Electromagnetic waves in lattice Boltzmann magnetohydrodynamics

P. J. Dellar

OCIAM, Mathematical Institute, 24–29 St Giles', Oxford, OX1 3LB, UK. *(Accepted 26 May 2010)*

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Abstract. - An existing lattice Boltzmann formulation of non-relativistic magnetohydrodynamics represents the magnetic field using a set of vector-valued distribution functions. By expressing the behaviour of this system using a basis of moments of the distribution functions, we derive an evolution equation for the electric field that coincides with the combination of Maxwell's equation, including the displacement current, and Ohm's law. Numerical experiments verify the propagation of electromagnetic waves radiated from an oscillating dipolar current.

Introduction. – The lattice Boltzmann approach to hydrodynamics has become a widely established alternative to conventional computational fluid dynamics that uses discrete approximations of the Navier–Stokes equations. Instead, the lattice Boltzmann approach formulates an approximation to the Boltzmann equation from the kinetic theory of gases that is easily discretised in space and time, leading to efficient and readily parallelisable algorithms [1–3]. Magnetohydrodynamics (MHD) is concerned with the behaviour of electrically conducting fluids and magnetic fields. The magnetic field is advected by, and diffuses through, the fluid, while itself exerting a force on the fluid. The fluid velocity is typically much smaller than the speed of light, so it is common to employ the quasi-static or MHD approximation and neglect Maxwell's displacement current [4]. This approximation was used in the design of the lattice Boltzmann MHD scheme that forms the topic of this Letter [5]. However, we show that the scheme in fact contains the full set of Maxwell's equations, including the displacement current. This greatly expands the range of phenomena that may be simulated using these schemes.

Lattice Boltzmann magnetohydrodynamics. – The equations for magnetohydrodynamics in an isothermal fluid with constant temperature θ are

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1a}$$
$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\theta\rho\mathbf{I} + \rho\mathbf{u}\mathbf{u}) = \mathbf{J}\times\mathbf{B} + \nabla\cdot\boldsymbol{\sigma}, \tag{1b}$$
$$\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \tag{1c}$$

where ρ is the fluid density, $\mathbf{u}$ the velocity, $\mathbf{I}$ the identity matrix, and $\boldsymbol{\sigma}$ the viscous stress. The magnetic field $\mathbf{B}$ exerts a Lorentz force $\mathbf{J}\times\mathbf{B}$ on the fluid. The evolution of $\mathbf{B}$ is given by combining Maxwell's equations,

$$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J}, \tag{2}$$

as simplified for non-relativistic motions ($|\mathbf{u}| \ll c$) by neglecting the displacement current, and the Ohm's law

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right). \tag{3}$$

Here σ is the conductivity, c the speed of light, μ_0 the permeability of free space, and $\eta = 1/(\sigma\mu_0)$ the resistivity. The momentum equation (1b) may be rewritten in conservation form by expressing the Lorentz force $\mathbf{J}\times\mathbf{B}$ as the divergence of the Maxwell stress. This is then readily implemented in lattice Boltzmann hydrodynamics [1–3] by changing the equilibrium momentum flux [5].

The evolution equation for $\mathbf{B}$ may be rewritten as

$$\partial_t\mathbf{B} + \nabla\cdot\Lambda = 0, \tag{4}$$

by introducing an electric field tensor Λ with components $\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}E_\gamma$. This cannot be derived from a Boltzmann equation with scalar distribution functions, as used in lattice Boltzmann hydrodynamics, because Λ would be required to be symmetric rather than antisymmetric [5].

The first lattice Boltzmann formulation of magnetohydrodynamics [6, 7] used distribution functions f_a^σ labelled by two indices that propagated with two different sets of particle velocities. This breaks the unwanted symmetry for Λ, and was inspired by the stochastic bidirectional streaming used in lattice gas cellular automata [8, 9]. A later variation by Mendoza & Muñoz [10] uses three sets of particle velocities, $\mathbf{v}_i^p$, $\mathbf{e}_{ij}^p$, and $\mathbf{b}_{ij}^p$, and two sets of distribution functions, f_i^p and G_{ij}^p. The particle velocities are related by $\mathbf{b}_{ij}^p = \mathbf{v}_i^p \times \mathbf{e}_{ij}^p$, and the velocity, electric, and magnetic fields are given by

$$\rho\mathbf{u} = \sum_{i,p} \mathbf{v}_i^p f_i^p, \ \mathbf{E} = \sum_{i,j,p} \mathbf{e}_{ij}^p G_{ij}^p, \ \mathbf{B} = \sum_{i,j,p} \mathbf{B}_{ij}^p G_{ij}^p. \quad (5)$$

This formulation was used to simulate Maxwell's equations coupled to a two-fluid plasma model. The current was given locally by $\mathbf{J} = \sum_s q_s n_s \mathbf{u}_s$ in terms of the two species' charges q_s, number densities n_s, and velocities $\mathbf{u}_s$.

Another approach writes $\mathbf{B} = \nabla\times(A_z\hat{\boldsymbol{z}})$ in two dimensions. The resulting advection-diffusion equation for A_z is readily incorporated into lattice gas cellular automata [11] and lattice Boltzmann schemes [1, 12]. The magnetic field is then recovered using finite difference approximations.

Dellar [5] introduced a separate set of distribution functions for the magnetic field alone. These distributions were allowed to take vector values, hence avoiding the symmetry problem, and postulated to evolve according to the vector Boltzmann equation

$$\partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau}\left(\mathbf{g} - \mathbf{g}_i^{(0)}\right), \quad (6)$$

where $i = 0, \ldots, N$. The $\boldsymbol{\xi}_i$ are a discrete set of particle velocities, such as those given in (34) below. The zeroth moment of (6) gives (4) for the magnetic field $\mathbf{B}$ and electric field tensor Λ given by

$$\mathbf{B} = \sum_{i=0}^{N} \mathbf{g}_i, \quad \Lambda = \sum_{i=0}^{N} \boldsymbol{\xi}_i \mathbf{g}_i. \quad (7)$$

The right hand side of (4) vanishes provided the equilibrium distributions $\mathbf{g}_i^{(0)}$ are chosen to satisfy

$$\sum_{i=0}^{N} \mathbf{g}_i^{(0)} = \mathbf{B}. \quad (8)$$

The combination of (6) and the equilibrium distributions

$$\mathbf{g}_i^{(0)} = w_i \left(\mathbf{B} + \theta^{-1}\boldsymbol{\xi}_i \cdot (\mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u})\right) \quad (9)$$

leads to the induction equation (1c) for resistive magnetohydrodynamics under a Chapman–Enskog expansion [5]. Each discrete velocity $\boldsymbol{\xi}_i$ has an associated weight w_i, as in (34) below, and the lattice constant θ is defined by

$$\sum_{i=0}^{N} w_i \boldsymbol{\xi}_i \boldsymbol{\xi}_i = \theta \mathbf{I}. \quad (10)$$

Moment equations for the magnetic distribution functions. – The analysis of lattice Boltzmann equations for hydrodynamics benefits from introducing a basis of moments of the distribution functions. The first few moments are hydrodynamic quantities such as ρ and $\mathbf{u}$. These may be completed in various ways to form a basis of moments [1, 13–16]. Evolution equations for the moments then offer a complete description of the lattice Boltzmann equation, one that leads naturally to a derivation of hydrodynamics. The collision operator is also more easily specified by its action on a basis of moments.

Dellar [17] introduced a similar basis of moments for the vector Boltzmann equation (6). The first two moments $\mathbf{B}$ and Λ are defined by (7). The electric field tensor Λ in turn evolves according to

$$\partial_t \Lambda + \nabla\cdot\mathsf{M} = \frac{1}{\tau}\left(\Lambda - \Lambda^{(0)}\right), \quad (11)$$

where the 3rd-rank tensor M is given by

$$\mathsf{M} = \sum_{i=0}^{N} \boldsymbol{\xi}_i \boldsymbol{\xi}_i \mathbf{g}_i. \quad (12)$$

The components $M_{\alpha\beta\gamma}$ with $\alpha \neq \beta$ vanish for the D2Q5 and D3Q7 lattices commonly used for the magnetic distribution functions [17]. The non-vanishing components of M, together with $\mathbf{B}$ and Λ, form a basis for the magnetic distribution functions. The non-vanishing components of M evolve according to (no implied summation on α)

$$\partial_t M_{\alpha\alpha\beta} + \partial_\alpha \Lambda_{\alpha\beta} = -\frac{1}{\tau}\left(M_{\alpha\alpha\beta} - M_{\alpha\alpha\beta}^{(0)}\right), \quad (13)$$

and the $\mathbf{g}_i$ may be reconstructed from the moments by [17]

$$\begin{aligned} g_{i\beta} &= \tfrac{1}{2}\left(\xi_{i\alpha}\Lambda_{\alpha\beta} + \xi_{i\gamma}\xi_{i\alpha}M_{\gamma\alpha\beta}\right) \text{ for } i \neq 0, \\ g_{0\beta} &= B_\beta - \left(M_{xx\beta} + M_{yy\beta} + M_{zz\beta}\right). \end{aligned} \quad (14)$$

Evolution of the electric field. – We use this system of moment equations to investigate the behaviour of the electric field tensor Λ in more detail than the earlier calculations that led merely to resistive magnetohydrodynamics [5, 17]. The components $\Lambda_{\alpha\beta}$ evolve according to

$$\partial_t \Lambda_{\alpha\beta} + \partial_\gamma M_{\gamma\alpha\beta} = -\frac{1}{\tau}\left(\Lambda_{\alpha\beta} - \Lambda_{\alpha\beta}^{(0)}\right). \quad (15)$$

The tensor Λ does not remain antisymmetric as it evolves, because M is not antisymmetric on its last two indices [17]. The physical electric field vector must thus be reconstructed from the antisymmetric component of Λ through $E_\gamma = -\frac{1}{2}\epsilon_{\gamma\alpha\beta}\Lambda_{\alpha\beta}$. This vector $\mathbf{E}$ evolves according to

$$\partial_t E_\gamma - \tfrac{1}{2}\epsilon_{\gamma\alpha\beta}\partial_\mu M_{\mu\alpha\beta} = -\frac{1}{\tau}\left(E_\gamma - E_\gamma^{(0)}\right). \quad (16)$$

So far we have retained the BGK or single-relaxation-time collision operator on the right hand side of the vector Boltzmann equation (6). However, we are free to choose a collision operator that assigns a relaxation time to M that is much shorter than the relaxation time for Λ, and hence for $\mathbf{E}$. To sufficient accuracy, we may then take

$$M_{\mu\alpha\beta} = M_{\mu\alpha\beta}^{(0)} = \theta\,\delta_{\mu\alpha}B_\beta. \quad (17)$$

This is justified more formally in the next section.

Substituting (17) into (16) gives

$$\partial_t \mathbf{E} - \tfrac{1}{2}\theta\nabla\times\mathbf{B} = -\frac{1}{\tau}\big(\mathbf{E} - \mathbf{E}^{(0)}\big). \qquad (18)$$

Further substituting $\mathbf{E}^{(0)} = -\mathbf{u}\times\mathbf{B}$, we obtain

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \frac{1}{c^2\tau}\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right), \qquad (19)$$

with the speed of light given by $c = (\frac{1}{2}\theta)^{1/2}$. We have thus recovered the combination of Maxwell's equation

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \mu_0 \mathbf{J}, \qquad (20)$$

including the displacement current term, and Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \qquad (21)$$

with conductivity

$$\sigma = \frac{1}{\mu_0 c^2 \tau} = \frac{\epsilon_0}{\tau}. \qquad (22)$$

Here ϵ_0 is the permittivity of free space, and $\epsilon_0\mu_0 = c^{-2}$.

Chapman–Enskog expansion. – More formally, (19) may be derived from a Chapman–Enskog expansion that treats $\mathbf{B}$ and Λ on an equal basis. The resistive MHD induction equation follows from posing the multiple scales expansion [5]

$$\mathbf{g}_i = \mathbf{g}_i^{(0)} + \tau\mathbf{g}_i^{(1)} + \cdots, \quad \partial_t = \partial_{t_0} + \tau\partial_{t_1} + \cdots, \qquad (23)$$

together with the solvability condition

$$\sum_{i=0}^{N} \mathbf{g}_i^{(n)} = 0, \text{ for } n = 1, 2, \ldots. \qquad (24)$$

In other words, the higher terms $\mathbf{g}_i^{(n)}$ for $n \geq 1$ make no contribution to the magnetic field $\mathbf{B}$.

The combination of the expansion (23) and the solvability condition (24) is equivalent to expanding the moments

$$\Lambda = \Lambda^{(0)} + \tau\Lambda^{(1)} + \cdots, \quad \mathsf{M} = \mathsf{M}^{(0)} + \tau\mathsf{M}^{(1)} + \cdots, \qquad (25)$$

while leaving $\mathbf{B}$ unexpanded. One then recovers the resistive MHD induction equation from the leading-order approximation to the evolution equation for Λ,

$$\Lambda^{(1)} = -\tau\left(\partial_{t_0}\Lambda^{(0)} + \nabla\cdot\mathsf{M}^{(0)}\right). \qquad (26)$$

In terms of van Kampen's theory for the elimination of fast variables, $\mathbf{B}$ is a slow variable while Λ and M are fast [18–20]. The elimination procedure, which is equivalent to the multiple-scales approach, constructs a closed evolution equation for $\mathbf{B}$ by finding successively more accurate expressions for the fast variables Λ and M in terms of the slow variable $\mathbf{B}$ and its spatial derivatives.

Maxwell's equations follow from treating both $\mathbf{B}$ and Λ as unexpanded slow variables, while M alone is fast. This is equivalent to adding the extra solvability condition

$$\sum_{i=0}^{N} \boldsymbol{\xi}_i \mathbf{g}_i^{(n)} = 0, \text{ for } n = 1, 2, \ldots, \qquad (27)$$

so the higher terms $\mathbf{g}_i^{(n)}$ for $n \geq 1$ make no contribution to either Λ or $\mathbf{B}$. Equation (18) then follows from substituting the leading order term $\mathsf{M}^{(0)}$ of the expansion

$$\mathsf{M} = \mathsf{M}^{(0)} + \tau_M \mathsf{M}^{(1)} + \cdots, \qquad (28)$$

into the exact evolution equation (15) for Λ. This is closer to the derivation of isothermal hydrodynamics from a lattice Boltzmann equation, in which both ρ and $\mathbf{u}$ are taken to be slow variables, while the momentum flux is fast.

Numerical implementation. – Integrating the Boltzmann equation (6) along its characteristics for a timestep Δt using the trapezium rule [21] gives

$$\overline{\mathbf{g}}_i(\mathbf{x} + \boldsymbol{\xi}_i\Delta t, t + \Delta t) = \overline{\mathbf{g}}_i(\mathbf{x}, t) - \frac{\Delta t}{\tau + \frac{1}{2}\Delta t}\left(\overline{\mathbf{g}}_i(\mathbf{x}, t) - \mathbf{g}_i^{(0)}(\mathbf{x}, t)\right), \qquad (29)$$

where

$$\overline{\mathbf{g}}_i = \mathbf{g}_i - \frac{1}{2}\frac{\Delta t}{\tau}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right). \qquad (30)$$

This change of variables, given by He *et al.* [21] for lattice Boltzmann hydrodynamics, leads to a second-order accurate explicit scheme that is linearly stable for $\tau \geq 0$. The combination (29) and (30) is readily generalised to matrix collision operators by replacing each eigenvalue τ with $\tau + \frac{1}{2}\Delta t$. For a more general formula see ref. [22].

The antisymmetric part of the tensor Λ contains the electric field, via $E_\gamma = -\frac{1}{2}\epsilon_{\gamma\alpha\beta}\Lambda_{\alpha\beta}$. Although the equilibrium value $\Lambda^{(0)}$ is exactly antisymmetric, a non-zero symmetric component of Λ arises during evolution under (15). The trace of Λ is related to the $\nabla\cdot\mathbf{B} = 0$ constraint, while the remaining symmetric, traceless part has no obvious physical interpretation [17].

It is therefore useful to choose a collision operator in which both M and the symmetric part of Λ are relaxed towards equilibrium with very short relaxation times τ_M and τ_s, while the antisymmetric part of Λ is relaxed on the time τ determined by the conductivity σ using (22). This collision operator is most easily computed through its action on the moments [15, 22, 23]. In discrete form, the post-collisional values of the moments are given by

$$\overline{\mathbf{E}}' = \overline{\mathbf{E}} - \left(\overline{\mathbf{E}} - \overline{\mathbf{E}}^{(0)}\right)\frac{\Delta t}{\tau + \frac{1}{2}\Delta t}, \qquad (31)$$

$$\overline{\Lambda}'_{(s)} = \overline{\Lambda}_{(s)} - \overline{\Lambda}_{(s)}\frac{\Delta t}{\tau_s + \frac{1}{2}\Delta t}, \qquad (32)$$

$$\overline{\mathsf{M}}' = \overline{\mathsf{M}} - \left(\overline{\mathsf{M}} - \mathsf{M}^{(0)}\right)\frac{\Delta t}{\tau_M + \frac{1}{2}\Delta t}. \qquad (33)$$

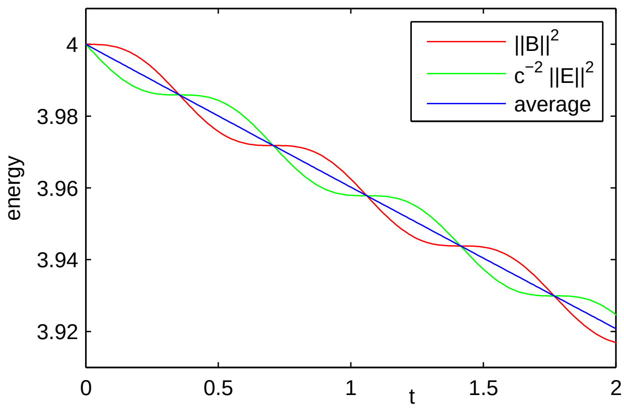

Fig. 1: Decaying electric and magnetic energies in a wave.

Here $\overline{\Lambda}_{(s)} = \frac{1}{2}\overline{\Lambda} + \frac{1}{2}\overline{\Lambda}^{\mathsf{T}}$ is the symmetric part of the tensor Λ. We then reassemble $\overline{\Lambda}'$ from $\overline{\mathbf{E}}'$ and $\overline{\Lambda}'_{(s)}$, and reconstruct the post-collisional distribution functions $\overline{\mathbf{g}}_i{}'$ from $\mathbf{B}$, $\overline{\Lambda}'$ and $\overline{\mathsf{M}}'$ using (14). This allows a slight refinement of the earlier analysis, in which $\mathbf{E}$ is kept as a slow variable while the symmetric part of Λ is fast.

The computations reported below were all performed using three copies of the D3Q7 lattice, one for each magnetic field component. The lattice velocity vectors are

$$\begin{aligned} \boldsymbol{\xi}_0 &= \mathbf{0}, \quad \boldsymbol{\xi}_1 = \hat{\boldsymbol{x}}, \quad \boldsymbol{\xi}_2 = \hat{\boldsymbol{y}}, \quad \boldsymbol{\xi}_3 = \hat{\boldsymbol{z}}, \\ \boldsymbol{\xi}_4 &= -\hat{\boldsymbol{x}}, \quad \boldsymbol{\xi}_5 = -\hat{\boldsymbol{y}}, \quad \boldsymbol{\xi}_6 = -\hat{\boldsymbol{z}}. \end{aligned} \tag{34}$$

and the weights are $w_0 = 1/4$ and $w_i = 1/8$ for $i \neq 0$. The lattice constant is $\theta = 1/4$. The algorithm is thus fully three-dimensional, even though the computations below are performed on $N \times 1 \times 1$ and $N \times N \times 1$ lattices with only one point in the suppressed dimensions.

Electromagnetic waves. – Taking the time derivative of (19) for a fluid at rest gives a telegraph equation [24] for the electric field after eliminating $\partial_t \mathbf{B} = -\nabla\times\mathbf{E}$,

$$\frac{1}{c^2}\partial_{tt}\,\mathbf{E} + \nabla\times\nabla\times\mathbf{E} = -\frac{1}{c^2\tau}\partial_t\,\mathbf{E}. \tag{35}$$

Taking the divergence of (35) shows that $\nabla\cdot\mathbf{E}$ remains zero if it is zero initially. In fact, plane electromagnetic waves have both $\mathbf{k}\cdot\mathbf{E} = 0$ and $\mathbf{k}\cdot\mathbf{B} = 0$. The dispersion relation for solutions proportional to $\exp[\mathrm{i}(kx - \omega t)]$ is

$$\omega = -\frac{\mathrm{i}}{2\tau} \pm \sqrt{k^2c^2 - \frac{1}{4\tau^2}}. \tag{36}$$

As $\tau \to \infty$ we recover the dispersion relation $\omega = \pm ck$ for undamped electromagnetic waves.

Figure 1 shows the interchange and decay of energy between the spatial averages $||\mathbf{B}||^2$ and $c^{-2}||\mathbf{E}||^2$ in a numerical simulation of an electromagnetic wave starting from the initial conditions $E_z = \sin(4\pi x)$ and $B_y = -E_z/c$. The total energy decays correctly in proportion to $\exp(-t/\tau)$. The simulation used 2048 points in the interval $[0, 1)$ with damping parameter $\tau = 100$ in dimensionless units where the speed of light $c = 8^{-1/2}$.

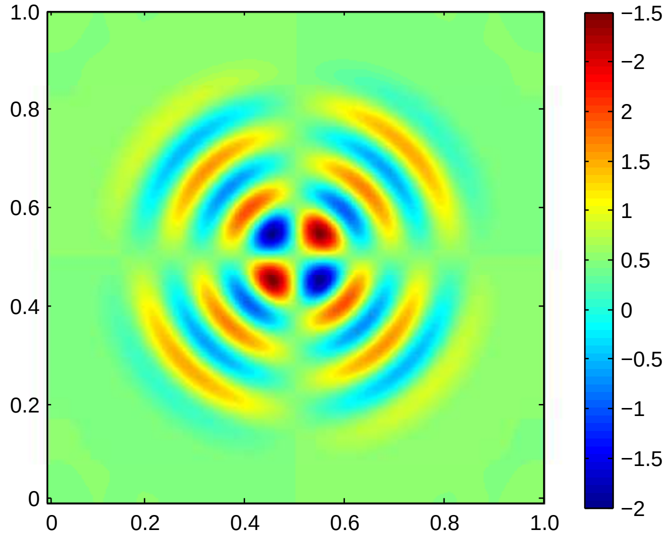

Fig. 2: Magnetic field component B_x at time $t = 1.0$

Radiation from an oscillating line dipole. – Next, we compute the electromagnetic field radiated by an oscillating current dipole extending along the z-axis,

$$\mu_0\mathbf{J} = \frac{2}{\sqrt{\pi}\,\ell^2}\,\frac{x}{\ell}\exp\left(-\frac{x^2+y^2}{\ell^2}\right)\Theta(t)\sin(\omega t)\,\hat{\boldsymbol{z}}. \tag{37}$$

This current distribution has $\nabla\cdot\mathbf{J} = 0$, so there are no accompanying charge oscillations, and is normalised so that the integral of $\mu_0 J_z$ over the half-plane $x > 0$ equals unity.

This current source was included by choosing the equilibrium electric field in the collision operator to be

$$\mathbf{E}^{(0)} = \mathbf{E} - (\tau/\epsilon_0)\,\mathbf{J}. \tag{38}$$

The evolution equation (18) for the electric field then coincides with Maxwell's equation (20) for the desired current distribution. The electric field $\overline{\mathbf{E}}$ defined by moments of the $\overline{\mathbf{g}}_i$ in the numerical implementation now differs from the physical electric field $\mathbf{E}$. The two are related by

$$\overline{\mathbf{E}} = \mathbf{E} + \tfrac{1}{2}(\Delta t/\epsilon_0)\,\mathbf{J}, \tag{39}$$

as given by a moment of (30). To achieve second-order accuracy we must reconstruct $\mathbf{E}$ from $\overline{\mathbf{E}}$ using this formula.

Figures 2 to 4 show the non-zero components B_x, B_y, and E_z of the electromagnetic field at dimensionless time $t = 1.0$. The domain is the unit square $x, y \in [0, 1)$ and $c = 8^{-1/2}$ as before. The fields shown were computed using the 3D lattice Boltzmann algorithm on $N \times N \times 1$ grids with parameters $\ell = 0.05$ and $\omega = 6\pi$. The other three components of the electromagnetic field remain 0 to round-off error. Figure 5 shows that these solutions

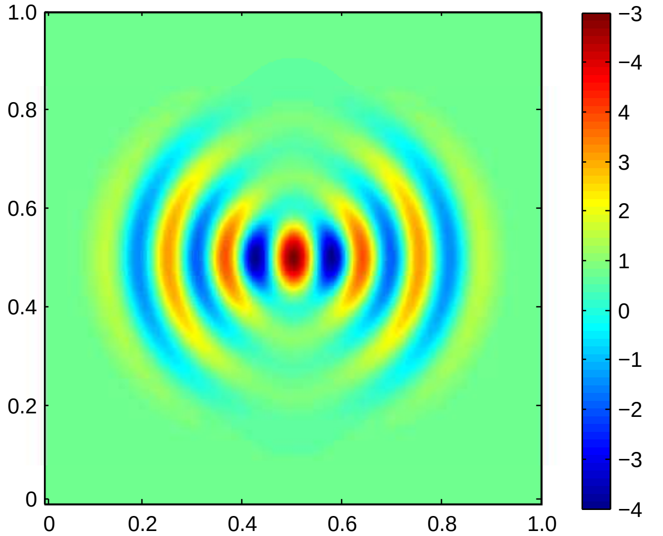

Fig. 3: Magnetic field component B_y at time $t = 1.0$

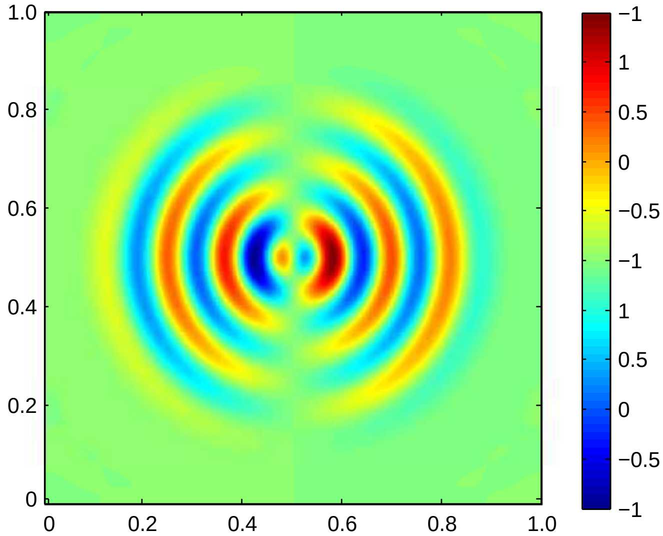

Fig. 4: Electric field component E_z at time $t = 1.0$

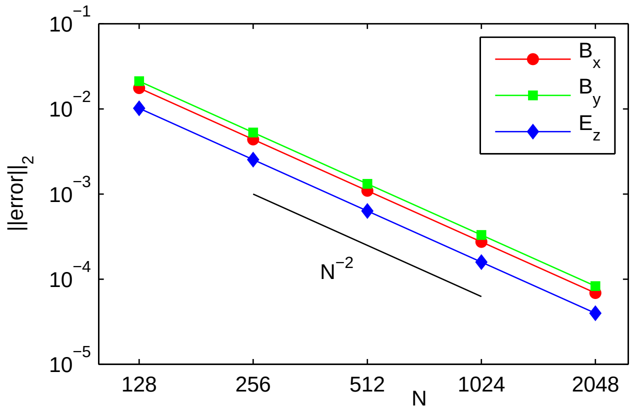

Fig. 5: Second order convergence of fields computed on $N \times N$ grids towards the spectrally accurate reference solutions.

converge with the expected second-order accuracy to reference solutions that are converged to 12 digits accuracy, and hence may be treated as exact in comparison.

Reference solutions. – The homogeneous Maxwell equations may be satisfied by writing $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$. If the scalar potential ϕ and vector potential $\mathbf{A}$ satisfy the Lorenz gauge condition $\nabla\cdot\mathbf{A} + (1/c^2)\partial_t\phi = 0$, the inhomogenous Maxwell equations decouple into separate wave equations for $\mathbf{A}$ and ϕ, [4]

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \nabla^2\mathbf{A} = \mu_0\mathbf{J}, \quad \frac{1}{c^2}\partial_{tt}\phi - \nabla^2\phi = \rho/\epsilon_0. \tag{40}$$

By Fourier transforming in space, and restricting the right hand side to have a time-dependence proportional to $\Theta(t)\sin(\omega t)$, where $\Theta(t)$ is the Heaviside step function, the solution of these wave equations may be expressed using the solution G of the ordinary differential equation

$$\frac{1}{c^2}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} + k^2 G = \sin(\omega t) \tag{41}$$

with initial conditions $G = G' = 0$ at $t = 0$. This solution is

$$G(t; k, c, \omega) = \begin{cases} \dfrac{1}{2k^2}\left(\sin(kct) - tkc\cos(kct)\right) & \text{if } \omega = kc, \\ \dfrac{c}{k}\,\dfrac{kc\sin(\omega t) - \omega\sin(kct)}{k^2c^2 - \omega^2} & \text{otherwise.} \end{cases} \tag{42}$$

The solution to the first of (40) may then be written as

$$\tilde{\mathbf{A}}(\mathbf{k}, t) = \mu_0\,\tilde{\mathbf{J}}_0(\mathbf{k})\,G(t; |\mathbf{k}|, c, \omega), \tag{43}$$

where $\tilde{\mathbf{A}}(\mathbf{k}, t)$ is the Fourier transform of $\mathbf{A}(\mathbf{x}, t)$, and $\tilde{\mathbf{J}}_0(\mathbf{k})$ is the Fourier transform of the *spatial*-only part of $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_0(\mathbf{x})\Theta(t)\sin(\omega t)$. The magnetic field is then computed from $\tilde{\mathbf{B}} = i\,\mathbf{k}\times\tilde{\mathbf{A}}$. The time derivative $\partial_t\tilde{\mathbf{A}}$ that appears in the electric field is given by

$$\partial_t\tilde{\mathbf{A}}(\mathbf{k}, t) = \mu_0\,\tilde{\mathbf{J}}_0(\mathbf{k})\,G'(t; |\mathbf{k}|, c, \omega), \tag{44}$$

where $G'(t; |\mathbf{k}|, c, \omega)$ is the time derivative of the function defined in (42). Accurate numerical solutions to Maxwell's equations may thus be computed using fast Fourier transforms (FFTs). In this example, solutions with 12 digits accuracy were achieved using just 128×128 Fourier modes.

Conclusion. – A lattice Boltzmann scheme that represents the magnetic field as the sum of a set of vector-valued distribution functions [5] has been widely used to simulate non-relativistic magnetohydrodynamics. The full behaviour of the distribution functions may be found from the evolution equations for a basis of moments [17]. We have shown that the evolution equation for the electric field, under a suitable collision operator, coincides completely with that given by Maxwell's equations, including Maxwell's displacement current, in combination with Ohm's law. We therefore recover the full relativistic set of

Maxwell's equations from a scheme designed only for non-relativistic MHD. This is essentially because a kinetic formulation must involve a first order hyperbolic system with source terms. Although derived for vacuum, the scheme may be extended to simulate media with varying permeability μ by adjusting the relation between $\mathsf{M}^{(0)}$ and $\mathbf{B}$.

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