Dispersive shallow water magnetohydrodynamics

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Shallow water magnetohydrodynamics (SWMHD) is a recently proposed model for a thin layer of incompressible, electrically conducting fluid. The velocity and magnetic field are taken to be nearly two dimensional, with approximate magnetohydrostatic balance in the perpendicular direction, leading to a reduced two dimensional model. The SWMHD equations have been found previously to admit unphysical cusp-like singularities in finite amplitude magnetogravity waves. This paper extends the Hamiltonian formulation of SWMHD to construct a dispersively regularized system, analogous to the Green–Naghdi equations of hydrodynamics, that supports smooth solitary waves and cnoidal wavetrains, and shares the potential vorticity conservation properties of SWMHD.

I. INTRODUCTION

The shallow water magnetohydrodynamics (SWMHD) equations were recently proposed by Gilman [1] as a model for phenomena in the solar tachocline [2], the thin layer between the outer turbulent convection zone, and the quiescent interior where heat transfer is predominantly radiative. The tachocline also marks a transition between an almost rigidly rotating interior, and an outer region where the angular velocity at fixed latitude is nearly independent of depth. The resulting strong shear across the tachocline may be expected to align any local magnetic field with the azimuthal direction.

The SWMHD equations comprise a hyperbolic system, and may be written in conservation form as [3]

$$\partial_t \begin{pmatrix} h\mathbf{u} \\ h \\ h\mathbf{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} h\mathbf{u}\mathbf{u} - h\mathbf{B}\mathbf{B} + \frac{1}{2}gh^2\mathsf{I} \\ h\mathbf{u} \\ h\mathbf{u}\mathbf{B} - h\mathbf{B}\mathbf{u} \end{pmatrix} = 0, \tag{1}$$

subject to the constraint $\nabla\cdot(h\mathbf{B}) = 0$. They describe a thin layer of incompressible, perfectly conducting fluid with a free surface. The variables $\mathbf{u}$ and $\mathbf{B}$ in Eq. (1) are the horizontal components of the fluid velocity and magnetic field, h is the layer depth, and g the gravitational acceleration. Although the unmagnetized shallow water equations (SWE) coincide with the Euler equations for a barotropic fluid with density h and equation of state $p = \frac{1}{2}gh^2$, the SWMHD equations differ from the barotropic fluid MHD equations through the omission of an isotropic magnetic pressure term $\frac{1}{2}\mathbf{B}^2\mathsf{I}$. The magnetic pressure is already included in the $\frac{1}{2}gh^2\mathsf{I}$ term because the height is determined by the *total* pressure, fluid plus magnetic, balancing gravity in the hydrostatic approximation [1]. Moreover, the total horizontal magnetic flux $h\mathbf{B}$ in a fluid column is conserved by Eq. (1), rather than the pointwise magnetic field intensity $\mathbf{B}$.

The SWMHD equations admit various families of waves that were investigated in Refs. [3] and [4]. The non-rotating SWMHD equations admit the self-similar shocks and rarefaction waves expected in a hyperbolic system [3]. The rotating SWMHD equations admit smooth periodic wavetrains [4] in which nonlinear steepening is balanced by the dispersive effects of the Coriolis force. However, in finite amplitude magnetogravity waves the maximum permissible height perturbation is finite, and the free surface develops cusps as this limit is approached [4]. This unphysical behavior, with an apparently infinite Lorentz force, was attributed to the neglect of small horizontal lengthscales in the derivation of the SWMHD equations. Only the locally vertical component of the rotation vector is retained in shallow water theories, the so-called traditional approximation [5, 6], and this component vanishes at the equator. Thus if SWMHD were used to describe a train of magnetogravity waves propagating from midlatitudes towards the solar equator, as suggested by the "butterfly diagram" of observed sunspot activity, these waves may be expected to break when the Coriolis force becomes too weak to balance nonlinear steepening. In a terrestrial context, breaking of upwardly propagating gravity waves at high altitudes contributes significantly to the general circulation of the atmosphere.

Moreover, the tachocline spans perhaps 2% of the Sun's radius [7]. While shallow, the tachocline is comparatively much less shallow than the Earth's atmosphere or oceans. For the parameters considered by Schecter *et al.* [4] for the tachocline's overshoot layer, the horizontal lengthscale set by the Rossby deformation radius (see Sec. VI A) may be as short as four layer depths. It therefore seems worthwhile to seek an extension to SWMHD that retains higher order terms in the aspect ratio h/ℓ, where ℓ is a typical horizontal lengthscale. Various such extensions of the shallow water equations for pure hydrodynamics have been proposed [8–12]. They typically postulate some simple vertical structure for the three dimensional variables, and

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integrate the three dimensional equations in the vertical to derive more complicated, but still two dimensional, equations such as the Green–Naghdi [8] and Boussinesq [9] equations. The one dimensional form of the Green–Naghdi equations was given previously by Su and Gardner [13], and the two dimensional form was rediscovered by Bazdenkov *et al.* [10]. These systems in turn reduce to the Camassa–Holm [11], Benjamin–Bona–Mahony (BBM) [14], and Korteweg–de Vries (KdV) [9] equations for unidirectional waves. The extra terms manifest themselves as dispersion on short lengthscales, so these various sets of equations have been generically named "dispersive shallow water" (DSW) equations [15].

In this paper we develop a magnetohydrodynamic analogue of the Green–Naghdi [8] equations, one that both retains terms of $O(h^2/\ell^2)$ and regularizes the unphysical cusps in the original SWMHD equations, while retaining the great simplification of eliminating one spatial coordinate. Following subsequent rederivations of the Green–Naghdi equations [6, 16–19] we substitute a columnar ansatz for the horizontal velocity and magnetic field into a Hamiltonian formulation of magnetohydrodynamics in Eulerian variables. Unlike a previous Hamiltonian formulation of SWMHD [20], we retain small contributions of $O(h^2/\ell^2)$ to the kinetic and magnetic energies from the vertical field components u_z and B_z. For simplicity we consider a fluid layer with a flat lower boundary at $z = 0$, but the theory readily extends to accomodate variable bottom topography [10, 12, 15–19].

In numerical experiments with two dimensional (one horizontal and one vertical coordinate) flow over topography, Nadiga *et al.* [15] found that the one dimensional Green–Naghdi equations agreed well with vertically averaged features of their two dimensional Euler solutions. They observed that the Green–Naghdi equations are formally just a small aspect ratio approximation of the Euler equations. There is no explicit assumption of weak nonlinearity, as required in the derivation of the KdV or Boussinesq equations. The main limitations of the Green–Naghdi equations, like any vertically averaged approximation, are that they cannot reproduce the effects of overturning surface waves; and that their derivation implicitly assumes an infinite density ratio across the free surface, whereas Nadiga *et al.*'s Euler computations used finite ratios of $100 : 1$ and $1000 : 1$. Ertekin *et al.* [21] also found good agreement between solutions of the Green–Naghdi equations and laboratory experiments for the generation of solitons by a moving pressure distribution in a shallow channel. Shields and Webster [22] compared solitary waves of the Green–Naghdi equations with exact potential flow results and with various higher order versions of the Green–Naghdi equations, and found good agreement for waves as short as three mean depths.

II. HAMILTONIAN FORMULATION

In their most general form, Hamilton's evolution equations are $\partial_t \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$ for all functionals $\mathcal{F}$, where $\{\mathcal{F}, \mathcal{H}\}$ denotes the Poisson bracket of the functional $\mathcal{F}$ with the Hamiltonian functional $\mathcal{H}$ [6, 18, 23, 24]. The Hamiltonian usually coincides with the total energy of a system. The Poisson bracket is required to be bilinear, antisymmetric, and to satisfy the Jacobi identity $\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{K}\} + \{\{\mathcal{G}, \mathcal{K}\}, \mathcal{F}\} + \{\{\mathcal{K}, \mathcal{F}\}, \mathcal{G}\} = 0$ for all functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$. These three properties express the usual notion of a Hamiltonian system, normally expresses in canonical coordinates, in a coordinate-free manner. The Jacobi identity is usually by far the most difficult property to verify.

Most continuum systems are only expressible in canonical Hamiltonian form using inconvenient Lagrangian variables, so it is common to use Eulerian variables in combination with a generalised "non-canonical" Poisson bracket [6, 18, 23, 24]. The Hamiltonian formulation offers a very compact derivation of the dispersive SWMHD equations, reflecting the general utility of Hamiltonian perturbation theory. We restrict the ubiquitous Lie–Poisson bracket for magnetohydrodynamics, Eqs. (5) and (6) below, to two spatial coordinates, and integrate the three dimensional energy density in the suppressed vertical coordinate to obtain the Hamiltonian. A longer derivation directly from the three dimensional incompressible ideal MHD equations is given in the appendix. The Lie–Poisson form of the bracket implies that the dispersive SWMHD equations could also be derived in the alternative Euler–Poincaré formulation [25–27] by approximating the Lagrangian $\mathcal{L}$, and hence the action, instead of the Hamiltonian as in Eq. (3). This approach would be closer to that of Miles and Salmon [16] who rederived the Green–Naghdi equations from Hamilton's principle, as expressed in Lagrangian variables, by approximating the action.

A. Hamiltonian

The Hamiltonian for shallow water magnetohydrodynamics is the total energy of a three dimensional layer of an incompressible, perfectly conducting fluid with unit density,

$$\mathcal{H} = \int dx \int dy \int_0^{h(x,y)} dz \; \frac{1}{2}(|\mathbf{u}_3|^2 + |\mathbf{B}_3|^2) + gz, \tag{2}$$

where $\mathbf{u}_3$ and $\mathbf{B}_3$ are the three dimensional velocity and magnetic fields. The fluid is confined to the region $0 \le z \le h(x, y)$, with an assumed free (constant pressure) surface at $z = h(x, y)$. The three terms in Eq. (2) correspond to kinetic, magnetic, and gravitational potential energies respectively.

The original SWMHD equations, like the shallow water equations, describe a thin layer whose depth h is much smaller than a typical horizontal lengthscale ℓ [1, 5, 6]. The three dimensional fluid velocity $\mathbf{u}_3$ and magnetic field $\mathbf{B}_3$ are assumed to be predominantly horizontal, and functions of the two horizontal coordinates x and y only. The two solenoidal constraints $\nabla_3\cdot\mathbf{u}_3 = 0$ and $\nabla_3\cdot\mathbf{B}_3 = 0$, and the boundary conditions $u_z = 0$ and $B_z = 0$ on $z = 0$, then imply that $u_z = -z(\nabla\cdot\mathbf{u})$ and $B_z = -z(\nabla\cdot\mathbf{B})$ are both linear in the vertical coordinate z. Here ∇, $\mathbf{u}$, and $\mathbf{B}$ denote the horizontal (x and y) components of the three dimensional objects ∇_3, $\mathbf{u}_3$, and $\mathbf{B}_3$ respectively.

This ansatz allows the z integration in Eq. (2) to be completed, so that

$$\mathcal{H} = \frac{1}{2}\int gh^2 + h(|\mathbf{u}|^2 + |\mathbf{B}|^2) + \frac{1}{3}h^3(|\nabla\cdot\mathbf{u}|^2 + |\nabla\cdot\mathbf{B}|^2)\, dxdy. \tag{3}$$

For a thin layer, $u_z = O(h/\ell)|\mathbf{u}|$ and $B_z = O(h/\ell)|\mathbf{B}|$ are both small, so the terms involving $|\nabla\cdot\mathbf{u}|^2$ and $|\nabla\cdot\mathbf{B}|^2$ are $O(h^2/\ell^2)$ smaller than the $|\mathbf{u}|^2$ and $|\mathbf{B}|^2$ terms. In other words, the vertical field components contribute $O(h^2/\ell^2)$ less to the kinetic and magnetic energies than the horizontal components. These contributions were previously discarded [20], and the remaining Hamiltonian coincides then with the total energy given by Gilman [1], but in this paper we retain the full Hamiltonian as given by Eq. (3).

B. Poisson bracket

The Poisson bracket is most naturally formulated in terms of the conserved variables. These are the layer depth h, the magnetic flux $\mathbf{Q} = h\mathbf{B}$ in a vertical column [1, 3, 20], and the conserved momentum

$$\mathbf{m} = \frac{\delta\mathcal{H}}{\delta\mathbf{u}} = h\mathbf{u} - \frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{u}) = \mathsf{L}_h\mathbf{u} \tag{4}$$

of the Green–Naghdi equations [17, 19, 28]. Equation (4) relates $\mathbf{m}$ to $\mathbf{u}$ via a positive definite, self-adjoint, and coercive linear operator L_h, provided the depth h satisfies $h \geq h_{\min} > 0$. Equation (4) may thus be inverted to determine $\mathbf{u} = \mathsf{L}_h^{-1}\mathbf{m}$ as a continuous function of $\mathbf{m}$, but in general there is no explicit formula for $\mathbf{u}$ in terms of $\mathbf{m}$.

The reconstructed three-dimensional magnetic field $\mathbf{B}_3$ must be tangential to the free surface. In other words, $\mathbf{B}_3\cdot\mathbf{n} = (B_x, B_y, -h\nabla\cdot\mathbf{B})\cdot(-\partial_x h, -\partial_y h, 1) = -\nabla\cdot(h\mathbf{B}) = 0$ on $z = h(x, y)$, with $\mathbf{n}$ being a vector normal to the free surface. This implies that $h\mathbf{B}$ is the conserved quantity associated with the magnetic field. Similarly, the condition that the free surface be a streamline of the three dimensional velocity field $\mathbf{u}_3$ reconstructed from $\mathbf{u}$ will lead to the continuity equation (14a). The asymmetry between $\mathbf{u}$ and $\mathbf{B}$ arises because $\nabla_3\cdot\mathbf{u}_3 = 0$ must be enforced by a pressure gradient, while $\nabla_3\cdot\mathbf{B}_3 = 0$ is an automatic consequence of antisymmetry in the three-dimensional induction equation. The direct derivation of dispersive SWMHD in the appendix confirms that the SWMHD induction equation is unmodified at $O(h^2/\ell^2)$, while the momentum equation acquires extra dispersive terms from the pressure gradient. It is of course possible to reformulate dispersive SWMHD to include an evolution equation for $h\mathbf{B} - \frac{1}{3}\nabla(h^3\nabla\cdot\mathbf{B})$ instead of $h\mathbf{B}$, just as Li [28] used $h\mathbf{u}$ instead of $\mathbf{m}$ as given by Eq. (4) in the Green–Naghdi equations, but the Poisson bracket would take a more complicated form involving L_h and its inverse.

In ($\mathbf{m}$, h, $\mathbf{Q}$) variables the Poisson bracket takes the form

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\frac{\delta\mathcal{F}}{\delta m_i}, \frac{\delta\mathcal{F}}{\delta h}, \frac{\delta\mathcal{F}}{\delta Q_i}\right) \mathsf{J}_{ij} \begin{pmatrix} \delta\mathcal{G}/\delta m_j \\ \delta\mathcal{G}/\delta h \\ \delta\mathcal{G}/\delta Q_j \end{pmatrix} dxdy, \tag{5}$$

in terms of the Poisson tensor (or cosymplectic operator)

$$\mathsf{J}_{ij} = -\begin{pmatrix} m_j\partial_i + \partial_j m_i & h\partial_i & Q_j\partial_i - \partial_k Q_k\delta_{ij} \\ \partial_j h & 0 & 0 \\ \partial_j Q_i - Q_k\partial_k\delta_{ij} & 0 & 0 \end{pmatrix}, \tag{6}$$

where partial derivatives act on everything to their right. This Poisson bracket is manifestly bilinear and antisymmetric (after an integration by parts). Here, and subsequently, the fluid variables are assumed to satisfy suitable boundary conditions, such as decaying sufficiently rapidly at infinity, to justify the neglect of surface terms arising from an integration by parts. The necessary boundary conditions for a finite domain are $\mathbf{u}\cdot\mathbf{n} = 0$ and $\mathbf{B}\cdot\mathbf{n} = 0$, or impermeable and perfectly conducting boundaries [29].

This Poisson bracket was shown by Morrison and Greene [30] to satisfy the Jacobi identity $\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + \{\{\mathcal{G}, \mathcal{H}\}, \mathcal{F}\} + \{\{\mathcal{H}, \mathcal{F}\}, \mathcal{G}\} = 0$ for all functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$. It is, however, not in canonical form. Instead, each term is linear in one of the conserved variables $(\mathbf{m}, h, \mathbf{Q})$, and contains one spatial derivative. This is typical for hydrodynamic systems expressed in Eulerian variables [24, 29, 31]. In fact, the Poisson bracket with J given by Eq. (6) is the natural non-canonical Lie–Poisson bracket for a fluid system with an advected scalar density h, and an advected magnetic field $\mathbf{Q}$, and was derived as such by Holm and Kupershmidt [32] from a canonical bracket expressed in Lagrangian variables using Clebsch potentials.

The advective, or “semi-direct product” structure is responsible for the block of zeros in J outside the first row and first column. The different arrangement of indices in the $\mathbf{Q}$ terms is because a magnetic field is most naturally treated as a “two-form”, a geometrical object defining the magnetic flux through surface elements, rather than as a vector like the momentum. The same Poisson bracket arose previously in conventional barotropic fluid magnetohydrodynamics [30, 32, 33], and in special relativistic MHD [34], for which $\mathbf{B}$ and $\mathbf{Q}$ coincide. The non-magnetic part of the Poisson bracket involving only $\mathbf{m}$ and h arose previously in various shallow water models [11, 12, 17, 19, 35], and was used in a disguised form by Li [28].

Hamilton’s evolution equations, $\partial_t\mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$ for all functionals $\mathcal{F}$, then correspond to [6, 23, 24]

$$\frac{\partial}{\partial t}\begin{pmatrix} m_i \\ h \\ Q_i \end{pmatrix} = \mathsf{J}_{ij}\begin{pmatrix} \delta\mathcal{H}/\delta m_j \\ \delta\mathcal{H}/\delta h \\ \delta\mathcal{H}/\delta Q_j \end{pmatrix}, \tag{7}$$

or in vector notation,

$$\partial_t\mathbf{m} = -m_j\nabla\frac{\delta\mathcal{H}}{\delta m_j} - \nabla\cdot\left(\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\mathbf{m}\right) - h\nabla\frac{\delta\mathcal{H}}{\delta h} + \left(\nabla\times\frac{\delta\mathcal{H}}{\delta\mathbf{Q}}\right)\times\mathbf{Q} + \frac{\delta\mathcal{H}}{\delta\mathbf{Q}}\nabla\cdot\mathbf{Q}, \tag{8a}$$

$$\partial_t h + \nabla\cdot\left(h\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\right) = 0, \tag{8b}$$

$$\partial_t\mathbf{Q} = \nabla\times\left(\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\times\mathbf{Q}\right) - \frac{\delta\mathcal{H}}{\delta\mathbf{m}}\nabla\cdot\mathbf{Q}. \tag{8c}$$

As the above system has no explicit dependence on the spatial coordinates x and y, Noether’s theorem implies that the total momentum is conserved. The momentum equation may thus be rewritten in conservation form as $\partial_t\mathbf{m} + \nabla\cdot\mathsf{T} = 0$. To actually compute the stress tensor T it is useful to rewrite Eq. (8a) as

$$\begin{aligned}\partial_t m_i = -\partial_j\left(m_i\frac{\delta\mathcal{H}}{\delta m_j} - Q_j\frac{\delta\mathcal{H}}{\delta Q_i}\right) - \partial_i\left(m_j\frac{\delta\mathcal{H}}{\delta m_j} + h\frac{\delta\mathcal{H}}{\delta h} + Q_j\frac{\delta\mathcal{H}}{\delta Q_j}\right)\\ + \left(\frac{\delta\mathcal{H}}{\delta m_j}\partial_i m_j + \frac{\delta\mathcal{H}}{\delta h}\partial_i h + \frac{\delta\mathcal{H}}{\delta Q_j}\partial_i Q_j\right).\end{aligned} \tag{9}$$

The first two terms are now in conservation form, while the last term differs by the divergence of a stress from the gradient $\partial_i H$ of the Hamiltonian density, the integrand H appearing in the Hamiltonian $\mathcal{H} = \int H dxdy$. This generalises a result from Holm and Kupershmidt [34] for purely algebraic Hamiltonian densities, for which the last term in Eq. (9) is precisely $\partial_i H$. Holm *et al.* [26] gave an equivalent manipulation for the Euler-Poincaré formulation involving the gradient of the Lagrangian density.

By contrast, the induction equation (8c) is not automatically in conservation form, but the non-conservative final term typically vanishes because $\nabla\cdot\mathbf{Q} = 0$, corresponding either to $\nabla\cdot\mathbf{B} = 0$ in conventional MHD, or to $\nabla\cdot(h\mathbf{B}) = 0$ in SWMHD [20]. The constraint $\nabla\cdot\mathbf{Q} = 0$ is preserved by the induction equation, since Eq. (8c) implies

$$\partial_t\nabla\cdot\mathbf{Q} + \nabla\cdot\left(\frac{\delta\mathcal{H}}{\delta\mathbf{m}}\nabla\cdot\mathbf{Q}\right) = 0, \tag{10}$$

so is most naturally imposed as an initial condition. In shallow water magnetohydrodynamics (with or without dispersion) the $\nabla\cdot(h\mathbf{B}) = 0$ constraint has a natural interpretation as the reconstructed three dimensional vector $\mathbf{B}_3$ being tangent to the free surface [1]. In other words, the free surface is a magnetic field line. However, the extra $-\mathbf{u}\nabla\cdot\mathbf{Q}$ term is needed in general (when $\nabla\cdot\mathbf{Q} \neq 0$) to make the system Galilean invariant [3, 20, 36–38], and to ensure that the Poisson bracket satisfies the Jacobi identity [30]. The difficulty with Galilean invariance arises because Eq. (10) would be simply $\partial_t\nabla\cdot\mathbf{Q} = 0$ without the non-conservative term proportional to $\nabla\cdot\mathbf{Q}$ in Eq. (8c).

III. DISPERSIVE SHALLOW WATER MHD EQUATIONS

With respect to the variables $\mathbf{m}$, h, and $\mathbf{Q}$, the Hamiltonian in Eq. (3) takes the form

$$\mathcal{H} = \frac{1}{2}\int gh^2 + \mathbf{m}\cdot\mathbf{u} + \mathbf{Q}\cdot\left[\mathbf{B} - \frac{1}{3h}\nabla(h^3\nabla\cdot\mathbf{B})\right] dxdy, \tag{11}$$

after integrating by parts. Substituting the variational derivatives,

$$\frac{\delta\mathcal{H}}{\delta\mathbf{m}} = \mathbf{u}, \quad \frac{\delta\mathcal{H}}{\delta\mathbf{Q}} = \mathbf{B} - \frac{1}{3h}\nabla(h^3\nabla\cdot\mathbf{B}), \tag{12}$$

$$\frac{\delta\mathcal{H}}{\delta h} = gh - \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{B}|^2) - \frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2 + \frac{1}{2}h^2(\nabla\cdot\mathbf{B})^2 + \frac{1}{3h}\mathbf{B}\cdot\nabla(h^3\nabla\cdot\mathbf{B}), \tag{13}$$

into the general expressions above, we obtain the dispersive shallow water magnetohydrodynamic equations in the form

$$\partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \tag{14a}$$

$$\partial_t(h\mathbf{B}) + \nabla\cdot(h\mathbf{u}\mathbf{B} - h\mathbf{B}\mathbf{u}) = 0, \tag{14b}$$

$$\begin{aligned} h\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + g\nabla h\right) - \nabla\cdot(h\mathbf{B}\mathbf{B}) &= \frac{1}{3}\nabla\left[h^2\mathcal{D}(h\nabla\cdot\mathbf{u})\right] + h\mathbf{B}\times\nabla\times\left[\frac{1}{3h}\nabla(h^3\nabla\cdot\mathbf{B})\right] \\ &\quad - h\nabla\left[\frac{1}{2}h^2(\nabla\cdot\mathbf{B})^2 + \frac{1}{3h}\mathbf{B}\cdot\nabla(h^3\nabla\cdot\mathbf{B})\right], \end{aligned} \tag{14c}$$

subject to $\nabla\cdot(h\mathbf{B}) = 0$, and where $\mathcal{D} = \partial_t + \mathbf{u}\cdot\nabla$ is the material time derivative. The left hand sides of Eqs. (14a-c) are the original SWMHD equations, while the right hand side of Eq. (14c) contains dispersive corrections due to finite layer depth. The $(1/3)\nabla h^2\mathcal{D}(h\nabla\cdot\mathbf{u})$ term in Eq. (14c) is the Green–Naghdi dispersion [8, 10, 13, 16, 17], sometimes rewritten as $-(1/3)\nabla h^2\mathcal{D}^2 h$, since $\mathcal{D}h = -h\nabla\cdot\mathbf{u}$ from Eq. (14a). One qualitative change in these equations is the appearance of a further time derivative on the right hand side of Eq. (14c). Equation (14c) may be further manipulated into the form

$$h\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + g\nabla h\right) - \nabla\cdot(h\mathbf{B}\mathbf{B}) = \frac{1}{3}\nabla\left[h^2\mathcal{D}(h\nabla\cdot\mathbf{u}) + h^3(\nabla\cdot\mathbf{B})^2 - h^3\mathbf{B}\cdot\nabla(\nabla\cdot\mathbf{B})\right] \tag{15}$$

that arises from a perturbative solution of the original three dimensional equations (see Appendix). Moreover, the height h may be eliminated between Eqs. (14a,b) to obtain the familiar frozen-flux equation from *incompressible* magnetohydrodynamics,

$$\partial_t\mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u} = 0, \tag{16}$$

although neither $\mathbf{u}$ nor $\mathbf{B}$ have zero divergence in shallow water magnetohydrodynamics.

IV. ALTERNATIVE VARIABLES

Like the original shallow water magnetohydrodynamic equations [20], the dispersive SWMHD equations may be simplified by a change of variables. However, the situation is slightly more complicated than before, because the variable $\mathbf{v} = h^{-1}\mathbf{m} = \mathbf{u} + O(h^2/\ell^2)$ no longer coincides with the fluid velocity $\mathbf{u}$ that appeared previously. In principle this distinction always arises in fluids with electromagnetic fields, because the Poynting flux $c^{-2}\mathbf{E}\times\mathbf{B}$ contributes to the momentum, but it is usually negligible in nonrelativistic MHD. The distinction also tends to appear in higher order Hamiltonian perturbation theories even for pure hydrodynamics, since $\delta\mathcal{H}/\delta\mathbf{u}$ is rarely just $h\mathbf{u}$, and is commonly met in a rapidly rotating context [39].

In $(\mathbf{v}, h, \mathbf{Q})$ variables, the Poisson bracket becomes [33]

$$\begin{aligned} \{\mathcal{F},\mathcal{G}\} &= \int \frac{1}{h}(\nabla\times\mathbf{v})\cdot\left(\frac{\delta\mathcal{F}}{\delta\mathbf{v}}\times\frac{\delta\mathcal{G}}{\delta\mathbf{v}}\right) + \left(\nabla\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}}\right)\frac{\delta\mathcal{G}}{\delta h} - \left(\nabla\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}}\right)\frac{\delta\mathcal{F}}{\delta h} \\ &\quad + \frac{\delta\mathcal{F}}{\delta\mathbf{v}}\cdot\left[\frac{1}{h}\mathbf{Q}\times\left(\nabla\times\frac{\delta\mathcal{G}}{\delta\mathbf{Q}}\right)\right] + \frac{\delta\mathcal{F}}{\delta\mathbf{Q}}\cdot\left[\nabla\times\left(\frac{1}{h}\mathbf{Q}\times\frac{\delta\mathcal{G}}{\delta\mathbf{v}}\right)\right] + \frac{1}{h}\nabla\cdot\mathbf{Q}\left(\frac{\delta\mathcal{F}}{\delta\mathbf{v}}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{Q}} - \frac{\delta\mathcal{F}}{\delta\mathbf{Q}}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}}\right)dxdy, \end{aligned} \tag{17}$$

where we have retained the last term proportional to $\nabla\cdot\mathbf{Q}$ that was omitted in Ref. [33]. This form emphasises the rôle of the potential vorticity $h^{-1}(\nabla\times\mathbf{v})$, and extends easily to include the Coriolis force in a frame rotating with angular velocity $\mathbf{\Omega}$ by replacing $\nabla\times\mathbf{v}$ with $\nabla\times\mathbf{v} + 2\mathbf{\Omega}$ inside the bracket [23]. Alternatively, the unmodified bracket may be used provided the momentum is taken to be $\mathbf{m} = h(\mathbf{v} + \mathbf{R})$, where $\mathbf{R}$ is any vector potential for the Coriolis force with $\nabla\times\mathbf{R} = 2\mathbf{\Omega}$ [35, 40].

As with the original hyperbolic SWMHD equations [20], the dispersive SWMHD equations take a particularly simple form when the $\nabla\cdot(h\mathbf{B}) = 0$ constraint is used to write $\mathbf{B} = h^{-1}\hat{\boldsymbol{z}}\times\nabla\psi = h^{-1}(-\psi_y, \psi_x, 0)$ in terms of a magnetic flux function ψ. The choice of ψ and the sign convention is the usual one in magnetohydrodynamics. The Poisson bracket then becomes [20]

$$\{\mathcal{F},\mathcal{G}\} = \int \left(\frac{\delta\mathcal{F}}{\delta u_x}, \frac{\delta\mathcal{F}}{\delta u_y}, \frac{\delta\mathcal{F}}{\delta h}, \frac{\delta\mathcal{F}}{\delta\psi}\right)\mathsf{J}\begin{pmatrix}\delta\mathcal{G}/\delta u_x\\ \delta\mathcal{G}/\delta u_y\\ \delta\mathcal{G}/\delta h\\ \delta\mathcal{G}/\delta\psi\end{pmatrix}dxdy, \tag{18}$$

with Poisson tensor

$$\mathsf{J} = -\begin{pmatrix} 0 & -q & \partial_x & -h^{-1}Q_y \\ q & 0 & \partial_y & h^{-1}Q_x \\ \partial_x & \partial_y & 0 & 0 \\ h^{-1}Q_y & -h^{-1}Q_x & 0 & 0 \end{pmatrix}, \tag{19}$$

where $q = h^{-1}\hat{\boldsymbol{z}}\cdot(\nabla\times\mathbf{v}+2\boldsymbol{\Omega})$ is the (scalar) potential vorticity. The upper left, purely hydrodynamic, 3×3 block was used previously by Shepherd [23]. The exact Poisson tensor (19) was used previously by Ripa [41] in a shallow water model with horizontal temperature gradients, with ψ playing the role of temperature. However, our derivation from a Lie–Poisson bracket via a change of variables offers a much more direct proof [40] of the essential Jacobi identity $\{\{\mathcal{F},\mathcal{G}\},\mathcal{H}\}+\{\{\mathcal{G},\mathcal{H}\},\mathcal{F}\}+\{\{\mathcal{H},\mathcal{F}\},\mathcal{G}\}=0$ than Ripa's long formal proof [41].

The dispersive SWMHD equations in these variables are

$$\begin{aligned}
&\partial_t h \;+\; \nabla\cdot(\mathbf{u}h)=0, \quad \partial_t\psi+\mathbf{u}\cdot\nabla\psi=0, \\
&\partial_t\mathbf{v} \;+\; (2\boldsymbol{\Omega}+\nabla\times\mathbf{v})\times\mathbf{u}+\mathbf{B}\times\nabla\times\left[\mathbf{B}-\frac{1}{3h}\nabla(h^3\nabla\cdot\mathbf{B})\right]+\nabla\left(gh+\frac{1}{2}|\mathbf{u}|^2-\frac{1}{2}|\mathbf{B}|^2\right.\\
&\qquad\left.-\frac{1}{2}h^2(\nabla\cdot\mathbf{u})^2+\frac{1}{2}h^2(\nabla\cdot\mathbf{B})^2+\frac{1}{3h}\mathbf{B}\cdot\nabla(h^3\nabla\cdot\mathbf{B})-\frac{1}{3h}\mathbf{u}\cdot\nabla(h^3\nabla\cdot\mathbf{u})\right)=0,
\end{aligned} \tag{20}$$

where $\mathbf{v}=\mathbf{m}/h=\mathbf{u}-(1/(3h))\nabla(h^3\nabla\cdot\mathbf{u})$, and the last term is the gradient of the Bernoulli function. The variational derivatives of the Hamiltonian are known from the variational chain rule,

$$\frac{\delta\mathcal{H}}{\delta\mathbf{v}}=\frac{1}{h}\frac{\delta\mathcal{H}}{\delta\mathbf{m}}, \quad \left(\frac{\delta\mathcal{H}}{\delta h}\right)_{\mathbf{v}}=\left(\frac{\delta\mathcal{H}}{\delta h}\right)_{\mathbf{m}}+\mathbf{v}\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{m}}, \quad \frac{\delta\mathcal{H}}{\delta\psi}=-\hat{\boldsymbol{z}}\cdot\nabla\times\frac{\delta\mathcal{H}}{\delta\mathbf{Q}}, \tag{21}$$

even though the Hamiltonian itself cannot be written explicitly in terms of $(\mathbf{v},h,\psi)$ because there is no explicit formula for the inverse operator L_h^{-1} that determines $\mathbf{u}$ from $\mathbf{m}$ or $\mathbf{v}$.

V. CONSERVATION PROPERTIES

The Hamiltonian structure of shallow water magnetohydrodynamics implies many conservation properties, and these are shared by the dispersive extension derived above. However, the velocity splitting causes some modifications. Materially conserved quantities like h and ψ are transported by the vertically averaged velocity $\mathbf{u}$, as seen in the first two of Eqs. (20), but the *definition* of the transported quantity that is the potential vorticity contains the different velocity $\mathbf{v}$ instead of $\mathbf{u}$. This velocity splitting occurs because the operations of taking a curl and vertical averaging do not commute [12]. In other words, the vertically averaged vorticity is not the curl of the vertically averaged velocity. The same phenomenon occurs in so-called "α-smoothed" models of incompressible fluids [25–27], where an average over (assumed isotropic) small-scale fluctuations takes the place of a vertical average.

The Poisson bracket has the Casimir functionals [20, 40, 41]

$$\mathcal{C}=\int hf(\psi)+hqg(\psi)\,dxdy, \tag{22}$$

where $f(\psi)$ and $g(\psi)$ are arbitrary functions of the magnetic flux function ψ. All such quantities are conserved by the SWMHD equations, with or without dispersion, because $\mathsf{J}\delta\mathcal{C}/\delta(\mathbf{m},h,\psi)=0$ for all such Casimir functionals [24, 29, 31] when J is given by Eq. (19). Thus $\partial_t\mathcal{C}=\{\mathcal{C},\mathcal{H}\}=0$, because $\{\mathcal{C},\mathcal{F}\}=0$ for all functionals $\mathcal{F}$.

The Casimirs in Eq. (22) imply conservation properties of the flux function and potential vorticity identical to those for the SWMHD equations given in Ref. [20], although the definition of q in terms of the primitive variables $\mathbf{u}$ and h has been modified by dispersion. In particular, the magnetic flux function ψ is materially conserved, as calculated explicitly in Eq. (20). The potential vorticity q is not materially conserved when magnetic fields are present, due to the source term on the right hand side of Eq. (23), but the total potential vorticity between any two magnetic field lines $\psi=cst$ is conserved [20].

Equations (20) imply the potential vorticity equation

$$\partial_t q+\mathbf{u}\cdot\nabla q=\mathbf{B}\cdot\nabla\left[\frac{1}{h}\hat{\boldsymbol{z}}\cdot\nabla\times\left(\mathbf{B}-\frac{1}{3h}\nabla(h^3\nabla\cdot\mathbf{B})\right)\right], \tag{23}$$

which shows that potential vorticity is materially conserved by the non-magnetic Green–Naghdi equations [16], just as $h^{-1}\hat{\boldsymbol{z}}\cdot\nabla\times\mathbf{u}$ is materially conserved by the shallow water equations [5, 6]. Equation (23) corresponds to a Kelvin circulation theorem (see Ref. [35] for a non-magnetic version)

$$\frac{d}{dt}\oint_C\mathbf{v}\cdot d\mathbf{l}=\oint_C\frac{1}{h}\frac{\delta\mathcal{H}}{\delta\psi}\nabla\psi\cdot d\mathbf{l}, \tag{24}$$

for the evolution of the circulation of $\mathbf{v}$ around any closed material curve C moving with the transport velocity $\mathbf{u}$. The circulation is equal to the surface integral of the vorticity $\nabla\times\mathbf{v}$ over any surface spanning the loop by Stokes' theorem. Unlike the pure fluid case, the right hand side contains a source term due to the magnetic field. However, this source term vanishes if the loop C is a closed magnetic field line, for which $d\mathbf{l}$ is perpendicular to $\nabla\psi$, which confirms that the total potential vorticity inside any closed field line is conserved [40].

VI. UNIDIRECTIONAL WAVES

We begin the study of dispersive SWMHD by considering unidirectional waves. If the dependent variables are assumed to depend on one spatial coordinate x only, Eqs. (14) simply to

$$\partial_t h + \partial_x(hu_x) = 0, \tag{25a}$$

$$\partial_x(hB_x) = 0, \tag{25b}$$

$$\partial_t(hB_y) + \partial_x\left(hu_xB_y - hu_yB_x\right) = 0, \tag{25c}$$

$$\partial_t(hu_y) + \partial_x\left(hu_xu_y - hB_yB_x\right) = -fhu_x, \tag{25d}$$

$$\partial_t(hu_x)+\partial_x\left\{hu_xu_x+\frac{1}{2}gh^2-hB_xB_x-\frac{1}{3}h^2\mathcal{D}(h\partial_xu_x)-\frac{1}{3}h^3(\partial_xB_x)^2+\frac{1}{3}h^3B_x\partial_{xx}B_x\right\} = fhu_y, \tag{25e}$$

where $\mathcal{D} = \partial_t + u_x\partial_x$. The right hand sides arise from a Coriolis force due to a possible rotation about the z-axis with angular velocity $\Omega = \frac{1}{2}f$, f being the Coriolis parameter [5, 6]. The initial conditions have been taken to satisfy $\nabla\cdot\mathbf{Q} = 0$, so $Q_x = hB_x$ is constant. In the absence of rotation ($f = 0$) Eqs. (25a,b,e) for h, B_x, and u_x decouple from Eqs. (25c,d) for B_y and u_y. In contrast to conventional MHD, there is no magnetic pressure contribution from B_y in the x-momentum equation (25e).

A. Linear waves

We consider solutions of Eqs. (25a-e) in the form $h = h_0 + h'\exp(i(kx - \omega t))$, and similarly for u_x, u_y, B_x, and B_y. We adopt a frame in which $\mathbf{u}_0 = \mathbf{0}$. As in Ref. [4], we exclude the possibility of shear instabilities in a rotating system due to a nonzero u_{y0}. A perpendicular magnetic field component B_{y0} is permitted, but does not feature in the linearised system. Following Li [28], we scale both horizontal and vertical lengths with the undisturbed layer depth h_0, and scale time such that the long gravity wave speed $\sqrt{gh_0}$ is unity.

The five perturbations h', u'_x, u'_y, B'_x, and B'_y are related by

$$u'_x = ch', \quad B'_x = -\frac{B_{x0}}{c}u'_x, \quad B'_y = -\frac{B_{x0}}{c}u'_y, \tag{26a}$$

$$\left[(B_{x0}^2 - c^2)(1 + k^2/3) + 1\right]u'_x + \frac{ifc}{k}u'_y = 0, \quad (B_{x0}^2 - c^2)u'_y - \frac{ifc}{k}u'_x = 0, \tag{26b}$$

where $c = \omega/k$ is the phase speed of the wave. Only the x-components are affected by dispersion, as manifested by the $(1 + k^2/3)$ factor. The other terms are all identical with Eqs. (4) of Ref. [4] for non-dispersive SWMHD.

Dispersive SWMHD, like non-dispersive SWMHD [3, 4], admits only four propagating waves despite having five dependent variables. The fifth wave associated with $\nabla\cdot\mathbf{Q}$ is degenerate [3], by analogy with the eighth wave associated with $\nabla\cdot\mathbf{B}$ in compressible MHD [36–38]. In the non-rotating case the four waves split into a pair of transverse Alfvén waves, involving u'_y and B'_y only, and a pair of longitudinal magnetogravity waves involving h', u'_x, and B'_x only. Being transverse, the Alfvén waves are unaffected by the dispersion, and so propagate with unchanged phase speed $|B_{x0}|$. By contrast, the dispersion relation for the two magnetogravity waves becomes

$$\omega^2 = \frac{k^2}{1 + k^2/3} + k^2B_{x0}^2 \tag{27}$$

in dispersive SWMHD. For long waves ($k \ll 1$) where dispersion is ineffective Eq. (27) becomes $\omega = k\sqrt{1 + B_{x0}^2}$, the phase speed $c = \omega/k$ being the usual combination of the surface gravity and Alfvén wave speeds.

The exact dispersion relation for magnetogravity waves on a fluid layer of unit depth is

$$\omega^2 = k\tanh(k) + k^2B_{x0}^2, \tag{28}$$

a formula that may be derived from the dispersion relation

$$\omega^2 = \frac{k(\rho_1 - \rho_2) + k^2B_{x0}^2[\coth kh_1 + \coth kh_2]}{\rho_1\coth kh_1 + \rho_2\coth kh_2} \tag{29}$$

given by Talwar [42] for two superposed fluid layers of finite densities ρ_1 and ρ_2, and depths h_1 and h_2, by letting $\rho_2 \to 0$ and $h_2 \to \infty$, while $\rho_1 = 1$ and $h_1 = 1$ in our dimensionless variables. The non-magnetic part of Eq. (29) may also be found in §231 of Lamb [43], and the infinite depth limit in §97 of Chandrasekhar [44]. The magnetic field appears as an anisotropic surface

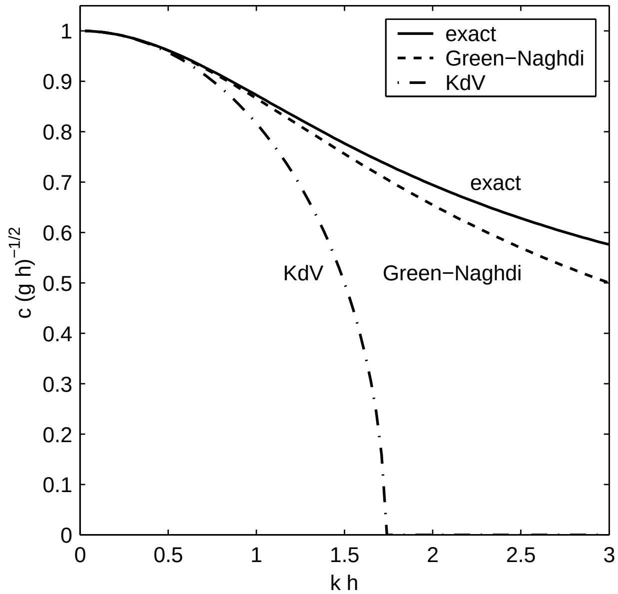

FIG. 1: The exact water wave phase speed $c^2 = gh \tanh kh$, and the Green–Naghdi and KdV approximations derived for $kh \ll 1$. The Green–Naghdi approximation is substantially more accurate than KdV for moderate kh, and its phase speed does not become negative.

tension, of magnitude $B_0^2 \cos^2 \theta$ for waves propagating at an angle θ to the magnetic field lines [44]. In particular, the $k^2 B_{x0}^2$ term is independent of the fluid layer depth, and both the phase speed ω/k and the group speed $d\omega/dk$ asymptote towards the Alvén speed B_{x0} for sufficiently short waves (as $k \to \infty$).

Dispersive SWMHD captures the magnetic contribution $k^2 B_{x0}^2$ to the dispersion relation exactly, while the $k^2/(1 + k^2/3)$ term in Eq. (27) is a Padé approximation to $k \tanh k$ from the exact gravity wave dispersion relation. Since $\tanh k$ has poles at $k = \pm\frac{1}{2} i\pi$, this Padé approximation is more accurate than the Taylor series approximation $k^2(1 - k^2/3)$ obtained from KdV dispersion [14, 19]. These three dispersion relations are plotted in Fig. 1. Benjamin *et al.* [14] also argued that the unbounded group speed $d\omega/dk$ obtained from KdV dispersion in the short wave limit is undesirable. As any shallow layer model will be inaccurate for sufficiently large wavenumbers k, one should aim for some "innocuous" behavior as $k \to \infty$.

In a rotating system, the two families of Alfvén and magnetogravity waves are coupled by the Coriolis force. However, the dispersion relation still has two branches $w_\pm$ given by

$$\omega_\pm^2 = k^2 B_{x0}^2 + \frac{k^2 + f^2}{2(1 + k^2/3)} \pm \frac{\sqrt{(k^2 + f^2)^2 + 4k^2 f^2 B_{x0}^2 (1 + k^2/3)}}{2(1 + k^2/3)}, \tag{30}$$

which coincides with Eq. (3) of Ref. [4] for $kh \ll 1$. For small k, the upper branch ω_+ emerges from the Coriolis frequency f, while the lower branch emerges from zero with $\omega_- \propto k^2$. The two branches are often called "fast" and "slow" waves, the term fast meaning waves whose frequencies are greater than the Coriolis frequency, and slow meaning waves whose frequencies become arbitrarily small for large wavelengths [5, 6]. Fast waves in rotating incompressible MHD are also called inertial waves, while slow waves are sometimes called MAC (magnetic Archimedean Coriolis) waves to reflect the dominant balance of Lorentz, buoyancy, and Coriolis forces [45]. They should not be confused with the fast and slow shock waves in non-rotating compressible MHD.

In rotating hyperbolic SWMHD, without dispersion, the only characteristic lengthscale is the Rossby deformation radius $R_\mathrm{d} = \sqrt{gh_0}/f$, the scale on which the Coriolis force becomes comparable to the horizontal pressure gradient [5, 6]. In dispersive SWMHD the layer depth h defines a second preferred lengthscale, one where $kh = O(1)$, so the dispersion relation in Eq. (30) contains two free parameters, shown as f and B_{x0}, rather than one as in SWMHD [4]. Figure 2 shows the frequency $\omega_\pm$ and phase speed $\omega_\pm/k$ for the case with $R_\mathrm{d} = h_0$, and also $B_{x0} = \sqrt{gh_0}$ as in Ref. [4]. The main effect of dispersion is that the fast branch ω_+ asymptotically approaches the Alfvén dispersion relation at high wavenumbers, rather than the long magneto-gravity wave dispersion relation as in SWMHD [4].

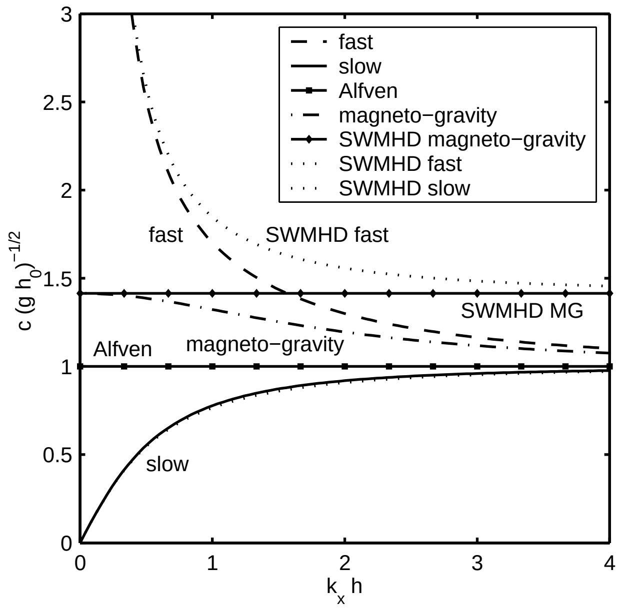

FIG. 2: Phase speeds of fast (ω_+ branch) and slow (ω_- branch) waves in rotating dispersive SWMHD. Slow waves are almost unaffected by dispersion. The fast waves asymptote to the non-rotating Alfvén speed, instead of the faster long magnetogravity wave speed, in line with the behavior of magnetogravity waves in non-rotating dispersive SWMHD. The Alfvén speed $|B_{x0}| = \sqrt{gh_0}$, and the Rossby deformation radius $R_\mathrm{d} = \sqrt{gh_0}/f = h_0$, the layer depth.

B. Nonlinear waves

The one dimensional non-rotating dispersive SWMHD equations also support finite amplitude sech^2 solitary waves analogous to those present in the one dimensional Green–Naghdi equations [13, 28, 46], and also in the finite depth irrotational water wave equations. These solutions are

$$h(x-ct) = 1 + (c^2 - Q^2 - 1)\mathrm{sech}^2\left[\frac{\sqrt{3(c^2-Q^2-1)}}{2\sqrt{c^2-Q^2}}(x-ct)\right], \tag{31}$$

provided the wave speed $|c| > \sqrt{1+Q^2}$, the speed of linear long magnetogravity waves from Eq. (27). The other variables are given by $u_x = c\,(1+1/h)$ and $B_x = Q/h$. When $Q = 0$ these formulas reduce to those given by Li [28] for the Green–Naghdi equations. More generally, the sech^2 waves are the infinite wavelength limit of a family of periodic cnoidal [9, 43] wave solutions given by

$$h(x-ct) = 1 + \alpha\,\mathrm{cn}^2\left[(x-ct)\left(\frac{3\beta}{4(1+\alpha)(1+\alpha-\beta)}\right)^{1/2}, \left(\frac{\alpha}{\beta}\right)^{1/2}\right], \tag{32}$$

where cn is the Jacobi elliptic function with modulus $\sqrt{\alpha/\beta}$. The parameter α is the wave amplitude, and β controls the wave length. The other variables are given by $B_x = Q/h$ and $u_x = c + M/h$. The mass flux M through the wave train is determined by $M^2 = Q^2 + (1+\alpha)(1+\alpha-\beta)$. As there is no natural preferred frame, analogous to the frame in which the fluid is at rest at infinity for solitary waves, the wave speed c may be chosen freely. The cnoidal and sech^2 waves coincide in the limit where $\beta = \alpha = M^2 - Q^2 - 1$ and the wavelength becomes infinite. The horizontal magnetic flux Q only appears in the combinations $M^2 - Q^2$, or $c^2 - Q^2$ for the solitary waves where $M = c$.

VII. CONCLUSION

The shallow water magnetohydrodynamics (SWMHD) equations have a Hamiltonian structure in terms of the ubiquitous non-canonical Lie–Poisson bracket describing barotropic fluids with magnetic fields in Eulerian variables. The SWMHD Hamiltonian results from integrating the three-dimensional energy density in the suppressed vertical coordinate and discarding terms involving the small aspect ratio. In this paper we have constructed a dispersively regularized extension of the SWMHD equations by retaining two small contributions to the energy from the vertical fluid velocity and magnetic field. The same dispersive SWMHD equations may also be derived directly from the three dimensional MHD equations (see appendix), and for unmagnetised fluids they coincide with the Green–Naghdi equations.

The unmodified SWMHD induction equation is already accurate to $O(h^2/\ell^2)$, the same as the Green–Naghdi equations, provided the horizontal velocity and magnetic field are interpreted as the layer averages $\overline{\mathbf{u}}$ and $\overline{\mathbf{B}}$ (see appendix). The momentum equation acquires further dispersive terms at $O(h^2/\ell^2)$ due to magnetic contributions to the non-hydrostatic pressure gradient in addition to those in the Green–Naghdi equations. In the absence of rotation, the dispersive SWMHD equations support smooth solitary waves and periodic wavetrains analogous to those in the Green–Naghdi and irrotational water wave equations, in contrast to the traveling shocks in the original SWMHD equations. The effect of rotation on these finite amplitude waves remains to be investigated.

Dispersive SWMHD shares the same Poisson bracket as SWMHD, and thus inherits many conservation properties. In particular, the potential vorticity inside closed magnetic field lines is still exactly conserved, but the definition of potential vorticity in dispersive SWMHD is changed at $O(h^2/\ell^2)$ by dispersion. It remains to be seen what implications these conservation properties have for nonlinear stability, such as analogues of the results obtained by Holm *et al.* [29] for two dimensional barotropic MHD, and whether the additional dispersion has the destabilising effects found by Bazdenkov *et al.* [10] for shear flows in the Green–Naghdi equations. Any instability associated with the variation of the vertical component of the rotation vector with latitude would be important for momentum transport by wave breaking in the solar tachocline, the scenario suggested in the introduction.

Dispersive SWMHD, like the Green–Naghdi equations, may be extended to include varying bottom topography, by replacing the boundary conditions $u_z = b_z = 0$ at $z = 0$ with no-normal-component boundary conditions at a spatially varying depth $z = \beta(x, y)$ [10, 12, 15–19]. The vertical velocity and magnetic field appearing in the Hamiltonian in Eq. (2) then become $u_z = \nabla\cdot(\beta\mathbf{u}) - z(\nabla\cdot\mathbf{u})$ and $B_z = \nabla\cdot(\beta\mathbf{B}) - z(\nabla\cdot\mathbf{B})$. Moreover, dispersive SWMHD is purely a small aspect ratio approximation, and require no assumptions about the magnitudes of velocities or height variations. Further assumptions would lead to magnetic analogues of the great lake [12] or generalised Boussinesq [15] equations for small amplitude fluctuations in the velocity and free surface height. The generalised Boussinesq equations may be obtained by replacing the $\frac{1}{3}h^3(\nabla\cdot\mathbf{u})^2$ in the Hamiltonian with $\frac{1}{3}h_0[\nabla\cdot(h\mathbf{u})]^2$, where h_0 is the undisturbed layer depth [19]. This modification has the effect of linearizing the dispersive term to just $-\frac{1}{3}h_0\nabla h_{tt}$ in the equation for $\partial_t\mathbf{u}$ [19]. Unfortunately, the magnetic dispersive terms vanish completely in this approximation because $\nabla\cdot(h\mathbf{B}) = 0$, so the resulting dispersion relation for linear magnetogravity waves does not match Eq. (27). However, a cruder linearization to $\frac{1}{3}h_0^3(\nabla\cdot\mathbf{B})^2$ does retain a magnetic contribution to the dispersion. A key simplification then arises because variations in h may be neglected in the operator L relating $\mathbf{m}$ to $\mathbf{u}$. This operator becomes $\mathsf{L}\mathbf{u} = \mathbf{u} - \frac{1}{3}\nabla\nabla\cdot\mathbf{u}$, and may be diagonalised by a Fourier transform. A further modification, replacing $\frac{1}{3}h_0^3|\nabla\cdot\mathbf{u}|^2$ with $\frac{1}{3}h_0^3|\nabla\mathbf{u}|^2$ in the Hamiltonian, leads to the operator $\mathsf{L}\mathbf{u} = \mathbf{u} - \frac{1}{3}h_0^3\nabla^2\mathbf{u}$ that acts on the vortical part of $\mathbf{u}$, not just its divergence. This operator appears in "α-smoothed" models of incompressible ideal fluid dynamics [25, 27], and recently incompressible ideal MHD [47], where the average over a thin layer is replaced by an average over small scale fluctuations.

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APPENDIX A: DERIVATION OF DISPERSIVE SWMHD BY AVERAGING

In this appendix we rederive the dispersive SWMHD equations from perturbative solutions to the full three dimensional MHD equations. We follow the approach of Camassa *et al.* [12] for their "great lake" equations, but we permit $O(1)$ displacements of the free surface, and include both a Coriolis force and a magnetic field. Consistent with a shallow layer scaling, we use the traditional approximation [5, 6] that takes the angular velocity vector $\mathbf{\Omega}$ to be vertical and independent of z, $2\mathbf{\Omega} = f(x, y)\hat{\boldsymbol{z}}$. This removes $\mathbf{\Omega}$ from the vertical momentum equation (A2b). We do allow $\mathbf{\Omega}$ to vary horizontally, as in a β-plane model [5, 6].

The three-dimensional rotating MHD equations for an incompressible fluid layer, confined between a rigid base at $z = 0$ and a free surface at $z = h(x, y)$, are

$$\partial_t\mathbf{u}_3 + \mathbf{u}_3\cdot\nabla_3\mathbf{u}_3 - \mathbf{B}_3\cdot\nabla_3\mathbf{B}_3 + 2\mathbf{\Omega}\times\mathbf{u}_3 = -\nabla_3 p - g\hat{\boldsymbol{z}}, \tag{A1a}$$

$$\partial_t\mathbf{B}_3 + \mathbf{u}_3\cdot\nabla_3\mathbf{B}_3 - \mathbf{B}_3\cdot\nabla_3\mathbf{u}_3 = 0, \tag{A1b}$$

$$\nabla_3\cdot\mathbf{B} = \nabla_3\cdot\mathbf{u}_3 = 0, \tag{A1c}$$

subject to the boundary conditions that $B_x = u_z = 0$ on $z = 0$. The free surface conditions are $\mathbf{B}\cdot\mathbf{n} = 0$ and $\partial_t h = \mathbf{u}\cdot\mathbf{n}$ on $z = h(x, y, t)$, where the (unnormalised) normal vector $\mathbf{n} = (-\partial_x h, -\partial_y h, 1)$ points upwards out of the fluid. Unlike Camassa *et al.* [12], we work with the unmodified total pressure p, fluid plus magnetic, that vanishes on the free surface.

The important step is to scale the vertical coordinate z with a small parameter $\delta = h/\ell$, the aspect ratio. The divergence-free conditions $\nabla_3 \cdot \mathbf{u}_3 = 0$ and $\nabla_3 \cdot \mathbf{B}_3 = 0$ then suggest scaling the vertical velocity u_z and vertical magnetic field B_z to be $O(\delta)$, so we set $\mathbf{u}_3 = (\mathbf{u}, \delta w)$ and $\mathbf{B}_3 = (\mathbf{B}, \delta b)$. Equations (A1) then become (as in Ref. [12] for the non-magnetic, non-rotating part)

$$
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + w\partial_z\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{B} - b\partial_z\mathbf{B} + 2\boldsymbol{\Omega}\times\mathbf{u} + \nabla(p+gz) &= 0, &\text{(A2a)}\\
\delta^2(\partial_t w + \mathbf{u}\cdot\nabla w + w\partial_z w - \mathbf{B}\cdot\nabla b - b\partial_z b) + \partial_z p + g &= 0, &\text{(A2b)}\\
\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} + w\partial_z\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u} - b\partial_z\mathbf{u} &= 0, &\text{(A2c)}\\
\partial_t b + \mathbf{u}\cdot\nabla b + w\partial_z b - \mathbf{B}\cdot\nabla w - b\partial_z w &= 0, &\text{(A2d)}\\
\nabla\cdot\mathbf{B} + \partial_z b = \nabla\cdot\mathbf{u} + \partial_z w &= 0. &\text{(A2e)}
\end{aligned}
$$

As before, ∇, $\mathbf{u}$, and $\mathbf{B}$ denote the horizontal (x and y) components of the three dimensional objects ∇_3, $\mathbf{u}_3$, and $\mathbf{B}_3$ respectively.

The δ^2 multiplying the vertical acceleration in Eq. (A2b) justifies the hydrostatic approximation leading to the usual (non-dispersive) shallow water equations [5, 6], and to shallow water MHD [1]. The Green–Naghdi equations, and their magnetic analog dispersive SWMHD, contain corrections to the pressure arising from the $O(\delta^2)$ term in the vertical momentum equation (A2b). Following Camassa *et al.* [12] we seek solutions of Eqs. (A2) via an asymptotic expansion in the small parameter δ,

$$
\begin{aligned}
\mathbf{u} &= \mathbf{u}^{(0)} + \delta^2\mathbf{u}^{(1)} + \cdots, \qquad \mathbf{B} = \mathbf{B}^{(0)} + \delta^2\mathbf{B}^{(1)} + \cdots,\\
w &= w^{(0)} + \delta^2 w^{(1)} + \cdots, \qquad b = b^{(0)} + \delta^2 b^{(1)} + \cdots,\\
p &= p^{(0)} + \delta^2 p^{(1)} + \cdots.
\end{aligned} \tag{A3}
$$

The $O(1)$ terms in Eq. (A2b) imply that $p^{(0)}$ is the hydrostatic pressure, $p^{(0)} = g\,[h(x,y,t) - z]$, with the property that $\nabla p^{(0)} = g\nabla h$ is independent of z. The horizontal momentum and induction equations are thus satisfied at leading order by a z-independent velocity $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x,y,t)$ and magnetic field $\mathbf{B}^{(0)} = \mathbf{B}^{(0)}(x,y,t)$. The continuity equations (A2e) then give $w^{(0)} = -z\nabla\cdot\mathbf{u}^{(0)}$ and $b^{(0)} = -z\nabla\cdot\mathbf{B}^{(0)}$. The integration constants have been chosen so that $w = 0$ and $b = 0$ on $z = 0$.

Having determined $\mathbf{u}^{(0)}$, $\mathbf{B}^{(0)}$, $w^{(0)}$, and $b^{(0)}$, the vertical momentum equation (A2b) gives

$$
\partial_z p^{(1)} = z\left[\partial_t\nabla\cdot\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\cdot\nabla(\nabla\cdot\mathbf{u}^{(0)}) - (\nabla\cdot\mathbf{u}^{(0)})^2 - \mathbf{B}^{(0)}\cdot\nabla(\nabla\cdot\mathbf{B}^{(0)}) + (\nabla\cdot\mathbf{B}^{(0)})^2\right], \tag{A4}
$$

at $O(\delta^2)$. Since the term in square brackets $[\cdot]$ is independent of z, Eq. (A4) integrates to give

$$
p^{(1)} = \frac{1}{2}(z^2 - h(x,y,t)^2)[\cdot], \tag{A5}
$$

using the free surface condition $p = 0$ on $z = h(x,y,t)$. Moreover,

$$
\nabla p^{(1)} = \frac{1}{2}(z^2 - h^2)\nabla[\cdot] - h(\nabla h)[\cdot], \tag{A6}
$$

and

$$
\begin{aligned}
\int_0^h \nabla p^{(1)} dz &= -\frac{1}{3}h^3\nabla[\cdot] - h^2(\nabla h)[\cdot] = -\frac{1}{3}\nabla(h^3[\cdot])\\
&= -\frac{1}{3}\nabla\left\{h^2\mathcal{D}(h\nabla\cdot\mathbf{u}^{(0)}) + h^3(\nabla\cdot\mathbf{B}^{(0)})^2 - h^3\mathbf{B}^{(0)}\cdot\nabla(\nabla\cdot\mathbf{B}^{(0)})\right\},
\end{aligned} \tag{A7}
$$

where the $\nabla\cdot\mathbf{u}^{(0)}$ term has been rewritten [8, 10, 13, 16, 17, 28] using $\mathcal{D} = \partial_t + \mathbf{u}^{(0)}\cdot\nabla$. Spatial differentiation does not commute with layer averaging because the layer depth h is itself a function of x and y.

In principle, the $O(\delta^2)$ corrections $\mathbf{u}^{(1)}$, $\mathbf{B}^{(1)}$, $w^{(1)}$, and $b^{(1)}$ may be computed from the $O(\delta^2)$ terms in Eqs. (A2). However, Camassa *et al.* [12], following Su and Gardner [13], preferred to derive equations for the layer mean velocity $\overline{\mathbf{u}}$ given by

$$
\overline{\mathbf{u}}(x,y,t) = \frac{1}{h(x,y,t)}\int_0^{h(x,y,t)} \mathbf{u}(x,y,z,t)dz, \tag{A8}
$$

where an overbar denotes a depth-averaged quantity. Wu [48] showed that

$$
h\,(\overline{\partial_t F + \mathbf{u}_3\cdot\nabla_3 F}) = \partial_t(h\overline{F}) + \nabla\cdot(h\,\overline{\mathbf{u}F}), \tag{A9}
$$

for general F, by integrating by parts in z and using the kinematic boundary conditions for u_z at the two material surfaces $z = 0$ and $z = h(x, y, t)$. Similarly,

$$h\,(\overline{\mathbf{B}_3 \cdot \nabla_3 F}) = \nabla\cdot(h\,\overline{\mathbf{B}F}), \tag{A10}$$

using the tangency conditions that $\mathbf{B}\cdot\mathbf{n} = 0$ on $z = 0$ and $z = h(x, y)$.

The layer averaged continuity equation

$$\partial_t h + \nabla\cdot(h\overline{\mathbf{u}}) = 0, \tag{A11}$$

is given by Eq. (A9) with $F = 1$, for which the left hand side vanishes. Equation (A2a) may be integrated using Eq. (A9) with $F = u_x$ and $F = u_y$, and Eq. (A10) with $F = B_x$ and $F = B_y$, to give

$$\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}\mathbf{u}} - h\overline{\mathbf{B}\mathbf{B}}) + 2\boldsymbol{\Omega}\times h\overline{\mathbf{u}} + \int_0^h \nabla p\; dz = 0, \tag{A12}$$

were the $O(\delta^2)$ correction to the hydrostatic pressure is given by Eq. (A7). The vertically averaged Reynolds stress factorizes as $\overline{\mathbf{u}\mathbf{u}} = \overline{\mathbf{u}}\,\overline{\mathbf{u}} + O(\delta^4)$, because the cross term $\overline{\mathbf{u}^{(1)}\mathbf{u}^{(1)}}$ in the z-integration is $O(\delta^4)$ [12, 13]. Thus Eq. (A12) becomes

$$\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}}\,\overline{\mathbf{u}} - h\overline{\mathbf{B}}\,\overline{\mathbf{B}}) + 2\boldsymbol{\Omega}\times h\overline{\mathbf{u}} + gh\nabla h - \frac{1}{3}\nabla\left\{h^2\mathcal{D}(h\nabla\cdot\overline{\mathbf{u}}) + h^3(\nabla\cdot\overline{\mathbf{B}})^2 - h^3\overline{\mathbf{B}}\cdot\nabla(\nabla\cdot\overline{\mathbf{B}})\right\} = O(\delta^4), \tag{A13}$$

where, to close the system, the $p^{(1)}$ term is evaluated using $\overline{\mathbf{u}}$ and $\overline{\mathbf{B}}$ instead of $\mathbf{u}^{(0)}$ and $\mathbf{B}^{(0)}$ by incurring a further error of $O(\delta^4)$. Similarly, the induction equation may be integrated to give

$$\partial_t(h\overline{\mathbf{B}}) + \nabla\cdot(h\overline{\mathbf{u}}\,\overline{\mathbf{B}}\,) = \nabla\cdot(h\overline{\mathbf{B}}\,\overline{\mathbf{u}}) + O(\delta^4). \tag{A14}$$

Note that it is is unnecessary to compute $\mathbf{u}^{(1)}$ or $\mathbf{B}^{(1)}$ explicitly. In particular, the structure in z need not be specified, although it would be natural to seek solutions involving a term proportional to z^2 plus a z-independent term.

The horizontal component of $\boldsymbol{\Omega}$ that we have omitted above was retained by Bazdenkov *et al.* [10], who computed the leading order corrections to the Green–Naghdi momentum equation, Eq. (A13) with $\overline{\mathbf{B}} = 0$. However, these corrections are formally $O(h/\ell)$, so a fully consistent treatment to $O(h^2/\ell^2)$ would require both an expansion in powers of δ instead of δ^2, $\mathbf{u} = \mathbf{u}^{(0)} + \delta\mathbf{u}^{(1)} + \delta^2\mathbf{u}^{(2)} + \cdots$ in place of Eq. (A3), and the explicit evaluation of the first correction $\mathbf{u}^{(1)}$.

In our Lie–Poisson Hamiltonian treatment, and in Miles and Salmon's [16] derivation of the Green–Naghdi equations from Hamilton's variational principle, the horizontal velocity appearing in the three-dimensional Hamiltonian or action integral is taken by fiat to be independent of z. In other words, it is $\mathbf{u}^{(0)}$ that appears in the Hamiltonian or in the action. The Hamiltonian structure then leads to the Green–Naghdi equations for $\mathbf{u}^{(0)}$. Similarly, Green and Naghdi's [8] derivation took the vertical velocity w to be precisely linear in z. On the other hand, in this derivation directly from the three-dimensional Euler equations, the horizontal velocity $\mathbf{u} = \mathbf{u}^{(0)} + \delta^2\mathbf{u}^{(1)} + \cdots$ varies in z, albeit not at leading order, and the vertical velocity w is not precisely linear in z. Thus the Green–Naghdi, or dispersive SWMHD, equations hold for the layer average $\overline{\mathbf{u}} = \mathbf{u}^{(0)} + \delta^2\overline{\mathbf{u}^{(1)}}$ instead of for $\mathbf{u}^{(0)}$.

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