I. THE STOKES FLOW REGIME

Stokes flow, or "slow viscous flow", describes fluid dynamics at low Mach number and low Reynolds number.
In Kinetic Theory we derived the compressible Navier–Stokes–Fourier equations with a body force $\mathbf{f}$,

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\rho\mathbf{u}) + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = \mathbf{f}, \text{ with } \boldsymbol{\Pi} = \rho\mathbf{u}\mathbf{u} + p\mathbf{I} - \boldsymbol{\tau}, \tag{1b}$$

$$\frac{3}{2}\rho\left(\partial_t\theta + \mathbf{u}\cdot\boldsymbol{\nabla}\theta\right) + \rho\theta\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\tau} : \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\cdot\mathbf{q}. \tag{1c}$$

The viscous stress $\boldsymbol{\tau}$ is proportional to the strain rate $\mathbf{e}$, which is the symmetric, traceless part of the velocity gradient:

$$\boldsymbol{\tau} = 2\mu\mathbf{e}, \quad \mathbf{e} = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^{\mathsf{T}} - \frac{2}{3}\mathbf{I}\,\boldsymbol{\nabla}\cdot\mathbf{u}\right). \tag{2}$$

A further simplification arises when the Reynolds number

$$Re = \frac{UL}{\mu/\rho} = \frac{UL}{\nu} \ll 1, \tag{3}$$

based on a characteristic velocity scale U and lengthscale L.
The most general variant is the unsteady Stokes equations with a body force $\mathbf{f}$:

$$\rho_0\partial_t\mathbf{u} = \mathbf{f} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{4}$$

We have used the smallness of the Reynolds number to discard the nonlinear $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ term, but retained the linear time derivative $\partial_t\mathbf{u}$. This is applicable to high frequency (large Strouhal number) regimes in which $U/T \gg U^2/L$. In other words, the characteristic timescale T is much shorter than the flow crossing time L/U.
The (quasi)steady Stokes equations with a body force $\mathbf{f}$ are

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{f} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{5}$$

There are no time derivatives, so the flow responds instantaneously as the body force and/or the boundary conditions evolve in time.
The "Stokes equations" (or homogeneous Stokes equations) are just

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{6}$$

sometimes written as

$$\boldsymbol{\nabla}p = \mu\boldsymbol{\nabla}^2\mathbf{u}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{7}$$

Taking the divergence of the left-hand equation gives $\boldsymbol{\nabla}^2 p = 0$, so the pressure is a harmonic function of $\mathbf{x}$.

II. THREE STOKES FLOW THEOREMS

A. Dissipation of kinetic energy

Suppose a flow is governed by the unsteady Stokes equations with a body force (4) in a space-fixed volume V with $\mathbf{u} = \mathbf{U}$ prescribed on the boundary ∂V. (For a viscous fluid we must prescribe the whole velocity vector $\mathbf{u}$ on the boundary, not just the normal component $\mathbf{u}\cdot\mathbf{n}$ as for an inviscid fluid). The kinetic energy inside this volume is

$$K = \frac{1}{2}\rho_0\int_v |\mathbf{u}|^2 \mathrm{d}V, \tag{8}$$

and its rate of change is

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int_V \rho_0\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{u}\,\mathrm{d}V = \int_V \mathbf{f}\cdot\mathbf{u} + u_i\partial_j\sigma_{ij}\,\mathrm{d}V = \int_V \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V} U_i\sigma_{ij}n_j\,\mathrm{d}S - \int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij}\,\mathrm{d}V. \tag{9}$$

The last term is the viscous dissipation

$$\Phi = \int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij}\,\mathrm{d}V = \int_V e_{ij}\sigma_{ij}\,\mathrm{d}V = 2\mu\int_V e_{ij}e_{ij}\,\mathrm{d}V \geq 0. \tag{10}$$

The (quasi)steady Stokes equations omit $\partial_t\mathbf{u}$ from (4), which causes the left hand side of (9) to vanish, leaving

$$\Phi = \int_V \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V}\mathbf{U}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S. \tag{11}$$

The viscous dissipation in Stokes flow is thus instantaneously equal to the sum of the rate of working by the body force and the rate of working by the surface traction $\boldsymbol{\sigma}\cdot\mathbf{n}$ at the boundary.

B. Minimum dissipation theorem

A Stokes flow minimises the viscous dissipation in a domain among incompressible vector fields with prescribed values on the boundary. Here is a more formal statement:

Suppose $\mathbf{u}^S$ solves the homogeneous ($\mathbf{f} = 0$) Stokes equations in a volume V, and $\mathbf{u}^S = \mathbf{U}$ on ∂V. Suppose $\mathbf{u}$ is some other vector field that satisfies $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\mathbf{u} = \mathbf{U}$ on ∂V. Let $\mathbf{e}^S$ and $\mathbf{e}$ be the strain rates of $\mathbf{u}^S$ and $\mathbf{u}$ respectively. Then

$$\int_V \mathbf{e}^S : \mathbf{e}^S \,\mathrm{d}V \le \int_V \mathbf{e} : \mathbf{e} \,\mathrm{d}V, \tag{12}$$

with equality if and only if $\mathbf{e}^S = \mathbf{e}$. A colon denotes a double contraction between two tensors, $\mathbf{e} : \mathbf{e} = e_{ij}e_{ij}$.

To prove this inequality, let $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}^S$, and $\delta\mathbf{e} = \mathbf{e} - \mathbf{e}^S$, with $\delta\mathbf{u} = 0$ on ∂V. Subtracting the left hand side of (12) from the right hand side, and multiplying by 2μ, gives

$$\begin{aligned}
2\mu \int_V e_{ij}e_{ij} - e^S_{ij}e^S_{ij} \,\mathrm{d}V &= 2\mu \int \delta e_{ij}\left(e_{ij} + e^S_{ij}\right)\,\mathrm{d}V, && \text{being the difference between two squares,}\\
&= 2\mu \int \delta e_{ij}\left(\delta e_{ij} + 2e^S_{ij}\right)\,\mathrm{d}V, && \text{by writing } \mathbf{e} = \delta\mathbf{e} + \mathbf{e}^S,\\
&= 2\mu \underbrace{\int \delta e_{ij}\delta e_{ij}\,\mathrm{d}V}_{\ge 0} + 4\mu \int \delta e_{ij}e^S_{ij}\,\mathrm{d}V.
\end{aligned}$$

The remaining term vanishes because

$$\begin{aligned}
4\mu \int \delta e_{ij}e^S_{ij}\,\mathrm{d}V &= 2\int \delta e_{ij}(\sigma^S_{ij} + p\delta_{ij})\,\mathrm{d}V,\\
&= 2\int_V (\partial_j \delta u_i)\sigma^S_{ij}\,\mathrm{d}V, && \text{as } \sigma^S_{ij} \text{ is symmetric and } \operatorname{Tr}\delta\mathbf{e} = 0,\\
&= 2\int_V \partial_j\left(\delta u_i \sigma^S_{ij}\right)\,\mathrm{d}V - 2\int_V \delta u_i \partial_j \sigma^S_{ij}\,\mathrm{d}V,\\
&= 2\int_{\partial V} \delta u_i \sigma^S_{ij} n_j \,\mathrm{d}S, && \text{as } \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^S = 0 \text{ in } V,\\
&= 0, && \text{as } \delta\mathbf{u} = 0 \text{ on } \partial V.
\end{aligned}$$

Thus changing the flow, for instance by introducing rigid particles that the fluid must flow around, will always increase the viscous dissipation. Later on we will calculate this increase for a dilute suspension of rigid spheres.

C. Uniqueness theorem

The minimum dissipation theorem implies that solutions of the (homogeneous) Stokes equations are unique, up to an additive constant in the pressure. Suppose ($\mathbf{u}^{(1)}$, $p^{(1)}$) and ($\mathbf{u}^{(2)}$, $p^{(2)}$) are two solutions of the homogeneoues ($\mathbf{f} = 0$) Stokes equations with the same boundary conditions, $\mathbf{u}^{(1)} = \mathbf{U}$ and $\mathbf{u}^{(2)} = \mathbf{U}$ on ∂V. Let $\Phi^{(1)}$ and $\Phi^{(2)}$ be the viscous dissipations of the flows.

The minimum dissipation theorem implies $\Phi^{(1)} \le \Phi^{(2)}$ and $\Phi^{(2)} \le \Phi^{(1)}$, so $\Phi^{(2)} = \Phi^{(1)}$.

This implies $e^{(1)}_{ij} = e^{(2)}_{ij}$ by the proof of the minimum dissipation theorem.

Hence $\mathbf{u}^{(1)}$ differs from $\mathbf{u}^{(2)}$ by at most a rigid body motion (a rotation and a translation). These are the only velocity fields for which $\mathbf{e} = 0$.

The boundary conditions require $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$ on ∂V, so $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$ in the whole volume V.

The homogenous Stokes equations then imply $\boldsymbol{\nabla}p^{(1)} = \boldsymbol{\nabla}p^{(2)}$, so $p^{(1)} = p^{(2)} + \text{constant}$.

D. Lorentz reciprocal theorem

Consider two Stokes flows ($\mathbf{u}^{(1)}$, $\boldsymbol{\sigma}^{(1)}$) and ($\mathbf{u}^{(2)}$, $\boldsymbol{\sigma}^{(2)}$) driven by body forces $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, and boundary conditions $\mathbf{u}^{(1)} = \mathbf{U}^{(1)}$ and $\mathbf{u}^{(2)} = \mathbf{U}^{(2)}$. Then

$$\int_V \mathbf{u}^{(2)} \cdot \mathbf{f}^{(1)} \, \mathrm{d}V + \int_{\partial V} \mathbf{U}^{(2)} \cdot \boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} \, \mathrm{d}S = \int_V \mathbf{u}^{(1)} \cdot \mathbf{f}^{(2)} \, \mathrm{d}V + \int_{\partial V} \mathbf{U}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n} \, \mathrm{d}S. \tag{13}$$

The rate of working of $\mathbf{u}^{(2)}$ against the force and surface traction on flow 1 equals the rate of working of $\mathbf{u}^{(1)}$ against the force and surface traction on flow 2.

Using the divergence theorem to write the second term as a volume integral, the left hand side of (13) is

$$\begin{aligned}\int_V f_j^{(1)} u_j^{(2)} + \partial_i \left(\sigma_{ij}^{(1)} u_j^{(2)}\right) \mathrm{d}V &= \int_V \left(f_j^{(1)} + \partial_i \sigma_{ij}^{(1)}\right) u_j^{(2)} + \sigma_{ij}^{(1)} \partial_i u_j^{(2)} \, \mathrm{d}V, \\ &= \int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} \, \mathrm{d}V, \\ &= 2\mu \int_V e_{ij}^{(1)} e_{ij}^{(2)} \, \mathrm{d}V, \end{aligned} \tag{14}$$

which is symmetric between $^{(1)}$ and $^{(2)}$, and hence equal to the right hand side of (13). The three steps used that $\mathbf{f}^{(1)} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(1)} = 0$ for a Stokes flow, that $\boldsymbol{\sigma}^{(1)}$ is symmetric, and lastly that $\mathbf{e}^{(2)}$ is traceless.

If flows (1) and (2) are the same, (14) is the same as the (11), which equates the viscous dissipation to the rate of working by the body force and boundary conditions.

This reciprocal theorem is the Stokes flow analogue of Green's theorem,

$$\int_V \phi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \phi \, \mathrm{d}V = \int_V \boldsymbol{\nabla}\cdot(\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi) \, \mathrm{d}V = \int_{\partial V} (\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi) \cdot \mathbf{n} \, \mathrm{d}S, \tag{15}$$

rewritten in the form

$$\int_V \phi \boldsymbol{\nabla}^2 \psi \, \mathrm{d}V - \int_{\partial V} \phi \boldsymbol{\nabla} \psi \cdot \mathbf{n} \, \mathrm{d}S = \int_V \psi \boldsymbol{\nabla}^2 \phi \, \mathrm{d}V - \int_{\partial V} \psi \boldsymbol{\nabla} \phi \cdot \mathbf{n} \, \mathrm{d}S \tag{16}$$

with ϕ undifferentiated on the left hand side, and ψ undifferentiated on the right hand side. Both sides are equal to

$$-\int_V \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \phi \, \mathrm{d}V, \tag{17}$$

which is symmetric between ϕ and ψ.

III. STOKES FLOWS AROUND A SINGLE SPHERE

A. Rotation

Consider a sphere of radius a rotating with angular velocity $\boldsymbol{\Omega}$ in unbounded fluid. We want to solve

$$\mu \boldsymbol{\nabla}^2 \mathbf{u} = \boldsymbol{\nabla} p, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0,$$

subject to the boundary conditions

$$\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x} \text{ on } r = a, \quad \text{and} \quad \mathbf{u}, p \to 0 \text{ as } r \to \infty.$$

We know that $\boldsymbol{\nabla}^2 p = 0$ from the divergence of the momentum equation. The spherically symmetric solution of Laplace's equation,

$$\boldsymbol{\nabla}^2 \varphi = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}r} \right) = 0,$$

that decays at infinity is

$$\varphi^{(0)} = \frac{1}{r}.$$

By taking spatial derivatives we can build further "solid spherical harmonics" (Batchelor 1967, page 121) that are also solutions of Laplace's equation. The next three of these are (with their overall signs chosen for convenience)

$$\varphi_i^{(1)} = -\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = \frac{x_i}{r^3}, \quad \varphi_{ij}^{(2)} = \frac{\partial^2}{\partial x_i x_j}\left(\frac{1}{r}\right) = \frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5},$$
$$\varphi_{ijk}^{(3)} = -\frac{\partial^3}{\partial x_i x_j x_k}\left(\frac{1}{r}\right) = \frac{\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ki}x_j}{r^5} - 5\frac{x_i x_j x_k}{r^7}.$$

The first solid spherical harmonic $\varphi^{(0)}$ is a scalar with zero indices, the second $\varphi_i^{(1)}$ is a vector with one index, and the third is a symmetric traceless tensor $\varphi_{ij}^{(2)}$ with two indices, and so on. This numbering convention is not universal.

These expressions are easier to work with than the spherical harmonics expressed in spherical polar coordinates

At first glance, it looks like we could have

$$p(\mathbf{x}) = \lambda\Omega_i\varphi_i^{(1)} = \lambda\frac{\Omega_i x_i}{r^3},$$

but this scalar would be formed from the dot product of an axial or pseudo-vector $\boldsymbol{\Omega}$ with a conventional vector $\mathbf{x}$. This involves an arbitrary choice of orientation, such as the right hand rule to convert a direction of rotation into a vector, which gives the dot product (and hence the pressure) an ambiguous sign. The coefficient λ must vanish to resolve this ambiguity.

This leaves $p = 0$ identically, and $\nabla^2\mathbf{u} = 0$ with $\mathbf{u} \to 0$ as $r \to \infty$. A suitable trial solution is

$$\mathbf{u}(\mathbf{x}) = \lambda\,\boldsymbol{\Omega}\times\mathbf{x}/r^3,$$

which is permissible because the definition of the cross product of two vectors involves the same arbitrary choice of orientation. In other words, we can represent the direction of rotation of the sphere by a vector $\boldsymbol{\Omega}$ unambiguously by requiring that $\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{x}$ on the boundary of the sphere.

Choosing λ to fit the boundary condition at $r = a$ gives the complete solution

$$\mathbf{u}(\mathbf{x}) = \left(\frac{a}{r}\right)^3 \boldsymbol{\Omega}\times\mathbf{x}, \quad p(\mathbf{x}) = 0.$$

B. Translation

We want the boundary condition on $r = a$ to be $\mathbf{u} = \mathbf{U}$, with everything else the same as above. Now the pressure can be

$$p(\mathbf{x}) = \lambda_1\mathbf{U}\cdot\mathbf{x}/r^3,$$

for some constant λ_1, since $\mathbf{U}$ and $\mathbf{x}$ are both standard vectors.

We can decompose $\mathbf{u}$ into a part driven by $\boldsymbol{\nabla}p$ and a harmonic part,

$$\mathbf{u} = \mathbf{u}^{(p)} + \mathbf{u}^{(h)},$$

where

$$\mathbf{u}^{(p)} = \frac{p}{2\mu}\mathbf{x} \text{ satisfies } \mu\boldsymbol{\nabla}^2\mathbf{u}^{(p)} = \boldsymbol{\nabla}p.$$

The remaining harmonic part $\mathbf{u}^{(h)}$ must also be linear in $\mathbf{U}$, so we try a linear combination of $\varphi^{(0)}$ and $\varphi^{(2)}$,

$$\mathbf{u}^{(h)} = \lambda_2\mathbf{U}\frac{1}{r} + \lambda_3\mathbf{U}\cdot\left(\frac{\mathsf{I}}{r^3} - 3\frac{\mathbf{x}\,\mathbf{x}}{r^5}\right).$$

Incompressibility imposes

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u} = \left(\frac{\lambda_1}{2\mu} - \lambda_2\right)\frac{\mathbf{U}\cdot\mathbf{x}}{r^3}$$

so $\lambda_2 = \lambda_1/(2\mu)$. Now applying the boundary condition $\mathbf{u} = \mathbf{U}$ on $r = a$, on which $\mathbf{x} = a\mathbf{n}$, gives

$$\frac{\lambda_1}{2\mu a}\left(\mathbf{U} + \mathbf{U}\cdot\mathbf{n}\,\mathbf{n}\right) + \frac{\lambda_3}{a^3}\mathbf{U}\cdot(\mathsf{I} - 3\,\mathbf{n}\,\mathbf{n}) = \mathbf{U}.$$

Equating coefficients of $\mathbf{U}$ and $\mathbf{U}\cdot\mathbf{n}\,\mathbf{n}$ determines

$$\lambda_1 = \frac{3}{2}\mu a, \quad \lambda_3 = \frac{1}{4}a^3,$$

so the complete solution is

$$\mathbf{u}(\mathbf{x}) = \frac{3a}{4}\mathbf{U}\cdot\left(\frac{\mathsf{I}}{r} + \frac{\mathbf{x}\,\mathbf{x}}{r^3}\right) + \frac{a^3}{4}\mathbf{U}\cdot\left(\frac{\mathsf{I}}{r^3} - 3\frac{\mathbf{x}\,\mathbf{x}}{r^5}\right) = \frac{3a}{4}\mathbf{U}\cdot\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\left(\frac{\mathsf{I}}{r} + \frac{\mathbf{x}\,\mathbf{x}}{r^3}\right) \tag{18}$$

and

$$p(\mathbf{x}) = \frac{3}{2}\mu a\mathbf{U}\cdot\mathbf{x}/r^3 \sim r^{-2}.$$

The velocity has a term that decays slowly like $1/r$, and a "finite size" correction that is order $(a/r)^2$ smaller, and thus decays like $1/r^3$. The response due to a point force, a dipole, follows from taking the limit $a \to 0$ and $|\mathbf{U}| \to \infty$ such that $a\mathbf{U}$ has a finite limit.

The tensor in (18) is sometimes called the Oseen tensor

$$\mathsf{G} = \frac{\mathsf{I}}{r} + \frac{\mathbf{x}\,\mathbf{x}}{r^3}.$$

To show that

$$\boldsymbol{\nabla}^2\mathsf{G} = 2\left(\frac{\mathsf{I}}{r^3} - 3\frac{\mathbf{x}\,\mathbf{x}}{r^5}\right),$$

on can either calculate directly, or observe that

$$G_{ij} = \frac{1}{r}\delta_{ij} + \frac{1}{r^3}x_i x_j = \varphi^{(0)}\delta_{ij} + x_i\varphi_j^{(1)}.$$

The solid spherical harmonics $\varphi^{(0)}$, $\varphi_j^{(1)}$ and x_j are all harmonic functions, so

$$\partial_{kk}G_{ij} = 2(\partial_k x_i)\partial_k\varphi_j^{(1)} = 2\partial_i\varphi_j^{(1)} = 2\frac{\partial}{\partial x_i}\left(\frac{x_j}{r^3}\right) = 2\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}\right).$$

One can also derive (18) from the Oseen tensor (as in Kim & Karrila) by noting that

$$\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathsf{G} = \left(1 + \frac{a^2}{3r^2}\right)\frac{\mathsf{I}}{r} + \left(1 - \frac{a^2}{r^2}\right)\frac{\mathbf{x}\,\mathbf{x}}{r^3}$$

so putting $r = a$ gives

$$\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathsf{G}\Big|_{r=a} = \frac{4}{3a}\mathsf{I}.$$

Now dot with $(3a/4)\mathbf{U}$ to get a Stokes flow that equals $\mathbf{U}$ on $r = a$, which we can write as

$$\mathbf{u}(\mathbf{x}) = 6\pi\mu a\mathbf{U}\cdot\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\frac{\mathsf{G}}{8\pi\mu}.$$

Moreover, since we know that G solves

$$\mu\boldsymbol{\nabla}^2\left(\frac{\mathsf{G}}{8\pi\mu}\right) + \boldsymbol{\nabla}\mathbf{p} = \mathsf{I}\delta(\mathbf{x}), \quad \boldsymbol{\nabla}\cdot\left(\frac{\mathsf{G}}{8\pi\mu}\right) = 0,$$

we get the effective force on the fluid due to the sphere as $6\pi\mu a\mathbf{U}$ with no integration requried. (The pressure here is a vector $\mathbf{p}$ because the other terms are tensors. Dotting the whole equation with a constant vector gives a scalar pressure.)

C. Strain

Now change the boundary condition on the sphere to $\mathbf{u} = \mathsf{E} \cdot \mathbf{x}$ on $r = a$, with E a symmetric traceless constant tensor (or matrix), just as $\mathsf{e} = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^{\mathsf{T}}\right)$ is a symmetric traceless tensor (traceless because $\mathrm{Tr}\,\mathsf{e} = \boldsymbol{\nabla}\cdot\mathbf{u} = 0$ for incompressible flow).

For the scalar pressure we can try contracting E with the harmonic $\varphi^{(2)}$,

$$p(\mathbf{x}) = \lambda_0 \mathsf{E} : \left(\frac{\mathsf{I}}{r^3} - 3\frac{\mathbf{x}\mathbf{x}}{r^5}\right).$$

However, $\mathsf{E} : \mathsf{I} = \mathrm{Tr}\,\mathsf{E} = 0$, which leaves

$$p(\mathbf{x}) = \lambda_1 \frac{\mathbf{x}\cdot\mathsf{E}\cdot\mathbf{x}}{r^5}.$$

As before, $\mathbf{u}^{(p)} = p\mathbf{x}/(2\mu)$ is a particular solution to $\mu\boldsymbol{\nabla}^2\mathbf{u}^{(p)} = \boldsymbol{\nabla}p$, and for the harmonic part we try

$$u_i^{(h)} = \lambda_2 E_{ij}\varphi_j^{(1)} + \lambda_3 E_{jk}\varphi_{ijk}^{(3)} = \lambda_2 E_{ij}\frac{x_j}{r^3} + \lambda_3 E_{jk}\left(\frac{\delta_{ij}x_k + \delta_{ki}x_j + \delta_{jk}x_i}{r^5} - 5\frac{x_i x_j x_k}{r^7}\right).$$

The contribution involving $\delta_{jk}x_i$ vanishes because $E_{jk}\delta_{jk} = \mathrm{Tr}\,\mathsf{E} = 0$. Incompressibility requires $\lambda_2 = 0$.

Imposing $\mathbf{u} = \mathsf{E}\cdot\mathbf{x}$ on $r = a$ determines

$$\lambda_1 = 5\mu a^3, \quad \lambda_3 = a^5/2$$

so the complete solution is

$$u_i = \frac{5}{2}a^3\frac{x_i x_i x_k}{r^5}E_{jk} + \frac{1}{2}a^5 E_{jk}\left(\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - 5\frac{x_i x_j x_k}{r^7}\right),$$

and

$$p(\mathbf{x}) = 5\mu a^3 \mathbf{x}\cdot\mathsf{E}\cdot\mathbf{x}/r^5 \sim 1/r^3.$$

Again, the velocity has a part that decays like $1/r^2$ and another part that is $O((a/r)^2)$ smaller, and decays like $1/r^4$. We can find the response to a point stress (a quadrupole) by taking $a \to 0$ with $a^3\mathsf{E}$ finite.

D. Force, torque, and stresslet

The force on the sphere is

$$\mathbf{F} = \int_{r=a} \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S, \text{ or } F_i = \int_{r=a} (-p\delta_{ij} + 2\mu e_{ij})\, n_j\,\mathrm{d}S.$$

From the above solution in Sec. III B for a translating sphere, we can calculate

$$-p\delta_{ij}n_j\Big|_{r=a} = -\frac{3\mu}{2a}U_k n_k n_i, \quad 2\mu e_{ij}n_j\Big|_{r=a} = -\frac{3\mu}{2a}U_k\left(\delta_{ik} - n_k n_i\right),$$

so the traction vector is uniform over the surface of the sphere:

$$\boldsymbol{\sigma}\cdot\mathbf{n} = -\frac{3\mu}{2a}\mathbf{U}.$$

Multiplying by the surface area $4\pi a^2$ gives the famous Stokes drag law

$$\mathbf{F} = -6\pi\mu a\mathbf{U}.$$

Similarly, the torque about the centre of a sphere rotating with angular velocity $\boldsymbol{\Omega}$ is

$$\mathbf{T} = \int_{r=a} (\mathbf{x}\times\boldsymbol{\sigma})\cdot\mathbf{n}\,\mathrm{d}S = -8\pi\mu a^3\boldsymbol{\Omega}.$$

However, the torque is only part of the complete first moment

$$M_{ij} = \int_{r=a} \sigma_{ik} n_k x_j \, \mathrm{d}S,$$

which we can decompose into symmetric and antisymmetric parts as

$$M_{ij} = \frac{1}{2}\left(M_{ij} + M_{ji}\right) + \frac{1}{2}\left(M_{ij} - M_{ji}\right) = S_{ij} - \frac{1}{2}\epsilon_{ijk} T_k.$$

The symmetric part is called the stresslet

$$S_{ij} = \frac{1}{2}\int_{r=a} \left(\sigma_{ik} x_j + \sigma_{jk} x_i\right) n_k \, \mathrm{d}S.$$

As before, we can calculate the stresslet for a sphere in a strain flow from the solution in Sec. III C which gives

$$S_{ij} = -\frac{20}{3}\pi\mu a^3 E_{ij}.$$

The stresslet characterises the response of a rigid, non-deformable object to a pure straining flow (like a four roller mill). This relation is the key to the enhanced, but still Newtonian, viscosity of a dilute suspension of spheres. Note that the left hand side is traceless because the right hand side is traceless.

E. Fáxen relations

Using the reciprocal theorem and the above solutions, we can derive the Fáxen relations for a sphere moving with velocity $\mathbf{U}$ and rotating with angular velocity $\boldsymbol{\Omega}$ relative to a general external background flow with velocity $\mathbf{u}^\infty$, vorticity $\boldsymbol{\omega}^\infty$ and strain rate E^∞. To be precise, $\mathbf{u}^\infty$ is a solution of the Stokes equations in the absence of the sphere, as generated by forces or boundary conditions far from the sphere, and $\boldsymbol{\omega}^\infty$ and E^∞ are the vorticity and strain rate for this flow.

If we insert a sphere into this flow at the origin, we need to add a second Stokes flow solution, one that decays to zero far from the sphere, to satisfy the boundary conditions on the sphere. The sphere experiences a force, torque, and stresslet given by:

$$\mathbf{F} = 6\pi\mu a\left[\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2\right)\mathbf{u}^\infty(\mathbf{x}=0) - \mathbf{U}\right],$$
$$\mathbf{T} = 8\pi\mu a^3\left[\frac{1}{2}\boldsymbol{\omega}^\infty(\mathbf{x}=0) - \boldsymbol{\Omega}\right],$$
$$\mathsf{S} = \frac{20}{3}\pi\mu a^3\left(1 + \frac{a^2}{10}\boldsymbol{\nabla}^2\right)\mathsf{E}^\infty(\mathbf{x}=0).$$

The notation $\mathbf{x}=0$ indicates a property of the undisturbed flow evaluated at the centre of the sphere.

The $1 + (a^2/6)\boldsymbol{\nabla}^2$ and $1 + (a^2/10)\boldsymbol{\nabla}^2$ operators arise from the second $(a/r)^2$ smaller terms in the previously calculated flow fields. This means that, for example, a sphere in Poiseuille flow moves with a velocity a little less than the velocity of the undisturbed Poiseuille flow at the centre of the sphere.

The torque $\mathbf{T}$ has no finite-size correction proportional to a^2 because the previous solution in Sec. III A had only one piece. Note the sign change in S because E^∞ is the strain rate of the external background flow, while previously E was the strain rate imposed on the surface of a sphere in fluid at rest at infinity. There is no second term to subtract, because the strain rate $\mathsf{e} = 0$ vanishes identically inside a rigid body.

The factor of $\frac{1}{2}$ in $\frac{1}{2}\boldsymbol{\omega}^\infty(\mathbf{x}=0) - \boldsymbol{\Omega}$ comes from the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\mathbf{x}) = 2\,\boldsymbol{\Omega}$, so the vorticity of a fluid in rigid-body rotation is twice the angular velocity.

IV. EINSTEIN VISCOSITY OF A DILUTE SUSPENSION OF SPHERICAL PARTICLES

Consider a volume average $\langle\cdots\rangle$ over a volume V containing $N \gg 1$ particles and their surrounding fluid. The volume-averaged stress is

$$\overline{\boldsymbol{\sigma}} = \langle\boldsymbol{\sigma}\rangle = \frac{1}{|V|}\int_V \boldsymbol{\sigma}\, \mathrm{d}V. \tag{19}$$

Let $V_p = V_1 \cup V_2 \cup \cdots \cup V_N$ be the volumes occupied by the particles themselves, and $V_f = V \setminus V_p$ the volume occupied by the fluid. Decomposing (19) into integrals over these two separate volumes gives

$$\begin{aligned}\overline{\boldsymbol{\sigma}} &= \frac{1}{|V|}\left(\int_{V_f} \boldsymbol{\sigma}\,\mathrm{d}V + \int_{V_p} \boldsymbol{\sigma}\,\mathrm{d}V\right), \\ &= \frac{1}{|V|}\left(\int_{V_f} (-p\mathbf{I} + 2\mu\mathbf{e})\,\mathrm{d}V + \int_{V_p} \boldsymbol{\sigma}\,\mathrm{d}V\right), \\ &= \frac{1}{|V|}\left(\int_{V} -p\mathbf{I}\,\mathrm{d}V + 2\mu\int_{V} \mathbf{e}\,\mathrm{d}V + \int_{V_p} \boldsymbol{\sigma} + p\mathbf{I}\,\mathrm{d}V\right).\end{aligned}$$

We have extended the integral of the strain rate $\mathbf{e}$ to the whole volume, since $\mathbf{e}$ vanishes inside the rigid particles (which cannot deform). We have also written the integral of p over the fluid region as an integral over the whole volume minus an integral over the particles.

In a Newtonian fluid the pressure p is minus 1/3 of the trace of the stress tensor, since $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e}$, and $\mathrm{Tr}\,\mathbf{e} = 0$ while $\mathrm{Tr}\,\mathbf{I} = 3$. We can adopt this definition more widely to define the pressure as $p = -\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\sigma}$ in the whole domain, as required for the first integral over V contributing to $\overline{\boldsymbol{\sigma}}$. The pressure defined in this way is sometimes called the "mechanical pressure" to distinguish it from the "thermodynamic pressure" defined by an equation of state.

The last term then becomes

$$\frac{1}{|V|}\int_{V_p} \boldsymbol{\sigma} + p\mathbf{I}\,\mathrm{d}V = \frac{1}{|V|}\int_{V_p} \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{Tr}\,\boldsymbol{\sigma})\mathbf{I}\,\mathrm{d}V \equiv \overline{\boldsymbol{\sigma}}^P,$$

so the volume-averaged stress is

$$\overline{\boldsymbol{\sigma}} = -\langle p\rangle\mathbf{I} + 2\mu\langle\mathbf{e}\rangle + \overline{\boldsymbol{\sigma}}^P.$$

It contains terms from the average pressure $\langle p\rangle$, the average strain rate $\langle\mathbf{e}\rangle$, and the contribution $\overline{\boldsymbol{\sigma}}^P$ from the particles.

Now the problem is that we do not know $\boldsymbol{\sigma}$ inside the particles, only that $\mathbf{e} = 0$ as the particles are rigid. This is analogous to saying that the pressure in an incompressible fluid is determined by $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, so $\mathrm{Tr}\,\boldsymbol{\sigma}$ is whatever it needs to be to make $\mathrm{Tr}\,\mathbf{e} = 0$. However, we do know that $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$ inside each particle, as inertia is negligible in the Stokes flow regime. For particle m, occupying the volume V_m, we can use the divergence theorem to write

$$\int_{V_m} \sigma_{ij}\,\mathrm{d}V = \int_{V_m} \frac{\partial}{\partial x_k}(\sigma_{ik}x_j) - x_j\frac{\partial}{\partial x_k}\sigma_{ik}\,\mathrm{d}V = \int_{\partial V_m} \sigma_{ik}x_j n_k\,\mathrm{d}S = M_{ij}^{(m)}.$$

The integral of the stress over the particle is thus the complete first moment of the stress on the boundary, as introduced in section III D.

We can decompose the tensor $\mathbf{M}^{(m)}$ into antisymmetric, symmetric-traceless, and isotropic pressure parts:

$$\int_{V_m} \sigma_{ij}\,\mathrm{d}V = \underbrace{\frac{1}{2}\int_{\partial V_m} (\sigma_{ik}x_j - \sigma_{jk}x_i)\,n_k\,\mathrm{d}S}_{A_{ij}^{(m)}} + \underbrace{\frac{1}{2}\int_{\partial V_m} \left(\sigma_{ik}x_j + \sigma_{jk}x_i - \frac{2}{3}\delta_{ij}\sigma_{kl}x_l\right) n_k\,\mathrm{d}S}_{S_{ij}^{(m)}} + \delta_{ij}\,\frac{1}{3}\underbrace{\int_{\partial V_m} \sigma_{kl}x_l n_k\,\mathrm{d}S}_{M_{ii}^{(m)}}.$$

The last term is proportional to

$$\int_{V_m} \mathrm{Tr}\,\boldsymbol{\sigma}\,\mathrm{d}V = \int_{\partial V_m} \sigma_{ik}x_i n_k\,\mathrm{d}S = M_{ii}^{(m)},$$

so the integral of the traceless part of the stress over particle m is

$$\int_{V_m} \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\,\mathrm{d}V = A_{ij}^{(m)} + S_{ij}^{(m)}.$$

The antisymmetric term is proportional to the torque $\mathbf{T}^{(m)}$ on the particle,

$$A_{ij}^{(m)} = \frac{1}{2}\int_{\partial V_m} (\sigma_{ik}x_j - \sigma_{jk}x_i)\,n_k\mathrm{d}S = -\frac{1}{2}\epsilon_{ijk}T_k^{(m)},$$

and so vanishes when no external (*e.g.* magnetic) torques are applied to the particles. This leaves $\overline{\boldsymbol{\sigma}}^P$ as the sum of contributions from the symmetric and traceless stresslets on the individual particles:

$$\overline{\sigma}^P_{ij} = \frac{1}{|V|}\sum_{m=1}^{N} S^{(m)}_{ij}, \text{ where } S^{(m)}_{ij} = \frac{1}{2}\int_{\partial V_m}\left(\sigma_{ik}x_j + \sigma_{jk}x_i - \frac{2}{3}\delta_{ij}\sigma_{kl}x_l\right) n_k \mathrm{d}S.$$

So far this calculation is exact for Stokes flow, and applies to particles of arbitrary shape. Now we assume that the particles are all spheres of radius a, and that the suspension is dilute, so the flow around each particle is described by our previous results for flows around isolated spheres in unbounded fluid, and the volume-averaged strain rate $\langle \mathbf{e}\rangle$ equals the applied strain $\mathbf{E}^\infty$. For example, we might imagine imposing tangential velocities on the upper and lower boundaries of a cuboidal domain V.

With these assumptions we have

$$S^{(m)}_{ij} = \frac{20}{3}\mu\pi a^3 E^\infty_{ij}$$

for each particle, so the particles' contribution to the total stress is

$$\overline{\boldsymbol{\sigma}}^P = \frac{1}{|V|}\sum_{m=1}^{N}\mathbf{S}^{(m)} = \frac{N}{|V|}\frac{20}{3}\mu\pi a^3\,\mathbf{E}^\infty = 5\mu\phi\,\mathbf{E}^\infty,$$

where $n = N/|V|$ is the number density of particles, and $\phi = \frac{4}{3}\pi a^3 n$ is the volume fraction of the domain occupied by particles. The volume-averaged stress is thus

$$\overline{\boldsymbol{\sigma}} = -\langle p\rangle\mathbf{I} + 2\mu\left(1+\frac{5}{2}\phi\right)\mathbf{E}^\infty,$$

which recovers Einstein's $\mu_E = \mu\big(1+(5/2)\phi\big)$ for a dilute suspension. The suspension still acts like a Newtonian fluid, but with an enhanced viscosity.

This is the modern version of the derivation. Einstein's original version used the instaneous energy balance (11) to determine the effective viscosity of a suspension from the rate of working of stresses on the boundary S of some measurement apparatus:

$$\frac{\mu_E}{\mu} = \frac{\displaystyle\int_S \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S}{\displaystyle\int_S \mathbf{u}^{(0)}\cdot\boldsymbol{\sigma}^{(0)}\cdot\mathbf{n}\,\mathrm{d}S} \tag{20}$$

for two different flows $\mathbf{u}$ and $\mathbf{u}^{(0)}$ that coincide on the boundary S of the apparatus. The difference between the two integrals can be reduced to an integral over the boundaries of the particles using the reciprocal theorem [[see Happel & Brenner page 435]]

The first effects of particle-particle interactions change the multiplier to

$$1+\frac{5}{2}\phi + 6.95\,\phi^2$$

for strain flow. This result was derived by taking account of the $O((a/R)^3)$ change in the strain rate around one sphere due to the disturbance created by another sphere distance R away. This leads to a logarithmic divergence at large R, since there are $4\pi R^2 dR$ spheres in a spherical shell of radius R and thickness dR. There is no divergence for small R because $R \geq 2a$ as the spheres cannot overlap. The divergence can be "renormalised" by treating the contribution from the far field as being those from a dilute suspension in fluid of viscosity $\mu_E = \mu(1+\frac{5}{2}\phi)$. See §8.4 of Kim & Karrila for details. The above result does not hold for simple shear flows, since there are closed relative trajectories that lead to repeated interactions between the same pairs of particles. This violates a necessary assumption about the particles being uncorrelated.

Continuing the renormalisation idea to argue that each sphere behaves as though it were in a fluid with the viscosity of the suspension, as modified to include the observation that the viscosity of ths supension should diverge at the maximum packing fraction $\phi_{\max} = 0.64$ for spheres gives the empirical Krieger–Dougherty viscosity for finite volume fractions:

$$\mu_S = \mu\left(1-\phi/\phi_{\max}\right)^{-2}.$$

There are also normal stress differences at finite ϕ, so the behaviour becomes non-Newtonian.

V. SPHERES WITH SPRINGS – THE UPPER CONVECTED MAXWELL MODEL FOR VISCOELASTIC LIQUIDS

Consider two spheres (beads) of radius a in Stokes flow, centres at $\mathbf{r}_1$ and $\mathbf{r}_2$ with separation vector $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$, joined by a Hookean (linear) spring exerting a force $H\mathbf{R}$.

Suppose further that the fluid flow is (locally) linear, with a spatially uniform gradient $\boldsymbol{\nabla}\mathbf{u}$, so

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{0}, t) + \mathbf{x} \cdot \boldsymbol{\nabla}\mathbf{u}, \tag{21}$$

and that $|\mathbf{R}| \gg a$ so the separation of the spheres is large compared with their radii. [[See note on $\boldsymbol{\nabla}\mathbf{u}$ convention.]]

If the spheres are small enough, they will experience stochastic Brownian forces S_i from collisions with molecules of the fluid surrounding them. Their centres thus move according to the coupled Langevin equations

$$m\ddot{\mathbf{r}}_1 = -\zeta\left(\dot{\mathbf{r}}_1 - \mathbf{u}(\mathbf{r}_1, t)\right) + H\mathbf{R} + \mathsf{S}_1, \tag{22}$$

$$m\ddot{\mathbf{r}}_2 = -\zeta\left(\dot{\mathbf{r}}_2 - \mathbf{u}(\mathbf{r}_2, t)\right) - H\mathbf{R} + \mathsf{S}_2, \tag{23}$$

where m is the mass of a sphere, and $\zeta = 6\pi\mu a$ is the Stokes drag coefficient. Consistent with the neglect of inertia in deriving the Stokes equations, we neglect the accelerations $m\ddot{\mathbf{r}}_i$ and set the net forces on the two spheres instantaneously to zero.

A suspension of many such, widely separately, bead-spring pairs may be described by a distribution function $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$. Again, consistent with the neglect of inertia, Ψ does not depend explicitly on $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$, because these are given instantaneously in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ by the instantaneous force balace.

Following Chandrasekhar (1943) we may formulate an evolution equation for Ψ by writing the stochastic Brownian forces as

$$\mathsf{S}_i = -k_B T\, \boldsymbol{\nabla}_{\mathbf{r}_i} \log\Psi, \tag{24}$$

where k_B is Boltzmann's constant and T is temperature. We will make sense of this equation, which seems to mix a stochastic LHS with a deterministic RHS, later as an ingredient for formulating an evolution equation for Ψ. It can be motivated by considering an equilibrium between the stochastic Brownian forces and an external deterministic force such as gravity acting on a population of particles.

To study a typical bead-spring pair it is more convenient to use the separation $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$ and centre of mass $\mathbf{x} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ instead of $\mathbf{r}_1$ and $\mathbf{r}_2$, and write

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = n(\mathbf{x}, t)\psi(\mathbf{R}, \mathbf{x}, t), \tag{25}$$

where n is the number density (bead-spring pairs per unit volume) and ψ is normalised (for each $\mathbf{x}$ and t) by

$$\int \psi(\mathbf{R}, \mathbf{x}, t)\, \mathrm{d}\mathbf{R} = 1. \tag{26}$$

In these coodinates the Brownian forces are proportional to

$$\boldsymbol{\nabla}_{\mathbf{r}_1} \log\Psi = \left(\frac{1}{2}\boldsymbol{\nabla}_{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{R}}\right)\log\psi, \quad \boldsymbol{\nabla}_{\mathbf{r}_2} \log\Psi = \left(\frac{1}{2}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{R}}\right)\log\psi. \tag{27}$$

Adding and subtracting the right hand sides of the two Langevin equations thus gives

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) - \frac{k_B T}{\zeta}\boldsymbol{\nabla}_{\mathbf{x}} \log\psi, \quad \dot{\mathbf{R}} = \mathbf{R} \cdot \boldsymbol{\nabla}\mathbf{u} - \frac{2H}{\zeta}\mathbf{R} - \frac{2k_B T}{\zeta}\boldsymbol{\nabla}_{\mathbf{R}} \log\psi. \tag{28}$$

To make sense of these equations, we substitute them into the Liouville equation for ψ

$$\partial_t\psi + \boldsymbol{\nabla}\cdot(\dot{\mathbf{x}}\psi) + \boldsymbol{\nabla}_{\mathbf{R}} \cdot \left(\dot{\mathbf{R}}\psi\right) = 0, \tag{29}$$

familiar from kinetic theory, to get the Fokker–Planck equation

$$\partial_t\psi + \mathbf{u} \cdot \boldsymbol{\nabla}\psi = \boldsymbol{\nabla}_{\mathbf{R}} \cdot \left(-\psi\mathbf{R} \cdot \boldsymbol{\nabla}\mathbf{u} + \psi\frac{2H}{\zeta}\mathbf{R} + \frac{2k_B T}{\zeta}\boldsymbol{\nabla}_{\mathbf{R}}\psi\right) + \boldsymbol{\nabla}_{\mathbf{x}} \cdot \left(\frac{k_B T}{2\zeta}\boldsymbol{\nabla}_{\mathbf{x}}\psi\right). \tag{30}$$

The last term gives a Brownian diffusion in physical space ($\mathbf{x}$). The diffusivity $k_B T/2\zeta$ is half the Stokes–Einstein diffusivity of a single sphere in Stokes flow. This last term is often omitted because its effect is typically much smaller than the effect of diffusion in orientation space ($\mathbf{R}$). The flow typically varies over lengthscales much larger than the typical separation $\mathbf{R}$ of pairs of spheres (a molecular lengthscale) so $|\boldsymbol{\nabla}_{\mathbf{x}}| \ll |\boldsymbol{\nabla}_{\mathbf{R}}|$.

By taking the second moment of this Fokker–Planck equation with respect to $\mathbf{R}$, and integrating by parts assuming $\psi \to 0$ as $|\mathbf{R}| \to \infty$, we find that the conformation tensor

$$\mathsf{C} = \langle \mathbf{R}\mathbf{R} \rangle \equiv \int \mathbf{R}\mathbf{R}\, \psi \, \mathrm{d}\mathbf{R} \tag{31}$$

obeys the closed evolution equation

$$\partial_t \mathsf{C} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathsf{C} - (\boldsymbol{\nabla} \mathbf{u})^{\mathsf{T}} \cdot \mathsf{C} - \mathsf{C} \cdot (\boldsymbol{\nabla} \mathbf{u}) = \frac{4 k_B T}{\zeta} \mathsf{I} - \frac{4H}{\zeta} \mathsf{C}. \tag{32}$$

The left hand side is called the upper convected derivative, often written $\overset{\triangledown}{\mathsf{C}}$, with components

$$\big[\overset{\triangledown}{\mathsf{C}}\big]_{ij} = \frac{\partial C_{ij}}{\partial t} + u_k \frac{\partial C_{ij}}{\partial x_k} - \frac{\partial u_i}{\partial x_k} C_{kj} - C_{ik} \frac{\partial u_j}{\partial x_k}. \tag{33}$$

The upper convected derivative is a material derivative for a second-rank tensor that responds to the local stretching and rotation of fluid elements by the velocity gradient, as well as to their advection by the velocity itself. It follows from noting that, in the absence of spring or Brownian forces, $\mathbf{R}$ evolves according to

$$\dot{\mathbf{R}} \equiv \partial_t \mathbf{R} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{R} = \mathbf{R} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{34}$$

which is the evolution equation for a material line element, or for vorticity in ideal hydrodynamics, or for magnetic field in ideal magnetohydrodynamics. The tensor $\mathbf{R}\mathbf{R}$ then evolves according to $(\overset{\triangledown}{\mathbf{R}\mathbf{R}}) = 0$.

The number density of bead-spring pairs crossing a unit area with normal vector $\mathbf{n}$ is $\mathbf{n} \cdot \mathbf{R}\psi$, so the momentum transferred across this area is

$$\int (H\mathbf{R})\, \mathbf{n} \cdot \mathbf{R}\psi \, \mathrm{d}\mathbf{R} \equiv \boldsymbol{\sigma}^{\text{spring}} \cdot \mathbf{n}. \tag{35}$$

Since this holds for all $\mathbf{n}$, the stress exerted by the springs is $\boldsymbol{\sigma}^{\text{spring}} = nH\langle \mathbf{R}\mathbf{R} \rangle = nH\mathsf{C}$. Equation (32) shows that $\mathsf{C} = (k_B T/H)\mathsf{I}$ at equilibrium (LHS vanishing), which corresponds to $\boldsymbol{\sigma}^{\text{spring}} = n k_B T \mathsf{I}$. This is just minus the pressure one would get for an ideal gas of particles with number density n, and in fact the beads also contribute a positive pressure of $2nk_BT$ for the same reason (the number density of beads is $2n$). However, the overall pressure is arbitrary in an incompressible fluid, so we can just write the stress due to the bead-spring pairs as

$$\boldsymbol{\sigma}^{\text{spring}} = nH\mathsf{C} = nk_BT\mathsf{I} + \boldsymbol{\sigma}^P, \tag{36}$$

where $\boldsymbol{\sigma}^P$ vanishes in the absence of a flow.

The upper convected derivative of the identity matrix is

$$\overset{\triangledown}{\mathsf{I}} \equiv \partial_t \mathsf{I} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathsf{I} - (\boldsymbol{\nabla} \mathbf{u})^{\mathsf{T}} \cdot \mathsf{I} - \mathsf{I} \cdot (\boldsymbol{\nabla} \mathbf{u}) = -(\boldsymbol{\nabla} \mathbf{u}) - (\boldsymbol{\nabla} \mathbf{u})^{\mathsf{T}} = -2\, \mathsf{e}. \tag{37}$$

It is not zero because the identity matrix does not deform with the flow. Using this expression to eliminate $\boldsymbol{\sigma}^{\text{spring}}$ gives

$$\overset{\triangledown}{\boldsymbol{\sigma}}{}^P + \frac{4H}{\zeta} \boldsymbol{\sigma}^P = 2nk_BT\, \mathsf{e}. \tag{38}$$

This is commonly written as the "upper convected Maxwell model"

$$\boldsymbol{\sigma}^P + \tau \overset{\triangledown}{\boldsymbol{\sigma}}{}^P = 2\mu' \mathsf{e}, \tag{39}$$

where $\mu' = nk_BT\zeta/(4H)$ is the steady state shear viscosity, and $\tau = \zeta/(4H)$ is the stress relaxation time.

This is a nonlinear generalisation of the linear Maxwell model

$$\boldsymbol{\sigma}^P + \tau \partial_t \boldsymbol{\sigma}^P = 2\mu' \mathsf{e} \tag{40}$$

proposed by Maxwell for rarefied gases, in which τ is the mean free time between collisions between particles. However, this linear Maxwell model is not *objective*, meaning that the stress in a fluid undergoing a rigid rotation as well as a deformation is not the same as the rotation of the stress in the fluid undergoing the same deformation without the rotation. The upper convected Maxwell model is objective.

This is similar to noting that the linear Maxwell model is not Galilean invariant, which can be cured by replacing $\partial_t \boldsymbol{\sigma}^P$ with $(\partial_t + \mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}^P$. To obtain an evolution equation that is objective one needs to replace $\partial_t \boldsymbol{\sigma}^P$ with $\overset{\triangledown}{\boldsymbol{\sigma}}{}^P$.

A. The Oldroyd-B viscoelastic model

The Oldroyd-B model is a more realistic description of a viscoelastic fluid. It takes the total stress in the fluid to be the sum of a particle stress $\boldsymbol{\sigma}^P$ evolving according to the upper convected Maxwell model and a Newtonian viscous stress with viscosity μ from from the solvent fluid:

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\mu\,\mathbf{e} + \boldsymbol{\sigma}^P. \tag{41}$$

As usual, the pressure p is ultimately determined by the incompressibility condition $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$.

We can combine $2\mu\,\mathbf{e} + \boldsymbol{\sigma}^P = \mathbf{T}$ into a single stress that evolves according to

$$\mathbf{T} + \tau\overset{\triangledown}{\mathbf{T}} = 2(\mu+\mu')\left(\mathbf{e} + \tau\frac{\mu}{\mu+\mu'}\overset{\triangledown}{\mathbf{e}}\right). \tag{42}$$

This looks like the upper convected Maxwell model, but with an extra term proportional to $\overset{\triangledown}{\mathbf{e}}$. Moreover, the effective relaxation time $\tau\mu/(\mu+\mu)$ for $\mathbf{e}$ is always shorter than the relaxation time τ for $\mathbf{T}$.

The linearised version of (42) with the upper convected derivatives $\triangledown$ replaced by simple partial time derivatives ∂_t is called a Jeffreys fluid (note different spelling from Jeffery in section VI D). The Jefferys fluid model describes a suspension of linear elastic spheres in a viscous fluid.

The Oldroyd-B model provides a reasonable description of some real materials, called "Boger fluids" or "constant viscosity elastic fluids". These fluids are distinguished by their steady state shear viscosity being independent of strain rate. In most complex fluids the stress in steady shear is a nonlinear function of the strain rate because the microstructure does not behave like a linear Hookean spring.

B. Warning: Tensor notation for the velocity gradient in rheology

The rheology literature often uses a very confusing convention for $\boldsymbol{\nabla}\mathbf{u}$. It is common to write a velocity field that depends linearly on the spatial coordinates $\mathbf{x}$ as

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{0}) + \mathbf{L}\,\mathbf{x}, \tag{43}$$

where $\mathbf{L}$ is a constant matrix (or second rank tensor). This follows the convention that $\mathbf{x}$ and $\mathbf{u}$ are column vectors, and operators are written on the left of the objects they operate on.

It is very tempting to call $\mathbf{L}$ the "velocity gradient". The problem arises if one wants to write this "velocity gradient" as $\nabla\mathbf{u}$, because then $[(\nabla\mathbf{u})\cdot\mathbf{x}]_i = (\partial_j u_i)x_j$ with the $\mathbf{x}$ contracting with the ∇. This is the opposite of the convention that tensor contractions are between adjacent symbols. In index notation, the rheology literature often defines

$[\boldsymbol{\nabla}\mathbf{u}]_{ij} = \partial_j u_i$ with the **left** index on the **right** symbol, not the more natural
$[\boldsymbol{\nabla}\mathbf{u}]_{ij} = \partial_i u_j$ with the left index on the left symbol.

VI. SUSPENSIONS OF SPHEROIDAL PARTICLES

A. Resistance matrix formulation for a single arbitrary body

Consider a single arbitrarily-shaped body in a linear background flow. Generalising the Fáxen relations to such a body leads to

$$\begin{pmatrix}\mathbf{F}\\ \mathbf{T}\\ \mathbf{S}\end{pmatrix} = \mu\begin{pmatrix}\mathbf{A} & \mathbf{B}^{\mathsf{T}} & \widetilde{\mathbf{G}}\\ \mathbf{B} & \mathbf{C} & \widetilde{\mathbf{H}}\\ \mathbf{G} & \mathbf{H} & \mathbf{M}\end{pmatrix}\begin{pmatrix}\mathbf{U}^\infty - \mathbf{U}\\ \boldsymbol{\Omega}^\infty - \boldsymbol{\Omega}\\ \mathbf{E}^\infty\end{pmatrix} \tag{44}$$

with a 3×3 block matrix called the resistance matrix. The reciprocal theorem imposes certain symmetry relations:

$$\mathbf{A} = \mathbf{A}^{\mathsf{T}},\quad \mathbf{C} = \mathbf{C}^{\mathsf{T}},\quad G_{ijk} = \widetilde{G}_{kij},\quad H_{ijk} = \widetilde{H}_{kij},\quad M_{ijkl} = M_{klij}. \tag{45}$$

The vector $\boldsymbol{\Omega}^\infty = \frac{1}{2}\boldsymbol{\omega}^\infty$ is the angular velocity of the background flow, equal to half the vorticity. The resistance matrix is diagonal for spheres, but not in general. For example, the torque is given by

$$T_i = \mu\left(B_{ij}(U_j^\infty - U_j) + C_{ij}(\Omega_j^\infty - \Omega_j) + \tilde{H}_{ijk}E_{jk}^\infty\right). \tag{46}$$

The torque thus depends, in general, on both the linear and angular relative velocities, and on the strain rate.

Any appearance of a torque raises the question: torque about where? It turns out that we can define a "hydrodynamic centre" $\mathbf{x}^{(0)}$ for which the matrix B is symmetric when $\mathbf{T}$ is the torque about $\mathbf{x}^{(0)}$.

Consider the torques $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(0)}$ on the body (with surface S) about two points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(0)}$. Their difference is

$$\begin{aligned}\mathbf{T}^{(1)} - \mathbf{T}^{(0)} &= \int_S (\mathbf{x}-\mathbf{x}^{(1)})\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{d}S - \int_S (\mathbf{x}-\mathbf{x}^{(0)})\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{d}S,\\ &= -\int_S (\mathbf{x}^{(1)}-\mathbf{x}^{(0)})\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{d}S,\\ &= -(\mathbf{x}^{(1)}-\mathbf{x}^{(0)})\times\mathbf{F}.\end{aligned} \tag{47}$$

For a body in pure translation, no rotation or strain, the force is $\mathbf{F} = \mu\mathsf{A}(\mathbf{U}^\infty - \mathbf{U})$ and the torques about the two points are $\mathbf{T}^{(0)} = \mu\mathsf{B}^{(0)}(\mathbf{U}^\infty - \mathbf{U})$ and $\mathbf{T}^{(1)} = \mu\mathsf{B}^{(1)}(\mathbf{U}^\infty - \mathbf{U})$. Equation (47) must hold for all constant vectors $\mathbf{U}^\infty - \mathbf{U}$, so

$$B_{ij}^{(1)} - B_{ij}^{(0)} = -\epsilon_{ikl}\left(x_k^{(1)} - x_k^{(0)}\right)A_{lj}. \tag{48}$$

Suppose $\mathbf{x}^{(0)}$ can be chosen so that $B_{ij}^{(0)}$ is symmetric. Taking the antisymmetric part of (48) then gives

$$\epsilon_{ijk}B_{ij}^{(1)} = (A_{kj} - A_{ll}\delta_{kj})(x_j^{(0)} - x_j^{(1)}), \tag{49}$$

which we can solve for

$$x_j^{(0)} = x_j^{(1)} + \left[(\mathsf{A} - (\mathrm{Tr}\,\mathsf{A})\,\mathsf{I})^{-1}\right]_{jk}\epsilon_{kpq}B_{pq}^{(1)}. \tag{50}$$

The matrix $\mathsf{A} - (\mathrm{Tr}\,\mathsf{A})\mathsf{I}$ is non-singular as A is symmetric, hence diagonalisable, and its eigenvalues are all positive because the drag force always opposes the motion. For this $\mathbf{x}^{(0)}$ we can show from (48) that $\mathsf{B}^{(0)}$ is symmetric.

B. Warning: angular velocity vector versus angular velocity tensor

Some references express angular velocities using an antisymmetric tensor Ω instead of a vector $\boldsymbol{\Omega}$. I have used an inconsistent non-bold font deliberately to help distinguish these symbols.

When reading material using the Ω tensor it is very important to check the convention for $\boldsymbol{\nabla}\mathbf{u}$ versus $(\boldsymbol{\nabla}\mathbf{u})^\mathsf{T}$.

In my conventions that always put the left index on the left symbol, so $[\nabla\mathbf{u}]_{ij} = \partial_i u_j$, the transposed velocity gradient is $(\boldsymbol{\nabla}\mathbf{u})^\mathsf{T} = \mathsf{E} + \Omega$, and $\boldsymbol{\Omega}\times\mathbf{p} = \Omega\cdot\mathbf{p}$.

Calculating $\frac{1}{2}\boldsymbol{\omega}\times\mathbf{p}$ in index notation gives

$$\frac{1}{2}\left[\boldsymbol{\omega}\times\mathbf{p}\right]_i = \frac{1}{2}\epsilon_{ijk}\left(\epsilon_{jlm}\partial_l u_m\right)p_k = \frac{1}{2}\epsilon_{jki}\epsilon_{jlm}\left(\partial_l u_m\right)p_k = \frac{1}{2}\left(\delta_{kl}\delta_{im} - \delta_{il}\delta_{km}\right)\left(\partial_l u_m\right)p_k = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)p_k. \tag{51}$$

Going back to tensor notation,

$$\boldsymbol{\Omega}\times\mathbf{p} = \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p} = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u})^\mathsf{T} - (\boldsymbol{\nabla}\mathbf{u})\right)\cdot\mathbf{p} = \Omega\cdot\mathbf{p}, \tag{52}$$

so the vorticity tensor is

$$\Omega = \frac{1}{2}\left((\boldsymbol{\nabla}\mathbf{u})^\mathsf{T} - (\boldsymbol{\nabla}\mathbf{u})\right), \tag{53}$$

in the convention that puts the left index on the left symbol in $\boldsymbol{\nabla}\mathbf{u}$.

C. Resistance matrix formulation for an axisymmetric body

For an axisymmetric body with its symmetry axis oriented parallel to a unit vector $\mathbf{p}$ we can exploit further symmetries. For example, the force due to a pure translation is

$$\mathbf{F} = \mu \left(A^{\parallel}\mathbf{pp} + A^{\perp}(\mathbf{I} - \mathbf{pp}) \right) \cdot (\mathbf{U}^{\infty} - \mathbf{U}) \,, \tag{54}$$

where $\mu A^{\parallel}$ and $\mu A^{\perp}$ are the drag coefficients for translations parallel and perpendicular to $\mathbf{p}$. As an interesting aside, $A^{\perp} = 2A^{\parallel}$ for very elongated bodies, so there is very little benefit from "streamlining" in Stokes flow.

Similarly,

$$\begin{aligned} B_{ij} &= Y^{B}\epsilon_{ijk}p_k, \\ C_{ij} &= X^{C}p_ip_j + Y^{C}(\delta_{ij} - p_ip_j), \\ H_{ijk} &= \frac{1}{2}Y^{H}(\epsilon_{ikl}p_j + \epsilon_{jkl}p_i)p_l, \\ G_{ijk} &= X^{G}(p_ip_j - \tfrac{1}{3}\delta_{ij})p_k + Y^{G}(p_i\delta_{jk} + p_j\delta_{ik} - 2p_ip_jp_k). \end{aligned}$$

where the X and Y scalar coefficients depend on the shape of the body.

These coefficients are given on pages 65 and 68 of Kim & Karrila's book for prolate and oblate spheroids. The expression for H_{ijk} in terms of Y^{H} in Kim & Karrila's book seems to be missing a factor of $1/2$. This factor is needed for the Bretherton constant defined below to be $\beta = Y^{H}/Y^{C}$ without a factor of 2.

The coefficient $Y^{B} = 0$ when the torque is taken about the hydrodynamic centre (since B must then be symmetric). Forces and translations then decouple from torques and rotations.

D. A single torque-free axisymmetric body

Consider a single axisymmetric body in a linear flow with no external torque about its hydrodynamic centre.

From the resistance matrix formulation we have

$$0 = \mathbf{T} = \mu \left[\mathsf{C}\,(\boldsymbol{\Omega}^{\infty} - \boldsymbol{\Omega}) + \widetilde{\mathsf{H}}\mathsf{E}^{\infty} \right], \tag{55}$$

which we can solve for

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\infty} + \mathsf{C}^{-1}\widetilde{\mathsf{H}}\mathsf{E}^{\infty}. \tag{56}$$

In component form this is

$$\Omega_i = \Omega_i^{\infty} + \left(\frac{1}{X^{C}}p_ip_l + \frac{1}{Y^{C}}(\delta_{il} - p_ip_l) \right) \frac{1}{2}Y^{H}\left(\epsilon_{jlm}p_k + \epsilon_{klm}p_j\right)p_m E_{jk}^{\infty}. \tag{57}$$

The orientation vector $\mathbf{p}$ thus evolves according to

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}\times\mathbf{p} = \boldsymbol{\Omega}^{\infty}\times\mathbf{p} + \beta\left(\mathsf{E}^{\infty}\cdot\mathbf{p} - \mathbf{p}\cdot\mathsf{E}^{\infty}\cdot\mathbf{p}\,\mathbf{p}\right), \tag{58}$$

where $\beta = Y^{H}/Y^{C}$. The last term involving $\mathbf{p}\cdot\mathsf{E}^{\infty}\cdot\mathbf{p}$ in (58) maintains $|\mathbf{p}| = 1$. The terms with p_ip_l in (57) describe rotation around the $\mathbf{p}$-axis, so they do not contribute to $\dot{\mathbf{p}}$.

This equation was derived by Jeffery (1922) for spheroidal particles. It was extended by Bretherton (1962) for (almost) all rigid particles with an axis of symmetry. The parameter β is called the Bretherton constant, and

$$\beta = \frac{r^2 - 1}{r^2 + 1} \tag{59}$$

for spheroids with aspect ratio $r = a/b$, where a is the length along the symmetry axis, and b is the length perpendicular to the axis. Thus $\beta = -1$ describes a flat disc, $\beta = 0$ a sphere, and $\beta = 1$ a slender rod.

Spherical particles with $\beta = 0$ rotate with the local fluid vorticity,

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}\times\mathbf{p} = \frac{1}{2}\boldsymbol{\omega}\times\mathbf{p}, \tag{60}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$. The factor of 1/2 arises from the vector identity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\frac{1}{2}\boldsymbol{\omega}\times\mathbf{x})$ for constant $\boldsymbol{\omega}$. The local rate of rotation $\boldsymbol{\Omega}$ is thus half the local vorticity $\boldsymbol{\omega}$.

Conversely, $\mathsf{E}\cdot\mathbf{p} + \boldsymbol{\Omega}\times\mathbf{p} = (\boldsymbol{\nabla}\mathbf{u})^{\mathsf{T}}\cdot\mathbf{p}$ for all vectors $\mathbf{p}$, so Jeffery's equation for slender rods with $\beta = 1$ is

$$\dot{\mathbf{p}} = \mathbf{p}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{p}\cdot(\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{p}\,\mathbf{p}. \tag{61}$$

Treating $\mathbf{p}(\mathbf{x}, t)$ as a material vector field, and $\dot{\mathbf{p}}$ as a Lagrangian time derivative following an individual rod, leads to

$$\partial_t\mathbf{p} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{p} - \mathbf{p}\cdot\boldsymbol{\nabla}\mathbf{u} = \mathbf{p}\cdot(\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{p}\,\mathbf{p}. \tag{62}$$

The left-hand side is an upper convected derivative for the vector field $\mathbf{p}$, with a normalising term on the right-hand side to preserve $|\mathbf{p}| = 1$. The left-hand side is the same evolution equation for a material line element $\boldsymbol{\ell}$,

$$\partial_t\boldsymbol{\ell} + \mathbf{u}\cdot\boldsymbol{\nabla}\boldsymbol{\ell} - \boldsymbol{\ell}\cdot\boldsymbol{\nabla}\mathbf{u} = 0, \tag{63}$$

or for vorticity $\boldsymbol{\omega}$ in an ideal fluid, or magnetic field $\mathbf{B}$ in ideal magnetohydrodynamics. This is also the equation for vectors that implies the upper convected derivative for the tensors $\boldsymbol{\ell\ell}$ or $\mathbf{pp}$, or for the magnetic tension $\mathbf{BB}$ in ideal MHD. Other values of β lead to other convected derivatives, notably the lower convected derivative for $\beta = -1$ (discs) and the Jaumann derivative for $\beta = 0$ (spheres).

Based on this observation, we can rewrite Jeffery's equation for general β as

$$\partial_t\mathbf{p} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{p} = \mathbf{p}\cdot\boldsymbol{\nabla}\mathbf{u} + (\beta - 1)\mathsf{E}\cdot\mathbf{p} - \beta\,\mathbf{p}\cdot\mathsf{E}\cdot\mathbf{p}\,\mathbf{p}, \tag{64}$$

or as

$$\partial_t\mathbf{p} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{p}) - \mathbf{u}\nabla\cdot\mathbf{p} + (\beta - 1)\mathsf{E}\cdot\mathbf{p} - \beta\,\mathbf{p}\cdot\mathsf{E}\cdot\mathbf{p}\,\mathbf{p}. \tag{65}$$

The first term on the right hand side is the same term that appears in the induction equation for ideal magnetohydrodynamics, or for the vorticity evolution equation in the Euler equations. Two of the remaining terms appear because we have replaced the divergence-free condition $\nabla\cdot\mathbf{p} = 0$ by the normalisation condition $|\mathbf{p}| = 1$.

One way to enforce the normalisation condition is to write $\mathbf{p} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ $\mathbf{p}$ in spherical polar coodinates. The time derivative of $\mathbf{p}$ is then

$$\dot{\mathbf{p}} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)\,\dot{\theta} + (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)\,\dot{\varphi}. \tag{66}$$

The right-hand side of Jeffery's equation will always lie in the span of these two vectors, as $|\mathbf{p}| = 1$ implies $\dot{\mathbf{p}}\cdot\mathbf{p} = 0$.

For the standard shear flow $\mathbf{u} = \dot{\gamma}y\hat{\boldsymbol{x}}$, Jeffery's equation then becomes

$$\dot{\varphi} = -\frac{\dot{\gamma}}{r^2+1}\left(r^2\cos^2\varphi + \sin^2\varphi\right), \tag{67}$$

$$\dot{\theta} = -\frac{\dot{\gamma}}{4}\frac{r^2-1}{r^2+1}\sin(2\theta)\sin(2\varphi), \tag{68}$$

with solutions

$$\tan\varphi = -r\tan\left(\frac{\dot{\gamma}t}{r+1/r} + k\right), \quad \tan\theta = \frac{Cr}{\sqrt{r^2\cos^2\varphi + \sin^2\varphi}}, \tag{69}$$

where C and k are constants. The vector $\mathbf{p}$ lies on the intersection of the unit sphere $|\mathbf{p}| = 1$ and the hyperboloid

$$r^2p_x^2 + p_y^2 = r^2C^2p_z^2, \tag{70}$$

which define periodic "Jeffery orbits".

E. Dilute suspension of spheroids

Now consider a dilute suspension of spheroids with number density n, and a distribution function $\Psi(\mathbf{p}, \mathbf{x}, t)$, normalised so that

$$\int_S \Psi(\mathbf{p}, \mathbf{x}, t)\,\mathrm{d}S = 1, \tag{71}$$

where the integration dS is taken over the unit sphere covered by $\mathbf{p}$.

Formally, a suspension being dilute for slender rods is very restrictive. We need $na^3 < 1$ so that the individual rods can each rotate freely without intersecting each other, so the volume fraction $\phi = \frac{4}{3}\pi nab^2 = O(r^{-2})$. However, the following results also apply (with minor changes to coefficients) in the "semi-dilute" regime with $r^{-2} < \phi < r^{-1}$, and for aligned rods we can allow $r^{-2} < \phi < 1$.

As with the earlier bead and spring model, we also add a stochastic Brownian torque

$$\mathbf{T}_{\mathrm{Br}} = -\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}\left(k_B T\log\Psi\right) \tag{72}$$

to the left hand side of the resistance matrix formulation. This is perpendicular to $\mathbf{p}$ so it is multiplied by the Y_C part of the C tensor. The expression for $\dot{\mathbf{p}}$ is given by Jeffery's equation for the local strain E and vorticity $\boldsymbol{\Omega}$, thus becomes

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}\times\mathbf{p} + \beta\left(\mathsf{E}\cdot\mathbf{p} - \mathbf{p}\cdot\mathsf{E}\cdot\mathbf{p}\,\mathbf{p}\right) - (1/Y^C)k_B T\boldsymbol{\nabla}_{\mathbf{p}}(\log\Psi). \tag{73}$$

The derivative $\nabla_{\mathbf{p}}$ is the tangential derivative on the surface of the unit sphere,

$$\nabla_{\mathbf{p}} = (\mathsf{I} - \mathbf{p}\,\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{p}}, \tag{74}$$

to preserve the normalisation $|\mathbf{p}| = 1$.

The Liouville equation Ψ,

$$\partial_t\Psi + \mathbf{u}\cdot\boldsymbol{\nabla}\psi + \nabla_{\mathbf{p}}\cdot(\dot{\mathbf{p}}\Psi) = 0, \tag{75}$$

then becomes

$$(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\Psi + (\boldsymbol{\Omega}\times\mathbf{p} + \beta\,\mathsf{E}\cdot\mathbf{p})\cdot\boldsymbol{\nabla}_{\mathbf{p}}\Psi - 3\,\beta\,\mathbf{p}\cdot\mathsf{E}\cdot\mathbf{p}\Psi = D_r\boldsymbol{\nabla}^2_{\mathbf{p}}\Psi, \tag{76}$$

with rotational Brownian diffusivity $D_r = k_B T/Y^C$. In principle there should also be a translational Brownian diffusion term, but, as for the earlier bead-spring model, this is usually negligibly small compared with the rotational Brownian diffusion. The translational diffusion involves macroscopic lengthscales, which are typically much larger than the lengthscale of the individual particles.

The stress exerted by the spheroids on the fluid is given generally by

$$\overline{\boldsymbol{\sigma}}^P = \langle\mathsf{m} : \mathsf{E}^\infty - \mathsf{h}\cdot\mathbf{T}_{\mathrm{Br}}\rangle, \tag{77}$$

where (as usual) $\langle\cdot\rangle$ denotes an average with the distribution Ψ. The tensors m and h come from the "mobility matrix" formulation for Stokes flow around a general particle:

$$\begin{pmatrix}\mathbf{U}^\infty - \mathbf{U}\\ \boldsymbol{\Omega}^\infty - \boldsymbol{\Omega}\\ \mu^{-1}\mathsf{S}\end{pmatrix} = \begin{pmatrix}\mathsf{a} & \mathsf{b}^\mathsf{T} & \tilde{\mathsf{g}}\\ \mathsf{b} & \mathsf{c} & \tilde{\mathsf{h}}\\ \mathsf{g} & \mathsf{h} & \mathsf{m}\end{pmatrix}\begin{pmatrix}\mu^{-1}\mathbf{F}\\ \mu^{-1}\mathbf{T}\\ \mathsf{E}^\infty\end{pmatrix}. \tag{78}$$

This is dual to the earlier resistance matrix formulation (44), except that S appears on the left hand side in both formulations. The tensors h and m are given in terms of the tensors in the resistance matrix by

$$\begin{pmatrix}\mathsf{g} & \mathsf{h}\end{pmatrix} = \begin{pmatrix}\mathsf{G} & \mathsf{H}\end{pmatrix}\begin{pmatrix}\mathsf{A} & \mathsf{B}^\mathsf{T}\\ \mathsf{B} & \mathsf{C}\end{pmatrix}, \quad\text{and}\quad \mathsf{m} = \mathsf{M} - \begin{pmatrix}\mathsf{G} & \mathsf{H}\end{pmatrix}\begin{pmatrix}\mathsf{A} & \mathsf{B}^\mathsf{T}\\ \mathsf{B} & \mathsf{C}\end{pmatrix}\begin{pmatrix}\tilde{\mathsf{G}}\\ \tilde{\mathsf{H}}\end{pmatrix}. \tag{79}$$

The second expression uses the symmetry $M_{ijkl} = M_{klij}$ of M required by the reciprocal theorem.

For particles with an axis of symmetry oriented along $\mathbf{p}$, the total stress simplifies to

$$\boldsymbol{\sigma} = -p\mathsf{I} + 2\mu\mathsf{e} + 2\mu\phi\Big(A\langle\mathbf{pppp}\rangle : \mathsf{e} + B(\langle\mathbf{pp}\rangle\cdot\mathsf{e} + \mathsf{e}\cdot\langle\mathbf{pp}\rangle) + C\mathsf{e} + FD_r\langle\mathbf{pp}\rangle\Big), \tag{80}$$

where the scalar constants A, B, C, and F are determined by the particle shape. This notation is standard, but these constants are not simply related to the sub-matrices of the 3×3 block resistance matrix.

For slender spheroidal particles with aspect ratio $r = a/b \gg 1$, the asymptotic forms of the constants are

$$A \sim \frac{r^2}{2(\log(2r) - 3/2)}, \quad B \sim \frac{6\log(2r) - 11}{r^2}, \quad C \to 2, \quad F \sim \frac{3r^2}{\log(2r) - 1/2}. \tag{81}$$

These formulae still hold in the semi-dilute limit (Batchelor 1971) if we replace $\log r$ with $-(1/2)\log\phi$ based on the volume fraction.

For large slender spheroidal particles (large enough to be non-Brownian) we only need to keep the A term, giving

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\left(\mathbf{e} + N\langle\mathbf{pppp}\rangle : \mathbf{e}\right). \tag{82}$$

The "non-Newtonian parameter" $N = \phi A \sim \phi r^2/(\log r)$ may be significant even when the volume fraction ϕ is small. If the particles are also aligned, the ensemble averaging brackets $\langle\cdots\rangle$ are unnecessary. This is then the same stress one finds for a common astrophysical approximation to "Braginskii MHD" in which the viscous stress is predominantly aligned with the magnetic field direction $\mathbf{p} = \mathbf{B}/|\mathbf{B}|$.

Alternatively, instead of omitting the ensemble averaging and the Brownian rotational diffusion, one may formulate equations for moments such as $\langle\mathbf{pp}\rangle$. This leads to some common models for fibre suspensions.

For example, the traceless part of the second moment $\mathbf{Q} = \langle\mathbf{pp} - \frac{1}{3}\mathbf{I}\rangle$ evolves according to

$$\overset{\triangledown}{\mathbf{Q}} = (\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{Q} - \mathbf{Q}\cdot(\boldsymbol{\nabla}\mathbf{u}) - (\boldsymbol{\nabla}\mathbf{u})^{\mathsf{T}}\cdot\mathbf{Q} = -6D_r\mathbf{Q} + \frac{2}{3}\mathbf{e} - 2\langle\mathbf{pppp}\rangle : \mathbf{e}. \tag{83}$$

While we recognise the upper convected derivative again on the left hand side, as for the pairs of spheres with springs, we no longer get a closed equation for $\langle\mathbf{pp}\rangle$.

A closure approximation based on simple shears and elongational flows is (Hinch & Leal 1976, JFM)

$$\langle\mathbf{pppp}\rangle : \mathbf{e} \approx \frac{1}{5}\left[6\langle\mathbf{pp}\rangle\cdot\mathbf{e}\cdot\langle\mathbf{pp}\rangle - \langle\mathbf{pp}\rangle\langle\mathbf{pp}\rangle : \mathbf{e} - 2\mathbf{I}\langle\mathbf{pp}\rangle^2 : \mathbf{e} + 2\mathbf{I}\langle\mathbf{pp}\rangle : \mathbf{e}\right]. \tag{84}$$

This closure gives more reasonable solutions for $\mathbf{Q}$ in the weak flow and strong flow limits than the "obvious" closure

$$\langle\mathbf{pppp}\rangle : \mathbf{e} \approx \langle\mathbf{pp}\rangle\langle\mathbf{pp}\rangle : \mathbf{e}. \tag{85}$$

VII. ROADMAP

A. Swimmers and active matter

A kinetic model that explains many experimental observations of swimming microorganisms assumes that the particles are self-propelled relative to the fluid in the direction $\mathbf{p}$. The advective velocity thus becomes $V_0\mathbf{p} + \mathbf{u}(\mathbf{x}, t)$ instead of just $\mathbf{u}(\mathbf{x}, t)$. There is a corresponding extra active stress $\boldsymbol{\sigma}^{\text{active}} = n\sigma_0\langle\mathbf{p}\,\mathbf{p} - \frac{1}{3}\mathbf{I}\rangle$, where n is the number density of swimmers and σ_0 a constant that may have either sign (positive for "pushers" and negative for "pullers"). This closely resembles the Maxwell stress $-\frac{1}{2}|\mathbf{B}|^2\mathbf{I} + \mathbf{B}\,Bv$ in magnetohydrodynamics.

It is common to add an integral reorientation term to describe "run and tumble" with timescale τ_r and kernel K,

$$\frac{\partial \Psi}{\partial t} + \boldsymbol{\nabla}\cdot((V_0\mathbf{p} + \mathbf{u})\Psi) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}}\Psi) = -\frac{1}{\tau_r}\left(\Psi - \int K(\mathbf{p}, \mathbf{p}')\Psi(\mathbf{x}, \mathbf{p}', t)d\mathbf{p}'\right) \tag{86}$$

It is possible to add a gravitational "gyrotactic" torque to $\dot{\mathbf{p}}$ that is important for some bacteria, so

$$\dot{\mathbf{p}} = \frac{1}{\tau_g}\left(\mathbf{k} - \mathbf{k}\cdot\mathbf{p}\,\mathbf{p}\right) + \boldsymbol{\Omega}\times\mathbf{p} + \beta\left(\mathbf{E}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{E}\cdot\mathbf{p}\,\mathbf{p}\right), \tag{87}$$

where $\mathbf{k}$ is a unit vector in the upward direction, and τ_g is the timescale of gravitational reorientation.

B. Liquid crystals

If the suspension is dense there are extra effects due to interactions between particles that are modelled using a Landau–de Gennes free energy that the gradient of $\mathbf{p}$. It is common to use one of the moments $\mathbf{n} = \langle\mathbf{p}\rangle$ or $\mathbf{Q} = \langle\mathbf{p}\,\mathbf{p} - \frac{1}{3}\mathbf{I}\rangle$. For example, a minimal model has

$$\mathcal{F} = \frac{1}{2}\int -A\,\mathrm{Tr}\,(\mathbf{Q}^2) + \frac{1}{2}C(\mathrm{Tr}\,(\mathbf{Q}^2))^2 + L(\partial_k Q_{ij})(\partial_k Q_{ij})dV. \tag{88}$$

This contributes an extra term in the evolution equation for Q_{ij} of the form

$$\Gamma H_{ij} \text{ where } H_{ij} = -\frac{\delta\mathcal{F}}{\delta Q_{ij}} = A^2 Q_{ij} - C^2(\mathrm{Tr}\,\mathbf{Q}^2)Q_{ij} + \lambda(\mathbf{x}, t)\delta_{ij} + L\boldsymbol{\nabla}^2 Q_{ij}. \tag{89}$$

The Lagrange multiplier term $\lambda(\mathbf{x}, t)\delta_{ij}$ is needed to enforce the constraint $\mathrm{Tr}\,\mathbf{Q} = 0$.

There are also products and gradients involving $\mathbf{Q}$ and $\mathbf{H}$ in the hydrodynamic stress $\boldsymbol{\sigma}$.

C. Braginskii magnetohydrodynamics

In Kinetic Theory we derived the Navier–Stokes equations from the Boltzmann equation with the BGK collision operator:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f = -\frac{1}{\tau}\left(f - f^{(0)}\right). \tag{90}$$

For particles with mass m and charge q in a magnetic field $\mathbf{B}$ there is also a Lorentz force term, giving

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f + q(\mathbf{v}\times\mathbf{B})\cdot\boldsymbol{\nabla}_{\mathbf{v}} f = -\frac{1}{\tau}\left(f - f^{(0)}\right). \tag{91}$$

Taking the second moment with respect to the peculiar velocity $\mathbf{w} = \mathbf{v} - \mathbf{u}$, so $\boldsymbol{\sigma} = -\int \mathbf{w}\mathbf{w} f d\mathbf{w}$, gives

$$\frac{\partial \sigma_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_i\sigma_{jk} - q_{ijk}\right) + \frac{\partial u_i}{\partial x_k}\sigma_{kj} + \frac{\partial u_i}{\partial x_k}\sigma_{kj} = \frac{q}{m}B_k\left(\epsilon_{ilk}\sigma_{lj} + \epsilon_{jlk}\sigma_{li}\right) + \frac{1}{\tau}\left(-p\delta_{ij} - \sigma_{ij}\right). \tag{92}$$

Expanding $\boldsymbol{\sigma} = -p\,\mathbf{I} + \boldsymbol{\sigma}^{(1)}$ with $\boldsymbol{\sigma}^{(1)}$ small, and treating the $(q/m)\mathbf{B}$ term as the same order as the collision term on the right hand side of (92) leads to the Braginskii magnetohydrodynamics viscous stress

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\mu\,\mathbf{b}\mathbf{b}\,\mathbf{b}\cdot\mathbf{e}\cdot\mathbf{b}, \tag{93}$$

where $\mu = \tau p$ is the usual dynamic viscosity and $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ is a unit vector in the direction of the magnetic field, at leading order when $\tau q|\mathbf{B}|/m \gg 1$. A plasma satisfying this last condition is called "strongly magnetised".