Moment equations for magnetohydrodynamics

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Abstract. In kinetic treatments of hydrodynamics the macroscopic variables such as density and momentum are given by moments of distribution functions. In discrete kinetic theory it is possible to construct a complete set of moments whose evolution provides a complete description of the dynamics of the underlying kinetic equation. Moreover, the collision operator is most easily specified by its action upon moments. This paper presents the equivalent moment system for a kinetic formulation of magnetohydrodynamics that uses vector-valued distribution functions to represent the magnetic field. Besides leading to modest improvements in numerical stability, this moment system is invaluable for designing new kinetic equations to model more realistic plasma processes such as current-dependent resistivity.

J. Stat. Mech. (2009) paper P06003

Received 4 November 2008
Accepted 21 December 2008
Published 8 June 2009

Online at `http://stacks.iop.org/JSTAT/2009/P06003`

`doi:10.1088/1742-5468/2009/06/P06003`

1. Introduction

Magnetohydrodynamics is a continuum description of an electrically conducting fluid interacting with a magnetic field. The magnetic field exerts a Lorentz force on the fluid, while the magnetic field is itself advected by, and diffuses through, the fluid. This paper presents a development of the author's earlier lattice Boltzmann formulation of magnetohydrodynamics (Dellar 2002) that uses a set of vector-valued distribution functions to represent the magnetic field. We construct bases of moments that offer equivalent descriptions of the two- and three-dimensional vector Boltzmann equations. These are analogous to the bases of moment for lattice Boltzmann hydrodynamics (without a magnetic field) proposed by Vergassola et al. (1990), Frisch (1991), and d'Humières (1992). Having constructed a basis of moments, we may specify a collision operator through its action on the basis. This allows us to construct matrix collision operators for the magnetic field that are analogous to the multiple relaxation time (MRT) collision operators used in lattice Boltzmann hydrodynamics (Lallemand & Luo 2000, d'Humières et al. 2002). These matrix collision operators offer improvements in numerical stability over the single relaxation time magnetic collision operator used in Dellar (2002) and later work based on the same formulation. More importantly, a matrix collision operator for the magnetic distribution functions is necessary for simulating more realistic plasma models, such as a current-dependent resistivity.

In the following subsections we review the construction of moment bases in discrete hydrodynamics, the equations of magnetohydrodynamics, and the chief obstacle to their lattice Boltzmann formulation. Section 2 introduces the evolution equation for the vector-valued distribution functions, and reviews the derivation of resistive magnetohydrodynamics. Section 3 constructs a basis of moments in two spatial dimensions, which is extended to three dimensions in section 4. Sections 5 and 6 present a decomposition of the electric field tensor, and an evolution equation for its trace. Section 7 formulates several MRT collision operators for the magnetic distribution functions, and numerical experiments are presented in section 8.

1.1. Discrete kinetic theory for hydrodynamics

In the discrete kinetic approach to hydrodynamics, the key dependent variables are a set of distribution functions $f_i(\mathbf{x}, t)$ that evolve according to an equation of the form

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = \mathcal{C}[f_i], \tag{1}$$

for $i = 0, \ldots, N$. Each f_i is the number densities of particles propagating with the corresponding velocity $\boldsymbol{\xi}_i$. Macroscopic quantities like the fluid density ρ, velocity $\mathbf{u}$, and momentum flux $\boldsymbol{\Pi}$ are given by moments of the f_i with respect to the discrete velocities. The particle velocities are confined to the discrete set $\{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_N\}$, so these moments are expressed as sums,

$$\rho = \sum_{i=0}^{N} f_i, \quad \rho\mathbf{u} = \sum_{i=0}^{N} \boldsymbol{\xi}_i f_i, \quad \boldsymbol{\Pi} = \sum_{i=0}^{N} \boldsymbol{\xi}_i \boldsymbol{\xi}_i f_i, \tag{2}$$

instead of the integrals of continuum kinetic theory where $\boldsymbol{\xi}$ is a continuous variable.

Taking moments of (1) gives evolution equations for the moments of the f_i, the first of which are

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbf{\Pi} = 0. \tag{3}$$

The right hand sides vanish if we assume that collisions conserve mass and momentum, $\sum_{i=0}^{N} \mathcal{C}[f_i] = 0$ and $\sum_{i=0}^{N} \boldsymbol{\xi}_i \mathcal{C}[f_i] = 0$. Higher moments are typically not conserved by collisions. For example, $\mathbf{\Pi}$ evolves according to

$$\partial_t \mathbf{\Pi} + \nabla\cdot\left(\sum_{i=0}^{N} \boldsymbol{\xi}_i\boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i\right) = \sum_{i=0}^{N} \boldsymbol{\xi}_i\boldsymbol{\xi}_i \mathcal{C}[f_i]. \tag{4}$$

More generally, the $\boldsymbol{\xi}_i \cdot \nabla f_i$ term in (1) implies that the evolution of each moment involves the divergence of a higher moment. In continuum kinetic theory, where the particle velocity is a continous variable $\boldsymbol{\xi}$, one may define infinitely many independent moments of the distribution function $f(\mathbf{x}, \boldsymbol{\xi}, t)$, for instance moments with respect to Grad's (1949) tensor Hermite polynomials.

However, in discrete kinetic theory there can only be $N + 1$ independent degrees of freedom at each point $(\mathbf{x}, t)$, since there are $N + 1$ distribution functions $f_0, \ldots, f_N$. The process of taking higher and higher moments of the f_i must therefore terminate, in the sense that the higher moments may be expressed as linear combinations of the existing lower moments. In other words, one may introduce a basis of $N+1$ independent moments, such that the distribution functions, and hence all higher moments, may be expressed in terms of the basis.

The closed set of evolution equations for the $N+1$ moments in the basis then offers a full description of the evolution of the f_i that is equivalent to the discrete Boltzmann equation (1). While the advection of particles is described simply by the left hand side of (1), the derivation of hydrodynamics is simpler using moments. Moreover, the collision operator $\mathcal{C}[f_i]$ is most easily studied through its action on moments. For example, $\mathcal{C}$ must annihilate the moments ρ and $\rho\mathbf{u}$ in a mass and momentum conserving theory. A linear collision operator, the kind commonly used in the lattice Boltzmann approach, is completely specified by its action on a basis.

The idea that one could understand the behaviour of a discrete Boltzmann equation like (1) through the evolution equations for a basis of moments appeared in papers by Vergassola et al. (1990), Frisch (1991), and d'Humières (1992). All three papers motivate their choices of moments using orthogonality, but differ in their inner products. Frisch (1991) and d'Humières (1992) use an unweighted (Euclidean) inner product, on hexagonal and D2Q9 square lattices respectively, while Vergassola et al. (1990) used a weighted inner product that assigned less weight to the diagonal directions of the D2Q9 lattice. As well the insight gained from replacing (1) with an equivalent system of evolution equations for moments, it is often advantageous to design the collision operator by specifying its action on a basis of moments. This approach has led to the so-called multiple relaxation time (MRT) collision operators. These operators offer improvements in numerical stability over the widely used single relaxation time or BGK collision operator.

1.2. Magnetohydrodynamics

Nonrelativistic magnetohydrodynamics is a continuum description of an electrically conducting fluid whose velocity is everywhere much smaller than the speed of light. We may thus neglect the time derivative of the electric field, or Maxwell's displacement current. The remaining Maxwell equations are

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad \mu_0\mathbf{J} = \nabla\times\mathbf{B}, \tag{5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{J}$ is the current, and μ_0 is the permittivity of free space. The electric field is now simply the flux that appears in the evolution equation for the magnetic field. It is related to other quantities by Ohm's law, which we take to be

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right), \tag{6}$$

with conductivity σ. Ohm's law arises from the electron momentum equation in kinetic treatments of plasmas. The version given here is one of the simplest, and omits many terms that become important in certain parameter regimes. For instance, it contains no time derivatives because the electrons' inertia has been neglected.

In theoretical treatments of magnetohydrodynamics it is common to absorb μ_0 into the definition of the magnetic field, so the Lorentz force becomes $(\nabla\times\mathbf{B})\times\mathbf{B}$. Ohm's law is then written as

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta\nabla\times\mathbf{B}, \tag{7}$$

with the resistivity η having the dimensions of a diffusivity (length^2/time). Combining equations (5) and (7) gives an evolution equation for the magnetic field,

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B} - \eta\nabla\times\mathbf{B}). \tag{8}$$

This is commonly rewritten as

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{9}$$

on the assumption that $\nabla\eta = 0$ (and also that $\nabla\cdot\mathbf{B} = 0$).

1.3. Discrete kinetic approaches to magnetohydrodynamics

The evolution equation for $\mathbf{B}$ may be rewritten using the divergence of an antisymmetric rank-2 tensor $\mathsf{\Lambda}$,

$$\partial_t \mathbf{B} + \nabla\cdot\mathsf{\Lambda} = 0, \quad \text{with } \Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}E_\gamma. \tag{10}$$

Our earlier evolution equation (3) for the momentum $\rho\mathbf{u}$, as derived from kinetic theory, contained the divergence of a tensor $\mathbf{\Pi} = \sum_i \boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i$ that is symmetric by construction. It is thus impossible to express the evolution of a magnetic field using scalar distribution functions analogous to the f_i used in hydrodynamics as described above.

Chen et al. (1991) introduced distribution functions f_a^σ that propagated with two different sets of particle velocities, $\mathbf{v}_a^\sigma$ and $\mathbf{B}_a^\sigma$. The two sets of velocities provide a continuum version of the stochastic bidirectional streaming used in lattice gas cellular automata by Chen et al. (1988) and Chen et al. (1992). Martínez et al. (1994) later reduced the number of variables by using a sparse set of distribution functions with f_a^σ identically zero for most combinations of a and σ. The macroscopic momentum and magnetic field in the latter model are expressed as

$$\rho\mathbf{u} = \sum_{a,\sigma} \mathbf{v}_a^\sigma f_a^\sigma, \quad \rho\mathbf{B} = \sum_{a,\sigma} \mathbf{B}_a^\sigma f_a^\sigma. \tag{11}$$

The electric field tensor is given by

$$\Lambda = \sum_{a,\sigma} \mathbf{v}_a^\sigma \mathbf{B}_a^\sigma f_a^\sigma, \tag{12}$$

which is no longer symmetric because $\mathbf{v}_a^\sigma$ and $\mathbf{B}_a^\sigma$ are different vectors. An extension of this approach by Mendoza & Munoz (2008) uses three sets of particle velocities, $\mathbf{v}_i^p$, $\mathbf{e}_{ij}^p$, and $\mathbf{b}_{ij}^p$, and two sets of distribution functions, f_i^p and G_{ij}^p. The particle velocities are related by $\mathbf{b}_{ij}^p = \mathbf{v}_i^p \times \mathbf{e}_{ij}^p$, and the velocity, electric, and magnetic fields are given by

$$\rho\mathbf{u} = \sum_{i,p} \mathbf{v}_i^p f_i^p, \quad \mathbf{E} = \sum_{i,j,p} \mathbf{e}_{ij}^p G_{ij}^p, \quad \mathbf{B} = \sum_{i,j,p} \mathbf{B}_{ij}^p G_{ij}^p. \tag{13}$$

Although the fields in this formulation contain three components, in the reported computations they only depended upon two spatial coordinates.

Another approach writes $\mathbf{B} = \nabla\times\mathbf{A}$ using a vector potential $\mathbf{A}$. In two dimensions, the induction equation reduces to an advection-diffusion equation for the z-component A_z, which is readily incorporated into a lattice gas cellular automaton (Montgomery & Doolen 1987) or a lattice Boltzmann scheme (Succi et al. 1991). In both cases the implementation of the Lorentz force requires a finite difference approximation to compute $\mathbf{B}$ from A_z.

In this paper we develop the author's earlier formulation of magnetohydrodynamics (Dellar 2002). This formulation uses a second, independent set of distribution functions to evolve the magnetic field. These distribution functions take vector values to avoid the symmetry restriction discussed under (10) above. The two sets of distribution functions are coupled only through macroscopic quantities, the fluid velocity and the magnetic field, at lattice points. This more modular approach makes it straightforward to adjust the fluid and magnetic Reynolds numbers independently, is easily implemented in three dimensions, and can be combined with any existing lattice Boltzmann scheme for three-dimensional hydrodynamics (Breyiannis & Valougeorgis 2004, Vahala et al. 2008, Pattison et al. 2008, Riley et al. 2008).

2. Discrete kinetic formulation using vector distribution functions

We express the magnetic field $\mathbf{B}$ as the sum of a set of vector-valued distribution functions $g_{i\beta}$,

$$B_\beta = \sum_{i=0}^{N} g_{i\beta}, \tag{14}$$

rather than by a first moment of some scalar distributions like fluid momentum. This enables us to circumvent the enforced symmetry of the second moment of a set of scalar distribution functions described above. The vector distribution functions are postulated to evolve according to the vector Boltzmann equation

$$\partial_t g_{i\beta} + \xi_{i\alpha}\partial_\alpha g_{i\beta} = -\frac{1}{\tau}\left(g_{i\beta} - g_{i\beta}^{(0)}\right). \tag{15}$$

The Bhatnagar, Gross & Krook (1954) or BGK collision operator on the right hand side will be generalised later.

Evolution equations for the moments of the $g_{i\beta}$ may be derived from (15). Summing equation (15) gives an evolution equation for $\mathbf{B}$ in the form

$$\partial_t B_\beta + \partial_\alpha \Lambda_{\alpha\beta} = 0, \tag{16}$$

where the electric tensor $\mathsf{\Lambda}$ is defined by

$$\Lambda_{\alpha\beta} = \sum_{i=0}^{N} \xi_{i\alpha} g_{i\beta}. \tag{17}$$

Equation (16) is intended to represent the Maxwell equation $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$, so we construct the collision operator to make the right hand side of (16) vanish. For the BGK collision operator above, we require

$$\sum_{i=0}^{N} g_{i\beta}^{(0)} = B_\beta. \tag{18}$$

Multiplying (15) by $\xi_{i\gamma}$ and summing gives an evolution equation for the tensor $\mathsf{\Lambda}$,

$$\partial_t \Lambda_{\alpha\beta} + \partial_\gamma M_{\gamma\alpha\beta} = -\frac{1}{\tau}\left(\Lambda_{\alpha\beta} - \Lambda_{\alpha\beta}^{(0)}\right), \tag{19}$$

where the tensor M defined by

$$M_{\gamma\alpha\beta} = \sum_{i=0}^{N} \xi_{i\gamma}\xi_{i\alpha} g_{i\beta}, \tag{20}$$

is symmetric on its first two indices, $M_{\gamma\alpha\beta} = M_{\alpha\gamma\beta}$. The superscript (0) on $\Lambda_{\alpha\beta}^{(0)}$ indicates that is is evaluated for the equilibrium distribution, and similarly for other moments,

$$\Lambda_{\alpha\beta}^{(0)} = \sum_{i=0}^{N} \xi_{i\alpha} g_{i\beta}^{(0)}, \quad M_{\alpha\gamma\beta}^{(0)} = \sum_{i=0}^{N} \xi_{i\alpha}\xi_{i\gamma} g_{i\beta}^{(0)}. \tag{21}$$

2.1. Derivation of resistive magnetohydrodynamics

The equilibria given in Dellar (2002) are

$$g_{i\beta}^{(0)} = w_i \left[B_\beta + \theta^{-1} \xi_{i\alpha} \left(u_\alpha B_\beta - B_\alpha u_\beta \right) \right]. \tag{22}$$

on the D2Q5 lattice with the weights $w_0 = 1/3$ and $w_{1,2,3,4} = 1/6$, for which $\theta = 1/3$ (see Appendix). The first few moments of the equilibria are

$$\sum_{i=0}^{N} g_{i\beta}^{(0)} = B_\beta, \quad \Lambda^{(0)} = u_\alpha B_\beta - B_\alpha u_\beta, \quad M^{(0)} = \theta\, \delta_{\gamma\alpha} B_\beta. \tag{23}$$

Substituting the equilibrium value $\mathsf{\Lambda}^{(0)}$ into (16) therefore gives the induction equation for ideal magnetohydrodynamics,

$$\partial_t B_\beta + \partial_\alpha \Lambda_{\alpha\beta}^{(0)} = 0, \quad \text{or} \quad \partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}). \tag{24}$$

The first correction $\Lambda^{(1)}$ may be obtained using a Chapman–Enskog multiple-scales expansion,

$$g_{i\beta} = g_{i\beta}^{(0)} + g_{i\beta}^{(1)} + \cdots, \quad \partial_t = \partial_{t_0} + \partial_{t_1} + \cdots. \tag{25}$$

These expansions are subject to the solvability condition that $g_{i\beta}^{(1)}$ and higher do not contribute to the magnetic field,

$$\sum_{i=0}^{N} g_{i\beta}^{(n)} = 0 \text{ for } n = 1, 2, \ldots. \tag{26}$$

The combination of the expansion of the $g_{i\beta}$ with the solvability condition is equivalent to expanding the higher moments,

$$\mathsf{\Lambda} = \mathsf{\Lambda}^{(0)} + \mathsf{\Lambda}^{(1)} + \cdots, \quad \mathsf{M} = \mathsf{M}^{(0)} + \mathsf{M}^{(1)} + \cdots \tag{27}$$

and so on, while leaving $\mathbf{B}$ unexpanded.

Substituting these expansions into the evolution equation (19) for $\mathsf{\Lambda}$ and truncating at leading order gives

$$\Lambda_{\alpha\beta}^{(1)} = -\tau \left(\partial_{t_0} \Lambda_{\alpha\beta}^{(0)} + \partial_\gamma M_{\gamma\alpha\beta}^{(0)} \right). \tag{28}$$

This simplifies to (Dellar 2002),

$$\Lambda_{\alpha\beta}^{(1)} = -\theta\tau \partial_\alpha B_\beta - \tau \partial_{t_0} \left(u_\alpha B_\beta - B_\alpha u_\beta \right) = -\theta\tau \partial_\alpha B_\beta + O(\mathrm{Ma}^3), \tag{29}$$

so the vector Boltzmann equation with the above equilibria solves the resistive MHD induction equation in the form

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{30}$$

with resistivity $\eta = \theta\tau$.

3. Moment system for D2Q5 MHD

The most common implementation of the above scheme uses a lattice with five velocities in two dimensions, as shown in figure 1. Two copies of this lattice are used, one for each component of the magnetic field. It is simplest to study the so-called D2Q5 lattice for a scalar distribution function first.

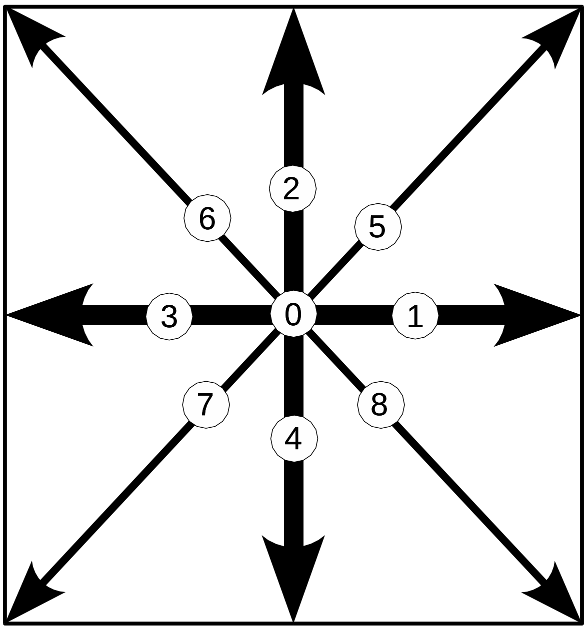

Figure 1. Two-dimensional discrete velocity lattice. The velocities in the D2Q5 lattice for the magnetic distribution functions are shown as thick lines. The additional velocities making up the D2Q9 lattice for the hydrodynamic distribution functions are shown as thin lines (see section 7).

3.1. Moments of the D2Q5 scalar lattice

Following the hydrodynamical terminology established in the Introduction, we consider a set of scalar distribution functions f_i for $i = 0, \ldots, 4$. The first few moments are given by

$$\rho = \sum_{i=0}^{4} f_i, \quad m_x = \sum_{i=0}^{4} \xi_{ix} f_i, \quad m_y = \sum_{i=0}^{4} \xi_{iy} f_i, \tag{31a}$$

$$\Pi_{xx} = \sum_{i=0}^{4} \xi_{ix}\xi_{ix} f_i, \quad \Pi_{yy} = \sum_{i=0}^{4} \xi_{iy}\xi_{iy} f_i. \tag{31b}$$

The quantity Π_{xy} is identically zero, because $\xi_{ix}\xi_{iy} = 0$ for every velocity in the lattice. In matrix notation,

$$\begin{pmatrix} \rho \\ m_x \\ m_y \\ \Pi_{xx} \\ \Pi_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}. \tag{32}$$

The 5×5 matrix above has full rank, its rows are linearly independent. The five moments defined in (31) therefore form a basis. The five f_i may be reconstructed uniquely from the values of these moments by inverting the above 5×5 matrix to obtain

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \rho \\ m_x \\ m_y \\ \Pi_{xx} \\ \Pi_{yy} \end{pmatrix}. \tag{33}$$

Treating f_0 as a special case, the distribution functions may be written compactly as

$$f_i = \tfrac{1}{2} \left(\boldsymbol{\xi}_i \cdot \mathbf{m} + \boldsymbol{\xi}_i \boldsymbol{\xi}_i : \boldsymbol{\Pi} \right) \text{ for } i \neq 0, \quad f_0 = \rho - (\Pi_{xx} + \Pi_{yy}). \tag{34}$$

This expression differs from the form (22) for the equilibria $g_{i\beta}^{(0)}$ that contained the weights w_i. As shown in the appendix, extending the moments for ρ and $\mathbf{m}$ to form an orthogonal basis requires linear combinations of Π_{xx} and Π_{yy}. It therefore seems preferable to use the nonorthogonal basis given by equations (31).

All higher moments may now be expressed in terms of the five moments defined above. For example, $\xi_{ix}^3 = \xi_{ix}$ for every velocity in the lattice, so

$$\sum_{i=0}^{4} \xi_{ix}\xi_{ix}\xi_{ix} f_i = \sum_{i=0}^{4} \xi_{ix} f_i = m_x. \tag{35}$$

Similarly, $\xi_{iy}^3 = \xi_{iy}$ while $\xi_{ix}^2 \xi_{iy} = \xi_{ix}\xi_{iy}^2 = 0$.

3.2. A basis of moments for D2Q5 MHD

Two copies of the D2Q5 lattice makes for ten degrees of freedom, in g_{ix} for $i = 0, \ldots, 4$ and g_{iy} for $i = 0, \ldots, 4$. Therefore we must find ten linearly independent moments to construct a basis.

The first two moments are the two components of the magnetic field,

$$B_x = \sum_{i=0}^{4} g_{ix}, \quad \text{and} \quad B_y = \sum_{i=0}^{4} g_{iy}. \tag{36}$$

Another four moments are given by the four components of the electric field tensor,

$$\Lambda_{xx} = \sum_{i=0}^{4} \xi_{ix} g_{ix},\ \Lambda_{xy} = \sum_{i=0}^{4} \xi_{ix} g_{iy},\ \Lambda_{yx} = \sum_{i=0}^{4} \xi_{iy} g_{ix},\ \Lambda_{yy} = \sum_{i=0}^{4} \xi_{iy} g_{iy}. \tag{37}$$

The evolution of $\mathsf{\Lambda}$ involves the third rank tensor

$$M_{\gamma\alpha\beta} = \sum_{i=0}^{4} \xi_{i\gamma} \xi_{i\alpha} g_{i\beta}, \tag{38}$$

as defined in (20) above. However, $\xi_{i\alpha}\xi_{i\gamma} = 0$ when $\alpha \neq \gamma$ for the D2Q5 lattice, so four of the eight possible components of M vanish, $M_{xyx} = M_{xyy} = M_{yxx} = M_{yxy} = 0$.

Our basis therefore comprises the ten quantities

$$B_x, B_y, \quad \Lambda_{xx}, \Lambda_{xy}, \Lambda_{yx}, \Lambda_{yy}, \quad M_{xxx}, M_{xxy}, M_{yyx}, M_{yyy}, \tag{39}$$

made up of the two components of $\mathbf{B}$, the four components of $\mathsf{\Lambda}$, and the four nonzero components of M. The vector distribution functions may be reconstructed from these moments by applying (34) to each component,

$$g_{i\beta} = \tfrac{1}{2}\left(\xi_{i\alpha}\Lambda_{\alpha\beta} + \xi_{i\gamma}\xi_{i\alpha}M_{\gamma\alpha\beta}\right) \text{ for } i \neq 0, \quad g_{0\beta} = B_\beta - (M_{xx\beta} + M_{yy\beta})\,. \quad (40)$$

3.3. A closed set of evolution equations

To obtain a complete description of the dynamics, we must express the evolution of the four nonzero components of M in terms of the lower moments. Multiplying the vector Boltzmann equation (15) by $\xi_{i\gamma}\xi_{i\mu}$ and summing, we obtain the generic evolution equation for the tensor M,

$$\partial_t M_{\gamma\alpha\beta} + \partial_\mu N_{\mu\gamma\alpha\beta} = -\frac{1}{\tau}\left(M_{\gamma\alpha\beta} - M^{(0)}_{\gamma\alpha\beta}\right), \quad (41)$$

which holds for any lattice. The fourth rank tensor N has components

$$N_{\mu\gamma\alpha\beta} = \sum_{i=0}^{N} \xi_{i\mu}\xi_{i\gamma}\xi_{i\alpha}g_{i\beta}, \quad (42)$$

and so is completely symmetric on its first three indices.

Specialising to the D2Q5 lattice, we note that $\xi_{i\mu}\xi_{i\gamma}\xi_{i\alpha} = 0$ unless $\mu = \gamma = \alpha$, so

$$N_{xxxx} = \sum_{i=0}^{4} \xi_{ix}g_{ix} = \Lambda_{xx}, \quad N_{xxxy} = \sum_{i=0}^{4} \xi_{ix}g_{iy} = \Lambda_{xy}, \quad (43a)$$

$$N_{yyyx} = \sum_{i=0}^{4} \xi_{iy}g_{ix} = \Lambda_{yx}, \quad N_{yyyy} = \sum_{i=0}^{4} \xi_{iy}g_{iy} = \Lambda_{yy}, \quad (43b)$$

with all other components vanishing.

We therefore have

$$\partial_t M_{xxx} + \partial_x \Lambda_{xx} = -\frac{1}{\tau}\left(M_{xxx} - M^{(0)}_{xxx}\right), \quad (44a)$$

$$\partial_t M_{xxy} + \partial_x \Lambda_{xy} = -\frac{1}{\tau}\left(M_{xxy} - M^{(0)}_{xxy}\right), \quad (44b)$$

$$\partial_t M_{yyx} + \partial_y \Lambda_{yx} = -\frac{1}{\tau}\left(M_{yyx} - M^{(0)}_{yyx}\right), \quad (44c)$$

$$\partial_t M_{yyy} + \partial_y \Lambda_{yy} = -\frac{1}{\tau}\left(M_{yyy} - M^{(0)}_{yyy}\right), \quad (44d)$$

which, together with the evolution equations for $\mathbf{B}$ and $\mathsf{\Lambda}$, forms a closed system.

The equations for $\mathsf{\Lambda}$ and M alone appear to form a closed system with eight degrees of freedom. However, $\mathbf{B}$ couples to the evolution equations for M through $\mathsf{M}^{(0)}$, since $M^{(0)}_{\gamma\alpha\beta} = \theta\delta_{\gamma\alpha}B_\beta$ for the equilibrium distributions given above.

4. Extension to the D3Q7 lattice

The same approach extends easily to three dimensions and the D3Q7 lattice. The particle velocities are again aligned with the coordinate axes,

$$\boldsymbol{\xi}_0 = \mathbf{0},\ \ \boldsymbol{\xi}_1 = \hat{\mathbf{x}},\ \ \boldsymbol{\xi}_2 = \hat{\mathbf{y}},\ \ \boldsymbol{\xi}_3 = \hat{\mathbf{z}},\ \ \boldsymbol{\xi}_4 = -\hat{\mathbf{x}},\ \ \boldsymbol{\xi}_5 = -\hat{\mathbf{y}},\ \ \boldsymbol{\xi}_6 = -\hat{\mathbf{z}}. \tag{45}$$

A basis of moments for the scalar distribution functions f_i is given by

$$\begin{pmatrix} \rho \\ m_x \\ m_y \\ m_z \\ \Pi_{xx} \\ \Pi_{yy} \\ \Pi_{zz} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}. \tag{46}$$

Inverting the abov 7×7 matrix leads to a compact formula like (34),

$$f_i = \tfrac{1}{2} \left(\boldsymbol{\xi}_i \cdot \mathbf{m} + \boldsymbol{\xi}_i \boldsymbol{\xi}_i : \boldsymbol{\Pi} \right) \ \text{for } i \neq 0, \quad f_0 = \rho - \left(\Pi_{xx} + \Pi_{yy} + \Pi_{zz} \right). \tag{47}$$

In fact, the two formulae are identical if one writes $f_0 = \rho - \operatorname{Tr} \boldsymbol{\Pi}$.

Turning to the vector distribution functions, we need three copies of the D3Q7 lattice, making 21 degrees of freedom in the $g_{i\alpha}$. As before, these are made up from three degrees of freedom in the magnetic field $\mathbf{B}$, nine degrees of freedom in the tensor $\mathsf{\Lambda}$, and another nine degrees of the freedom in the nonzero components of the M tensor. These last are again the components for which the first two indices are equal,

$$M_{xxx},\, M_{xxy},\ \ M_{xxz},\quad M_{yyx},\ \ M_{yyy},\ \ M_{yyz},\quad M_{zzx},\ \ M_{zzy},\ \ M_{zzz}. \tag{48}$$

These moments evolve according to (no implied summation on α)

$$\partial_t M_{\alpha\alpha\beta} + \partial_\alpha \Lambda_{\alpha\beta} = -\frac{1}{\tau} \left(M_{\alpha\alpha\beta} - M^{(0)}_{\alpha\alpha\beta} \right), \tag{49}$$

the three-dimensional generalisation of (44). The reconstruction of the $g_{i\beta}$ from the moments is given by

$$\begin{aligned} g_{i\beta} &= \tfrac{1}{2} \left(\xi_{i\alpha} \Lambda_{\alpha\beta} + \xi_{i\gamma} \xi_{i\alpha} M_{\gamma\alpha\beta} \right) \ \text{for } i \neq 0, \\ g_{0\beta} &= B_\beta - \left(M_{xx\beta} + M_{yy\beta} + M_{zz\beta} \right). \end{aligned} \tag{50}$$

5. Decomposition of the tensor $\mathsf{\Lambda}$

This discrete kinetic formulation for the magnetic field was motivated by the equivalence of two different expressions for the evolution of the magnetic field,

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \quad \Leftrightarrow \quad \partial_t \mathbf{B} + \nabla \cdot \mathsf{\Lambda} = 0, \tag{51}$$

when $\mathsf{\Lambda}$ is a purely antisymmetric tensor with components $\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} E_\gamma$. The equilibrium value $\Lambda^{(0)}_{\alpha\beta} = u_\alpha B_\beta - B_\alpha u_\beta$ is indeed antisymmetric.

However, carrying out the Chapman–Enskog expansion to first order in section 2.1 gave

$$\begin{aligned}\Lambda_{\alpha\beta} &= \Lambda^{(0)}_{\alpha\beta} + \Lambda^{(1)}_{\alpha\beta} + O(\tau^2),\\ &= (u_\alpha B_\beta - B_\alpha u_\beta) - \theta\tau\partial_\alpha B_\beta - \tau\partial_{t_0}\left(u_\alpha B_\beta - B_\alpha u_\beta\right) + O(\tau^2),\end{aligned} \tag{52}$$

and the term $-\theta\tau\partial_\alpha B_\beta$ is not antisymmetric.

We should thus decompose $\mathsf{\Lambda}$ into its symmetric and antisymmetric parts, and identify the electric field with the antisymmetric part. Thus a more accurate statement is

$$E_\gamma = -\tfrac{1}{2}\epsilon_{\gamma\alpha\beta}\Lambda_{\alpha\beta}. \tag{53}$$

The remaining symmetric part, given by $\frac{1}{2}\left(\mathsf{\Lambda} + \mathsf{\Lambda}^{\mathsf{T}}\right)$ may be further decomposed into an isotropic part, proportional to $\mathrm{Tr}\,\mathsf{\Lambda}$, and a symmetric traceless tensor. Thus we may write

$$\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}E_\gamma + \tfrac{1}{3}\,\delta_{\alpha\beta}\Lambda_{\gamma\gamma} + \tfrac{1}{2}\left(\Lambda_{\alpha\beta} + \Lambda_{\beta\alpha} - \tfrac{2}{3}\delta_{\alpha\beta}\Lambda_{\gamma\gamma}\right). \tag{54}$$

This is the decomposition of a general rank-2 tensor that is irreducible under rotations.

6. Evolution of the trace of $\mathsf{\Lambda}$

Taking the trace of the Chapman–Enskog expansion (52) of the electric field tensor gives

$$\mathrm{Tr}\,\mathsf{\Lambda} = \Lambda_{\alpha\alpha} = -\theta\tau\nabla\cdot\mathbf{B} + O(\tau^2). \tag{55}$$

The trace of $\mathsf{\Lambda}$ therefore acts as a proxy for the $\nabla\cdot\mathbf{B} = 0$ constraint. In numerical experiments, $\mathrm{Tr}\,\mathsf{\Lambda}^{(0)} \approx 0$ is maintained up to numerical round-off error (Dellar 2002).

Collecting together the previous equations gives a system for the evolution of the six moments Λ_{xx}, Λ_{yy}, Λ_{zz}, M_{xx}, M_{yy}, M_{zz},

$$\partial_t\Lambda_{xx} + \partial_x M_{xxx} = -\frac{1}{\tau}\Lambda_{xx}, \tag{56a}$$

$$\partial_t M_{xxx} + \partial_x\Lambda_{xx} = -\frac{1}{\tau}\left(M_{xxx} - \theta B_x\right),, \tag{56b}$$

and similarly with x replaced by y or z. Taking the time derivative of (56a), eliminating $\partial_{xt}M_{xxx}$ using (56b), then eliminating $\partial_x M_{xxx}$ using (56a) again gives

$$\Lambda_{xx} + 2\tau\partial_t\Lambda_{xx} + \tau^2\partial_{tt}\Lambda_{xx} = -\tau\theta\,\partial_x B_x + \tau^2\partial_{xx}\Lambda_{xx}. \tag{57}$$

Adding this to the equivalent equations for Λ_{yy} and Λ_{xx} gives

$$(1+\tau\partial_t)^2\,\mathrm{Tr}\,\mathsf{\Lambda} = -\tau\theta\,\nabla\cdot\mathbf{B} + \tau^2\left(\partial_{xx}\Lambda_{xx} + \partial_{yy}\Lambda_{yy} + \partial_{zz}\Lambda_{zz}\right). \tag{58}$$

The differential operator $(1+\tau\partial_t)^2$ has homogeneous solutions $\exp(-t/\tau)$ and $t\exp(-t/\tau)$. The Chapman–Enskog expansion seeks solutions that vary slowly over timescales much longer than τ, for which it is valid to replace $(1+\tau\partial_t)^2 T$ by T.

Substituting the leading order approximation $\Lambda_{xx} = \Lambda^{(1)}_{xx} = -\tau\theta\partial_x B_x$, and similarly for Λ_{yy} and Λ_{zz}, gives

$$T = -\tau\theta\,\nabla\cdot\mathbf{B} - \tau^3\theta\left(\partial_{xxx}B_x + \partial_{yyy}B_y + \partial_{zzz}B_z\right) + O(\tau^4), \tag{59}$$

consistent with the findings of section 6 of Dellar (2002). The latter were derived by continuing the Chapman–Enskog expansion of the $g_{i\beta}$ to third order.

7. An algorithm for magnetohydrodynamics with a matrix collision operator for the magnetic field

Magnetohydrodynamics is the study of the coupled behaviour of electrically conducting fluids and magnetic fields. The magnetic field exerts a Lorentz force $\mathbf{J}\times\mathbf{B}$ on the fluid, as well as being advected by the fluid. Thus the complete set of equations describing isothermal magnetohydrodynamics with temperature θ and dynamic viscosity μ is

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{60a}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \theta\rho\,\mathbf{I}) = \mathbf{J}\times\mathbf{B} + \nabla\cdot\left(\mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}\right)\right), \tag{60b}$$

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}. \tag{60c}$$

The Lorentz force may be written as (minus) the divergence of the Maxwell stress, so the inviscid momentum equation takes the standard conservation form

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbf{\Pi}^{(0)} = 0, \text{ with } \mathbf{\Pi}^{(0)} = \theta\rho\,\mathbf{I} + \rho\mathbf{u}\mathbf{u} + \tfrac{1}{2}|\mathbf{B}|^2\,\mathbf{I} - \mathbf{B}\mathbf{B}. \tag{61}$$

The standard lattice Boltzmann formulation of isothermal hydrodynamics may therefore be readily adapted to include the Lorentz force by choosing the equilibria $f_i^{(0)}$ to satisfy

$$\sum_{i=0}^{N} f_i^{(0)} = \rho, \quad \sum_{i=0}^{N} \boldsymbol{\xi}_i f_i^{(0)} = \rho\mathbf{u}, \quad \sum_{i=0}^{N} \boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i^{(0)} = \mathbf{\Pi}^{(0)}. \tag{62}$$

Suitable equilibria for the D2Q9 lattice illustrated in figure 1 are given by

$$f_i^{(0)} = W_i\left[\rho\left(2 - \frac{3}{2}|\boldsymbol{\xi}_i|^2\right) + 3\,(\rho\mathbf{u})\cdot\boldsymbol{\xi}_i + \frac{9}{2}\,\mathbf{\Pi}^{(0)}\!:\!\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \frac{3}{2}\,\mathrm{Tr}\,\mathbf{\Pi}^{(0)}\right] \tag{63}$$

where $\mathbf{\Pi}^{(0)}$ is given by (61). The weights for the D2Q9 lattice are $W_0 = 4/9$, $W_{1,2,3,4} = 1/9$, and $W_{5,6,7,8} = 1/36$, and the temperature $\theta = 1/3$ in lattice units. The equilibria in (63) coincide with the expressions given in Dellar (2002), and with the standard D2Q9 isothermal equilibria from Qian et al. (1992) when $\mathbf{B} = 0$.

The lattice Boltzmann algorithm for magnetohydrodynamics therefore involves the simultaneous solution of the two discrete Boltzmann equations

$$\partial_t f_i + \boldsymbol{\xi}_i\cdot\nabla f_i = -\frac{1}{\tau}\left(f_i - f_i^{(0)}\right), \quad \partial_t\mathbf{g}_i + \boldsymbol{\xi}_i\cdot\nabla\mathbf{g}_i = -\frac{1}{\tau_{\mathrm{b}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right). \tag{64}$$

The two equations are coupled because the $f_i^{(0)}$ contain $\mathbf{B} = \sum_i \mathbf{g}_i$, and the $\mathbf{g}_i^{(0)}$ contain $\mathbf{u} = (1/\rho)\sum_i \boldsymbol{\xi}_i f_i$.

Equations (64) are usually implemented computationally as

$$\overline{f}_i(\mathbf{x} + \boldsymbol{\xi}_i\Delta t, t + \Delta t) = \overline{f}_i(\mathbf{x}, t) - \frac{\Delta t}{\tau + \frac{1}{2}\Delta t}\left(\overline{f}_i(\mathbf{x}, t) - f_i^{(0)}(\mathbf{x}, t)\right), \tag{65a}$$

$$\overline{\mathbf{g}}_i(\mathbf{x} + \boldsymbol{\xi}_i\Delta t, t + \Delta t) = \overline{\mathbf{g}}_i(\mathbf{x}, t) - \frac{\Delta t}{\tau_{\mathrm{b}} + \frac{1}{2}\Delta t}\left(\overline{\mathbf{g}}_i(\mathbf{x}, t) - \mathbf{g}_i^{(0)}(\mathbf{x}, t)\right). \tag{65b}$$

These expressions were derived by integrating equations (64) along characteristics for a timestep Δt, followed by the change of variables proposed by He et al. (1998)

$$\overline{f}_i = f_i - \frac{1}{2}\frac{\Delta t}{\tau}\left(f_i - f_i^{(0)}\right), \quad \overline{\mathbf{g}}_i = \mathbf{g}_i - \frac{1}{2}\frac{\Delta t}{\tau_{\mathrm{b}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right). \tag{66}$$

Although these equations are linearly stable for any positive values of τ and τ_{b}, the $\overline{f}_i$ and $\overline{\mathbf{g}}_i$ oscillate around their equilibrium values when $\tau < \frac{1}{2}\Delta t$ or $\tau_{\mathrm{b}} < \frac{1}{2}\Delta t$. In other words, the $\overline{f}_i$ and $\overline{\mathbf{g}}_i$ are *over*-relaxed past equilibrium. This over-relaxation can trigger the onset of nonlinear instability.

Greater stability may be obtained in lattice Boltzmann simulations of hydrodynamics by noting that most discrete velocity lattices contain additional degrees of freedom, call non-hydrodynamic or "ghost" variables, alongside the density, momentum, and momentum flux. The momentum flux $\mathbf{\Pi}$ must be over-relaxed to achieve low viscosities, but the ghosts may be safely relaxed monotonically towards equilibrium. One may design a collision operator that applies a short relaxation time to the momentum flux, giving a low viscosity, but a longer relaxation time to the ghosts. This is the essence of the so-called multiple relaxation time (MRT) collision operators (Lallemand & Luo 2000, d'Humières et al. 2002). An earlier and simpler idea simply sets the ghosts to their equilibrium values after each collision (Higuera et al. 1989, McNamara et al. 1995). This corresponds to a relaxation time of $\frac{1}{2}\Delta t$ for the ghosts in the discrete formulation of (65).

Applying the second idea to magnetohydrodynamics, we relax the momentum flux towards its equilbrium value $\mathbf{\Pi}^{(0)}$,

$$\tilde{\mathbf{\Pi}} = \mathbf{\Pi} - \frac{\Delta t}{\tau + \frac{1}{2}\Delta t}\left(\mathbf{\Pi} - \mathbf{\Pi}^{(0)}\right), \tag{67}$$

then reconstruct the post-collision distribution functions $\tilde{f}_i$ from ρ, $\mathbf{u}$ and $\tilde{\mathbf{\Pi}}$ using

$$\tilde{f}_i = W_i\left[\rho\left(2 - \frac{3}{2}|\boldsymbol{\xi}_i|^2\right) + 3\,(\rho\mathbf{u})\cdot\boldsymbol{\xi}_i + \frac{9}{2}\,\tilde{\mathbf{\Pi}}\!:\!\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \frac{3}{2}\,\mathrm{Tr}\,\tilde{\mathbf{\Pi}}\right]. \tag{68}$$

Collisions conserve ρ and $\mathbf{u}$ so there are no tildes on these variables. Finally, we stream the post-collision distribution functions by setting

$$\overline{f}_i(\mathbf{x} + \boldsymbol{\xi}_i\Delta t, t + \Delta t) = \tilde{f}_i(\mathbf{x}, t). \tag{69}$$

The magnetic field only enters through the definition of $\mathbf{\Pi}^{(0)}$, everything else is as it would be in pure hydrodynamics.

So far, this is similar to the work of Pattison et al. (2008) and Riley et al. (2008). These authors used an MRT collision operator for the hydrodynamic distribution functions, but retained the BGK or single relaxation time collision operator of (65b) for the magnetic distribution functions.

However, exactly the same steps may be applied to the vector distribution functions that carry the magnetic field. We relax the electric field tensor towards its equilibrium value,

$$\tilde{\Lambda} = \Lambda - \frac{\Delta t}{\tau_{\mathrm{b}} + \frac{1}{2}\Delta t}\left(\Lambda - \Lambda^{(0)}\right), \tag{70}$$

reconstruct the post-collision distribution functions $\tilde{\mathbf{g}}_i$ from $\mathbf{B}$ and $\tilde{\mathsf{\Lambda}}$,

$$\tilde{\mathbf{g}}_i = w_i \left(\mathbf{B} + 3\,\boldsymbol{\xi}_i \cdot \tilde{\mathsf{\Lambda}}\right), \tag{71}$$

and stream

$$\overline{\mathbf{g}}_i(\mathbf{x} + \boldsymbol{\xi}_i \Delta t, t + \Delta t) = \tilde{\mathbf{g}}_i(\mathbf{x}, t). \tag{72}$$

The reconstruction (71) implicitly sets the moment M to its equilibrium value $\mathsf{M}^{(0)}$, just as (68) implicitly sets the three ghost variables on the D2Q9 lattice to their equilibrium values (which are zero in the approach of Dellar (2003)). More generally, we may also relax M towards its equilibrium value,

$$\tilde{\mathsf{M}} = \mathsf{M} - \frac{\Delta t}{\tau_{\rm m} + \frac{1}{2}\Delta t}\left(\mathsf{M} - \mathsf{M}^{(0)}\right), \tag{73}$$

using a relaxation time $\tau_{\rm m}$ that differs from $\tau_{\rm b}$, and then reconstruct using (40) or (50),

$$\mathbf{g}_i = \frac{1}{2}\left(\boldsymbol{\xi}_i \cdot \tilde{\mathsf{\Lambda}} + \boldsymbol{\xi}_i\boldsymbol{\xi}_i : \tilde{\mathsf{M}}\right) \text{ for } i \neq 0, \quad \mathbf{g}_0 = \mathbf{B} - \operatorname{Tr}\tilde{\mathsf{M}}. \tag{74}$$

In a slight abuse of notation, we write $\operatorname{Tr}\tilde{\mathsf{M}}$ for the vector with components $[\operatorname{Tr}\tilde{\mathsf{M}}]_\beta = \tilde{M}_{\alpha\alpha\beta}$ obtained by contracting $\tilde{\mathsf{M}}$ on its first two indices.

8. Numerical experiments

We tested the schemes described above in simulations of the reconnection of magnetic islands through the doubly-periodic coalescence instability, as described in Longcope & Strauss (1993), Marliani & Strauss (1999), and Dellar (2002). In two-dimensional incompressible magnetohydrodynamics it is convenient to express the velocity and magnetic field as $\mathbf{u} = (-\partial_y\varphi, \partial_x\varphi, 0)$ and $\mathbf{B} = (-\partial_y\psi, \partial_x\psi, 0)$, where φ and ψ are the streamfunction and magnetic flux function respectively. These vector fields automatically satisfy $\nabla\cdot\mathbf{u} = 0$ and $\nabla\cdot\mathbf{B} = 0$. The numerical experiments reported below began with the initial conditions

$$\psi = \sin(\pi(x+y))\sin(\pi(x-y)), \quad \varphi = 2\times 10^{-3}\exp(-10(x^2+y^2)), \tag{75}$$

in the doubly periodic domain $-1 \leq x, y \leq 1$. The initial magnetic field corresponds to an array of islands, with currents directed alternately into and out of the xy plane. The velocity perturbation disturbs the symmetry of the array, and neighbouring pairs of islands with aligned currents then attact each other, and eventually merge through resistive reconnection. The initial velocity perturbation above is 20 times larger than that used previously. This reduces the time spent in the initial linear phase of the instability.

Figure 2 shows the evolution of the magnetic field lines during a typical numerical experiment on a 128×128 grid with Mach number $\mathrm{Ma} = \sqrt{3}/64 \approx 0.027$, and diffusivities $\nu = \eta = 1/150$. The magnetic flux function ψ was reconstructed with spectral accuracy from the magnetic field components B_x and B_y at grid points, as described in Dellar (2002). Figure 3 shows the corresponding evolution of the peak current and peak

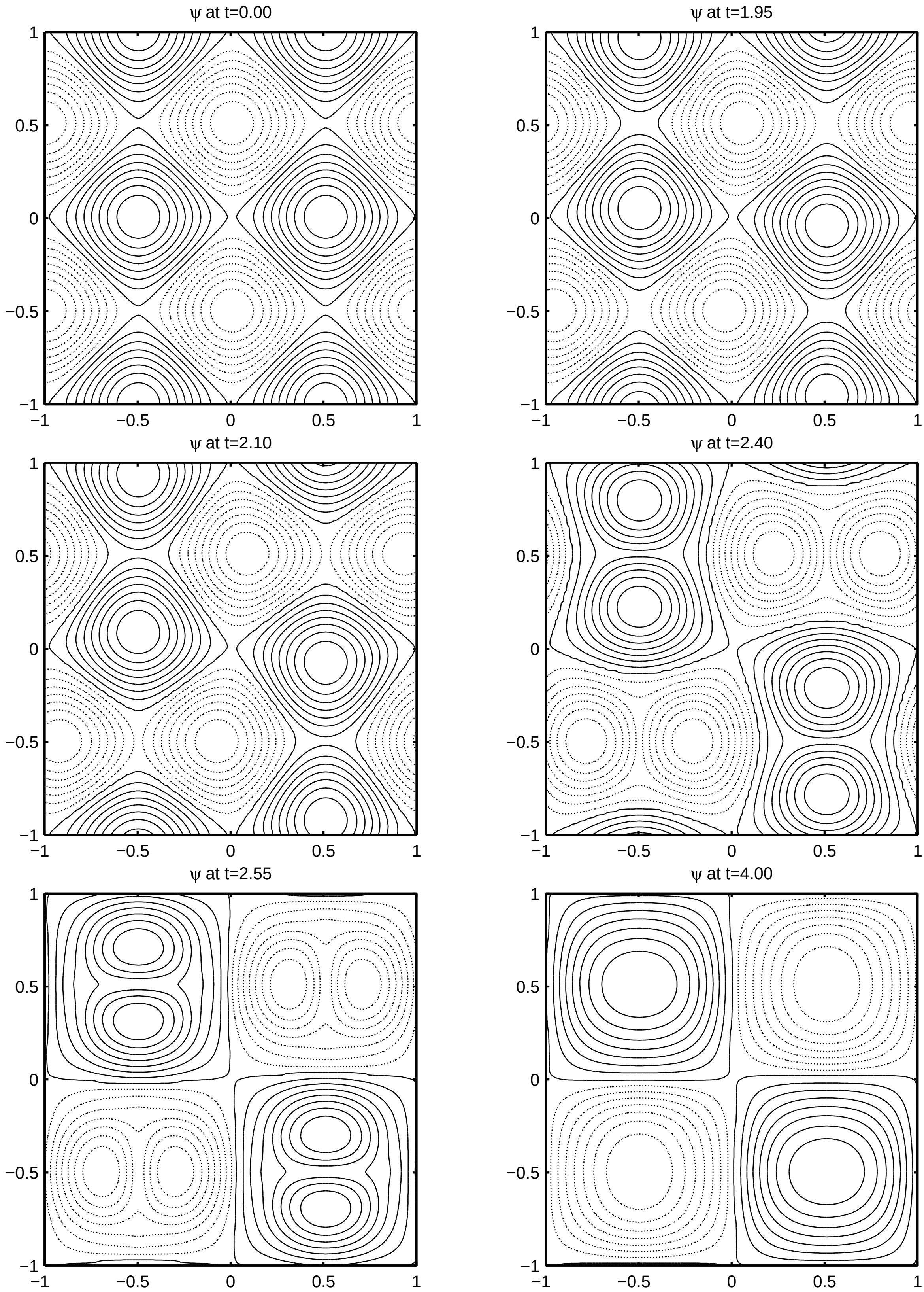

Figure 2. Evolution of the magnetic field lines during coalescence. Positive contours are shown solid (—) and negative contours are shown dotted (- - -).

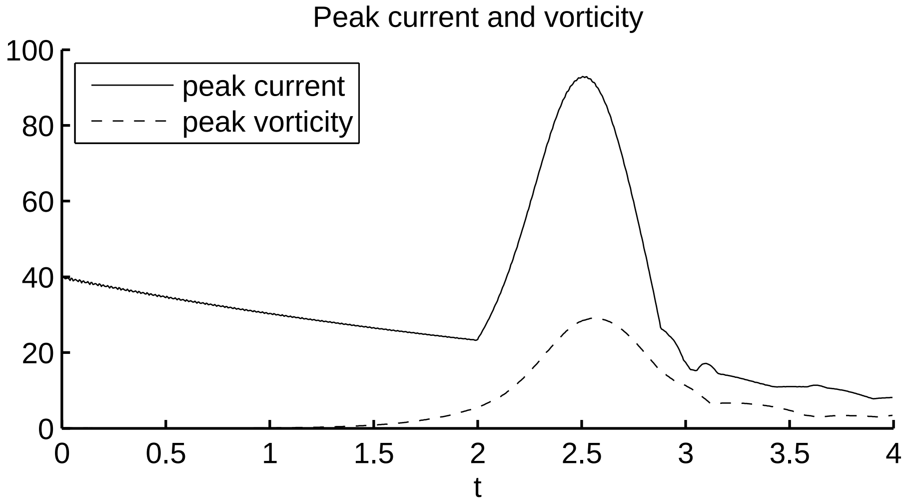

Figure 3. Evolution of the peak current and peak vorticity during coalescence with diffusivities $\nu = \eta = 1/150$.

vorticity. The nonlinear phase of the instability triggers a large growth in these peak values, corresponding to regions of intense current where the magnetic field lines are squeezed together before merging through resistive diffusion. The value $\nu = \eta = 1/150$ for the diffusivities is close to the stability limit on a 128^2 grid with BGK collision operators for both the hydrodynamic and the magnetic distribution functions.

Applying the multiple relaxation time (MRT) collision operator described above to the hydrodynamic distribution functions offers a large gain in stability. This collision operator sets the three ghost modes on the D2Q9 lattice to their equilibrium values (zero) at every timestep. The simulation then remains stable with $\nu = \eta = 1/575$, with the same Mach number and grid resolution as before, and with the same BGK collision operator applied to the magnetic distribution functions. While this offers close to a factor of four decrease in the diffusivities, it is worth emphasising that the experiments with $\nu = \eta = 1/575$ are stable on a 128^2 grid, but they are not converged. Figure 4 shows the peak current for numerical experiments on grids with 128^2, 256^2, 512^2, and 1024^2 points. The peak current on the 128^2 grid is artifically lowered by the finite spatial resolution.

Perhaps surprisingly, setting the M tensor to equilibrium at every timestep as well causes a substantial *reduction* in stability. The stability threshold is then around $\nu = \eta = 1/175$, which is only a small improvement over the original implementation using BGK collision operators for both the fluid and the magnetic distribution functions.

By carefully tuning the relaxation time for the M tensor in (73) it is possible to improve slightly upon the stability of the hybrid scheme that uses an MRT collision operator for the hydrodynamic distribution functions and the BGK collision operator for the magnetic distribution functions. Figure 5 shows the evolution of the peak current and vorticity for a simulation with $\nu = \eta = 1/600$, which was made stable by choosing

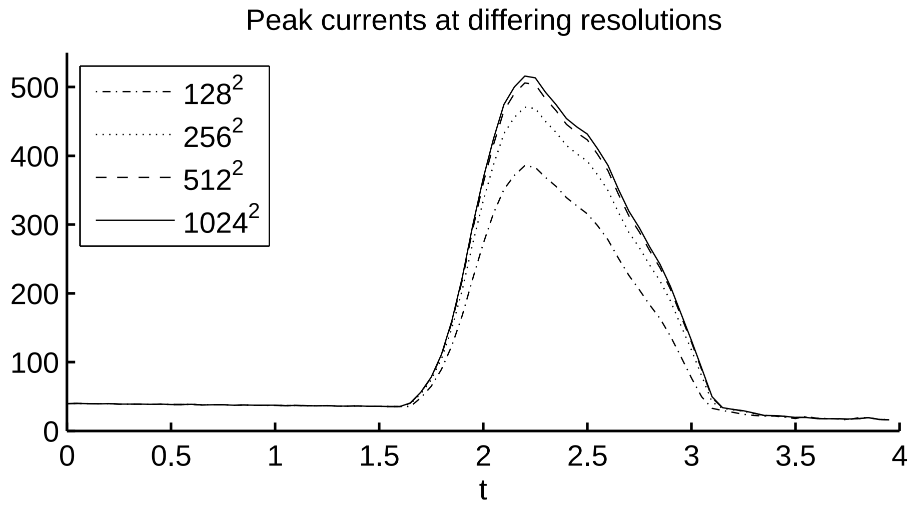

Figure 4. Evolution of the peak current with diffusivities $\nu = \eta = 1/575$ at various spatial resolutions. The 128^2 simulation is stable, but it is not well-resolved.

$\tau_{\mathrm{m}} = 0.05$ as the relaxation time for M. For comparison, the relaxation time for Λ was $\tau_{\mathrm{b}} = 0.005$ for this grid resolution and Mach number.

One small benefit of adopting an MRT collision operator for the magnetic distribution functions is a lowering of the already low value for $\nabla\cdot\mathbf{B}$, as measured by the trace of the electric field tensor. Figure 6 shows the evolution of the peak value of $\mathrm{Tr}\,\Lambda$, scaled in relation to the peak of $\Lambda_{xy} - \Lambda_{yx}$. The current is given by scaling $\Lambda_{xy} - \Lambda_{yx}$ with a constant factor that depends upon the collision time τ_{b}, the grid resolution, and the Mach number. The data in figure 6 therefore serve as a consistent approximation to the relative magnitude of $\nabla\cdot\mathbf{B}$ and $|\nabla\times\mathbf{B}|$. Both schemes are thus shown to maintain $\nabla\cdot\mathbf{B} \approx 0$ with a very small error, much smaller than the spatial truncation error, but the magnetic MRT collision operator leads to even smaller values of $\nabla\cdot\mathbf{B}$.

Although these computations are all two-dimensional, two-dimensional MHD in some ways more closely resembled three-dimensional hydrodynamics than two-dimensional hydrodynamics. The Lorentz force provides a source of vorticity, analogous to the vortex stretching term in three-dimensional hydrodynamics, that is absent in two-dimensional hydrodynamics. The configuration in the numerical simulations is designed to create large peak vorticities and currents through the action of the coalescence instability. For example, the numerical simulations with $\eta = \nu = 1/575$ achieve peak currents and vorticities roughly four times larger than the simulations with $\eta = \nu = 1/150$.

9. Conclusions

We have developed an equivalent moment system for the vector Boltzmann equation that was designed to evolve a magnetic field. The basis of moments comprises the magnetic

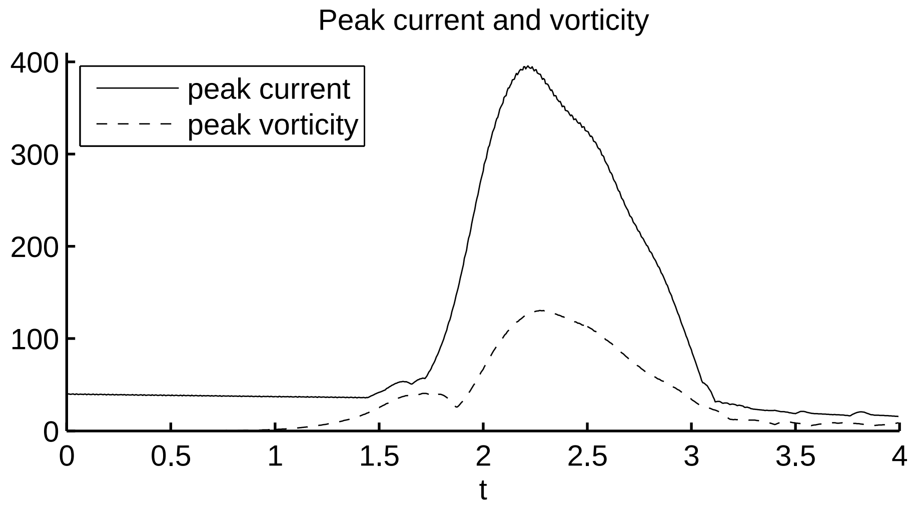

Figure 5. Evolution of the peak current and peak vorticity during coalescence with diffusivities $\nu = \eta = 1/600$ and MRT collision operators for both fluid and magnetic distribution functions.

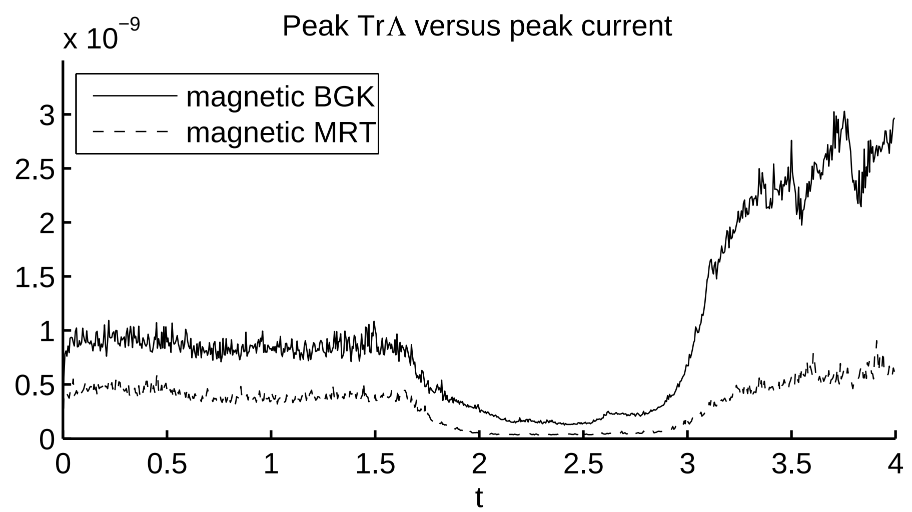

Figure 6. Evolution of the ratio of the peak of $\Lambda_{xx} + \Lambda_{yy}$ to the peak of $\Lambda_{xy} - \Lambda_{yx}$, which represents the ratio of $\nabla\cdot\mathbf{B}$ to $|\nabla\times\mathbf{B}|$. Both simulations were performed on a 128^2 grid with $\nu = \eta = 1/575$, and an MRT collision operator for the hydrodynamic distribution functions. The second MRT collision operator for the magnetic distribution function lowers the already low value of $\mathrm{Tr}\,\Lambda$, which serves as a consistent approximation to $\nabla\cdot\mathbf{B}$.

field $\mathbf{B}$, the electric field tensor $\mathsf{\Lambda}$, and the non-vanishing components of the third rank tensor M. We have also derived formulae for reconstructing the vector distribution functions $g_{i\beta}$ from these moments. These formulae allow the collision operator to be specified in the basis of moments.

The original formulation of the vector Boltzmann equation was motivated by the correspondence

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0 \quad \Leftrightarrow \quad \partial_t \mathbf{B} + \nabla\cdot\mathsf{\Lambda} = 0 \text{ with } \Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}E_\gamma. \tag{76}$$

In other words, $\mathsf{\Lambda}$ was assumed to be a purely antisymmetric tensor with components derived from $\mathbf{E}$. The equilibrium value $\mathsf{\Lambda}^{(0)} = \mathbf{u}\,\mathbf{B} - \mathbf{B}\,\mathbf{u}$ is indeed antisymmetric, but the antisymmetry of $\mathsf{\Lambda}$ is not preserved by its evolution under the vector Boltzmann equation. In particular, the Chapman–Enskog expansion enabled us to calculate the first correction $\mathsf{\Lambda}^{(1)}$ with components

$$\Lambda^{(1)}_{\alpha\beta} = -\tau\theta\partial_\alpha B_\beta + O(\mathrm{Ma}^3), \tag{77}$$

so $\mathsf{\Lambda}^{(0)} + \mathsf{\Lambda}^{(1)}$ contains a mix of symmetric and antisymmetric parts. Thus a more accurate statement for reconstructing the electric field from the $\mathsf{\Lambda}$ tensor is

$$E_\gamma = -\tfrac{1}{2}\epsilon_{\gamma\alpha\beta}\Lambda_{\alpha\beta}. \tag{78}$$

The non-zero symmetric component of $\mathsf{\Lambda}$ was harmless in the author's earlier formulation of resistive MHD with constant resistivity. The resulting evolution equation for $\mathbf{B}$ only differs from its intended form by terms proportional to $\nabla\cdot\mathbf{B}$, which is maintained at a vanishingly small level. However, when attempting to simulate a more realistic plasma, it is necessary to separate out the antisymmetric part of $\mathsf{\Lambda}$, corresponding to the physically relevant electric field, from the symmetric part. This may be achieved using a more general matrix collision operator in place of the simple BGK collision operator.

For example, a simple extension of Ohm's law (7) would allow the resistivity η to be a function of the local current density,

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta(|\nabla\times\mathbf{B}|)\,\nabla\times\mathbf{B}, \tag{79}$$

analogous to the dependence of the viscosity on the strain rate in a generalised Newtonian fluid. Inserting a spatially varying η, and hence a spatially varying τ_b, into the earlier BGK or single-relaxation-time version of the vector Boltzmann equation leads to an evolution equation of the form

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\cdot(\eta\nabla\mathbf{B}). \tag{80}$$

This differs by a term proportional to $\nabla\eta$ from the physically correct form

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}). \tag{81}$$

A lattice Boltzmann formulation of (81) without the spurious $\nabla\eta$ term may be achieved by using a matrix collision operator that applies a spatially varying relaxation time only to the antisymmetric part of the $\mathsf{\Lambda}$ tensor. A detailed implementation will be presented in a future paper.

Acknowledgments

The author's research is supported by an Advanced Research Fellowship, grant number EP/E054625/1, from the UK's Engineering and Physical Sciences Research Council.

Appendix: Orthogonal bases for the D2Q5 lattice

Lattice Boltzmann equilibria are commonly expressed as polynomials in the particle velocities $\boldsymbol{\xi}_i$, multiplied by some weights w_i. For example, the equilibria for the magnetic distribution functions were given by

$$g_{i\beta}^{(0)} = w_i \left(B_\beta + \theta^{-1} \xi_{i\alpha} \Lambda_{\alpha\beta}^{(0)} \right), \tag{A.1}$$

where the lattice constant θ is determined by the relation

$$\sum_{i=0}^{4} w_i \xi_{i\alpha} \xi_{i\beta} = \theta \delta_{\alpha\beta}. \tag{A.2}$$

The weights for the D2Q5 lattice are $w_0 = 1/3$ and $w_{1,2,3,4} = 1/6$, for which $\theta = 1/3$.

In lattice Boltzmann formulations of hydrodynamics it is conventional to use second moments with respect to the polynomials $\xi_{i\alpha}\xi_{i\beta} - \theta\delta_{\alpha\beta}$. These polynomials are orthogonal to unity, which gives the density moment, through property (A.2). For example, the equilibria for the hydrodynamic distribution functions f_i are given in section 7 by contracting the desired moments with the tensor Hermite polynomials 1, $\boldsymbol{\xi}_i$, $\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \frac{1}{3}\mathbf{I}$.

However, no choice of weights in the D2Q5 lattice makes the second moments $\xi_{ix}^2 - \theta$ and $\xi_{iy}^2 - \theta$ orthogonal to each other, as well as orthogonal to the density. Mutually orthogonal second moments may only be found using linear combinations of ξ_{ix}^2 and ξ_{iy}^2,

$$(1+\lambda)\xi_{ix}^2 - \lambda\xi_{iy}^2 - \tfrac{1}{3}, \quad (1+\lambda)\xi_{ix}^2 - \lambda\xi_{iy}^2 - \tfrac{1}{3}, \tag{A.3}$$

with $\lambda = \frac{1}{2}(\pm 3^{-1/2} - 1)$. It thus seems preferable to use non-orthogonal moments with respect to ξ_{ix}^2 and ξ_{iy}^2, as in section 3.1 onwards.

References

Bhatnagar P L, Gross E P & Krook M 1954 *Phys. Rev.* **94**, 511–525.
Breyiannis G & Valougeorgis D 2004 *Phys. Rev. E* **69**, 065702–4.
Chen H, Matthaeus W H & Klein L W 1988 *Phys. Fluids* **31**, 1439–1455.
Chen S, Chen H, Martínez D O & Matthaeus W H 1991 *Phys. Rev. Lett.* **67**, 3776–3779.
Chen S, Martínez D O, Matthaeus W H & Chen H D 1992 *J. Statist. Phys.* **68**, 533–556.
Dellar P J 2002 *J. Comput. Phys.* **179**, 95–126.
Dellar P J 2003 *J. Comput. Phys.* **190**, 351–370.
d'Humières D 1992 *in* B. D Shizgal & D. P Weaver, eds, 'Rarefied Gas Dynamics: Theory and Simulations' Vol. 159 of *Prog. Astronaut. Aeronaut.* AIAA Washington, D.C. pp. 450–458.
d'Humières D, Ginzburg I, Krafczyk M, Lallemand P & Luo L S 2002 *Phil. Trans. R. Soc. Lond. A* **360**, 437–451.
Frisch U 1991 *Physica D* **47**, 231–232.
Grad H 1949 *Commun. Pure Appl. Math.* **2**, 331–407.
He X, Chen S & Doolen G D 1998 *J. Comput. Phys.* **146**, 282–300.

Higuera F J, Succi S & Benzi R 1989 *Europhys. Lett.* **9**, 345–349.
Lallemand P & Luo L S 2000 *Phys. Rev. E* **61**, 6546–6562.
Longcope D W & Strauss H R 1993 *Phys. Fluids B* **5**, 2858–2869.
Marliani C & Strauss H R 1999 *Phys. Plasmas* **6**, 495–502.
Martínez D O, Chen S & Matthaeus W H 1994 *Phys. Plasmas* **1**, 1850–1867.
McNamara G R, Garcia A L & Alder B J 1995 *J. Statist. Phys.* **81**, 395–408.
Mendoza M & Munoz J D 2008 *Phys. Rev. E* **77**, 026713–12.
Montgomery D & Doolen G D 1987 *Phys. Lett. A* **120**, 229–231.
Pattison M, Premnath K, Morley N & Abdou M 2008 *Fusion Eng. Design* **83**, 557–572.
Qian Y H, d'Humières D & Lallemand P 1992 *Europhys. Lett.* **17**, 479–484.
Riley B, Richard J & Girimaji S S 2008 *Int. J. Mod. Phys. C* **19**, 1211–1220.
Succi S, Vergassola M & Benzi R 1991 *Phys. Rev. A* **43**, 4521–4524.
Vahala G, Keating B, Soe M, Yepez J, Vahala L, Carter J & Ziegeler S 2008 *Commun. Comput. Phys.* **4**, 624–646.
Vergassola M, Benzi R & Succi S 1990 *Europhys. Lett.* **13**, 411–416.