Variations on a beta-plane: derivation of non-traditional beta-plane equations from Hamilton's principle on a sphere

By Paul J. Dellar†

OCIAM, Mathematical Institute, 24–29 St Giles', Oxford, OX1 3LB, UK

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Starting from Hamilton's principle on a rotating sphere, we derive a series of successively more accurate β-plane approximations. These are Cartesian approximations to motion in spherical geometry that capture the change with latitude of the angle between the rotation vector and the local vertical. Being derived using Hamilton's principle, the different β-plane approximations each conserve energy, angular momentum, and potential vorticity. They differ in their treatments of the locally horizontal component of the rotation vector, the component that is usually neglected under the traditional approximation. In particular, we derive an extended set of β-plane equations in which the locally vertical and locally horizontal components of the rotation vector both vary linearly with latitude. This was previously thought to violate conservation of angular momentum and potential vorticity. We show that the difficulty in maintaining these conservation laws arises from the need to express the rotation vector as the curl of a vector potential while approximating the true spherical metric by a flat Cartesian metric. Finally, we derive depth-averaged equations on our extended β-plane with topography, and show that they coincide with the extended non-traditional shallow water equations previously derived in Cartesian geometry.

1. Introduction

Geophysical fluid dynamics is concerned with the wide range of complex behaviour exhibited on many different scales by stratified fluids moving over the surface of a rotating planet. Simplified or model equations designed to capture particular phenomena or processes thus play an important rôle. In this paper we consider the simplications that replace spherical, or even spheroidal, coordinates with Cartesian coordinates while retaining the latitude-dependence of the angle between the local vertical and the planetary rotation vector. These approximations are known as β-planes, after the symbol used for the latitude-dependence of the locally vertical component of the rotation vector. They were first proposed by Rossby *et al.* (1939) as a purely conceptual model for motion on a sphere. Only later, beginning with Veronis (1963), were the β-planes investigated as possible rational approximations to the equations of motion formulated in spherical geometry. In particular, a valid set of β-plane equations should inherit the conservation laws of the underlying spherical equations. These include conservation of energy and angular momentum, and material conservation of another quantity called potential vorticity. The last is probably the most important because potential vorticity is tied to individual fluid elements, while energy and momentum may be transported over long distances by waves without any net displacement of fluid elements.

Following Rossby *et al.* (1939), we seek equations that capture the change with latitude of the components of the planetary rotation vector with respect to local axes, but in all other respects appear as though they were formulated in Cartesian coordinates. We thus seek equations of the form

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \tilde{f}w - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} - \tilde{f}u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{1.1}$$

in axes with x eastwards, y northwards, and z upwards anti-parallel to the local gravitational acceleration. The velocity vector is $\boldsymbol{u} = (u, v, w)$, and the material time derivative is $\mathrm{D}/\mathrm{D}t = \partial_t + \boldsymbol{u}\cdot\nabla$. The fluid density is ρ, the pressure is p, and g is the acceleration due to gravity (see below). This orientation of axes with x eastwards conveniently reduces the number of Coriolis terms in (1.1) from six to four, with two Coriolis parameters f and $\tilde{f}$. In a sufficiently small region around a reference latitude ϕ_0, the two Coriolis parameters are $f = 2\,\Omega\sin\phi_0$ and $\tilde{f} = 2\,\Omega\cos\phi_0$. Thus f and $\tilde{f}$ are equal to twice the locally vertical and locally horizontal components of the rotation vector respectively. To create a complete β-plane approximation we require relations between the x, y, z coordinates and spherical polar coordinates, and expressions for f and $\tilde{f}$ as functions of x, y, and z that are valid over a larger region of the sphere.

The standard β-plane approximation replaces the exact spherical relation $f = 2\,\Omega\sin\phi$ with a truncated Taylor expansion in the variable $\tilde{y} = (\phi - \phi_0)/R$,

$$f = 2\Omega\sin\left(\phi_0 + \frac{\tilde{y}}{R}\right) \approx 2\Omega\sin\phi_0 + \beta\tilde{y} \text{ with } \beta = \frac{2\Omega\cos\phi_0}{R}, \tag{1.2}$$

† Email: dellar@maths.ox.ac.uk

where ϕ_0 is a reference latitude, and R is the planetary radius. The other Coriolis parameter $\tilde{f}$ is set to zero under the widely-used traditional approximation, as described below. The combination of (1.1) with f given by (1.2) and $\tilde{f} = 0$ is widely used in theoretical studies of the wind-driven ocean circulation (e.g. Gill 1982; Pedlosky 1987; Salmon 1998). However, this system of equations cannot be derived as a rational approximation to the equations of motion on a sphere merely by expanding these equations in ratios of lengthscales (see Veronis 1963, 1981, and §5 below). The difficulty arises because free particles on a non-rotating sphere move along great circles, which typically appear as curved paths in a coordinate system. The equations of motion in those coordinates thus contain curvature terms, like those in (5.3) below, that are not included in (1.1).

Many of these curvature terms vanish at the equator, where the latitude and longitude lines are themselves great circles, so equatorial or near-equatorial β-planes may be derived without further assumptions. Alternatively, one may invoke an additional assumption regarding smallness of a Rossby number, as in Phillips (1973). For this reason the mid-latitude β-plane approximation is sometimes put forward only in a quasigeostrophic context (e.g. Pedlosky 1987; Holton 1992), although it is widely used outside quasigeostrophic theory. Verkley (1990) proposed an alternative approach that constructs a second system of spherical polar coordinates whose equator passes through the reference latitude ϕ_0. The curvature terms thus vanish locally in this coordinate system, but the expressions for the Coriolis terms become much more complex, and there is no natural conservation law for angular momentum.

The neglect of terms involving $\tilde{f}$, the locally horizontal component of the rotation vector, was named the traditional approximation by Eckart (1960*a*), on the grounds that it was widely used, but otherwise lacked theoretical justification. Phillips (1966) gave a derivation based on expressing $2\boldsymbol{\Omega}$ as the curl of a vector potential (see §2), and exploiting the smallness of the fluid's vertical lengthscale H relative to the planetary radius R to approximate both the vector potential and the curl operator. Although this is a derivation of the traditional approximation, it does not necessarily *justify* the traditional approximation, as established in the subsequent exchange between Veronis (1968) and Phillips (1968). For example, when $|\phi - \phi_0| \ll |\phi_0|$ one could use a similar argument to further approximate the true latitude ϕ by a reference latitude ϕ_0 in the vector potential. However, this further approximation leads to a constant vector potential, and the Coriolis force disappears completely. The essential difficulty is that these approximations do not commute with differentiation. Replacing r by R or ϕ by ϕ_0 in a function may change its derivatives at leading order, even if $|r - R| \ll R$ or $|\phi - \phi_0| \ll |\phi_0|$. Some further examples are given in Appendix B.

More recent work (e.g. Gerkema *et al.* 2008) has typically presented the traditional approximation is a consequence of the vertical lengthscale H being much smaller than the horizontal lengthscale L. The vertical velocity w is then expected to be much smaller than the horizontal velocity components u and v, so the $\tilde{f}w$ term is small compared with fv in the first of equations (1.1). Similarly, both terms on the left hand side of the last of equations (1.1) are then small compared with the right hand side. Taking $H \ll L$ is also the basis for the hydrostatic approximation that further omits the vertical acceleration from the vertical momentum equation. The pressure p is then determined from setting $\rho^{-1}\partial_z p + g = 0$. Another argument in favour of the traditional approximation in a stratified fluid relies upon the buoyancy or Brunt–Väisälä frequency N being much larger than the inertial frequency (Queney 1950; Phillips 1968, 1973; Thuburn *et al.* 2002).

Interest has recently grown in retaining the non-traditional terms proportional to $\tilde{f}$, as reviewed by Gerkema *et al.* (2008). Numerical simulations now routinely resolve smaller horizontal lengthscales for which the assumption $H \ll L$ becomes less clearly valid. From 1992 the UK Meteorological Office has retained the non-traditional terms in its forecasting model (Cullen 1993). The non-traditional terms are likely to be even more significant in the oceans. The deep oceans contain regions of very weak stratification with buoyancy frequency $N \lesssim 10f$ in the deep oceans, and much activity at near-inertial frequencies (Munk & Phillips 1968; Fu 1981). van Haren & Millot (2005) report observations of gyroscopic waves in areas of the Mediterranean whose stratification vanishes to within the uncertainty of their measurements ($N = 0 \pm 0.4f$). Non-traditional effects are required to explain these gyroscopic waves. On larger scales, PEQUOD data for mean zonal velocities in the equatorial ocean show a depth-dependence that is consistent with non-traditional effects (Hua, Moore & Le Gentil 1997).

Grimshaw (1975) derived an extended set of β-plane equations that retain the non-traditional terms proportional to $\tilde{f}$ in (1.1). However, $\tilde{f}$ was treated as a constant, rather than expanded as

$$\tilde{f} = 2\Omega\cos\left(\phi_0 + \frac{y}{R}\right) \approx 2\Omega\cos\phi_0 \,+\, \tilde{\gamma}y \text{ with } \tilde{\gamma} = -\frac{2\Omega\sin\phi_0}{R} \tag{1.3}$$

by analogy with the expansion of $f = 2\,\Omega\sin\phi$ in (1.2). Allowing $\tilde{f}$ to vary with latitude appeared to violate conservation of angular momentum and potential vorticity. Perhaps for this reason, Grimshaw's (1975) equations with constant $\tilde{f}$ have become known as "the" non-traditional β-plane equations (LeBlond & Mysak 1978; Gerkema & Shrira 2005; Gerkema *et al.* 2008).

In this paper we derive a set of β-plane equations that allow $\tilde{f}$ to vary with latitude while preserving all the expected conservation properties. We also show that Grimshaw's (1975) non-traditional β-plane, and the traditional β-plane, may both be obtained by making further approximations in our derivation. We begin with the three-dimensional equations of motion on a sphere, as formulated using Hamilton's principle of least action, and make approximations in the action. We formulate Hamilton's principle using Lagrangian variables, for which the treatment of inertia, and hence of the Coriolis force in a rotating frame, most closely resembles that of classical particle mechanics. This enables us to exploit Ripa's (1997) variational approach to the motion of a particle on a sphere and a β-plane, and Müller's (1989) derivation of the primitive equations in spherical geometry by approximating the inertial terms in a Lagrangian based on particle mechanics (see also Hinkelmann 1969; White *et al.* 2005; Zdunkowski & Bott 2003).

It was once common to evaluate the merits of competing models, derived from truncated expansions of the underlying equations in a small parameter, by determining whether they could be manipulated to give results resembling conservation of energy, or material conservation of potential vorticity (e.g. Phillips 1966, 1968; Veronis 1968). However, all these conservation properties may be derived from a formulation of the underlying equations using Hamilton's principle, together with Noether's theorem that relates conservation laws to the invariance of variational principles under symmetries (e.g. Goldstein 1980). Conservation of energy and angular momentum are a consequence of invariance under translations in time and axial rotations, while material conservation of potential vorticity arises from a more subtle particle relabelling symmetry (Salmon 1982*a*; Ripa 1982). This motivates our approach, following Salmon (1983) of making approximations in the variational principle, instead of approximating the equations directly. This approach guarantees the preservation of conservation laws, provided nothing is done to jeopardise the symmetries of the variational principle.

Unlike the work mentioned in the previous two paragraphs, our final equations are expressed in a carefully constructed set of pseudo-Cartesian coordinates instead of spherical polar coordinates. By "pseudo-Cartesian" we mean curvilinear coordinates in which deviations of the metric away from a Cartesian form are neglected. Curvilinear coordinates are necessary to ensure that horizontal coordinate lines lie within surfaces of constant geopotential, as described in §4 below. Moreover, we do not assume a shallow fluid layer, and do not approximate the true spherical radius r by a constant R. We thus obtain non-traditional terms in our approximated equations, as in White *et al.*'s (2005) extension of Müller's (1989) approach to encompass White & Bromley's (1995) quasi-hydrostatic equations in spherical geometry. We also obtain our pressure gradient terms consistently from the variational principle, rather than putting them in by hand as in Müller (1989) and White *et al.* (2005). Our work highlights the essential rôle of a vector potential $\boldsymbol{R}$ for the Coriolis force, through writing $\boldsymbol{\Omega} = (1/2)\nabla\times\boldsymbol{R}$, and the interaction of the resulting constraint $\nabla\cdot\boldsymbol{\Omega} = 0$ with the approximation of the spherical metric by a flattened pseudo-Cartesian metric. The derivation given below begins in spherical geometry, but a purely Cartesian procedure for allowing Ω_y to depend on y is also given in Appendix B.

2. Hamilton's variational principle for an ideal fluid

The three-dimensional Euler equations for an incompressible fluid in an inertial frame may be derived (Herivel 1955) from the Lagrangian

$$\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\ \frac{1}{2}\left|\frac{\partial \boldsymbol{x}}{\partial \tau}\right|^2 - \mathit{\Phi}(\boldsymbol{x}) + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right). \tag{2.1}$$

The first two terms in (2.1) are the kinetic energy $(1/2)|\boldsymbol{x}_\tau|^2$ and the gravitational potential energy $\mathit{\Phi}(\boldsymbol{x})$. The integral is expressed over Lagrangian particle labels $\boldsymbol{a} = (a,b,c)$, and the particle positions $\boldsymbol{x} = (x,y,z)$ should be treated as functions of $\boldsymbol{a}$ and time τ. We use the variable τ for time to emphasise that partial time derivatives $\partial/\partial\tau$ are taken at fixed particle labels $\boldsymbol{a}$, instead of at fixed spatial coordinates $\boldsymbol{x}$. Thus $\partial/\partial\tau = \mathrm{D}/\mathrm{D}t$ in Eulerian variables. Applying this relation to $\boldsymbol{x}$ gives the Eulerian fluid velocity $\boldsymbol{u} = \partial\boldsymbol{x}/\partial\tau$. Salmon (1982*b*, 1983, 1988, 1998) gives many more details of this form of Hamilton's principle using particle labels, and its geophysical applications.

The last term in (2.1) is a constraint that enforces incompressibility using the Lagrange multiplier $p(\boldsymbol{a},\tau)$. It is convenient to assign the labels $\boldsymbol{a}$ so that the density ρ is given by the reciprocal of the Jacobian of the label to particle map,

$$\frac{1}{\rho} = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}. \tag{2.2}$$

The last expression is the determinant of a 3×3 matrix of partial derivatives, $x_a = \partial x/\partial a$ etc. The last term in (2.1) thus enforces incompressiblity by imposing $\rho = \rho_0$ is constant.

According to Hamilton's principle of least action, the equations of motion render the action $\mathcal{S}$ stationary,

$$\delta\mathcal{S} = \delta\int \mathrm{d}\tau\mathcal{L} = 0, \tag{2.3}$$

under independent variations of the particle positions $\boldsymbol{x}$ and pressure p. The variations $\delta\boldsymbol{x}$ and δp should vanish at the endpoints of the τ integration, as in classical particle mechanics (e.g. Goldstein 1980) to allow integrations by parts with respect to τ in the action. The variation in the pressure term due to a variation δx with y and z fixed is

$$\int \mathrm{d}\boldsymbol{a}\, p(\boldsymbol{a},\tau)\frac{\partial(\delta x,y,z)}{\partial(a,b,c)} = -\int \mathrm{d}\boldsymbol{a}\,\frac{\partial(p,y,z)}{\partial(a,b,c)}\delta x = -\int \mathrm{d}\boldsymbol{a}\,\frac{\partial(p,y,z)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(a,b,c)}\delta x = -\int \mathrm{d}\boldsymbol{a}\,\frac{\partial p}{\partial x}\frac{1}{\rho}\delta x. \tag{2.4}$$

The variation of the whole Lagrangian in (2.1) thus yields the three-dimensional incompressible Euler equations with a gravitational potential,

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho}\nabla p = -\nabla\varPhi, \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.5}$$

This variational approach may be extended to compressible fluids (Serrin 1959; Eckart 1960*b*) by replacing the $\rho = \rho_0$ constraint and its Lagrange multiplier with a potential energy $U(\rho, s)$. In general U depends upon both density and an additional advected scalar s, the entropy. Since $\mathrm{D}s/\mathrm{D}t = 0$ we can write $s = s(\boldsymbol{a})$ with no τ dependence. The same approach encompasses Boussinesq fluids with an advected scalar buoyancy instead of the entropy.

The Coriolis force may be included by adding a further term to the Lagrangian that is linear in the velocity,

$$\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\;\frac{1}{2}\left|\frac{\partial\boldsymbol{x}}{\partial\tau}\right|^2 + \boldsymbol{R}\cdot\frac{\partial\boldsymbol{x}}{\partial\tau} - \varPhi(\boldsymbol{x}) + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right). \tag{2.6}$$

The $\boldsymbol{R}$ notation was introduced by Abarbanel & Holm (1987), although equivalent special cases may be found in Salmon (1982*b*, 1983) and the particle mechanics version of (2.6) may be found in Landau & Lifshitz (1976). Taking variations of the action for this Lagrangian gives

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + (\nabla\times\boldsymbol{R})\times\boldsymbol{u} + \frac{1}{\rho}\nabla p = -\nabla\varPhi, \tag{2.7}$$

which coincides with the Euler equation in a rotating frame if $\nabla\times\boldsymbol{R} = 2\boldsymbol{\varOmega}$. Thus $\boldsymbol{R}$ is a vector potential for the rotation vector $\boldsymbol{\varOmega}$. The Coriolis force $2\boldsymbol{\varOmega}\times\boldsymbol{u}$ is mathematically equivalent to the Lorentz force $q\boldsymbol{u}\times\boldsymbol{B}$ experienced by a particle with charge q, and $\boldsymbol{R}$ is the equivalent of the vector potential $\boldsymbol{A}$ that appears in the Lagrangian for a charged particle in the magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ (e.g. Goldstein 1980).

When $\boldsymbol{\varOmega}$ is spatially uniform, $\boldsymbol{R} = \boldsymbol{\varOmega}\times\boldsymbol{x}$ serves as a vector potential. The Lagrangian in (2.6) may then be interpreted as computing the kinetic energy from the velocity with respect to an inertial frame, $\boldsymbol{x}_\tau + \boldsymbol{\varOmega}\times\boldsymbol{x}$, and combining the centrifugal force with gravity into a single geopotential $\varPhi = \varPhi_g + (1/2)|\boldsymbol{\varOmega}\times\boldsymbol{x}|^2$,

$$\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\;\frac{1}{2}\left|\frac{\partial\boldsymbol{x}}{\partial\tau} + \boldsymbol{\varOmega}\times\boldsymbol{x}\right|^2 - \left\{\varPhi_g(\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{\varOmega}\times\boldsymbol{x}|^2\right\} + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right). \tag{2.8}$$

Alternatively, one may take the coordinates $\boldsymbol{x}$ with respect to a rotating frame as being a particular choice of generalised coordinates. The Lagrangian (2.8) then arises from rewriting the kinetic and potential energies with respect to an inertial frame using these generalised coordinates. The more general form (2.6) follows from the observation that the action is invariant under replacing $\boldsymbol{\varOmega}\times\boldsymbol{x}$ by $\boldsymbol{R} = \boldsymbol{\varOmega}\times\boldsymbol{x} + \nabla\varphi$, where φ is any scalar field. The action $\mathcal{S}$, and hence the equations of motion, are invariant under a gauge transformation that adds $\nabla\varphi$ to $\boldsymbol{R}$. Again, this result is precisely analogous to a result for the Lagrangian of a charged particle in a magnetic field (e.g. Goldstein 1980).

3. The vector potential and conservation of circulation

We have added the most general possible term that is linear in the particle velocity to the Lagrangian. However, the equations of motion derived from Hamilton's principle contain the Coriolis force generated by $\boldsymbol{\varOmega} = \frac{1}{2}\nabla\times\boldsymbol{R}$, so $\boldsymbol{\varOmega}$ is subject to the constraint $\nabla\cdot\boldsymbol{\varOmega} = 0$. Even without involving Hamilton's principle, the same constraint is needed to derive conservation of circulation from the Euler equation with a Coriolis force,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\varOmega}\times\boldsymbol{u}. \tag{3.1}$$

The circulation of the velocity $\boldsymbol{u}$ around a closed material loop $\mathcal{C}$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell} = -\oint_{\mathcal{C}} \frac{1}{\rho}\nabla p\cdot\mathrm{d}\boldsymbol{\ell} - \oint_{\mathcal{C}} 2\boldsymbol{\varOmega}\times\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell}, \tag{3.2}$$

where the left hand side was derived using the formula $\mathrm{D}(\mathrm{d}\boldsymbol{\ell})/\mathrm{D}t = (\mathrm{d}\boldsymbol{\ell}\cdot\nabla)\boldsymbol{u}$ from Batchelor (1967) for the evolution of a material line element $\mathrm{d}\boldsymbol{\ell}$.

Using further formula from Batchelor (1967) for the evolution of material surface elements, we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathcal{S}}\boldsymbol{\Omega}\cdot\mathbf{n}\,\mathrm{d}S &= \iint_{\mathcal{S}}(\boldsymbol{u}\cdot\nabla\boldsymbol{\Omega}+\boldsymbol{\Omega}\nabla\cdot\boldsymbol{u}-\boldsymbol{\Omega}\cdot\nabla\boldsymbol{u})\cdot\mathbf{n}\,\mathrm{d}S,\\ &= \iint_{\mathcal{S}}\big[\nabla\times(\boldsymbol{\Omega}\times\boldsymbol{u})+\boldsymbol{u}\nabla\cdot\boldsymbol{\Omega}\big]\cdot\mathbf{n}\,\mathrm{d}S,\\ &= \oint_{\mathcal{C}}(\boldsymbol{\Omega}\times\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{\ell}+\iint_{\mathcal{S}}\nabla\cdot\boldsymbol{\Omega}\,\boldsymbol{u}\cdot\mathbf{n}\,\mathrm{d}S,\end{aligned}\tag{3.3}$$

for any time-independent vector field $\boldsymbol{\Omega}$ and material surface $\mathcal{S}$ spanning the material curve $\mathcal{C}$. The last step uses Stokes' theorem. Equation (3.2) for the evolution of circulation may thus be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\oint_{\mathcal{C}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell}+\iint_{\mathcal{S}}2\boldsymbol{\Omega}\cdot\mathbf{n}\,\mathrm{d}S\right]=-\oint_{\mathcal{C}}\frac{1}{\rho}\nabla p\cdot\mathrm{d}\boldsymbol{\ell}+2\iint_{\mathcal{S}}\nabla\cdot\boldsymbol{\Omega}\;\boldsymbol{u}\cdot\mathbf{n}\,\mathrm{d}S.\tag{3.4}$$

We require $\nabla\cdot\boldsymbol{\Omega}=0$ to eliminate the source term on the right hand side of (3.4). All the other terms involving $\boldsymbol{\Omega}$ have been expressed as a time derivative on the left hand side. Imposing $\nabla\cdot\boldsymbol{\Omega}=0$ also removes the dependency of the left hand side on the particular choice of spanning surface $\mathcal{S}$. Writing $2\boldsymbol{\Omega}=\nabla\times\boldsymbol{R}$ and using Stokes' theorem transforms (3.4) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}}(\boldsymbol{u}+\boldsymbol{R})\cdot\mathrm{d}\boldsymbol{\ell}=-\oint_{\mathcal{C}}\frac{1}{\rho}\nabla p\cdot\mathrm{d}\boldsymbol{\ell}.\tag{3.5}$$

We have thus absorbed the torque exerted by the Coriolis force into an evolution equation for the circulation of $\boldsymbol{u}+\boldsymbol{R}$ around the material curve $\mathcal{C}$. The remaining right hand side is the baroclinic torque due to the pressure gradient. The result (3.5) is generally known as Kelvin's theorem in the absence of the Coriolis force, and as Bjerknes' theorem when $\boldsymbol{R}=\boldsymbol{\Omega}\times\boldsymbol{x}$ (e.g. Holton 1992; Zdunkowski & Bott 2003). The loop integral appearing on the left hand side of (3.5) is invariant under gauge transformations of $\boldsymbol{R}$ by $\nabla\varphi$, because the closed loop integral of the gradient of a scalar φ vanishes, which motivates this more general form of Bjerknes' theorem given by Abarbanel & Holm (1987). Again, the replacement of $\boldsymbol{u}$ by $\boldsymbol{u}+\boldsymbol{R}$ appears completely general, since $\boldsymbol{R}$ is an arbitrary vector field, but the resulting Coriolis force is generated by the divergence-free vector field $\boldsymbol{\Omega}=(1/2)\nabla\times\boldsymbol{R}$.

In a stratified fluid, an evolution equation for potential vorticity follows from applying (3.4) to a material curve $\mathcal{C}$ and spanning surface $\mathcal{S}$ that lie in a surface of constant entropy (e.g. Pedlosky 1987; White 2002; Zdunkowski & Bott 2003; Vallis 2006). The loop integral of the baroclinic torque then vanishes, and we may use Stokes' theorem to transform the remaining loop integral into a surface integral,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathcal{S}}(\nabla\times\boldsymbol{u}+2\boldsymbol{\Omega})\cdot\mathbf{n}\,\mathrm{d}S=2\iint_{\mathcal{S}}\nabla\cdot\boldsymbol{\Omega}\;\boldsymbol{u}\cdot\mathbf{n}\,\mathrm{d}S.\tag{3.6}$$

Writing $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}+2\boldsymbol{\Omega}$, we obtain

$$\iint_{\mathcal{S}}\left(\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t}+\boldsymbol{\omega}\nabla\cdot\boldsymbol{u}-\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}-2\nabla\cdot\boldsymbol{\Omega}\;\boldsymbol{u}\right)\cdot\frac{\nabla s}{|\nabla s|}\,\mathrm{d}S=0,\tag{3.7}$$

since $\mathbf{n}=\nabla s/|\nabla s|$ is the unit normal to a constant entropy surface. This integral becomes

$$\iint_{\mathcal{S}}\left(\frac{\mathrm{D}}{\mathrm{D}t}\left(\boldsymbol{\omega}\cdot\nabla s\right)+\left(\boldsymbol{\omega}\cdot\nabla s\right)\nabla\cdot\boldsymbol{u}-2\nabla\cdot\boldsymbol{\Omega}\;\boldsymbol{u}\cdot\nabla s\right)\frac{1}{|\nabla s|}\,\mathrm{d}S=0,\tag{3.8}$$

after using $\mathrm{D}\left(\partial_i s\right)/\mathrm{D}t=-(\partial_i u_j)(\partial_j s)$ in index notation. Equation (3.8) holds for all material surfaces $\mathcal{S}$ contained within a constant entropy surface, so it implies the pointwise evolution equation

$$\frac{\mathrm{D}q}{\mathrm{D}t}=\frac{2}{\rho}\nabla\cdot\boldsymbol{\Omega}\;\boldsymbol{u}\cdot\nabla s\tag{3.9}$$

for the potential vorticity

$$q=\frac{\boldsymbol{\omega}\cdot\nabla s}{\rho}.\tag{3.10}$$

We have removed the term proportional to $\nabla\cdot\boldsymbol{u}$ from (3.8) using the continuity equation $\mathrm{D}\rho/\mathrm{D}t+\rho\nabla\cdot\boldsymbol{u}=0$, so this derivation applies equally to compressible and incompressible fluids. In conclusion, $\nabla\cdot\boldsymbol{\Omega}=0$ is necessary for material conservation of potential vorticity, as observed by Grimshaw (1975).

4. Hamilton's principle in spherical geometry

Transforming (2.6) into spherical polar coordinates, the Lagrangian for an incompressible fluid on a rotating sphere may be written as

$$\mathcal{L} = \int \mathrm{d}\boldsymbol{a} \left[\frac{1}{2} r^2 \dot{\phi}^2 + \frac{1}{2} r^2 \cos^2\phi \dot{\lambda}^2 + r^2 \cos^2\phi \, \Omega \dot{\lambda} + \varPhi(\boldsymbol{x}) + p(\boldsymbol{a}, \tau) \left(r^2 \cos\phi \, \frac{\partial(\lambda, \phi, r)}{\partial(a, b, c)} - \frac{1}{\rho_0} \right) \right], \tag{4.1}$$

where λ, ϕ, and r are longitude, latitude, and radius. This expression arises from using the vector potential

$$\boldsymbol{\Omega} \times \boldsymbol{x} = \Omega r \cos\phi \, \hat{\boldsymbol{\lambda}} \tag{4.2}$$

for the Coriolis force. The velocity vector is

$$\dot{\boldsymbol{x}} = r \cos\phi \, \dot{\lambda} \, \hat{\boldsymbol{\lambda}} + r \dot{\phi} \, \hat{\boldsymbol{\phi}} + \dot{r} \, \hat{\boldsymbol{r}}, \tag{4.3}$$

where dots denote derivatives with respect to time τ.

This Lagrangian is exact for a sphere. However, the gradient of the combined geopotential $\varPhi = \varPhi_g + (1/2)|\boldsymbol{\Omega} \times \boldsymbol{x}|^2$ has a horizontal component due to the centrifugal force even when the gravitational potential $\varPhi_g$ is a function of r only. For geophysically relevant parameters this leads to $\nabla\varPhi$ being the dominant term in the horizontal momentum equations. It is therefore common to omit the centrifugal potential, leaving only the gravitational potential $\varPhi(r)$ whose gradient has no horizontal component. The justification for this omission is that the Earth's surface is much closer to a surface of constant geopotential, or an oblate spheroid, than to a sphere. One then re-interprets the coordinate r as labelling surfaces of constant geopotential, instead of surfaces of constant geometrical distance from the centre of a sphere. The Lagrangian (4.1) is then an approximation to the exact Lagrangian, with the full geopotential, formulated in oblate spheroidal coordinates. This exact Lagrangian gives the equations derived by Gates (2004), and its approximation omits terms of order $(d/r)^2$ from the metric, where d is the distance between the two foci of the spheroid. This approximation of a spheroid by what appear to be spherical polar coordinates is employed by Phillips (1973); Veronis (1973); Gill (1982); Müller (1989); White (2002); White *et al.* (2005) and described in particular detail by van der Toorn & Zimmerman (2008).

Following Ripa's (1997) study of motion on a spherical surface, we introduce the horizontal coordinates x and y defined by

$$\lambda = \sec\phi_0 \frac{x}{R}, \quad \sin\phi = \sin\phi_0 + \frac{y}{R} \cos\phi_0. \tag{4.4}$$

This relation between y and ϕ in (4.4) leads to the quantity β appearing in the equations derived below with its conventional value, $\beta = (2/R)\Omega \cos\phi_0$ as in (1.3). The widely used coordinate $\tilde{y} = (\phi - \phi_0)/R$ leads to a different expression for β, as does the Mercator latitude coordinate used by Grimshaw (1975). Calculations using both these coordinates are given in Appendix A.

As we are concerned with three-dimensional motions, unlike Ripa (1997), we also introduce a vertical coordinate z. This coordinate is constructed to make the volume element for the x, y, z coordinates equal to the Cartesian volume element,

$$\begin{aligned} \mathrm{d}V = r^2 \cos\phi \mathrm{d}\lambda \mathrm{d}\phi \mathrm{d}r &= r^2 \cos\phi \frac{\mathrm{d}\lambda}{\mathrm{d}x} \frac{\mathrm{d}\phi}{\mathrm{d}y} \frac{\mathrm{d}r}{\mathrm{d}z} \mathrm{d}x \mathrm{d}y \mathrm{d}z \\ &= r^2 \cos\phi \left(\frac{\sec\phi_0}{R} \right) \left(\frac{\cos\phi_0}{R \cos\phi} \right) \left(\frac{R^2}{r^2} \right) \mathrm{d}x \mathrm{d}y \mathrm{d}z = \mathrm{d}x \mathrm{d}y \mathrm{d}z. \end{aligned} \tag{4.5}$$

Ripa's (1997) choice of y coordinate eliminates the ϕ-dependence of the volume element. To also eliminate the r-dependence, we require

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{R^2}{r^2}. \tag{4.6}$$

The solution of this ordinary differential equation determines

$$r = R(1 + 3z/R)^{1/3} \sim R + z - \frac{z^2}{R} + \cdots. \tag{4.7}$$

The constant volume element $\mathrm{d}V = \mathrm{d}x \mathrm{d}y \mathrm{d}z$ ensures that the pressure gradient terms in the equations of motion obtained from Hamilton's principle take their expected forms.

However, the x, y, z coordinates are not Cartesian coordinates, and the individual components of the metric tensor are not constant. In x, y, z coordinates the Lagrangian (4.1) becomes

$$\begin{aligned}\mathcal{L} = \int d\boldsymbol{a} \frac{1}{2}\dot{z}^2 \left\{1 + 3\frac{z}{R}\right\}^{-4/3} + \frac{1}{2}\dot{y}^2 \left\{1 + 3\frac{z}{R}\right\}^{2/3} \left\{1 - 2\tan\phi_0 \frac{y}{R} - \frac{y^2}{R^2}\right\}^{-1} \\ + \left(\frac{1}{2}\dot{x}^2 + \dot{x}\,\Omega\cos\phi_0\right) \left\{1 + 3\frac{z}{R}\right\}^{2/3} \left\{1 - 2\tan\phi_0 \frac{y}{R} - \frac{y^2}{R^2}\right\} \\ + \Phi(z) + p(\boldsymbol{a}, \tau) \left(\frac{\partial(x, y, z)}{\partial(a, b, c)} - \frac{1}{\rho_0}\right). \qquad (4.8)\end{aligned}$$

The factors in braces $\{\cdot\}$ that multiply $\dot{x}^2$, $\dot{y}^2$, $\dot{z}^2$ are the diagonal components g_{xx}, g_{yy}, g_{zz} of the metric. Veronis (1963) showed that the spatial dependence of the metric components leads to discrepancies between a rational expansion of the spherical equations in a ratio of lengthscales and what are normally called the β-plane equations, as described below.

The coordinate x only appears in the Lagrangian (4.8) through the Jacobian $\partial(x, y, z)/\partial(a, b, c)$ in the incompressibility constraint. Otherwise x is an ignorable coordinate, so Noether's theorem gives a conservation law for angular momentum in the absence of pressure torques, as in equation (5.8) below. The x and y coordinate lines defined by (4.4) are tangent to the stereographic coordinate lines $\tilde{x}$ and $\tilde{y}$ used by Phillips (1973). Taking a stereographic projection from a point with arbitrary latitude ϕ_0 and longitude $\lambda = 0$ gives

$$\sin\phi = \sin\phi_0 \frac{4R^2 - \tilde{x}^2 - \tilde{y}^2}{4R^2 + \tilde{x}^2 + \tilde{y}^2} + \cos\phi_0 \frac{4R\,\tilde{y}}{4R^2 + \tilde{x}^2 + \tilde{y}^2}, \qquad (4.9)$$

where $\tilde{x}$ and $\tilde{y}$ are the stereographic coordinates. In this coordinate system the $\tilde{x}$ line is a great circle tangent to the latitude circle $\phi = \phi_0$. In other words, equation (4.9) asymptotes to the second of equations (4.4) as $\tilde{x}/R$ and $\tilde{y}/R$ tend to zero. However, the lines of constant $\tilde{x}$ becomes inclined relative to the latitude circles away from the origin of the projection. There is thus no ignorable coordinate analogous to longitude λ, or to the x defined by (4.4), and hence no obvious angular momentum conservation law available from Noether's theorem.

5. Beta-plane equations from an approximate Lagrangian

A β-plane approximation arises from exploiting the smallness of x, y, z compared with the planetary radius R. In principle there are two independent small parameters,

$$\epsilon = L/R, \quad \text{and} \quad \delta = H/L, \qquad (5.1)$$

where L and H are typical horizontal and vertical lengthscales for the fluid motion. The traditional approximation arises from a small δ limit, in addition to the small ϵ limit that gives a β-plane. In geophysical fluid dynamics it is conventional to perform derivations using dimensional variables (e.g. Grimshaw 1975; Veronis 1981; Shutts 1989; White 2002; Gerkema *et al.* 2008). We therefore retain explicit factors of R instead of absorbing them into dimensionless coordinates.

Expanding the Lagrangian (4.8) in L/R for fixed H/L and keeping terms of up to order $(L/R)^3$ gives

$$\begin{aligned}\mathcal{L} = \int d\boldsymbol{a}\ \frac{1}{2}\left(\frac{\partial x}{\partial\tau}\right)^2 \left\{1 - 2\frac{y}{R}\tan\phi_0 + 2\frac{z}{R}\right\} + \frac{1}{2}\left(\frac{\partial y}{\partial\tau}\right)^2 \left\{1 + 2\frac{y}{R}\tan\phi_0 + 2\frac{z}{R}\right\} \\ + \frac{1}{2}\left(\frac{\partial z}{\partial\tau}\right)^2 \left\{1 - 4\frac{z}{R}\right\} + \Phi(z) + p(\boldsymbol{a}, \tau) \left(\frac{\partial(x, y, z)}{\partial(a, b, c)} - \frac{1}{\rho_0}\right) \\ + \Omega R \frac{\partial x}{\partial\tau} \left[\left(1 + 2\frac{z}{R} - \frac{z^2}{R^2} - \frac{y^2}{R^2}\right)\cos\phi_0 - \left(2\frac{y}{R} + 4\frac{yz}{R^2}\right)\sin\phi_0\right]. \qquad (5.2)\end{aligned}$$

The same expression may be derived by substituting $x = \epsilon R x'$, $y = \epsilon R y'$, $z = \epsilon R z'$, expanding up to and including term of order ϵ^3, then rewriting the resulting expression in terms of the original variables x, y, and z.

The terms in braces $\{\cdot\}$ are the linearised metric coefficients from (4.8). If considered purely as an expansion in lengthscales, the y-dependence of the metric terms in braces $\{\cdot\}$ is comparable in magnitude to the y-dependence of the term responsible for the β effect in the Coriolis force. The equations of motion obtained from the Lagrangian (5.2) therefore contain curvature terms proportional to the velocity squared, and metric factors multiplying the time derivatives, as described by Veronis (1963, 1981),

$$\left(1-2\frac{y}{R}\tan\phi_0+2\frac{z}{R}\right)\frac{\mathrm{D}u}{\mathrm{D}t}+\frac{2}{R}u\left(w-v\tan\phi_0\right)+2\,\Omega_y w-2\,\Omega_z v=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{5.3a}$$

$$\left(1+2\frac{y}{R}\tan\phi_0+2\frac{z}{R}\right)\frac{\mathrm{D}v}{\mathrm{D}t}+\frac{1}{R}\left(u^2+v^2\right)\tan\phi_0+\frac{2}{R}uw+2\,\Omega_z u=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{5.3b}$$

$$\left(1-4\frac{z}{R}\right)\frac{\mathrm{D}w}{\mathrm{D}t}-\frac{1}{R}\left(u^2+v^2+2w^2\right)-2\,\Omega_y u=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g. \tag{5.3c}$$

The curvature terms appear because a free particle moving on the surface of a non-rotating sphere follows a great circle. This is a geodesic path on the surface of a sphere, but a curved path in the x, y, z coordinates. However, the pressure gradient appears in a Cartesian form because the volume element is $\mathrm{d}x\mathrm{d}y\mathrm{d}z$. The gravitational acceleration is $g=-\Phi'(z)$. Expressions for Ω_y and Ω_z are given in (5.9) below.

Some additional assumption is needed to justify discarding the curvature terms while retaining the terms that give rise to the β effect. Following Veronis (1963, 1981) one may restrict attention to near-equatorial β-planes, since the $\tan\phi_0$ terms become negligibly small when $|\phi_0|\ll 1$, and vanish exactly at the equator. In our three-dimensional treatment a small aspect ratio approximation ($H\ll L$) is also needed to justify discarding the z-dependence of the metric. The derivation in LeBlond & Mysak (1978) neglects terms of order $\tan\phi_0(L/R)$ while retaining terms of order $\cos\phi_0(L/R)$. This is valid only for $\phi_0\ll 1$, so being close to the equator is implicitly one of their geometric assumptions. At mid-latitudes $\tan\phi_0(L/R)$ and $\cos\phi_0(L/R)$ are comparable in magnitude, so it becomes inconsistent to neglect one while retaining the other.

Alternatively, following Phillips (1973), one may introduce the additional assumption

$$\Omega y\gg\left(\frac{\partial x}{\partial\tau},\frac{\partial y}{\partial\tau}\right) \tag{5.4}$$

to justify neglect of the curvature terms. The conventional mid-latitude β-plane equations then arise as a distinguished limit in which the Rossby number is comparable to the ratio of lengthscales,

$$Ro=\frac{U}{2\Omega L}\sim\frac{L}{R}. \tag{5.5}$$

In this limit, the leading order terms $(1/2)x_\tau^2$ and $(1/2)y_\tau^2$ in the kinetic energy are comparable in magnitude to the $\Omega x_\tau y^2/R$ term that is responsible for the β effect.

Using one of these further appproximations to discard the metric terms in braces $\{\cdot\}$ from (5.2) leads to the simplified Lagrangian

$$\begin{aligned}\mathcal{L}=\int\mathrm{d}\boldsymbol{a}\;&\frac{1}{2}\left(\frac{\partial x}{\partial\tau}\right)^2+\frac{1}{2}\left(\frac{\partial y}{\partial\tau}\right)^2+\frac{1}{2}\left(\frac{\partial z}{\partial\tau}\right)^2+\Phi(z)+p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)}-\frac{1}{\rho_0}\right)\\&+\Omega R\frac{\partial x}{\partial\tau}\left[\left(1+2\frac{z}{R}-\frac{z^2}{R^2}-\frac{y^2}{R^2}\right)\cos\phi_0-\left(2\frac{y}{R}+4\frac{yz}{R^2}\right)\sin\phi_0\right].\end{aligned} \tag{5.6}$$

The Euler–Lagrange equations now take the form of equations of motion in Cartesian coordinates,

$$\frac{\mathrm{D}u}{\mathrm{D}t}+2\,\Omega_y w-2\,\Omega_z v=-\frac{1}{\rho}\frac{\partial p}{\partial x},\quad\frac{\mathrm{D}v}{\mathrm{D}t}+2\,\Omega_z u=-\frac{1}{\rho}\frac{\partial p}{\partial y},\quad\frac{\mathrm{D}w}{\mathrm{D}t}-2\,\Omega_y u=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g, \tag{5.7}$$

although x, y, z are actually curvilinear coordinates, as described above.

The term in square brackets $[\cdots]$ multiplying $\partial x/\partial\tau$ in (5.6) is the effective vector potential R_x, and the first of equations (5.7) may be rewritten as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(u+R_x\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{5.8}$$

This shows that the zonal momentum $u+R_x$ is conserved in the absence of pressure torques, as obtained by applying Noether's theorem to (5.6) with its ignorable coordinate x. Evaluating $\boldsymbol{u}\cdot\nabla R_x=2\,\Omega_y w-2\,\Omega_z v$, we find

$$2\,\Omega_y=2\,\Omega\cos\phi_0\left(1-2\tan\phi_0\frac{y}{R}-\frac{z}{R}\right)=2\,\Omega\cos\phi_0\left(1-\frac{z}{R}\right)+\gamma y, \tag{5.9a}$$

$$2\,\Omega_z=2\,\Omega\sin\phi_0\left(1+\cot\phi_0\frac{y}{R}+2\frac{z}{R}\right)=2\,\Omega\sin\phi_0\left(1+2\frac{z}{R}\right)+\beta y. \tag{5.9b}$$

The proportionality constants for the change of $\boldsymbol{\Omega}$ with y are thus

$$\beta=\frac{2\Omega\cos\phi_0}{R},\quad\gamma=-\frac{4\Omega\sin\phi_0}{R}. \tag{5.10}$$

The constant γ for the y-dependence of Ω_y has twice the value one would obtain by simply expanding the expression for the rotation vector $\boldsymbol{\Omega}$ in spherical polar coordinates as a function of y, as in (1.3).

More importantly, Ω_z acquires a z-dependence proportional to γ that compensates for the y-dependence of Ω_y. The vector field $\boldsymbol{\Omega}$ is thus divergence-free with respect to x, y, z when treated as pseudo-Cartesian coordinates with an Euclidean metric,

$$\nabla\cdot\boldsymbol{\Omega} = \frac{\partial \Omega_y}{\partial y} + \frac{\partial \Omega_z}{\partial z} = 0. \tag{5.11}$$

It is given by $\boldsymbol{\Omega} = (1/2)\nabla\times\boldsymbol{R} = (1/2)(0, \partial_z R_x, -\partial_y R_x)$, where $\boldsymbol{R} = \hat{\boldsymbol{x}} R_x$ is the vector potential, and $\nabla\times$ is the curl operator in pseudo-Cartesian coordinates with an Euclidean metric.

6. Consistent further approximations

Making further approximations in the Lagrangian (5.6) leads to additional simplifications, including the traditional approximation β-plane, and Grimshaw's (1975) non-traditional β-plane. The largest term in the vector potential in (5.6) appears as part of the exact time derivative $\partial_\tau(\Omega x R\cos\phi_0)$ and so does not contribute to the action (see Ripa 1997). Equivalently, a constant may be removed from the vector potential without changing its curl, and hence without changing the Coriolis force. We therefore omit the $\Omega R x_\tau \cos\phi_0$ term in the further simplified Lagrangians below.

6.1. *The traditional approximation*

Dropping all terms involving z/R from the vector potential leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\; \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 + \Phi(z) + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right) \\ + \Omega R \frac{\partial x}{\partial \tau}\left[-\frac{y^2}{R^2}\cos\phi_0 - 2\frac{y}{R}\sin\phi_0\right].\end{aligned} \tag{6.1}$$

Taking variations gives the traditional approximation β-plane equations. There are no terms involving Ω_y, and

$$2\,\Omega_z = 2\,\Omega\sin\phi_0 + \beta y. \tag{6.2}$$

In this approximation the vertical velocity z_τ also disappears from the kinetic energy, so the vertical momentum equation reduces to the hydrostatic balance $g + p_z = 0$.

6.2. *The Grimshaw (1975) non-traditional β-plane*

Retaining the terms proportional to z/R and y^2/R^2, but not yz/R^2, leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\; \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 + \Phi(z) + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right) \\ + \Omega R \frac{\partial x}{\partial \tau}\left[\left(-\frac{y^2}{R^2} + 2\frac{z}{R}\right)\cos\phi_0 - 2\frac{y}{R}\sin\phi_0\right].\end{aligned} \tag{6.3}$$

This therefore corresponds to a distinguished limit in which $\epsilon = \delta$, so the y^2 and the z terms are both $O(\epsilon^2)$. The yz term is $O(\epsilon^3)$ and may be discarded. In dimensional variables, this distinguished limit is

$$\frac{H}{L} \sim \frac{L}{R} \sim \frac{U}{2\,\Omega L}, \tag{6.4}$$

while the earlier equations permitted $H/L = O(1)$. Taking variations of (6.3) leads to the β-plane equations proposed by Grimshaw (1975) with

$$2\,\Omega_y = 2\,\Omega\cos\phi_0, \quad 2\,\Omega_z = 2\,\Omega\sin\phi_0 + \beta y. \tag{6.5}$$

The horizontal component Ω_y is retained, but with a spatially uniform value. The vertical component Ω_z varies with latitude, as in the traditional β-plane above. Again, $\mathrm{D}w/\mathrm{D}t$ disppears from the vertical momentum equation, but this time we obtain a *quasihydrostatic* balance (White & Bromley 1995; White *et al.* 2005)

$$-2\,\Omega_y u = -\frac{\partial p}{\partial z} - g, \tag{6.6}$$

because Ω_y appears in the vertical momentum equation.

6.3. *A quasihydrostatic β-γ-plane*

A slight simplification in the β-γ-plane equations may be achieved by dropping the z^2/R^2 terms from the Lagrangian, leaving

$$\mathcal{L} = \int \mathrm{d}\boldsymbol{a}\ \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 + \Phi(z) + p(\boldsymbol{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right) \\ +\Omega R \frac{\partial x}{\partial \tau}\left[\left(-\frac{y^2}{R^2} + 2\frac{z}{R}\right)\cos\phi_0 - \left(2\frac{y}{R} + 4\frac{yz}{R^2}\right)\sin\phi_0\right]. \tag{6.7}$$

This is the Lagrangian that we shall use to derive shallow water equations below. It leads to the simplest set of equations in which Ω_y and Ω_z both vary with latitude,

$$2\,\Omega_y = 2\,\Omega\cos\phi_0 + \gamma y, \quad 2\,\Omega_z = 2\,\Omega\sin\phi_0\left(1 + 2\frac{z}{R}\right) + \beta y. \tag{6.8}$$

The z-dependence of Ω_z is essential to restore $\nabla\cdot\boldsymbol{\Omega} = 0$ with respect to a constant metric, and arises from the same yz term in the Lagrangian that gives a y-dependence of Ω_y. However, the earlier z-dependence of Ω_y in (5.9*a*) has been eliminated.

7. Depth-averaged equations

Depth-averaged descriptions, such as the shallow water and Green & Naghdi (1976) equations, may be derived from a three-dimensional Lagrangian by restricting the fluid elements to move in columns (Salmon 1983; Miles & Salmon 1985). We use this approach to derive a non-traditional analogue of the shallow water equations, and show that these equations coincide with those derived by Dellar & Salmon (2005) and Stewart & Dellar (2010) using purely Cartesian geometry.

We assume that the fluid lies in a layer between a rigid base at $z = B(x,y)$ and an upper free surface at $z = B(x,y) + h(x,y,\tau)$. The labels c may be assigned so that $c = 0$ on $z = B(x,y)$, and $c = 1$ on the free surface $z = B(x,y) + h(x,y,\tau)$. We also assume that the map from labels $\boldsymbol{a}$ to coordinates $\boldsymbol{x}$ takes the restricted form

$$x = x(a,b,\tau), \quad y = y(a,b,\tau), \tag{7.1}$$

with no dependence on the third label c. The incompressibility constraint for the three-dimensional Jacobian then factorises into

$$\frac{1}{\rho_0} = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial(x,y)}{\partial(a,b)}\frac{\partial z}{\partial c}, \tag{7.2}$$

which may be solved to give

$$z = \left(\rho_0\frac{\partial(x,y)}{\partial(a,b)}\right)^{-1} c + B(x,y) = h(x,y,\tau)c + B(x,y). \tag{7.3}$$

The dynamic boundary condition of zero pressure on the free surface is implicit in the Lagrangian, because there is no contribution from the work done by an external pressure at the free surface (see Miles & Salmon 1985). The kinetic energy due to vertical motions is usually neglected in shallow water theory, since it is order $(H/L)^2$ smaller than that due to horizontal motions. Miles & Salmon (1985) showed that retaining this term gives the Green & Naghdi (1976) equations in place of the shallow water equations.

Substituting $z = hc + B$ into the Lagrangian (6.7) and completing the integration over c gives the shallow water Lagrangian

$$\mathcal{L} = \int \mathrm{d}a\mathrm{d}b\,\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 - g\left(B + \frac{1}{2}h\right) \\ +\Omega R\frac{\partial x}{\partial \tau}\left[\left(-\frac{y^2}{R^2} + \frac{h+2B}{R}\right)\cos\phi_0 - 2\frac{y}{R}\left(1 + \frac{h+2B}{R}\right)\sin\phi_0\right]. \tag{7.4}$$

The expression in square brackets $[\cdots]$ is the vector potential R_x evaluated at the midsurface $z = B + h/2$. Similarly, the gravitational term $g(B + h/2)$ is the gravitational potential $\Phi = gz$ evaluated at the midsurface. Both expressions also correspond to their averages over the layer, as defined in (7.13) below, $\overline{R_x} = R_x(x,y,z = B + h/2)$ and $g(B + h/2) = g\overline{z}$.

The variational derivatives of (7.4) give the zonal momentum p_x,

$$p_x = \frac{\partial \mathcal{L}}{\partial x_\tau} = \frac{\partial x}{\partial \tau} - 2\Omega\, y\sin\phi_0\left(1 + \frac{h+2B}{R}\right) + \Omega R\cos\phi_0\left(-\frac{y^2}{R^2} + \frac{h+2B}{R}\right), \tag{7.5}$$

the meridional momentum p_y, and the Montgomery potential

$$p_y = \frac{\partial \mathcal{L}}{\partial y_\tau} = \frac{\partial y}{\partial \tau}, \quad \frac{\partial \mathcal{L}}{\partial h} = -\frac{1}{2}\, g + \frac{\partial x}{\partial \tau}\Omega \left(\cos\phi_0 - 2\frac{y}{R}\sin\phi_0\right). \tag{7.6}$$

The Euler–Lagrange equations for stationarity of the action under variations $\delta \boldsymbol{x}$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\delta \mathcal{L}}{\delta x_{i\tau}}\right) - \frac{\delta \mathcal{L}}{\delta x_i} = (\partial_t + \boldsymbol{u}\cdot\nabla)\, p_i - \frac{1}{h}\frac{\partial}{\partial x_i}\left(h^2 \frac{\delta \mathcal{L}}{\delta h}\right) - \left.\frac{\delta \mathcal{L}}{\delta x_i}\right|_h = 0, \tag{7.7}$$

using formulae from Miles & Salmon (1985). The last two terms separate the implicit dependence on the coordinate x and y through the height h from any explicit dependence x and y through the topography B or vector potential $\boldsymbol{R}$. The height h in (7.4) is a shorthand for the reciprocal of the Jacobian of the map $(a, b) \mapsto (x, y)$ from labels to particles as defined in (7.3).

The Euler–Lagrange equations (7.7) may be written more simply as

$$u_t + \boldsymbol{u}\cdot\nabla u - f_{\mathit{eff}}\, v + \partial_x\left[g(h+B) - hu\Omega_y\right] - \Omega_y \nabla\cdot(h\boldsymbol{u}) = 0, \tag{7.8a}$$

$$v_t + \boldsymbol{u}\cdot\nabla v + f_{\mathit{eff}}\, u + \partial_y\left[g(h+B) - hu\Omega_y\right] = 0, \tag{7.8b}$$

where the effective traditional Coriolis parameter f_{eff} is given by

$$f_{\mathit{eff}} = 2\Omega\sin\phi_0 + \beta y - \frac{\partial}{\partial y}\left(\Omega_y (B + h/2)\right). \tag{7.9}$$

The last term arises from the dependence of the zonal angular momentum of a fluid column on its mean distance from the rotation axis. The non-traditional component Ω_y of the rotation vector also alters the pressure away from its hydrostatic value. The pressure is now determined by a quasi-hydrostatic balance with both gravity and the non-traditional part of the Coriolis force. The combination of the vertical velocity and the non-traditional part of the Coriolis force leads to the $-\Omega_y\nabla\cdot(h\boldsymbol{u})$ term in (7.8*a*). The height evolves according to the usual shallow water continuity equation $h_t + \nabla\cdot(h\boldsymbol{u}) = 0$, as derived by differentiating (7.3) with respect to τ. Equations (7.8) coincide with the non-traditional shallow water equations derived by Dellar & Salmon (2005) in Cartesian coordinates, as amended by Stewart & Dellar (2010) to correct the case when the horizontal part of the rotation vector has non-zero horizontal divergence.

These extended non-traditional shallow water equations materially conserve the potential vorticity

$$q = \frac{1}{h}\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right), \tag{7.10}$$

using a general result derived from the particle relabelling symmetry by Ripa (1981) and Salmon (1982*a*). For the Lagrangian in (7.4) this expression evaluates to

$$q = \frac{1}{h}\left[2\,\Omega\sin\phi_0\left(1 + \frac{h+2B}{R} + 2\frac{y}{R}\frac{\partial}{\partial y}\left(B + \frac{h}{2}\right)\right) + 2\,\Omega\cos\phi_0\left(\frac{y}{R} - \frac{\partial}{\partial y}\left(B + \frac{h}{2}\right)\right)\right], \tag{7.11}$$

which may be written more compactly as

$$q = \frac{1}{h}\left[2\,\overline{\Omega}_z - 2\,\overline{\Omega}_y\frac{\partial}{\partial y}\left(B + \tfrac{1}{2}h\right) + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right], \tag{7.12}$$

by introducing the vertically-averaged rotation vector

$$\overline{\boldsymbol{\Omega}}(x, y, t) = \frac{1}{h}\int_B^{h+B} \boldsymbol{\Omega}(x, y, z)\,\mathrm{d}z. \tag{7.13}$$

Thus $2\,\overline{\Omega}_z = 2\,\Omega\sin\phi_0 + \beta y - \gamma(B + h/2)$, and $2\,\overline{\Omega}_y = 2\,\Omega\cos\phi_0 + \gamma y$. The vertical average $\overline{\boldsymbol{\Omega}}$ is time dependent because the layer height $h(x, y, t)$ is time dependent. Again, this expression for q coincides with the expression given previously by Dellar & Salmon (2005) when $\partial_y\Omega_y = 0$, and with the amended form given by Stewart & Dellar (2010) when $\partial_y\Omega_y \neq 0$.

8. Conclusion

Starting from Hamilton's principle for an ideal fluid expressed in spherical polar coordinates, we introduced a set of pseudo-Cartesian coordinates x, y, z with the important properties that the horizontal x and y coordinate lines lie in curved surfaces of constant geopotential, while the volume element is precisely $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ due to cancellations between the metric coefficients. We then assumed that the coordinates x, y, z were all small compared with the planetary radius R, expanded the Lagrangian in powers of x/R, y/R, z/R, and truncated. An additional assumption, equivalent to a small Rossby number, was introduced to justify the neglect of metric terms in the kinetic energy due to the non-Cartesian nature of the x, y, z coordinates. This additional assumption may be omitted when the coordinate system is centred around a point on, or sufficiently close to, the equator.

From the truncated Lagrangian in these x, y, z coordinates we derived sets of equations that include the traditional-approximation β-plane equations, Grimshaw's (1975) non-traditional β-plane, and an extended β-γ-plane that allows the non-traditional Coriolis parameter to vary with latitude. These different approximations arose from different treatments of the vertical coordinate z relative to the horizontal coordinates x and y. The traditional β-plane equations are a consequence of neglecting all terms proportional to z in the Lagrangian, while Grimshaw's (1975) non-traditional β-plane arises from retaining a term proportional to z while neglecting a term proportional to yz. Retaining this additional term leads to a non-traditional β-γ-plane in which the non-traditional Coriolis parameter $\tilde{f}$ varies with latitude. We rewrite the resulting equations (5.7) and (6.8) in slightly different notation here for emphasis,

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \tilde{f}w - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} - \tilde{f}u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{8.1}$$

where the two Coriolis parameters are

$$f = 2\,\Omega\sin\phi_0\left(1 + 2\frac{z}{R}\right) + 2\,\Omega\cos\phi_0\frac{y}{R}, \quad \tilde{f} = 2\,\Omega\cos\phi_0 - 4\,\Omega\sin\phi_0\frac{y}{R}, \tag{8.2}$$

and the x, y, z coordinates are defined by (4.4) and (4.7). The parameter β thus takes its conventional value $\beta = (2/R)\Omega\cos\phi_0$, while $\gamma = -(4/R)\Omega\sin\phi_0$. The additional z-dependence of f restores $\nabla\cdot\boldsymbol{\Omega} = 0$ in pseudo-Cartesian coordinates, which is essential for conservation of circulation and potential vorticity. The z-dependence of f may be interpreted as giving a pseudo-Cartesian representation of the dependence of the true angular momentum on spherical radius r. The z-dependence of f and the y-dependence of $\tilde{f}$ together restore the order H/R terms to the zonal momentum equation from (8.1),

$$\underbrace{\frac{\mathrm{D}u}{\mathrm{D}t}}_{\frac{U}{2\Omega L}} + \Big[\underbrace{2\Omega\cos\phi_0}_{\frac{H}{L}} \underbrace{-\,4\,\Omega\sin\phi_0\frac{y}{R}}_{\frac{H}{R}}\Big]w - \Big[2\,\Omega\sin\phi_0\big(\underbrace{1}_{1} + \underbrace{2\frac{z}{R}}_{\frac{H}{R}}\big) + \underbrace{2\,\Omega\cos\phi_0\frac{y}{R}}_{\frac{L}{R}}\Big]v = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{8.3}$$

The magnitudes of the various terms are shown relative to the traditional Coriolis term. The traditional β-plane term is order L/R, and the non-traditional Coriolis terms are naturally order H/L smaller than the traditional terms under the shallow layer velocity scaling $w \sim (H/L)u$. The additional terms of order H/R extend the β-plane equations into the deep atmosphere regime (White *et al.* 2005) where H/L is treated as an $O(1)$ quantity. White & Bromley (1995) estimate the change in relative velocity incurred by an ascending air parcel due to non-traditional effects. The additional order H/R terms may become relevant for meridional overturning circulations (in the ocean) or Hadley circulations (in the atmosphere) where fluid parcels ascend and descend at different latitudes. Conversely, the near-inertial waves studied by Gerkema & Shrira (2005) depend critically on $\omega - f$, the difference between the wave frequency ω and the local inertial frequency. These waves are thus very sensitive to changes in f with latitude, but relatively insensitive to changes in $\tilde{f}$ with latitude.

Being derived from Hamilton's principle, each of our equation sets conserves energy, angular momentum, and potential vorticity. These properies are guaranteed by Noether's theorem from the symmetries of the truncated Lagrangians. Moreover, introducing the generalised coordinates x, y, z into Hamilton's principle does not involve resolving forces or performing vector calculus in the generalised coordinates. A minor drawback of the variational approach is the need to calculate small terms one order higher in the Lagrangian than in the equations of motion, since the latter arise from variational derivatives of the Lagrangian. For example, to obtain terms that are linear in the small parameter y/R correctly in the equations of motion, the Lagrangian must be accurate to order $(y/R)^2$. To obtain the conventional value for β we must use the coordinate y introduced by Ripa (1997), rather than the conventional latitude coordinate $\tilde{y} = (\phi - \phi_0)/R$, or the Mercator latitude coordinate described in Appendix A, even those these coordinates only differ at order $(\phi - \phi_0)^2$. The consequences of using these other latitude coordinates are described in Appendix A.

The constraint $\nabla\cdot\boldsymbol{\Omega} = 0$ is essential for conservation of circulation and potential vorticity, as recognised by Grimshaw (1975), and for the existence of a variational formulation. The essential difficulty in allowing Ω_y to vary with the latitude y arises from the interaction of the $\nabla\cdot\boldsymbol{\Omega} = 0$ constraint with the approximation of the spherical metric, with its spatially varying coefficients, by a flat Cartesian metric. Moreover, approximations of the true spherical radius r by a constant R, and of true latitude ϕ by a constant ϕ_0, do not commute with differentiations with respect to r and ϕ, as described in Appendix B. Both these difficulties may be avoided by performing approximations in Hamilton's principle, rather than directly in the equations of motion. Our derivation from Hamilton's principle also shows that the coefficient γ for the y-dependence of Ω_y differs by a factor of two from the value one obtains by approximating $\boldsymbol{\Omega}$ alone.

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Appendix A. The choice of latitude coordinate

The β-plane equations derived in the main text have $f = 2\,\Omega_z$ varying linearly with the coordinate y introduced by Ripa (1997) and defined by

$$\sin\phi = \sin\phi_0 + \frac{y}{R}\cos\phi_0. \tag{A 1}$$

The derivative $\mathrm{d}f/\mathrm{d}y$ takes its conventional value $\beta = (2/R)\Omega\cos\phi_0$. If instead we adopt the coordinate $\tilde{y}$ defined by

$$\phi = \phi_0 + \frac{\tilde{y}}{R}, \tag{A 2}$$

as used by LeBlond & Mysak (1978), Veronis (1981), Pedlosky (1987), and Gerkema *et al.* (2008), a calculation analogous to that in the main text leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = \int d\boldsymbol{a}\; \frac{1}{2}\left(\frac{\partial x}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\tilde{y}}{\partial\tau}\right)^2 &+ \mathit{\Phi}(z) + p(\boldsymbol{a},\tau)\left(\frac{\cos(\phi_0+\tilde{y}/R)}{\cos\phi_0}\frac{\partial(x,\tilde{y},z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right)\\ &+ \Omega R\frac{\partial x}{\partial\tau}\left(-2\frac{\tilde{y}}{R}\sin\phi_0 + \frac{\tilde{y}^2}{R^2}(\sec\phi_0 - 2\cos\phi_0)\right)\end{aligned} \tag{A 3}$$

in the traditional approximation (*i.e.* dropping all terms involving z in the vector potential and kinetic energy). Besides an anomaly in the pressure gradient due to the $\tilde{y}$-dependence of the volume element, the resulting equations of motion contain

$$\tilde{\Omega}_z = \Omega\sin\phi_0 + \Omega\frac{\tilde{y}}{R}\left(2\cos\phi_0 - \sec\phi_0\right). \tag{A 4}$$

The proportionality constant is thus $\tilde{\beta} = (2/R)\Omega(2\cos\phi_0 - \sec\phi_0)$. The appearance of a term involving $\sec\phi_0$ is reminiscent of Cushman-Roisin's (1982) comparison of the oscillatory motion of a free particle between a sphere and a β-plane defined using $\tilde{y}$. The two coordinates y and $\tilde{y}$ coincide at the equator ($\phi_0 = 0$) and hence so do β and $\tilde{\beta}$.

Grimshaw (1975) defined x, y, z coordinates using

$$\lambda = \frac{x}{R}\sec\phi_0, \quad \mu = \mu_0 + \frac{y}{R}\sec\phi_0, \quad r = R\exp\left(\frac{z}{R}\right), \tag{A 5}$$

where μ is the Mercator latitude coordinate defined by the relations

$$\operatorname{sech}\mu = \cos\phi, \quad \tanh\mu = \sin\phi, \quad \sinh\mu = \tan\phi. \tag{A 6}$$

Substituting these expressions into the spherical Lagrangian (4.1) and expanding up to $O((L/R)^3)$ leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = \int d\boldsymbol{a}\; &\frac{1}{2}\left[\left(\frac{\partial x}{\partial\tau}\right)^2 + \left(\frac{\partial y}{\partial\tau}\right)^2\right]\left\{1 - 2\frac{y}{R}\tan\phi_0 + 2\frac{z}{R}\right\} + \frac{1}{2}\left(\frac{\partial z}{\partial\tau}\right)^2\left\{1 + 2\frac{z}{R}\right\}\\ &+ \mathit{\Phi}(z) + p(\boldsymbol{a},\tau)\left(\frac{\exp(3z/R)}{\cos^2\phi_0}\left(1 - \tanh^2\left(\mu_0 + \frac{y}{R}\sec\phi_0\right)\right)\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0}\right)\\ &+ \Omega R\frac{\partial x}{\partial\tau}\left[\left(1 + 2\frac{z}{R} + 2\frac{z^2}{R^2} - 3\frac{y^2}{R^2}\right)\cos\phi_0 - 4\frac{y}{R}\sec\phi_0 - \left(2\frac{y}{R} + 4\frac{yz}{R^2}\right)\sin\phi_0\right],\end{aligned} \tag{A 7}$$

where $\mu_0 = \tanh^{-1}\sin\phi_0$. The horizontal kinetic energy becomes isotropic in these coordinates, but there is a non-constant term multiplying the Jacobian $\partial(x,y,z)/\partial(a,b,c)$ in the volume element. A term proportional to $\sec\phi_0$ also appears in the vector potential, just as it does for the earlier choice of coordinates leading to (A 3) above. The equations of motion obtained from the Lagrangian (A 7) contain the Coriolis terms

$$2\,\Omega_y = 2\,\Omega\cos\phi_0\left(1 + 2\frac{z}{R}\right) - 4\Omega\frac{y}{R}\sin\phi_0, \tag{A 8a}$$

$$2\,\Omega_z = 2\,\Omega\sin\phi_0\left(1 + 2\frac{z}{R}\right) + 2\Omega\frac{y}{R}\left(3\cos\phi_0 - 2\sec\phi_0\right). \tag{A 8b}$$

Again, the proportionality constant for the linear dependence of $2\,\Omega_z$ on y does not take its expected value $\beta = (2\Omega/R)\cos\phi_0$ except at the equator.

Appendix B. A purely Cartesian approach

Following the heuristic derivation of the β-plane approximation, as sketched in the Introduction, the natural rotation vector for a non-traditional β-plane should be

$$2\tilde{\boldsymbol{\Omega}} = (0, \tilde{f}, f) = (2\Omega\cos\phi_0 \,+\, \tilde{\gamma}y)\,\hat{\boldsymbol{y}} + (2\Omega\sin\phi_0 + \beta y)\,\hat{\boldsymbol{z}}, \tag{B 1}$$

where f and $\tilde{f}$ are given by expanding $2\,\Omega\sin\phi$ and $2\,\Omega\cos\phi$ as in (1.3). However, this vector field has non-zero divergence, $\nabla\cdot(2\tilde{\boldsymbol{\Omega}}) = \tilde{\gamma} \neq 0$, away from the equator. Therefore, it cannot be written as the curl of a vector potential, $2\tilde{\boldsymbol{\Omega}} = \nabla\times\boldsymbol{R}$, as required for conservation of circulation and potential vorticity. This is why Grimshaw (1975) set $\tilde{\gamma} = 0$ in his non-traditional β-plane.

The non-zero divergence of $\tilde{\boldsymbol{\Omega}}$ arises from the approximation of the spherical metric by a constant Cartesian metric. In spherical geometry, the vector field $\boldsymbol{\Omega} = \Omega\,(\sin\phi\,\hat{\boldsymbol{r}} + \cos\phi\,\hat{\boldsymbol{\phi}})$ has zero divergence, even though its ϕ component is a function of ϕ. The r-dependence of the spherical volume element $\mathrm{d}V = r^2\cos\phi\,\mathrm{d}\lambda\,\mathrm{d}\phi\,\mathrm{d}r$ allows the net influx of Ω_ϕ to a control volume $\mathrm{d}\lambda\mathrm{d}\phi\mathrm{d}r$ to be balanced by a net outflux of Ω_r, because the upper surface of a control volume has a larger area than the lower surface. Equivalently, we calculate

$$\nabla\cdot\boldsymbol{\Omega} = \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}\left(\Omega_\phi\cos\phi\right) + \frac{1}{r^2}\left(r^2\Omega_r\right) = 0, \tag{B 2}$$

when $\Omega_r = \Omega\sin\phi$, $\Omega_\phi = \Omega\cos\phi$, and $\Omega_\lambda = 0$. The non-zero divergence arises when one approximates r by R (constant) and $\cos\phi$ by $\cos\phi_0$ (constant), however good an approximation this may seem in the sense of $r - R$ and $\phi - \phi_0$ being small compared with R and ϕ_0.

However, one may restore $\nabla\cdot\boldsymbol{\Omega} = 0$ in Cartesian coordinates by adding a compensating z-dependence to Ω_z,

$$2\boldsymbol{\Omega} = (2\Omega\cos\phi_0 \,+\, \tilde{\gamma}y)\,\hat{\boldsymbol{y}} + (2\Omega\sin\phi_0 + \beta y - \tilde{\gamma}z)\,\hat{\boldsymbol{z}}. \tag{B 3}$$

A convenient vector potential is

$$\boldsymbol{R} = \left[2\Omega(z\cos\phi_0 - y\sin\phi_0) - \tfrac{1}{2}\beta y^2 + \tilde{\gamma}yz\right]\hat{\boldsymbol{x}}. \tag{B 4}$$

This satisfies $\nabla\times\boldsymbol{R} = 2\boldsymbol{\Omega}$, has no z-component, which is convenient for deriving a depth-averaged shallow water theory (as in Dellar & Salmon 2005), and has no dependence on the x coordinate. The resulting Lagrangian therefore has no explicit dependence on x, so Noether's theorem yields a conserved zonal momentum p_x, as in Ripa (1993). In fact, this Lagrangian coincides with the Lagrangian for the simplified β-γ-plane derived in (6.7), except the value of $\tilde{\gamma}$ differs by a factor of two from the γ derived from spherical geometry.

Alternatively, we might try to approximate the vector potential directly. The rotation vector $\boldsymbol{\Omega}$ is given in spherical polar coordinates by the vector potential

$$\boldsymbol{R}_{\mathrm{sphere}} = \boldsymbol{\Omega}\times\boldsymbol{x} = \Omega r\cos\phi\;\hat{\boldsymbol{\lambda}}, \tag{B 5}$$

which is denoted $\boldsymbol{v}_e$ by Phillips (1966). Putting $r = R + z$, $\phi = \phi_0 + y/R$, and expanding up to second order in y/R and z/R, we obtain

$$\boldsymbol{R}_{\mathrm{sphere}} = \Omega R\left(\cos\phi_0 - \frac{y}{R}\sin\phi_0 + \frac{z}{R}\cos\phi_0 - \frac{1}{2}\frac{y^2}{R^2}\cos\phi_0 - \frac{yz}{R^2}\sin\phi_0\right)\hat{\boldsymbol{x}}. \tag{B 6}$$

Dropping the constant term $\Omega R\cos\phi_0$, which has zero curl, we are left with precisely half the Cartesian vector potential $\boldsymbol{R}$ given in (B 4). The extra factor of two is required to compensate for the approximation of the spherical metric by a constant metric. The curl of the vector potential $\boldsymbol{R}_{\mathrm{sphere}}$ in spherical polar coordinates is given by

$$\nabla\times\left(R_\lambda\hat{\boldsymbol{\lambda}}\right) = \frac{1}{r}\frac{\partial}{\partial r}(rR_\lambda)\;\hat{\boldsymbol{\phi}} - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,R_\lambda)\;\hat{\boldsymbol{r}} = 2\,\boldsymbol{\Omega}, \tag{B 7}$$

since $R_\lambda = \Omega r\cos\phi$. Replacing r with r_0 (constant) and $\cos\phi$ with $\cos\phi_0$ (constant) has the effect of halving the quantity calculated,

$$\frac{1}{r_0}\frac{\partial}{\partial r}(r_0R_\lambda)\,\hat{\boldsymbol{\phi}} - \frac{1}{r_0\cos\phi_0}\frac{\partial}{\partial\phi}(\cos\phi_0\,R_\lambda)\,\hat{\boldsymbol{r}} = \Omega\left(\cos\phi\,\hat{\boldsymbol{\phi}} + \sin\phi\,\hat{\boldsymbol{r}}\right) = \boldsymbol{\Omega}. \tag{B 8}$$

Again, this is an $O(1)$ change, even if $|r - r_0| \ll r_0$ and $|\phi - \phi_0| \ll \phi_0$.

Phillips's (1966) derivation of the traditional approximation replaces r by r_0 (constant) in both the curl operator and the vector potential, which becomes

$$\boldsymbol{R}_{\mathrm{trad}} = \Omega r_0\cos\phi\;\hat{\boldsymbol{\lambda}}, \tag{B 9}$$

but leaves ϕ unapproximated. The approximated curl of this vector potential is thus

$$-\frac{1}{r_0\cos\phi}\frac{\partial}{\partial\phi}(\Omega r_0\cos^2\phi)\,\hat{\boldsymbol{r}} = 2\Omega\sin\phi\,\hat{\boldsymbol{r}} = \boldsymbol{\Omega}\cdot\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}, \tag{B 10}$$

which is the locally vertical part of $\boldsymbol{\Omega}$, as it appears in the traditional approximation.

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