Common Hamiltonian structure of the shallow water equations with horizontal temperature gradients and magnetic fields

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The Hamiltonian structure of the inhomogeneous layer models for geophysical fluid dynamics devised by Ripa [Geophys. Astrophys. Fluid Dyn. **70** p. 85 (1993)] involves the same Poisson bracket as a Hamiltonian formulation of shallow water magnetohydrodynamics in velocity, height, and magnetic flux function variables. This Poisson bracket becomes the Lie–Poisson bracket for a semidirect product Lie algebra under a change of variables, giving a simple and direct proof of the Jacobi identity in place of Ripa's long outline proof. The same bracket has appeared before in compressible and relativistic magnetohydrodynamics. The Hamiltonian is the integral of the three dimensional energy density for both the inhomogeneous layer and magnetohydrodynamic systems, which provides a compact derivation of Ripa's models.

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I. INTRODUCTION

Many oceanic phenomena may be investigated using layered models, in which the continuous vertical structure is approximated by a small stack of layers with varying thicknesses [1]. The fluid variables within each layer, such as density and horizontal velocity, are taken to be vertically uniform. The simplest layer model is the shallow water equations for a single layer of incompressible fluid with a free surface [1, 2]. Baroclinic effects due to unaligned density and pressure gradients in a continuously stratified fluid may be modeled using two or more layers [1]. Many phemonena outside the tropics may be captured by just two layer ocean models [2], but layer models in general have difficulty representing thermodynamic phenomena such as solar heating or fresh water forcing that become important in the tropics [3].

Ripa [4] considered a family of layered models that permitted horizontal variations in the fluid density within each layer. These variations may be attributed to horizontal temperature gradients. Ripa's family extends an earlier single layer model introduced by Lavoie [5], and used by Anderson [6] to study the effect of diurnal heating on a tropical jet. A two layer version was used by Schopf and Cane [7] to study the effects of solar heating and wind stresses on the upper equatorial ocean. Further applications have been listed by Ripa [3].

Many models in geophysical fluid dynamics have a Hamiltonian structure, which conveys important qualitative properties [2, 8–10] like a direct link between symmetries and conservation laws via Noether's theorem, and a framework for deriving nonlinear stability criteria via the energy-Casimir method [11]. Ripa [4] gave a Hamiltonian structure for his family of layered models. Ripa's structure is a non-canonical one involving a generalised Poisson bracket (see Morrison [9] or Shepherd [10] for an introduction) as is standard for hydrodynamic systems in Eulerian variables. Generalised Hamiltonian structures take the form

$$\partial_t \boldsymbol{\eta} = \mathsf{J}\frac{\delta \mathcal{H}}{\delta \boldsymbol{\eta}}, \tag{1}$$

as evolution equations for the field variables $\boldsymbol{\eta}$, where $\mathcal{H}$ is the Hamiltonian, and J the Poisson tensor. This equation is equivalent to $\partial_t \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$ for any functional $\mathcal{F}$, where the Poisson bracket of two functionals is given in terms of J by the volume integral

$$\{\mathcal{F}, \mathcal{G}\} = \int \frac{\delta \mathcal{F}}{\delta \boldsymbol{\eta}} \mathsf{J} \frac{\delta \mathcal{G}}{\delta \boldsymbol{\eta}} dV. \tag{2}$$

This bracket is automatically bilinear in $\mathcal{F}$ and $\mathcal{G}$, and is antisymmetric when J is antisymmetric, or when J is an anti-selfadjoint differential operator. To be a Poisson bracket, the bracket given by Eq. (2) must also satisfy the Jacobi identity, $\{\{\mathcal{F},\mathcal{G}\},\mathcal{K}\} + \{\{\mathcal{G},\mathcal{K}\},\mathcal{F}\} + \{\{\mathcal{K},\mathcal{F}\},\mathcal{G}\} = 0$ for any three functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$ [8–10]. These three properties: bilinearity, antisymmetry, and the Jacobi identity, together create a generalised Hamiltonian structure in a sense that is independent of any particular choice of coordinates [8–10]. The Jacobi identity is usually by far the most difficult property to establish.

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Ripa [4] gave a long outline proof of the Jacobi identity for his Poisson bracket, written in the form of Eq. (2) with J given by Eq. (6) below. In this paper we observe that Ripa's Poisson bracket is the same as that arrived at by Dellar [12] as a Hamiltonian form for Gilman's [13] shallow water magnetohydrodynamic equations. Dellar arrived at this bracket by a change of variables from Morrison and Greene's [14] Poisson bracket for barotropic compressible magnetohydrodynamics. This bracket is of Lie–Poisson form, and so inherits its Jacobi identity automatically from the Jacobi identity satisfied by the Lie algebra giving rise to the bracket [15], which in this case is of the semidirect product type ubiquitous in Hamiltonian formulations of compressible fluid dynamics in Eulerian variables [11, 16].

II. SHALLOW WATER EQUATIONS WITH AN INHOMOGENEOUS LAYER

In its simplest case, with one active layer and no bottom topography, Ripa's [4] model reduces to Lavoie's [5] rotating shallow water equations with forcing due to a horizontally varying potential temperature field θ. Specifically, θ is the reduced gravity $g\Delta\Theta/\overline{\Theta}$ computed as the potential temperature difference $\Delta\Theta$ in the layer from some reference value $\overline{\Theta}$ [6]. The evolution equations for the primitive variables h, $\mathbf{u}$, θ are then [17]

$$\partial_t h + \nabla\cdot(\mathbf{u}h) = 0, \tag{3a}$$

$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta = 0, \tag{3b}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\nabla(\theta h) + \frac{1}{2}h\nabla\theta, \tag{3c}$$

where $\mathbf{u}$ is the depth-averaged fluid velocity, and h the layer depth. The layer depth is supposed to be small compared with a typical horizontal lengthscale ℓ. In other words the aspect ratio $h/\ell \ll 1$, so the vertical fluid acceleration in the layer may be neglected [1, 2].

Equations (3a-c) hold either in a finite domain $\mathcal{D}$, subject to the boundary conditions

$$\mathbf{n}\cdot\mathbf{u} = 0, \quad \mathbf{n}\times\nabla\theta = 0 \text{ on } \partial\mathcal{D}, \tag{4}$$

or in an unbounded domain subject to sufficiently rapid decay towards a uniform rest state at infinity. The boundary condition $\mathbf{n}\times\nabla\theta = 0$ implies that θ is constant along each connected component of the boundary [4]. This condition, along with the other boundary condition $\mathbf{n}\cdot\mathbf{u} = 0$, implies that the circulation around each connected component of the boundary is constant in time [4]. In the usual shallow water equations, only the $\mathbf{n}\cdot\mathbf{u} = 0$ condition is necessary for constant circulations [1].

In a rotating system, $\boldsymbol{\Omega}$ is the angular velocity vector that gives rise to the Coriolis force $2\boldsymbol{\Omega}\times\mathbf{u}$. In geophysical applications, $\boldsymbol{\Omega}$ is often taken to be just the component of the angular velocity parallel to the local gravity vector, and thus varies with latitude; either over the surface of a sphere or on a β-plane [1, 2]. If $\theta = g'$ is constant, these equations reduce to the usual shallow water equations with reduced gravity g'.

III. HAMILTONIAN FORM

Ripa [4] showed that Eqs. (3a-c) may be written in non-canonical Hamiltonian form as

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{u}\\ h\\ \theta\end{pmatrix} = \mathsf{J}\begin{pmatrix}\delta\mathcal{H}/\delta\mathbf{u}\\ \delta\mathcal{H}/\delta h\\ \delta\mathcal{H}/\delta\theta\end{pmatrix}, \tag{5}$$

with Poisson tensor

$$\mathsf{J} = -\begin{pmatrix} 0 & -q & \partial_x & -h^{-1}\theta_x \\ q & 0 & \partial_y & -h^{-1}\theta_y \\ \partial_x & \partial_y & 0 & 0 \\ h^{-1}\theta_x & h^{-1}\theta_y & 0 & 0 \end{pmatrix}, \tag{6}$$

where $\theta_x = \partial_x\theta$, $\theta_y = \partial_y\theta$, and $q = h^{-1}(2|\boldsymbol{\Omega}| + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})$ is the potential vorticity, for the Hamiltonian

$$\mathcal{H} = \int \frac{1}{2}h|\mathbf{u}|^2 + \frac{1}{2}\theta h^2 \, dxdy. \tag{7}$$

This Hamiltonian coincides with the total energy, comprising both kinetic energy and gravitational potential energy for the reduced gravity θ. It may be derived [4] by integrating the three dimensional energy density in the vertical direction, neglecting the contribution from the vertical velocity u_z to the kinetic energy on the grounds that it is $\mathcal{O}(h^2/\ell^2)$ smaller than the horizontal contributions for a shallow layer with small aspect ratio h/ℓ [1, 2]. Substituting the variational derivatives of the Hamiltonian,

$$\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = h\mathbf{u}, \quad \frac{\delta\mathcal{H}}{\delta\theta} = \frac{1}{2}h^2, \quad \frac{\delta\mathcal{H}}{\delta h} = \frac{1}{2}|\mathbf{u}|^2 + \theta h, \tag{8}$$

into Eq. (5) leads to the evolution equations (3a-c).

In particular, this Hamiltonian formulation leads to an evolution equation for $\mathbf{u}$ in the form

$$\partial_t\mathbf{u} + hq\,\hat{\mathbf{z}}\times\mathbf{u} = \frac{1}{h}(\nabla\theta)\frac{\delta\mathcal{H}}{\delta\theta} - \nabla\frac{\delta\mathcal{H}}{\delta h}, \tag{9}$$

whose curl implies an evolution equation for the potential vorticity $q = h^{-1}(2|\mathbf{\Omega}| + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})$,

$$\partial_t q + \mathbf{u}\cdot\nabla q = h^{-1}\hat{\mathbf{z}}\cdot\nabla\left(\frac{1}{h}\frac{\delta\mathcal{H}}{\delta\theta}\right)\times\nabla\theta = \frac{1}{2}h^{-1}[h,\theta], \tag{10}$$

where $[f,g] = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)$ is the Jacobian of two functions. The latter form is as given by Ripa [4]. The right hand side involving $[h,\theta]$ is reminiscent of the baroclinic vorticity generation term $\rho^{-2}\nabla\rho\times\nabla p$ in the meteorological primitive equations [1]. Equation (10) reduces to potential vorticity conservation on fluid elements ($\partial_t q + \mathbf{u}\cdot\nabla q = 0$) in the pure shallow water equations where $\nabla\theta = 0$ [1, 2]. This very important qualitative property of the shallow water equations is a consequence of the particle relabelling symmetry of the shallow water equations formulated in Lagrangian variables [2, 8, 10]. This symmetry is broken in Ripa's models, as expressed by the source term on the right hand side of Eq. (10), because particles with different values of θ (which is itself a Lagrangian tracer) cannot be interchanged without altering the dynamics.

The integral form of Eq. (10) is a version of Kelvin's circulation theorem [1, 2],

$$\frac{d}{dt}\oint_C \mathbf{u}\cdot d\mathbf{l} = \oint_C \frac{1}{h}\frac{\delta\mathcal{H}}{\delta\theta}\nabla\theta\cdot d\mathbf{l}, \tag{11}$$

for the evolution of the circulation around any closed material curve C. The circulation is equal to the surface integral of vorticity over any surface spanning the loop by Stokes' thorem. The right hand side of Eq. (11) again contains a source term, one that vanishes for the pure shallow water equations where $\nabla\theta = 0$. However, the right hand side also vanishes if the curve C is chosen to be a line of constant θ, for which $\nabla\theta\cdot d\mathbf{l} = 0$. Thus, although potential vorticity is no longer conserved on individual fluid particles in the thermal shallow water equations, the total potential vorticity inside any contour of constant θ is conserved, in the sense that

$$\frac{d}{dt}\int_{S(\theta_0)} hq\,dxdy = 0, \tag{12}$$

where $S(\theta_0)$ is the material surface bounded by the $\theta = \theta_0$ contour.

In terms of particle labels, this conservation property arises because the labels along any line of constant θ may be freely exchanged without altering the dynamics. The same conservation property was obtained by Dellar [12] in a different way by computing the Casimirs, or conserved integrals, of the Poisson bracket [9, 10]. These Casimirs are the functionals [4, 11, 12]

$$\mathcal{C} = \int hf(\theta) + hqg(\theta)dxdy, \tag{13}$$

for any functions $f(\theta)$ and $g(\theta)$. The Casimir functionals are all conserved by the evolution equations (5) because $\mathsf{J}\delta\mathcal{C}/\delta\boldsymbol{\eta} = 0$. The previous result in Eq. (12) follows from taking $f = 0$ and g to be the step function $H(\theta - \theta_0)$, so that $g(\theta) = 1$ for $\theta \geq \theta_0$ and $g(\theta) = 0$ otherwise.

IV. SHALLOW WATER MAGNETOHYDRODYNAMICS

The upper left 3×3 block of the Poisson tensor is identical to that given by Shepherd [10] for the shallow water equations. In fact, the same complete Poisson tensor was obtained by Dellar [12] for the shallow water magnetohydrodynamic equations,

$$\partial_t h + \nabla\cdot(\mathbf{u}h) = 0, \tag{14a}$$
$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \tag{14b}$$
$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} = -g'\nabla h + \mathbf{B}\cdot\nabla\mathbf{B}. \tag{14c}$$

The reduced gravity g' in Eq. (14c) is now constant, and the extra term $\mathbf{B}\cdot\nabla\mathbf{B}$ is the vertically averaged Lorentz force due to the magnetic field. The first two equations are the same as before, except the advected scalar θ is now a flux function for the horizontal magnetic field $\mathbf{B}$ given by

$$\mathbf{B} = h^{-1}\hat{\mathbf{z}}\times\nabla\theta = h^{-1}(-\theta_y, \theta_x, 0). \tag{15}$$

This representation automatically satisfies the constraint $\nabla\cdot(h\mathbf{B}) = 0$. This is the shallow water analogue [13] of the $\nabla\cdot\mathbf{B} = 0$ constraint in conventional magnetohydrodynamics that excludes magnetic monopoles. The sign convention in Eq. (15) is the standard one for plasma physics and geophysical fluid dynamics, although it differs from that used in Ref. [11]. The previous boundary condition $\mathbf{n}\times\nabla\theta = 0$ becomes the usual no normal flux boundary condition for magnetohydrodynamics, $\mathbf{n}\cdot\mathbf{B} = 0$.

Shallow water magnetohydrodynamics is also a Hamiltonian system for the same Poisson bracket given by Eq. (6), rewritten in terms of $\mathbf{B}$ as [12]

$$\mathsf{J} = -\begin{pmatrix} 0 & -q & \partial_x & -B_y \\ q & 0 & \partial_y & B_x \\ \partial_x & \partial_y & 0 & 0 \\ B_y & -B_x & 0 & 0 \end{pmatrix}, \tag{16}$$

but with the different Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int h|\mathbf{u}|^2 + \frac{1}{h}|\nabla\theta|^2 + g'h^2 \; dxdy. \tag{17}$$

This Hamiltonian is again the total energy, with an additional magnetic energy contribution of $\frac{1}{2}h|\mathbf{B}|^2$ per unit area. It may be derived by integrating the three dimensional energy density $\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{B}|^2 + g'z$ for an incompressible fluid with unit density in the vertical from $z = 0$ to $z = h(x, y)$ [12].

V. CONSERVATION FORM AND THE LIE–POISSON BRACKET

Without the Coriolis force, the shallow water magnetohydrodynamics equations (14a-c) may be rewritten in conservation form as the hyperbolic system [18]

$$\partial_t \begin{pmatrix} h\mathbf{u} \\ h \\ h\mathbf{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} h\mathbf{u}\mathbf{u} - h\mathbf{B}\mathbf{B} + \frac{1}{2}gh^2\mathsf{I} \\ h\mathbf{u} \\ h\mathbf{u}\mathbf{B} - h\mathbf{B}\mathbf{u} \end{pmatrix} = 0, \tag{18}$$

where I is the 2×2 identity matrix, subject to the constraint $\nabla\cdot(h\mathbf{B}) = 0$. This constraint is the shallow water analogue of the $\nabla\cdot\mathbf{B} = 0$ constraint in conventional magnetohydrodynamics that expresses the absence of magnetic monopoles, and allows the use of a flux function as in Eq. (15).

In terms of the conserved variables $\mathbf{m} = h\mathbf{u}$, h, and $\mathbf{Q} = h\mathbf{B} = \hat{\mathbf{z}}\times\nabla\theta$, the Hamiltonian system given by Eqs. (5) and (6) transforms to [14]

$$\frac{\partial}{\partial t}\begin{pmatrix} m_i \\ h \\ Q_i \end{pmatrix} = \mathsf{J}_{ij}\begin{pmatrix} \delta\mathcal{H}/\delta m_j \\ \delta\mathcal{H}/\delta h \\ \delta\mathcal{H}/\delta Q_j \end{pmatrix}, \tag{19}$$

where

$$\mathsf{J}_{ij} = -\begin{pmatrix} m_j\partial_i + \partial_j m_i & h\partial_i & Q_j\partial_i - \partial_k Q_k\delta_{ij} \\ \partial_j h & 0 & 0 \\ \partial_j Q_i - Q_k\partial_k\delta_{ij} & 0 & 0 \end{pmatrix}, \tag{20}$$

and partial derivatives act on everything to their right. This Poisson tensor is antisymmetric (after an integration by parts) and satisfies the Jacobi identity for all $\mathbf{m}$, h, and $\mathbf{Q}$ [14, 15]. A slightly simpler form was given previously by Morrison and Greene [19] that only satisfies the Jacobi identity when $\nabla\cdot\mathbf{Q} = 0$.

Every term in Eq. (20) is linear in one of the conserved variables $\mathbf{m}$, h, and $\mathbf{Q}$, and involves one spatial derivative. In fact, this Poisson tensor is of Lie–Poisson form, which means the Poisson bracket may be written as

$$\{\mathcal{F}, \mathcal{G}\} = \int \boldsymbol{\eta}\cdot\left[\frac{\delta\mathcal{F}}{\delta\boldsymbol{\eta}}, \frac{\delta\mathcal{G}}{\delta\boldsymbol{\eta}}\right] dV, \tag{21}$$

the inner product of the conserved variable $\boldsymbol{\eta} = (\mathbf{m}, h, \mathbf{Q})$ with the Lie bracket $[\mathcal{F}_{\boldsymbol{\eta}}, \mathcal{G}_{\boldsymbol{\eta}}]$ of the variational derivatives of the two functionals $\mathcal{F}$ and $\mathcal{G}$. Poisson brackets of this form automatically inherit their Jacobi identity from the Jacobi identity satisfied by the Lie algebra appearing in Eq. (21) [15]. Ripa's Poisson bracket given by Eq. (6) is just a rewriting of the Lie–Poisson bracket given by Eq. (20) under the change of variables $\mathbf{m} = h\mathbf{u}$, $\mathbf{Q} = \hat{\mathbf{z}}\times\nabla\theta$, and so still satisfies the Jacobi identity.

Equations (21) and (20) may be rewritten as [14]

$$\{\mathcal{F},\mathcal{G}\} = \int m_i \left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta m_i} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta m_i}\right) + h\left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta h} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta h}\right)$$
$$+Q_i\left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta Q_i} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta Q_i}\right) - Q_i\left(\frac{\delta\mathcal{G}}{\delta Q_j}\partial_i\frac{\delta\mathcal{F}}{\delta m_j} - \frac{\delta\mathcal{F}}{\delta Q_j}\partial_i\frac{\delta\mathcal{G}}{\delta m_j}\right)dV, \tag{22}$$

after integrating by parts. The first term in Eq. (22) involves the natural Lie bracket $[\mathbf{f},\mathbf{g}] = \mathbf{f}\cdot\nabla\mathbf{g} - \mathbf{g}\cdot\nabla\mathbf{f}$ for the Lie algebra $\mathfrak{v}$ of vector fields. The other three terms arise from the extension of this Lie bracket to the semidirect product Lie algebra $\mathfrak{v}\ltimes(\Lambda^3\oplus\Lambda^2)$, in which vector fields act separately on three-forms or densities h, and two-forms $\mathbf{B}$ [11, 16].

This use of two-forms and three-forms requires the bracket Eq. (22) to be formulated first in three spatial dimensions, as it was by Morrison and Greene [14, 19], before suppressing one spatial coordinate for the shallow water system. In three dimensions, three-forms are natural objects for masses per unit volume (densities), and two-forms for fluxes per unit surface (magnetic fields). This structure is ubiquitous for a frozen-in density and magnetic field, so the same Poisson bracket appears in compressible [14, 19], shallow water [12], and special relativistic magnetohydrodynamics [20]. The non-magnetic part involving only $\mathbf{m}$ and h appears in various extended shallow water models [21, 22]. An intermediate form of the same bracket, using the variables $\mathbf{m}$, h, θ, was given by Holm *et al.* [11] for two dimensional compressible magnetohydrodynamics with a barotropic equation of state,

$$\{\mathcal{F},\mathcal{G}\} = \int m_i \left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta m_i} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta m_i}\right) + h\left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta h} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta h}\right)$$
$$+\theta\nabla\cdot\left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\frac{\delta\mathcal{F}}{\delta\theta} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\frac{\delta\mathcal{G}}{\delta\theta}\right)dV. \tag{23}$$

This form of the bracket is also of Lie–Poisson type, and is related to Ripa's form by an algebraic change of variables.

VI. RIPA'S MODEL IN LIE–POISSON FORM

There is no explicit formula for Ripa's Hamiltonian given by Eq. (7) in terms of the variables $\mathbf{m}$, h, $\mathbf{Q}$, since θ appears algebraically. The same situation occurs in higher order shallow water models with nonhydrostatic dispersion, where the relation $\mathbf{m} = h\mathbf{u} - \frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{u})$ between momentum and velocity cannot be inverted in closed form [8, 21]. By contrast, only $|\nabla\theta| = |\mathbf{Q}|$ appears in the Hamiltonian given by Eq. (17) for shallow water magnetohydrodynamics.

However, the variational derivatives of the Hamiltonian may still be computed using the variational chain rule,

$$\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = \frac{1}{h}\frac{\delta\mathcal{H}}{\delta\mathbf{m}}, \quad \frac{\delta\mathcal{H}}{\delta\theta} = -\hat{\mathbf{z}}\cdot\nabla\times\frac{\delta\mathcal{H}}{\delta\mathbf{Q}}, \quad \left(\frac{\delta\mathcal{H}}{\delta h}\right)_{\mathbf{u}} = \left(\frac{\delta\mathcal{H}}{\delta h}\right)_{\mathbf{m}} + \mathbf{u}\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{m}}. \tag{24}$$

For instance, the momentum equation obtained from Eqs. (19) and (20) simplifies to

$$\partial_t m_i = -(hu_j\partial_i u_j + \partial_j h u_i u_j) - h\partial_i\left(h\theta - \frac{1}{2}u_j u_j\right) + \left[\left(\nabla\times\frac{\delta\mathcal{H}}{\delta\mathbf{Q}}\right)\times\mathbf{Q}\right]_i, \tag{25}$$

because $\nabla\cdot\mathbf{Q} = 0$ [19]. The second of Eqs. (24) implies $\nabla\times(\delta\mathcal{H}/\delta\mathbf{Q}) = -\hat{\mathbf{z}}\delta\mathcal{H}/\delta\theta$, so Eq. (25) becomes the expected shallow water momentum equation

$$\partial_t(h\mathbf{u}) = -\nabla\cdot(h\mathbf{u}\mathbf{u}) - \frac{1}{2}\nabla(\theta h^2). \tag{26}$$

This form shows that θ is the spatially varying reduced gravity g, due to a spatially varying density jump across the layer [4]. Similarly, the evolution equation for $\mathbf{Q}$ simplifies to [19]

$$\partial_t\mathbf{Q} = \nabla\times\left(\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\times\mathbf{Q}\right), \tag{27}$$

or $-\partial_t\nabla\times(\theta\hat{\mathbf{z}}) = \nabla\times(\mathbf{u}\times(\hat{\mathbf{z}}\times\nabla\theta)) = \nabla\times(\mathbf{u}\cdot\nabla\theta\hat{\mathbf{z}})$, coinciding with the curl of Eqs. (3b) and (14b).

VII. CORIOLIS FORCE AND GEOSTROPHIC MOMENTUM

In the Hamiltonian formulations of Ripa's model and shallow water magnetohydrodynamics using $(h, \mathbf{u}, \theta)$ variables, the background rotation $\mathbf{\Omega}$ only appears as a contribution to the potential vorticity $q = h^{-1}(2|\mathbf{\Omega}| + \hat{\mathbf{z}} \cdot \nabla\times\mathbf{u})$ inside the Poisson bracket Eq. (6). The Lie–Poisson bracket Eq. (22) may be modified to include a Coriolis term by adding a similar term

$$-\begin{pmatrix} 0 & -2h\Omega \\ 2h\Omega & 0 \end{pmatrix} \tag{28}$$

to the top left 2×2 block of the Poisson tensor. In suffix notation, the top left block becomes $-(m_j\partial_i + \partial_j m_i - 2h\Omega\epsilon_{ij})$, where ϵ_{ij} is the rank two alternating tensor with $\epsilon_{12} = -\epsilon_{21} = 1$.

Alternatively, the unmodified bracket may be used provided $\mathbf{m}$ is taken to be the "geostrophic momentum" $\mathbf{m} = h(\mathbf{u} + \mathbf{R})$, where $\mathbf{R}$ is a vector potential for the Coriolis force such that $\nabla\times\mathbf{R} = 2\mathbf{\Omega}$ [10, 22]. However, the Hamiltonian should still contain the kinetic energy $\frac{1}{2h}|\mathbf{m} - h\mathbf{R}|^2$ computed in the rotating frame, thus

$$\mathcal{H} = \int \frac{|\mathbf{m}|^2}{2h} - \mathbf{m}\cdot\mathbf{R} + \frac{1}{2}h|\mathbf{R}|^2 + \frac{1}{2}\theta h^2 \; dxdy \tag{29}$$

for Ripa's model, with variational derivatives

$$\frac{\delta\mathcal{H}}{\delta\mathbf{m}} = \frac{\mathbf{m}}{h} - \mathbf{R} = \mathbf{u}, \quad \frac{\delta\mathcal{H}}{\delta h} = \theta h - \mathbf{u}\cdot\mathbf{R} - \frac{1}{2}|\mathbf{u}|^2. \tag{30}$$

For a suitable choice of $\mathbf{R}$, the geostrophic momentum $\mathbf{m}$ then coincides with the momentum as calculated in a non-rotating inertial frame. For instance, taking $\mathbf{R} = -(f_0 y + \frac{1}{2}\beta y^2)\hat{\mathbf{x}}$ so that $\nabla\times\mathbf{R} = (f_0 + \beta y)\hat{\mathbf{z}}$ recovers Ripa's zonal momentum functional [4] for a β plane

$$\mathcal{M}_x = \int m_x \; dxdy = \int h(u - f_0 y - \frac{1}{2}\beta y^2) \; dxdy. \tag{31}$$

Functionals like $\mathcal{M}_x$ are useful for establishing stability results by the energy-Casimir method. For instance, $\mathcal{M}_x$ is conserved if the topography $h_0(\mathbf{x})$ and domain $\mathcal{D}$ are invariant under translations in x.

VIII. COMBINED THERMAL SHALLOW WATER MAGNETOHYDRODYNAMICS

The two different effects, horizontal temperature gradients and magnetic fields, may be combined in the same shallow water model with Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int h|\mathbf{u}|^2 + \frac{1}{h}|\nabla\psi|^2 + \theta h^2 \; dxdy, \tag{32}$$

where ψ is now used for the magnetic flux function following the usual convention in magnetohydrodynamics. The Poisson tensor becomes

$$\mathsf{J} = -\begin{pmatrix} 0 & -q & \partial_x & -h^{-1}\theta_x & -h^{-1}\psi_x \\ q & 0 & \partial_y & -h^{-1}\theta_y & -h^{-1}\psi_y \\ \partial_x & \partial_y & 0 & 0 & 0 \\ h^{-1}\theta_x & h^{-1}\theta_y & 0 & 0 & 0 \\ h^{-1}\psi_x & h^{-1}\psi_y & 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

by extending the semidirect product structure to two advected scalar functions and a density h. The evolution equations are

$$\partial_t h + \nabla\cdot(\mathbf{u}h) = 0, \tag{34a}$$
$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \tag{34b}$$
$$\partial_t\psi + \mathbf{u}\cdot\nabla\psi = 0, \tag{34c}$$
$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} = -\nabla(\theta h) + \frac{1}{2}h\nabla\theta + \mathbf{B}\cdot\nabla\mathbf{B}, \tag{34d}$$

and the potential vorticity equation for q becomes

$$\partial_t q + \mathbf{u}\cdot\nabla q = h^{-1}\hat{\mathbf{z}}\cdot\left[\nabla\left(\frac{1}{h}\frac{\delta\mathcal{H}}{\delta\theta}\right)\times\nabla\theta + \nabla\left(\frac{1}{h}\frac{\delta\mathcal{H}}{\delta\psi}\right)\times\nabla\psi\right]. \tag{35}$$

There will be no potential vorticity conservation properties in general, because the interlocking lines of constant θ and lines of constant ψ determine unique Lagrangian labels, destroying any relabelling symmetry. Further properties of this combined system remain to be investigated, especially whether it provides an improved model for the solar tachocline, the physical system motivating Gilman's [13] introduction of shallow water magnetohydrodynamics.

IX. CONCLUSION

The inhomogeneous shallow water equations and shallow water magnetohydrodynamics both have Hamiltonian formulations in terms of an active scalar θ, either the potential temperature or the magnetic flux function. Starting from the unmodified shallow water equations in Hamiltonian form, the advection equation for θ uniquely determines the extra row in the Poisson tensor, and antisymmetry then determines the extra column. The two systems therefore share the same Poisson tensor and Poisson bracket. Dellar [12] arrived at Ripa's Poisson bracket by changing variables in the natural Lie–Poisson bracket for magnetohydrodynamics. Thus the Poisson bracket in Ripa's [4] form automatically inherits the Jacobi identity from an underlying Lie algebra [15]. This provides a much simpler and direct proof of the Jacobi identity than Ripa's original formal proof [4].

Ripa's [4] model may contain any number of layers, each with a thickness h_j, stratification θ_j, and horizontal velocity $\mathbf{u}_j$, and may include a spatially varying bottom topography. The shallow water-like system described above is only the simplest version, equivalent to a single active layer with a quiescent lower layer, a rigid lid, and no bottom topography [4, 17]. Ripa refers to this as a $1\frac{1}{2}$ layer model, the $\frac{1}{2}$ layer being the quiescent lower layer.

However, in suitably chosen variables the Poisson tensor takes a block diagonal form, with a 4×4 block equivalent to Eq. (6) involving the variables h_j, q_j and θ_j for each layer. Thus the transformation to Lie–Poisson form described above may be applied layer by layer to each 4×4 block. This formulation also offers a very compact derivation of Ripa's model, since the Hamiltonian is just the three dimensional integral of the kinetic plus gravitational potential energies [4],

$$\mathcal{H} = \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} \frac{1}{2}\overline{\rho}\,|\mathbf{u}_j|^2 + g\rho_j z \; dz, \tag{36}$$

where $z_j = \sum_{k=0}^{j} h_k$ is the height of the upper surface of layer j. The model could be extended to allow linear variations with z in each layer [23] by including additional scalar variables for the z derivatives.

Finally, the existence of a Lie–Poisson Hamiltonian structure implies an analogous Euler–Poincaré structure for Ripa's model. Euler–Poincaré structures are based on a Lagrangian functional and Hamilton's variational principle, instead of a Hamiltonian functional and a Poisson bracket [24]. In order to work with Eulerian variables, the variations in Hamilton's principle are restricted to those generated by a Lie algebra. This is the dual of the Lie algebra appearing in Eq. (21), because variational derivatives like $\delta\mathcal{F}/\delta\boldsymbol{\eta}$ occupy the dual space to the field variables $\boldsymbol{\eta}$, although the distinction is often glossed over by identifying the two spaces via an L_2 inner product as in Eq. (21). Thus the Euler–Poincaré approach to approximation, for example, is closer to the direct use of Hamilton's variational principle in Lagrangian variables, and offers a more direct route to generalised forms of Kelvin's circulation theorem analogous to Eq. (11). For example, Salmon [2, 25] derived the shallow water equations by substituting a columnar motion Ansatz, in which the horizonal fluid velocity is independent of z, into Hamilton's variational principle, but this computation had to be performed using Lagrangian variables.

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