = \begin{pmatrix} -\lambda_2 - \lambda_3 & 0 & 0 \\ 0 & -\lambda_1 - \lambda_3 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$$

with no zeros on the diagonal (as all $\lambda_i > 0$).

Now we can check that $\odot$ defines a symmetric $\underline{\underline{B}}^{(0)}$ for this $\underline{x}^{(0)}$.

Resistance matrix formulation for axisymmetric bodies

Suppose the body is axisymmetric with a symmetry axis $\underline{p}$ (a unit vector).



[We'll see later that what we get is invariant under $\underline{p} \mapsto -\underline{p}$]

Now we can exploit further symmetries, e.g. the force due to pure translation is

$$\underline{F} = \mu\left(A^{\parallel}\underline{p}\,\underline{p} + A^{\perp}(\underline{\underline{I}} - \underline{p}\,\underline{p})\right)\cdot(\underline{U}^{\infty} - \underline{U})$$

where $\mu A^{\parallel}$ and $\mu A^{\perp}$ are drag coefficients for translations parallel and perpendicular to $\underline{p}$. For very elongated bodies $A^{\perp} = 2A^{\parallel}$ so very little benefit from streamlining in Stokes flow.

Similarly,

$$B_{ij} = Y^{B}\varepsilon_{ijk}p_k$$

$$C_{ij} = X^{C}p_ip_j + Y^{C}(\delta_{ij} - p_ip_j)$$

$$H_{ijk} = \tfrac{1}{2}Y^{H}(\varepsilon_{ikl}p_j + \varepsilon_{jkl}p_i)$$

$$G_{ijk} = X^{G}(p_ip_j - \tfrac{1}{3}\delta_{ij})p_k + Y^{G}(p_i\delta_{jk} + p_j\delta_{ik} - 2p_ip_jp_k)$$

where the X^{s} and Y^{s} are scalar coefficients that depend on the body shape.

$Y^{B} = 0$ for torques about the hydrodynamic centre ($\underline{\underline{B}}^{(c)}$ must be symmetric and antisymmetric) so forces & translations decouple from torques & rotations.

A single torque-free axisymmetric body:

$$0 = \underline{T} = \mu\left(\underline{\underline{C}}(\underline{\Omega}^{\infty} - \underline{\Omega}) + \underline{\underline{\tilde{H}}}\,\underline{\underline{E}}^{\infty}\right)$$

The body rotates with angular velocity

$$\underline{\Omega} = \underline{\Omega}^{\infty} + \underline{\underline{C}}^{-1}\underline{\underline{\tilde{H}}}\,\underline{\underline{E}}^{\infty}$$

so that the torque is zero.

The orientation vector $\underline{p}$ evolves according to

$$\dot{\underline{p}} = \underline{\Omega}\times\underline{p}$$

Jeffery's equation (1922)

This maintains $|\underline{p}| = 1$

$$= \underline{\Omega}^{\infty}\times\underline{p} + \beta\left(\underline{\underline{E}}^{\infty}\cdot\underline{p} - \underline{p}\cdot\underline{\underline{E}}^{\infty}\cdot\underline{p}\,\underline{p}\right)$$

where $\beta = Y^{H}/Y^{C}$ is called the Bretherton parameter (defined for almost all axisymmetric bodies).

For spheroids $\beta = \dfrac{(a/b)^2 - 1}{(a/b)^2 + 1}$

with semiaxes a & b.