HIGHER DIMENSIONAL DIVERGENCE FOR MAPPING CLASS GROUPS

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Abstract. In this paper we investigate the higher dimensional divergence functions of mapping class groups of surfaces. We show that these functions exhibit phase transitions at the quasi-flat rank (as measured by $3 \cdot$ genus + number of punctures $- 3$).

1. Introduction

In this paper we study *higher dimensional divergence functions*, a type of filling function particularly significant in spaces with an interesting geometry on the boundary at infinity, such as spaces of non-positive curvature. In some sense, these functions describe the spread of geodesics, and the filling close to the boundary at infinity. They provide a powerful quasi-isometric invariant with which to study the large-scale geometry of a space or a group.

Given a fixed point x_0 in a metric space X and two constants $A > 0$ and $\lambda \in (0, 1)$, the *k–dimensional divergence function* (or *k–divergence*) *in r*, roughly speaking, measures the minimal filling of k–dimensional spheres avoiding the ball $B(x_0, r)$, and of area at most Ar^k, by $(k+1)$–dimensional balls that are outside the open ball $B(x_0, \lambda r)$. The choices of the center x_0 and of the parameters A and λ do not affect the order of the k–divergence. Versions of divergence functions were defined for non-positively curved manifolds in [BF] (there, the spheres and the balls filling them are Lipschitz maps of the corresponding Euclidean spheres and balls into the given manifold, and the volumes are computed using the fact that such maps are differentiable almost everywhere), and for $CAT(0)$–spaces and integral currents in [Wen2].

In the present paper, we investigate the behavior of higher dimensional divergence functions for mapping class groups. Since we work in the setting of finitely generated groups, this allows us to use the classifying space for the group and a version of divergence defined using combinatorial filling functions. The k–divergence functions, for k at least 1, are also called *higher dimensional* divergence functions, by contrast with the 0–divergence functions which by their very nature are less of

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isoperimetric filling functions, and more of distortion functions of exteriors of large balls.

A central notion in non-positively curved geometry is that of *rank*, which is often taken as the largest dimension at which Euclidean behavior can still be detected, where the meaning of "behavior" can vary as we see below; in a sense, the rank gives a dimension above which hyperbolic features are bound to appear. There are several ways to define the rank, and the connections between the different definitions are not fully understood (see for instance [Gro2, §6.B_2], where seven possible definitions of rank are listed). To begin with, there exists the notion of *(quasi-)flat rank*, i.e. the maximal dimension of a Euclidean space that can be (quasi-)isometrically embedded in the given space. In the case of a locally compact complete simply connected $CAT(0)$–space that admits a cocompact action by isometries, the quasi-flat rank equals the flat rank [AS, Gro2, Kle]; thus, this notion of rank has the two-fold advantage of being consistent with the classical notion of rank for symmetric spaces, and of being easily transferred to groups acting properly cocompactly on $CAT(0)$–spaces, *via* its quasi-flat version. The quasi-flat rank of a mapping class group $\mathcal{MCG}(S)$ equals the maximal rank of a free abelian subgroup of $\mathcal{MCG}(S)$, i.e., the *complexity* of S, as measured by $\xi(S) = 3g+p-3$, where g is the genus and p is the number of boundary components of S. (We do not require that mapping classes fix the "boundary", so formally each boundary component should be considered as a "puncture"). That the algebraic rank is equal to $\xi(S)$ was proven by [BLM]; the equality between the algebraic rank and the quasi-flat rank was proven in [BM, Ham].

Another notion of rank is defined using isoperimetric functions. One considers the rank in this sense to be the maximal dimension for which these functions are of the same order as those in Euclidean spaces.

Finally, a third notion of rank is obtained by considering the dimension in which there is a phase transition for the higher dimensional divergence functions.

All these ranks agree for mapping class groups. The equality between the isoperimetric rank and the quasi-flat rank follows from Theorems 3.1 and 3.2. The main result of this paper is a description of the behavior of the higher dimensional divergence functions of the mapping class group; in particular, of the fact that they display a "phase transition", when the dimension equals the quasi-flat rank, whence providing the equality of the first two notions of rank with the third. Such phase transitions can also be seen in the setting of Hadamard manifolds, see e.g., [Hin, Leu, Wen2].

Theorems 3.3 and 3.4 *Let S be a compact orientable surface and let Div_k be the k–dimensional divergence of $\mathcal{MCG}(S)$.*

If $k < \xi(S)$ then $\mathrm{Div}_k \succeq x^{k+2}$.

If $k \geq \xi(S)$ then $\mathrm{Div}_k(x) = o(x^{k+1})$; furthermore, if S is either genus 0 or 1, or genus 2 with empty boundary, then $\mathrm{Div}_k(x) \asymp x^k$.

The relation of one function being "asymptotically bounded" by another, $\succeq$, or two functions being asymptotically equivalent, $\asymp$, are made precise in Section 2.1.

A special case of the above result is that it establishes that the 1-dimensional divergence is at least cubic for surfaces with $\xi(S) \geq 2$, while it is proved in [ABDDY] that for such surfaces the 1–dimensional divergence is at most quartic. It would be interesting to know the exact order of the 1-dimensional divergence. Besides the estimate in [ABDDY], the only other previously known result about divergence in mapping class groups is that the 0–divergence is quadratic [Beh].

One of the methods we develop in this work is a technique to obtain lower bounds for higher divergence from lower bounds on zero–divergence; we apply this to the study of mapping class groups and expect it to be useful in many other contexts as well.

The plan of the paper is as follows. In Section 2 we recall some basic notions and establish notation which we will use in the paper, in particular some combinatorial terminology and results on mapping class groups. Also, we recall background from [BD2] on combinatorial formulations of isoperimetric and divergence functions and their behavior in the presence of a combing. Section 3 contains the theorem describing the behavior of divergence functions in mapping class groups.

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2. Preliminaries

2.1. **General terminology.** Given two functions f, g which both map $\mathbb{R}_+$ to itself, we write $f \preceq_{C,k} g$ for some constant $C \geq 1$ and integer $k \geq 1$, if

$$f(x) \leq Cg(Cx + C) + Cx^k + C \quad \text{for all } x \in \mathbb{R}_+.$$

We write $f \asymp_{C,k} g$ if and only if $f \preceq_{C,k} g$ and $g \preceq_{C,k} f$. Two functions $\mathbb{R}_+ \to \mathbb{R}_+$ are said to be k*–asymptotically equal* if there exists $C \geq 1$ s.t. $f \asymp_{C,k} g$. This is an equivalence relation.

When at least one of the two functions f, g involved in the relations above is an n–dimensional isoperimetric or divergence function, we automatically consider only relations where $k = n$, therefore k will no longer appear in the subscript of the relation. When irrelevant, we do not mention the constant C either and likewise remove the corresponding subscript.

We use the notations $f = O(g)$ and $f = o(g)$, for f and g real-valued functions of one real variable, with their standard meaning.

In a metric space (X, dist), the *open* R*–neighborhood* of a subset A, i.e. $\{x \in X : \mathrm{dist}(x, A) < R\}$, is denoted by $\mathcal{N}_R(A)$. In particular, if $A = \{a\}$ then $\mathcal{N}_R(A) = B(a, R)$ is the open R–ball centered at a. We use the notation $\overline{\mathcal{N}}_R(A)$ and $\bar{B}(a, R)$ to designate the corresponding *closed neighborhood* and *closed ball* defined by non-strict inequalities. We make the convention that $B(a, R)$ and $\bar{B}(a, R)$ are the empty set for $R < 0$ and any $a \in X$.

Fix two constants $L \geq 1$ and $C \geq 0$. A map $\mathfrak{q}\colon Y \to X$ is said to be

- (L, C)*–quasi-Lipschitz* if
$$\mathrm{dist}(\mathfrak{q}(y), \mathfrak{q}(y')) \leq L\mathrm{dist}(y, y') + C, \text{ for all } y, y' \in Y\,;$$
- an (L, C)*–quasi-isometric embedding* if moreover
$$\mathrm{dist}(\mathfrak{q}(y), \mathfrak{q}(y')) \geq \frac{1}{L}\mathrm{dist}(y, y') - C \text{ for all } y, y' \in Y\,;$$
- an (L, C)-*quasi-isometry* if it is an (L, C)–quasi-isometric embedding $\mathfrak{q}\colon Y \to X$ satisfying the additional assumption that $X \subset \mathcal{N}_C(\mathfrak{q}(Y))$.

- an (L,C)–*quasi-geodesic* if it is an (L,C)–quasi-isometric embedding defined on an interval of the real line;
- a *bi-infinite* (L,C)*–quasi-geodesic* when defined on the entire real line.

In the last two cases the terminology is extended to the image of $\mathfrak{q}$. When the constants L,C are irrelevant they are not mentioned.

We call $(L,0)$–quasi-isometries (quasi-geodesics) *L–bi-Lipschitz maps (paths)*. If an (L,C)–quasi-geodesic $\mathfrak{q}$ is L–Lipschitz then $\mathfrak{q}$ is called an (L,C)*–almost geodesic*. Every (L,C)-quasi-geodesic in a geodesic metric space is at bounded (in terms of L,C) distance from an $(L+C,C)$–almost geodesic with the same end points, see e.g. [BBI, Proposition 8.3.4]. Therefore, without loss of generality, we assume in this text that all quasi-geodesics are in fact almost geodesics, in particular that they are continuous.

Given two subsets $A,B\subset\mathbb{R}$, a map $f\colon\ A\to B$ is said to be *coarsely increasing* if there exists a constant D such that for each a,b in A satisfying $a+D<b$, we have that $f(a)\leq f(b)$. Similarly, we define *coarsely decreasing* and *coarsely monotonic* maps. A map between quasi-geodesics is coarsely monotonic if it defines a coarsely monotonic map between suitable nets in their domain.

A metric space is called

- *proper* if all its closed balls are compact;
- *cocompact* if there exists a compact subset K in X such that all the translations of K by isometries of X cover X;
- *periodic* if it is geodesic and for fixed constants $L\geq 1$ and $C\geq 0$ the image of some fixed ball under (L,C)–quasi-isometries of X covers X;
- a *Hadamard space* if X is geodesic, complete, simply connected and satisfies the CAT(0) condition;
- a *Hadamard manifold* if moreover X is a smooth Riemannian manifold.

2.2. **Combinatorial terminology.** We use the standard terminology related to simplicial complexes as it appears in [Hat]. When we speak of simplicial complexes in what follows, we mean their topological realisation most of the time. Throughout the paper, we assume that all simplicial complexes are connected. We consider every simplicial complex endowed with a "large scale metric structure" defined by assuming that all edges have length one and taking the shortest path metric on the 1-skeleton.

We say that a simplicial complex X has a *bounded* (L,C)*–quasi-geodesic combing*, where $L\geq 1$ and $C\geq 0$, if for every $x\in X^{(1)}$ there exists a way to assign to every element $y\in X^{(1)}$ an (L,C)–quasi-geodesic $\mathfrak{q}_{xy}$ connecting y to x in $X^{(1)}$, such that

$$\mathrm{dist}(\mathfrak{q}_{xy}(i),\mathfrak{q}_{xa}(i))\leq L\mathrm{dist}(y,a)+L\,,$$

for all $x,y,a\in X^{(1)}$ and $i\in\mathbb{R}$. Here the quasi-geodesics are assumed to be extended to $\mathbb{R}$ by constant maps.

Given an n–dimensional simplicial complex $\mathcal{C}$, we call the closed simplices of dimension n the *chambers* of $\mathcal{C}$.

Given a simplicial map $f\colon\ X\to Y$, where X,Y are simplicial complexes, X of dimension n, we call *f–non-collapsed chambers in X* the chambers whose images by f stay of dimension n. We denote by X_{Vol} the set of f–non-collapsed chambers. We define the *volume of f* to be the (possibly infinite) cardinality of X_{Vol}.

Recall that a group G is *of type* $\mathcal{F}_k$ if it admits an Eilenberg-MacLane space $K(G,1)$ whose k-skeleton is finite.

Proposition 2.1 ([AWP], Proposition 2)**.** *If a group G acts cellularly on a CW-complex X, with finite stabilizers of points and such that $X^{(1)}/G$ is finite then G is finitely generated and quasi-isometric to X. Moreover, if X is n-connected and $X^{(n+1)}/G$ is finite then G is of type $\mathcal{F}_{n+1}$.*

Conversely, it is easily seen that for a group of type $\mathcal{F}_{n+1}$ one can define an $(n+1)$-dimensional n-connected simplicial complex X on which G acts properly discontinuously by simplicial isomorphisms, with trivial stabilizers of vertices, such that X/G has finitely many cells. Any two such complexes X, Y are quasi-isometric, and the quasi-isometry, which can initially be seen as a bi-Lipschitz map between two subsets of vertices, can be easily extended to a simplicial map $X \to Y$ [AWP, Lemma 12].

A group is *of type* $\mathcal{F}_\infty$ if and only if it is of type $\mathcal{F}_k$ for every $k \in \mathbb{N}$. It was proven in [ECH$^+$, Theorem 10.2.6] that every combable group is of type $\mathcal{F}_\infty$.

2.3. **Higher dimensional filling functions.** Since a finite $(n+1)$–presentation of a group composed only of simplices can always be found, it suffices to restrict to simplicial complexes when discussing filling problems. Throughout the paper, we make use of the terminology related to the n–dimensional filling functions in the simplicial setting as described in full detail in [BD2]. We briefly recall a number of relevant notions and results.

For the rest of the paper, we assume that all the simplicial complexes that we consider are universal covers of compact simplicial complexes. We also assume that, in the study of filling functions up to dimension n, the simplicial complexes considered have dimension at least $n+1$ and are n–connected.

When we speak of manifolds we always mean *manifolds with a simplicial complex structure.*

We denote by V an arbitrary m–dimensional connected compact sub-manifold of $\mathbb{R}^m$, where $m \geq 2$ is an integer and V is smooth or piecewise linear, and with boundary. We denote its boundary by ∂V. Unless otherwise stated, the standing assumption is that ∂V is connected.

Given V as above, a *domain modelled on V in X* (also called a *V–domain*) is a simplicial map $\mathfrak{d}$ of $\mathcal{D}$ to $X^{(m)}$, where $\mathcal{D}$ is a simplicial structure on V. When the manifold V is irrelevant we simply call $\mathfrak{d}$ a *domain of dimension m* (somewhat abusively, since it might have its entire image inside $X^{(m-1)}$); we also abuse notation by using $\mathfrak{d}$ to denote both the map and its image.

A *hypersurface in X modelled on ∂V* (also called a *∂V–hypersurface*) is a simplicial map $\mathfrak{h}$ of $\mathcal{M}$ to $X^{(m-1)}$, where $\mathcal{M}$ is a simplicial structure of the boundary ∂V. Again, we abuse notation by letting $\mathfrak{h}$ also denote the image of the above map, and we also call both $\mathfrak{h}$ and its image a *hypersurface of dimension $m-1$*.

According to the terminology introduced previously, $\mathcal{D}_{\mathrm{Vol}}$, respectively $\mathcal{M}_{\mathrm{Vol}}$, is the set of $\mathfrak{d}$–non-collapsed chambers (respectively $\mathfrak{h}$–non-collapsed chambers). The *volume* of $\mathfrak{d}$ (respectively $\mathfrak{h}$) is the cardinality of $\mathcal{D}_{\mathrm{Vol}}$, respectively $\mathcal{M}_{\mathrm{Vol}}$.

We sometimes say that the domain $\mathfrak{d}$ is a *V–domain*, and $\mathfrak{h}$ is a *∂V–hypersurface*. When V is a closed ball in $\mathbb{R}^m$, we call $\mathfrak{d}$ an *m–dimensional ball* and $\mathfrak{h}$ an *$(m-1)$–dimensional sphere.*

Definition 2.2. A k–dimensional hypersurface $\mathfrak{h}$ is called *η–round* for a constant $\eta > 0$ if $\mathrm{diam}(\mathfrak{h}) \le \eta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$.

We say that a domain $\mathfrak{d}$ *fills* a hypersurface $\mathfrak{h}$ if this pair corresponds to a $(k+1)$–dimensional connected compact smooth sub-manifold with boundary V in $\mathbb{R}^{k+1}$ satisfying $\mathcal{D} \cap \partial V = \mathcal{M}$ and $\mathfrak{d}|_{\mathcal{M}} = \mathfrak{h}$, possibly after pre-composing $\mathfrak{h}$ with a simplicial equivalence of $\mathcal{M}$.

The *filling volume* of the hypersurface $\mathfrak{h}$, $\mathrm{FillVol}(\mathfrak{h})$, is the minimum of all the volumes of domains filling $\mathfrak{h}$. If no domain filling $\mathfrak{h}$ exists then we set $\mathrm{FillVol}(\mathfrak{h}) = \infty$.

Definition 2.3. The *k–th isoperimetric function*, also known as the *k–th filling function*, of a simplicial complex X is the function $\mathrm{Iso}_k \colon \mathbb{R}_+^* \to \mathbb{R}_+ \cup \{\infty\}$ such that $\mathrm{Iso}_k(x)$ is the supremum of the filling volume $\mathrm{FillVol}(\mathfrak{h})$ over all k–dimensional spheres $\mathfrak{h}$ of volume at most Ax^k.

In what follows the constant $A > 0$ from Definition 2.3 is fixed, but not made explicit. Two filling functions corresponding to two different values of A are equivalent in the sense of the relation $\asymp$.

The isoperimetric function has a more general version, using instead of the sphere and its filling with a ball, a hypersurface and its filling with a domain, both modelled on a $(k+1)$–dimensional submanifold with boundary V in $\mathbb{R}^{k+1}$. We then define as above the filling function, denoted Iso_V.

According to [AWP, Theorem 1, Corollary 3], if two simplicial complexes, X_1 and X_2, of dimension at least $k+1$, are quasi-isometric, then $\mathrm{Iso}_V^{X_1} \asymp \mathrm{Iso}_V^{X_2}$ for every domain V. Therefore the corresponding isoperimetric functions Iso_k and Iso_V are well defined, up to the equivalence relation $\asymp$, for every group of type $\mathcal{F}_{n+1}$.

In [BD2] we proved that under certain conditions, which are satisfied in the presence of a bounded quasi-geodesic combing, an arbitrary sphere has a partition into round spheres, such that the sum of the volumes of the spheres in the partition is bounded by a multiple of the volume of the initial sphere. Here a *partition* consists of the following: finitely many spheres (more generally, hypersurfaces) $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ compose a partition of a sphere $\mathfrak{h}$ if by filling all of them one obtains a ball filling $\mathfrak{h}$ (see [BD2]). The hypersurfaces $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ are called *contours of the partition.*

Theorem 2.4 ([BD2])**.** *Let X be a simplicial complex with a bounded quasi-geodesic combing, and let $k \ge 2$ be an integer.*

Then for every $\varepsilon > 0$ there exists a constant $\eta > 0$ such that every k–dimensional sphere $\mathfrak{h}$ has a partition with contours $\mathfrak{h}_1, \dots \mathfrak{h}_n$ that are η–round spheres, and contours $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ that are hypersurfaces of volume and filling volume zero such that

(1) $\sum_{i=1}^n \mathrm{Vol}(\mathfrak{h}_i) \le 2 \cdot 6^{k+1} \mathrm{Vol}(\mathfrak{h})$.

(2) $\mathfrak{h}_1, \dots \mathfrak{h}_n$ *and* $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ *are contained in the tubular neighborhood* $\mathcal{N}_R(\mathfrak{h})$, *where* $R = \varepsilon \mathrm{Vol}(\mathfrak{h})^{1/k}$.

A consequence of the above is that isoperimetric functions are bounded by the corresponding Euclidean isoperimetric functions.

Corollary 2.5 (The Federer-Fleming inequality for groups; [BD2])**.** *Assume that the simplicial complex X has a bounded quasi-geodesic combing. Then for every $k \ge 1$, $\mathrm{Iso}_k(x) \preceq x^{k+1}$. Moreover for $k = 2$ the supremum of $\mathrm{Iso}_V(x)$ over all handlebodies V is $\preceq x^3$.*

The inequality $\mathrm{Iso}_k(x) \preceq x^{k+1}$ was proved by Federer–Fleming [FF] for integral currents, in Euclidean spaces, and it was later extended by S. Wenger to complete metric spaces with a cone-type inequality [Wen1]. For Lipschitz fillings we refer to [ECH+]. For fillings of Riemannian hypersurfaces in Banach spaces, $\mathrm{Iso}_k(x) \preceq x^{k+1}$ was proved by Gromov [Gro1]. We were informed by S. Wenger that the simplicial version of the inequality can also be deduced from [Wen1] and [Wh, Theorem 1, p. 435].

The reduction to the subset of round spheres also applies to another type of filling function, the divergence. We briefly recall the definition of divergence, and refer to [BD2] for more details.

We fix a constant $0 < \delta < 1$ and an integer $2 \leq k \leq n-1$, where n is the dimension of the simplicial complex X we work in. Given a vertex c in X, a k–dimensional hypersurface $\mathfrak{h}\colon \mathcal{M} \to X$ modelled on ∂V such that $k \leq n-1$, and a number $r > 0$ that is at most the distance from c to $\mathfrak{h}(\mathcal{M}^{(1)})$, *the divergence* of this quadruple, denoted $\mathrm{div}(\mathfrak{h}, c; r, \delta)$, is the infimum of all volumes of domains modelled on V filling $\mathfrak{h}$ and disjoint from $B(c, \delta r)$. If no such domain exists then we set $\mathrm{div}(\mathfrak{h}, c; r, \delta) = \infty$.

Definition 2.6. The *divergence function modelled on V* of the complex X, denoted $\mathrm{Div}_V(r, \delta)$, is the supremum of all finite values of $\mathrm{div}(\mathfrak{h}, c; r, \delta)$, where $\mathfrak{h}$ is a hypersurface modelled on ∂V with the distance from c to $\mathfrak{h}(\mathcal{M}^{(1)})$ at least r and $\mathrm{Vol}(\mathfrak{h})$ at most Ar^k.

When V is the $(k+1)$–dimensional unit ball, $\mathrm{Div}_V(r, \delta)$ is denoted $\mathrm{Div}^{(k)}(r, \delta)$, and it is called the *$k$–dimensional divergence function* (or the *k–th divergence function*) of X.

In the definition of divergence, as for the isoperimetric function, we fix the constant $A > 0$ once and for all, and we do not mention it anymore.

Proposition 2.7 ([BD2])**.** *Let ε and δ be small enough positive constants. Assume that $\mathrm{Iso}_V(x) \leq \varepsilon x^{k+1}$. Then $\mathrm{Div}_V(x, \delta) = \mathrm{Iso}_V(x)$ for every x large enough.*

Arguments similar to those used for the usual isoperimetry allow to reduce the problem of estimating the divergence to spheres that are round, when a bounded combing exists.

Theorem 2.8 ([BD2])**.** *Assume that X is a simplicial complex of dimension n endowed with a bounded quasi-geodesic combing. Let V be a $(k+1)$–dimensional connected compact sub-manifold of $\mathbb{R}^{k+1}$ with connected boundary, where $2 \leq k \leq n-1$.*

For every $\varepsilon > 0$ there exists $\eta > 0$ such that the following holds.

Consider the restricted divergence function $\mathrm{Div}_V^r(x, \delta)$, obtained by taking the supremum only over hypersurfaces modelled on ∂V that are η–round, of volume at most $2Ax^k$ and situated outside balls of radius x.

Assume that $\mathrm{Div}_V^r(x, \delta) \leq Br^\beta$ for some $\beta \geq k+1$ and $B > 0$ universal constant. Then the general divergence function $\mathrm{Div}_V(x, \delta(1-\varepsilon))$ is at most $B' r^\beta$ for some $B' > 0$ depending on B, ε, η and X.

2.4. **Mapping class groups and marking complexes.** This section contains relevant background material on the mapping class group. Here we introduce the notion used throughout the paper and recall the relevant facts we need about the marking complex and hierarchy paths. For convenience, we sketch the constructions of the main objects we will use below, details and arguments, as referenced, below can be found in [Beh, BKMM, BM, MM].

Let S be a compact oriented surface of genus g, with p boundary components. The *curve complex* of S, denoted $\mathcal{C}(S)$, is the simplicial flag complex with a vertex for every homotopy class of essential non-peripheral simple closed curve on S, and an edge between each pair of vertices whose corresponding curves can be realized disjointly on the surface. The mapping class group of S is denoted $\mathcal{MCG}(S)$. If $\mu \in \mathcal{MCG}(S)$ and $Y \subseteq S$, we write $\pi_{\mathcal{C}(Y)}(\mu)$ to denote the *subsurface projection* of μ to $\mathcal{C}(Y)$. We will use a quasi-isometric model of $\mathcal{MCG}(S)$, the *marking complex*, $\mathcal{K}(S)$, defined as follows (see [MM] for details). Its vertices, called *markings*, consist of the following pair of data:

- *base curves*: a multicurve consisting of $3g+p-3$ components, i.e., a maximal simplex in $\mathcal{C}(S)$. This collection is denoted $\mathrm{base}(\mu)$.
- *transversal curves*: to each curve $\gamma \in \mathrm{base}(\mu)$ is associated an essential curve. Letting T denote the complexity 1 component of $S \setminus \bigcup_{\alpha \in \mathrm{base}\,\mu, \alpha \neq \gamma} \alpha$, the transversal curve to γ is a curve $t(\gamma) \in \mathcal{C}(T)$ with $\mathrm{dist}_{\mathcal{C}(T)}(\gamma, t(\gamma)) = 1$.

Two vertices $\mu, \nu \in \mathcal{K}(S)$ are connected by an edge if either of the two conditions hold:

(1) *Twist*: μ and ν differ by a Dehn twist along one of the base curves: $\mathrm{base}(\mu) = \mathrm{base}(\nu)$ and all their transversal curves agree except for $t_\mu(\gamma)$, obtained from $t_\nu(\gamma)$ by twisting once about the curve γ.
(2) *Flip*: The base curves and transversal curves of μ and ν agree except for one pair $(\gamma, t(\gamma)) \in \mu$ for which the corresponding pair consists of the same pair but with the roles of base and transversal reversed.

For the careful reader, we note that the technicality that after performing one Flip the new base curve may intersect several transversal curves. Nevertheless by [MM, Lemma 2.4], there is a finite set of natural ways to resolve this issue, yielding a uniformly bounded number of flip moves which can be obtained by flipping the pair $(\gamma, t(\gamma)) \in \mu$; an edge connects each of these possible flips to μ.

The key fact we need about the marking complex is the following:

Theorem 2.9 ([MM])**.** *The graph $\mathcal{K}(S)$ is locally finite and the mapping class group acts cocompactly and properly discontinuously on it. In particular, the orbit map yields a quasi-isometry from $\mathcal{MCG}(S)$ to $\mathcal{K}(S)$.*

A quasi-geodesic $\mathfrak{g}$ in $\mathcal{K}(S)$ is $\mathcal{C}(S)$*–monotonic* if one can associate a geodesic $\mathfrak{t}$ in $\mathcal{C}(S)$ which *shadows* $\mathfrak{g}$ in the sense that if the endpoints of $\mathfrak{g}$ are μ and ν, then the endpoints of $\mathfrak{t}$ are vertices of $\pi_{\mathcal{C}(S)}(\mathrm{base}(\mu))$ and respectively $\pi_{\mathcal{C}(S)}(\mathrm{base}(\nu))$, moreover there is a coarsely monotonic map $v\colon\, \mathfrak{g} \to \mathfrak{t}$ such that $v(\rho)$ is a vertex in $\pi_{\mathcal{C}(S)}(\mathrm{base}(\rho))$ for every vertex $\rho \in \mathfrak{g}$.

Let g be a pseudo-Anosov on S. A fundamental fact about the action of the mapping class group on the curve complex, established in [MM, Proposition 7.6], is that there exists a quasi-invariant axis of g in $\mathcal{C}(S)$, that is, a bi-infinite geodesic $\mathfrak{a}$ in $\mathcal{C}(S)$ such that $g^n\mathfrak{a}$ is at Hausdorff distance $O(1)$ from $\mathfrak{a}$ for all $n \in \mathbb{Z}$. The set of distances $\mathrm{dist}_{\mathcal{K}(S)}(\nu, g\nu)$ with $\pi_{\mathcal{C}(S)}(\nu)$ at distance $O(1)$ from $\mathfrak{a}$ admits a minimum. Let μ be a point such that $\mathrm{dist}_{\mathcal{K}(S)}(\mu, g\mu)$ is this minimum, and let $\mathfrak{h}$ be a hierarchy path (i.e. a particular type of quasi-geodesic in $\mathcal{K}(S)$ as constructed in [MM]) joining μ and $g\mu$, such that $\mathfrak{h}$ shadows a tight geodesic $\mathfrak{t}$ in $\mathcal{C}(S)$ at distance $O(1)$

from $\mathfrak{a}$. We call the bi-infinite quasi-geodesic $\mathfrak{p} = \bigcup_{n\in\mathbb{Z}} g^n\mathfrak{h}$ a *quasi-axis* of g in $\mathcal{K}(S)$.

It follows from [Beh] that there exists a constant $C > 0$ such that for every $\delta > 0$ small enough, if the surface S has complexity at least 2, then g^{-n} and g^n may be joined in $\mathcal{MCG}(S)$ by a path whose length is of order n^2 which is disjoint from the δn–ball around 1. Equivalently, for every $\mu \in \mathfrak{p}$, $g^{-n}\mu$ and $g^n\mu$ may be joined in $\mathcal{K}(S)$ by a path of length of order n^2 and disjoint from the δn–ball around μ.

We now recall some terminology from [BKMM, §2.1.8]. Let Δ be an arbitrary simplex in the curve graph $\mathcal{C}(S)$; vertices of a simplex are often referred to as a *multicurve* on S. The *open subsurface* open(Δ) determined by Δ is the union of all the components of $S \setminus \Delta$ with complexity at least 1 and of all the annuli homotopic to curves in Δ. We call *components of an open subsurface* the list of all the components and annuli defined as above by a multicurve Δ. For $\Delta = \emptyset$ the whole surface S is the unique component of open(Δ).

In what follows we use the notation $W_1, \ldots, W_k$, with $0 < k \le \xi(S)$, for the list of components of open(Δ), and for every $i \in \{1, 2, \ldots, k\}$ we denote $S \setminus W_i$ by W_i^c. Let Δ_i denote the boundary ∂W_i; note that Δ is the union of the Δ_i, after removing any duplicate curves.

Define $\mathcal{Q}(\Delta)$ to be the set of elements of $\mathcal{K}(S)$ whose base curves contain Δ. Note that this is coarsely the subset of the marking complex associated to the left coset of the stabilizer of the multicurve given by the curves in Δ.

Lemma 2.10. (1) *If μ and ν are two markings in $\mathcal{Q}(\Delta)$ then there exist hierarchy paths joining them and entirely contained in $\mathcal{Q}(\Delta)$.*

(2) *The set $\mathcal{Q}(\Delta)$ is quasi-isometric to $\mathcal{K}(W_1) \times \cdots \times \mathcal{K}(W_k)$.*

Proof. (1) follows from the construction of hierarchy paths [MM], while (2) follows from [BM, Lemma 2.1]. □

Given subsets $A_i \subseteq \mathcal{K}(W_i)$, for $i \in \{1, 2, \ldots, k\}$, the product $A_1 \times \cdots \times A_k$ can be identified with a subset of $\mathcal{K}(W_1) \times \cdots \times \mathcal{K}(W_k)$, and by Lemma 2.10, (2), it can also be (quasi-)identified with a subset of $\mathcal{Q}(\Delta)$. In what follows we use the same notation $A_1 \times \cdots \times A_k$ for the subsets in $\mathcal{K}(W_1) \times \cdots \times \mathcal{K}(W_k)$ and in $\mathcal{Q}(\Delta_i)$.

Let Z be a subsurface of S; throughout, all subsurfaces we consider are implicitly assumed to be essential subsurfaces. Following [Beh], we consider a projection $\pi_{\mathcal{K}(Z)}\colon \mathcal{K}(S) \to 2^{\mathcal{K}(Z)}$ defined as follows. For an arbitrary $\mu \in \mathcal{K}(S)$ we can build a marking on Z by first choosing an element $\gamma_1 \in \pi_{\mathcal{C}(Z)}(\mu)$, and then recursively choosing γ_n from $\pi_{\mathcal{C}(Z\setminus \cup_{i<n}\gamma_i)}(\mu)$, for each $n \le \xi(Z)$. Take these γ_i to be the base curves of a marking on Z. For each γ_i we define its transversal $t(\gamma_i)$ to be an element of $\pi_{\mathcal{A}(\gamma_i)}(\mu)$, where $\mathcal{A}(\gamma_i)$ is the annulus with core curve γ_i. This process yields a marking, see [Beh] for details. Arbitrary choices were made in this construction, but two choices in building $\pi_{\mathcal{K}(Z)}(\mu)$ lead to elements of $\mathcal{K}(Z)$ whose distance is $O(1)$, where the bound depends only on $\xi(S)$ [Beh].

Given a marking μ and a multicurve Δ, the projection $\pi_{\mathcal{K}(S\setminus\Delta)}(\mu)$ can be defined as above. This allows one to construct a point $\mu' \in \mathcal{Q}(\Delta)$ which, up to a uniformly bounded error, is closest to μ. See [BM] for details. The marking μ' is obtained by taking the union of the (possibly partial collection of) base curves Δ with transversal curves given by $\pi_\Delta(\mu)$ together with the base curves and transversals given by $\pi_{\mathcal{K}(S\setminus\Delta)}(\mu)$. Note that the construction of μ' requires, for each subsurface W determined by the multicurve Δ, the construction of a projection $\pi_{\mathcal{K}(W)}(\mu)$. As explained previously, each $\pi_{\mathcal{K}(W)}(\mu)$ is determined up to uniformly bounded distance in $\mathcal{K}(W)$, thus μ' is well defined up to uniformly bounded distance, depending only on the topological type of S.

The following is an immediate corollary of the Masur–Minsky distance formula in [MM, Theorem 6.12].

Corollary 2.11. *There exist $A \geq 1$ and $B \geq 0$ depending only on S such that for any subsurface $Z \subset S$, the projection of $\mathcal{K}(S)$ onto $\mathcal{K}(Z)$ is an (A,B)–quasi-Lipschitz map, that is for any two markings $\mu, \nu \in \mathcal{K}(S)$ the following holds:*

$$\mathrm{dist}_{\mathcal{K}(Z)}\left(\pi_{\mathcal{K}(Z)}(\mu)\,,\, \pi_{\mathcal{K}(Z)}(\nu)\right) \leq A\mathrm{dist}_{\mathcal{K}(S)}(\mu,\nu) + B\,.$$

Consequently the nearest point projection onto $\mathcal{Q}(\Delta)$ is a quasi-Lipschitz map.

Let g_i be an element in $\mathcal{MCG}(S)$ that is pseudo-Anosov when restricted to W_i and is the identity on W_i^c, $i \in \{1,2,\ldots,k\}$ (this includes the case of pseudo-Anosov, where $\Delta = \emptyset$). Let $\mathfrak{p}_i$ be a quasi-axis of g_i in $\mathcal{K}(W_i)$.

Proposition 2.12. *There exists a quasi-Lipschitz map $\Phi_i\colon \mathcal{K}(S) \to \mathfrak{p}_i$ with the following properties:*

1. Φ_i *is* coarsely locally-constant *in the complement of $\mathfrak{p}_i \times \mathcal{K}(W_i^c)$, i.e. there exists constants $\lambda > 0$ and $r_0 > 0$ with the property that for any point μ at distance $r \geq r_0$ from $\mathfrak{p}_i \times \mathcal{K}(W_i^c)$, the diameter of $\Phi_i(B(\mu, \lambda r))$ is at most a uniform constant, c, which depends only on g_i;*
2. *Φ_i restricted to $\mathfrak{p}_i \times \mathcal{K}(W_i^c)$ is at uniformly bounded distance from the projection onto the first component.*

Proof. Follows immediately from the proofs of Theorems 3.1 and 3.5 in [BM]. □

The map $\Phi\colon \mathcal{K}(S) \to \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k$ defined by $\Phi(\mu) = (\Phi_1(\mu)\,,\,\ldots\,,\,\Phi_k(\mu))$ is also quasi-Lipschitz.

The inclusion $\langle g_1\rangle \times \cdots \langle g_k\rangle \to \mathcal{MCG}(S)$ is easily seen to be a quasi-isometric embedding by the Masur–Minsky distance formula in [MM, Theorem 6.12]. Moreover, the orbit map yields a quasi-isometric embedding $\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k \to \mathcal{K}(S)$, with constants depending on the chosen elements $g_1, \ldots, g_k$. The composition of the latter inclusion with the map Φ is at uniformly bounded distance from the identity, since Φ is a sort of (quasi-)nearest point projection.

2.5. **Mapping class groups and simplicial complexes.** The mapping class group itself and finite index subgroups of it act properly discontinuously cocompactly on various other CW–complexes. Indeed, all finite index torsion-free subgroups of a mapping class group have classifying spaces given by finite CW-complexes, see [Iva1, Iva2]. For the mapping class groups themselves, concrete constructions of CW-complexes on which they act properly discontinuously and with compact quotient (complexes that are moreover cocompact models for classifying spaces for proper actions) are described in [JW, Mis]. Any of these CW-complexes can be used to define filling functions for the mapping class group.

Another approach is to apply Theorem 10.2.6 in [ECH$^+$]. The mapping class groups are automatic [Mos], hence combable, so they are $\mathcal{F}_\infty$. In particular, for every $n \geq 0$, they act by simplicial isomorphisms, properly discontinuously, with trivial stabilizers of vertices, on a simplicial complex X of dimension $n+1$ and n-connected, such that the quotient has finitely many cells.

For every surface S we denote by X_S a simplicial complex with properties as above. Note that we allow the case when S has several connected components.

Using the previous section and [AWP, Lemma 12], the following properties can be established about these simplicial complexes, up to repeated barycentric subdivisions. We use the same generic objects defined in the previous section, with the same notation and terminology.

Given a pseudo-Anosov $g \in \mathcal{MCG}(S)$, there exists a bi-infinite almost geodesic $\widehat{\mathfrak{p}}$ in $X_S^{(1)}$ such that g acts on it with compact quotient, and the points on it quasi-minimize the displacement by g. Such a path is a *quasi-axis* of g. Moreover, for every $x \in \widehat{\mathfrak{p}}$, $g^{-n}x$ and $g^n x$ may be joined in X_S by a curve of length of order n^2 which is disjoint from the δn–ball around x.

For $\Delta, W_1, \ldots, W_k$ defined as previously, there exists a quasi-isometric embedding which is also a simplicial map $\Lambda_\Delta \colon X_{W_1} \times \cdots \times X_{W_k} \to X_S$, with image a subcomplex such that every two points in it can be joined in it by a path that is an almost geodesic in X_S.

As before, given subsets B_i in X_{W_i}, $i = 1, 2, \ldots, k$, we let $B_1 \times \cdots \times B_k$ denote the subset in $X_{W_1} \times \cdots \times X_{W_k}$ and its image in X_S.

There exists a bounded perturbation of the nearest point projection $\widehat{\pi}_\Delta \colon X_S \to X_{W_1} \times \cdots \times X_{W_k}$ which is a simplicial map, and a quasi-Lipschitz map. This allows to define a map with the same properties $\widehat{\pi}_{W_i} \colon X_S \to X_{W_i}$ for $i = 1, \ldots, k$. Consider pure reducible elements g_i as before, and their quasi-axes $\widehat{\mathfrak{p}}_i$ in X_{W_i}.

The projections Φ_i defined in Proposition 2.12 allow to define a simplicial map $\widehat{\Phi}_i \colon X_S \to \mathbb{R}$ that is quasi-Lipschitz and coarsely locally-constant in the complement of $\widehat{\mathfrak{p}}_i \times X_{W_i^c}$, while its restriction to $\widehat{\mathfrak{p}}_i \times \mathcal{K}(W_i^c)$ is at uniformly bounded distance from the projection onto the first component. We can then define the map $\widehat{\Phi} \colon X_S \to \mathbb{R}^k$, $\widehat{\Phi} = \left(\widehat{\Phi}_1, \ldots, \widehat{\Phi}_k\right)$ which is also quasi-Lipschitz.

The almost geodesics $\widehat{\mathfrak{p}}_i \colon \mathbb{R} \to X_{W_i}$ (which can also be seen as simplicial maps) define an inclusion $\Upsilon_\Delta \colon \mathbb{R}^k \to X_{W_1} \times \cdots \times X_{W_k}$ which is simplicial, a quasi-isometric embedding, and equivariant with respect to the action of $\langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_k \rangle$. Hence, we have an inclusion $\Upsilon \colon \mathbb{R}^k \to X_S$,$\Upsilon = \Lambda_\Delta \circ \Upsilon_\Delta$ with the same properties; moreover, we also have that $\widehat{\Phi} \circ \Upsilon$ is at uniformly bounded distance from the identity.

3. Higher dimensional filling and divergence in mapping class groups

In this section we fix an arbitrary surface S with $\xi(S) > 0$. As in Section 2.5, we consider a simplicial complex X_S on which $\mathcal{MCG}(S)$ acts properly discontinuously and such that the quotient of each skeleton $X_S^{(m)}$ is composed of finitely many m–simplices. From the fact that mapping class groups are automatic [Mos] it follows that X_S has a bounded (L, C)–quasi-geodesic combing.

3.1. **Isoperimetry in mapping class groups.** Known facts about mapping class groups imply that for each dimension k below the quasi-flat rank, the isoperimetric function in X_S is asymptotically equal to the *k–dimensional Euclidean isoperimetric function*, and that above the rank the isoperimetric function is sub-Euclidean.

Theorem 3.1. *The k–th isoperimetric function in the mapping class group of a surface satisfies* $\mathrm{Iso}_k(x) \asymp x^{k+1}$ *for* $k < \xi(S)$ *and* $\mathrm{Iso}_k(x) = o(x^{k+1})$ *for* $k \geq \xi(S)$.

Proof. By Theorem 2.5, for every integer $k \geq 1$ we have $\mathrm{Iso}_k(x) \preceq x^{k+1}$. Moreover, the presence of quasi-flats of dimension k inside the mapping class groups (see Proposition 3.5 for a construction of some such quasi-flats and Corollary 2.11 for relevant results about their geometry) and [AWP, Theorem 2] imply that for $k < \xi(S)$ we have $\mathrm{Iso}_k(x) \asymp x^{k+1}$.

For $k \geq \xi(S)$, the Theorem follows from [Wh, Theorem 1, p. 435], [Wen3], and the fact that the maximal dimension of locally compact subsets in an arbitrary asymptotic cone of $\mathcal{MCG}(S)$ is $\xi(S)$ [BM].

For the convenience of the reader, we also provide a more explicit argument. It suffices to prove that the filling function is $o(x^{k+1})$ only for spheres (respectively surfaces) that are round and unfolded, see [BD2]. We argue for a contradiction and assume that there exists a sequence $\mathfrak{r}_n$ of k-dimensional spheres for $k \geq 3$ (respectively of surfaces for $k = 2$) that are round and unfolded, of volume $\asymp x_n^k$ and of filling volume at least λx_n^{k+1}, where λ is a positive constant and $x_n \to \infty$. Let $\mathfrak{d}_n$ be filling $(k+1)$-dimensional balls (respectively filling handlebodies) realizing $\mathrm{FillVol}(\mathfrak{r}_n)$ and with a minimal number of chambers in the domain.

The argument in [Wen3, pp. 263–264] with $T_n = \mathfrak{r}_n$ and $S_n = \mathfrak{d}_n$ implies that the sequence $(\mathfrak{d}_n)$ yields a compact subset of dimension $k+1$ in an asymptotic cone of X_S, a contradiction. □

For $k \geq \xi(S)$ we conjecture $\mathrm{Iso}_k(x) \asymp x^k$. The sharp result holds in low genus:

Theorem 3.2. *Given a surface S of genus* 0 *or* 1, *or of genus* 2 *and without boundary, the k–th isoperimetric function in the mapping class group of S satisfies* $\mathrm{Iso}_k(x) \asymp x^k$ *for* $k \geq \xi(S)$.

Proof. It was recently established that the mapping class group of a surface has a cocompact classifying space for proper actions of dimension equal to the virtual cohomological dimension [HOP, AMP]. The virtual cohomological dimension for the mapping class group of S is: $p-3$ if $g=0$; $4g-5$ if $p=0$; and $4g+p-4$ if both g and p are positive [Har].

Since the surfaces in the hypothesis of the theorem thus have $vcd(\mathcal{MCG}(S)) = \xi(S)$, the result then follows from the fact that if a group has a cocompact classifying space for proper actions of dimension r, then $\mathrm{Iso}_k(x) \asymp x^k$ for all $k \geq r$ [AWP, Corollary 9]. □

3.2. **Divergence in mapping class groups.** In the mapping class group, the value for rank analogous to that in a symmetric space is the quasi-flat rank, i.e. the maximal dimension of a quasi-flat in the Cayley graph of the group. As discussed previously, for $\mathcal{MCG}(S)$ this rank is given by $\xi(S) = 3g+p-3$ [BM, Ham]. Below we show how, analogous to the case of symmetric spaces, this rank plays a critical role for divergence in mapping class groups as well.

Theorem 3.3. *Given a surface S and an arbitrary integer $k \geq \xi(S)$, the k-dimensional divergence in $\mathcal{MCG}(S)$ satisfies* $\mathrm{Div}_k(x) = o(x^{k+1})$.

If moreover S is of genus 0 *or* 1, *or of genus* 2 *and without boundary, then* $\mathrm{Div}_k(x) \asymp x^k$.

Proof. The result follows from Proposition 2.7, and from Theorems 3.1 and 3.2. $\square$

Theorem 3.4. *For any S and for any integer $0 \leq k < \xi(S)$, the k-dimensional divergence in $\mathcal{MCG}(S)$ satisfies $\mathrm{Div}_k \succeq x^{k+2}$.*

The rest of the section is devoted to the proof of Theorem 3.4. We use the terminology introduced in Sections 2.4 and 2.5. In particular we consider a compact connected orientable surface S of complexity $m = \xi(S) \geq 2$, and an m-connected simplicial complex X_S of dimension $m+1$ on which $\mathcal{MCG}(S)$ acts by simplicial isomorphisms, properly discontinuously, with trivial stabilizers of vertices and such that the quotient has finitely many simplices. Theorem 3.4 is a direct consequence of Proposition 3.5 below. Indeed, in this proposition we show that for every $0 < k \leq \xi(S)$ there exist naturally arising $(k-1)$–dimensional spheres in X_S which have divergence $\asymp x^{k+1}$.

Proposition 3.5. *Let Δ be a multicurve on S and let $W_1, \dots, W_k$ be the components of $\mathrm{open}(\Delta)$, $0 < k \leq \xi(S)$. For $i = 1, \dots, k$, consider g_i elements in $\mathcal{MCG}(S)$ that are pseudo-Anosov when restricted to W_i and the identity on $W_i^c = S \setminus W_i$. Let $\Upsilon \colon \mathbb{R}^k \to X_S$ be a simplicial map defining a quasi-flat and equivariant for the action of $\langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_k \rangle$, and let $x_0 = \Upsilon(0)$.*

Let $\mathcal{S}_n^{(k-1)}$ denote the boundary of the minimal simplicial subcomplex of $\mathbb{R}^k$ covering the cube $\{(x_1, \dots, x_k) \in \mathbb{R}^k \ ; \ |x_i| \leq n, i = 1, 2, \dots k\}$ and let $\mathfrak{s}_n^{(k-1)}$ be the $(k-1)$–dimensional sphere in X_S defined by the restriction of Υ to $\mathcal{S}_n^{(k-1)}$.

(1) *There exists a constant $C_1 > 0$, such that for every small enough $\delta \in (0,1)$, every k–disk $\mathfrak{d}$ filling the sphere $\mathfrak{s}_n^{(k-1)}$ in X_S and disjoint from the δn–ball around x_0 has area at least $C_1 n^{k+1}$.*

(2) *There exists a constant $C_2 > 0$ such that, if at least one of the surfaces W_i has complexity ≥ 2 (which implies, in particular, that $k < 3g + p - 3$), then for every $\delta > 0$ small enough there exists a k–disk $\mathfrak{d}$ filling the sphere $\mathfrak{s}_n^{(k-1)}$, disjoint from the δn–ball around x_0, and with area at most $C_2 n^{k+1}$.*

Remark 3.6. Proposition 3.5, (1), implies that the k–dimensional quasi-flats $\Upsilon(\mathbb{R}^k)$ in X_S and $\langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_k \rangle$ in $\mathcal{MCG}(S)$ are maximal in the sense that no quasi-flat of strictly larger dimension quasi-contains them. This is a subtle point, even though it is obvious when considering quasi-flats of the same type.

Proof. (1) Let $\mathfrak{d} : \mathcal{D} \to X_S$ be a k–dimensional disk which fills $\mathfrak{s}_n^{(k-1)}$ in X_S and is disjoint from the δn–ball around x_0. Then $\widehat{\Phi} \circ \mathfrak{d}$ is a disk filling $\widehat{\Phi} \circ \mathfrak{s}_n^{(k-1)}$ in $\mathbb{R}^k$. Since $\widehat{\Phi} \circ \Upsilon$ is at uniformly bounded distance from the identity, $\widehat{\Phi} \circ \mathfrak{s}_n^{(k-1)}$ is at uniformly bounded distance from $\mathcal{S}_n^{(k-1)}$.

In particular, for $\epsilon > 0$ small enough, the image of $\widehat{\Phi} \circ \mathfrak{d}$ covers the full cube in $\mathbb{R}^k$ centered in x_0 and of edge length ϵn. We denote the latter by Cube_ϵ and the initial full cube of edge length $2n$ by Cube_2.

For ease of notation, we explain the argument in the case $k = 2$ and then, after each step, how it can be modified to yield the general case, when the required modifications are not obvious.

For any choice of $\eta < n/2$ we may consider a grid subdividing Cube_2 into squares with edge length 2η (to simplify the discussion we will assume that both n and ϵn are integral multiples of 2η; otherwise an additional discussion of boundary rectangles is needed, which adds only notational complications, see, e.g. [LS]) and for each of the squares composing it consider the sub-square of edge η obtained by shrinking with factor $\frac{1}{2}$ around the center. Let Π be one such full square that is moreover contained in Cube_ϵ. Consider the ℓ_1–geodesic $\mathfrak{g}$ starting in the midpoint a of the upper horizontal edge of the square, going through the center of the square and ending in the midpoint b of the right hand side vertical edge.

In what follows we denote the map $\widehat{\Phi} \circ \mathfrak{d} : \mathcal{D} \to \mathbb{R}^k$ by f.

Step 1. The first step is to prove that for some constant $\lambda \ll \eta$ the set $f^{-1}(\mathfrak{g})$ has a connected component $\mathfrak{C}$ for which $f(\mathfrak{C})$ intersects the λ–ball around each of the two points a and b respectively. This argument is inspired from the proof of [LS, Proposition 3.2].

Assume on the contrary that this is not true. Since the map f is simplicial, defined on a triangulation of $\mathbb{D}^2$, there are finitely many connected components of $f^{-1}(\mathfrak{g})$. According to our hypothesis all the images of components that intersect $B(a,\lambda)$ do not intersect $B(b,\lambda)$. Let G be the union of the components with image by f intersecting $B(a,\lambda)$ but not $B(b,\lambda)$. Since both G and $f^{-1}(\mathfrak{g})\setminus G$ are compact there exists $\varepsilon > 0$ such that $U_\varepsilon = \{x \in \mathbb{D}^2\ ;\ \mathrm{dist}(x,G) < \varepsilon\}$ intersects $f^{-1}(\mathfrak{g})$ only in G, and $f(U_\varepsilon)$ intersects the boundary of the square Π only in the upper horizontal edge of the square.

In what follows we construct a continuous map F from the 2-dimensional disk $\mathbb{D}^2$ to $\mathbb{R}^2$, which coincides with f outside U_ε and on $f^{-1}(\partial\Pi)$ (here $\partial\Pi$ is the boundary of Π) and with image not containing the geodesic segment $\mathfrak{c} = [\mathfrak{g} \setminus \{a\}] \cap B(a,\lambda)$. This will finish this step, since it will imply that $\partial\Pi$, contractible in $F(\mathbb{D}^2)$, is therefore contractible in $\mathbb{R}^2 \setminus \mathfrak{c}$, a contradiction.

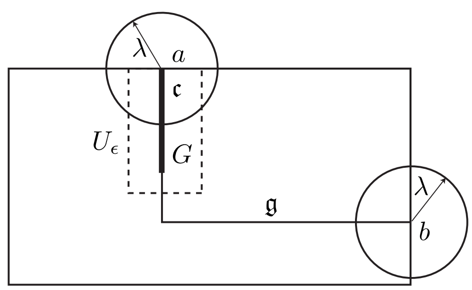

FIGURE 1. The set-up for Step 1

Consider the map $F\colon \mathbb{D}^2 \to \mathbb{R}^2$ defined as follows. For $x \notin U_\varepsilon$ let $F(x) = f(x)$. For x in U_ε let $\mathfrak{g}_x$ be the ℓ_1–geodesic through $f(x)$, contained in Π, composed of a vertical and a horizontal segment with common endpoint on the diagonal from the upper right corner to the lower left corner. Let a_x be the intersection point of $\mathfrak{g}_x$ with the upper horizontal edge of $\partial\Pi$, and let $d(x)$ be the ℓ^1–distance from a_x to $f(x)$.

Let ℓ be the length of the geodesic $\mathfrak{g}$ and let $t(x) = \ell\left[1 - \frac{\mathrm{dist}(x,G)}{\varepsilon}\right]$. We then define $F(x)$ to be the point on $\mathfrak{g}_x$ with distance to the upper horizontal edge along $\mathfrak{g}_x$ equal to the maximum between 0 and $d(x) - t(x)$.

In other words, if $d(x)$ is at most $t(x)$, then $F(x)$ equals a_x; while if $d(x)$ is larger than $t(x)$, then $F(x)$ equals the point on $\mathfrak{g}_x$ at distance $d(x) - t(x)$ from a_x.

The map F is continuous, coincides with f outside U_ε and on the boundary of Π, and its image does not contain the piece of geodesic $\mathfrak{c} = [\mathfrak{g} \setminus \{a\}] \cap B(a, \lambda)$. This provides a contradiction which, as noted above, finishes the step when $k = 2$.

In the general case of dimension k one may consider, instead of the upper right square bounded by $\mathfrak{g}$ and the boundary of Π, a cube of edge length half the length of the edge of Π with one vertex the center of Π and half of its faces contained in $\partial\Pi$. The geodesic $\mathfrak{g}$ is to be replaced by the ℓ_1 geodesic joining the endpoints of a big diagonal not containing the center; denote this new geodesic also by $\mathfrak{g}$. It suffices to prove that $f^{-1}(\mathfrak{g})$ has one connected component $\mathfrak{C}$ such that $f(\mathfrak{C})$ intersects the λ neighborhoods of both endpoints of $\mathfrak{g}$. This is completed as above by showing that for each cube Π there exists a connected set $\mathfrak{C}_\Pi$ in $f^{-1}(\mathfrak{g})$ such that $f(\mathfrak{C}_\Pi)$ covers the ℓ^1 geodesic constructed as above, except eventually the λ–neighborhoods of its endpoints.

Step 2. We denote by $\mathfrak{g}_\Pi$ the subgeodesic of $\mathfrak{g}$ covered by $f(\mathfrak{C}_\Pi)$. Let V_Π be the set of vertices of simplices in the triangulation $\mathcal{D}$ of $\mathbb{D}^2$ that intersect $\mathfrak{C}_\Pi$. Since the image by f of every simplex has diameter at most 1 it follows that the Hausdorff distance between $f(V_\Pi)$ and $\mathfrak{g}_\Pi$ is at most 1.

Note that since C_Π is a connected set, the set $\mathfrak{d}(V_\Pi)$ is 1–coarsely connected in the sense that: for every two vertices $x, y \in \mathfrak{d}(V_\Pi)$, there exists a finite sequence of vertices $z_1 = x, z_2, ..., z_n = y$ such that for each i we have $z_i \in \mathfrak{d}(V_\Pi)$ and $\mathrm{dist}(z_i, z_{i+1}) \leq 1$.

In each V_Π we fix a vertex x_Π such that $\widehat{\Phi} \circ \mathfrak{d}(x_\Pi)$ is at distance at most 1 from the center of Π. Let $\mathcal{P}$ be the set of all shrunken squares Π that appear in the grid and are contained in Cube_ϵ, and let $\mathcal{V}$ be the set of vertices $\{x_\Pi \,;\, \Pi \in \mathcal{P}\}$. Note that there are approximately $\frac{\epsilon^2 n^2}{\eta^2}$ elements in $\mathcal{V}$. Fix a small constant $\beta > 0$.

Case 1. Assume that at least half of the points in $\mathcal{V}$ have the property that their images by $\mathfrak{d}$ are at distance $\geq \frac{\beta}{2} n$ from $X_{W_1} \times \cdots \times X_{W_k}$. Denote this subset of $\mathcal{V}$ by $\mathcal{V}'$.

If a vertex w is at distance $\geq \frac{\beta}{2} n$ from $X_{W_1} \times \cdots \times X_{W_k}$ then there exists $i \in \{1, 2, ...k\}$ such that w is at distance $\geq \alpha n$ from $X_{W_i} \times X_{W_i^c}$, where α is a fixed constant depending only on β and $\xi(S)$. This follows from the corresponding statement in the marking complex: if a point μ is at distance $\geq \frac{\beta'}{2} n$ from $\mathcal{Q}(\Delta)$ then there exists $i \in \{1, 2, ...k\}$ such that it is at distance $\geq \alpha' n$ from $\mathcal{Q}(\Delta_i)$, where α' is a fixed constant depending only on β' and $\xi(S)$. The latter statement can be proved by induction on $\xi(S)$ and the fact that if μ is at distance $\leq \alpha n$ from all $\mathcal{Q}(\Delta_i)$ then its projection on $\mathcal{Q}(\Delta_1)$ is at distance $\leq L\alpha n$ from $\mathcal{Q}(\Delta_1 \cup \Delta_i)$ and hence μ is at distance $\leq (L + 1)\alpha n$ from $\mathcal{Q}(\Delta_1 \cup \Delta_i)$.

We may thus assume that at least $\frac{1}{k}$ of the points in $\mathcal{V}'$ are such that their image by $\mathfrak{d}$ is at distance $\geq \alpha n$ from $X_{W_i} \times X_{W_i^c}$ for a fixed i. Note that for each of these vertices x_Π the corresponding set $\mathfrak{d}(V_\Pi)$ contains a point such that its image by $\widehat{\Phi}_i$ is at distance at least η from the $\widehat{\Phi}_i$–image of x_Π. This, the fact that $\widehat{\Phi}_i$ is coarsely locally-constant (see Proposition 2.12) and the fact that x_Π is at distance at least αn from $X_{W_i} \times X_{W_i^c}$ implies that $\mathfrak{d}(V_\Pi)$ contains a point at distance at least $\lambda\alpha n$ from x_Π. Since $\mathfrak{d}(V_\Pi)$ is 1–coarsely connected, it follows that $\mathfrak{d}(V_\Pi)$ contains at least $\lambda\alpha n$ distinct points. Note that if Π and Π' are two disjoint cubes, then for η large enough $\mathfrak{d}(V_\Pi)$ and $\mathfrak{d}(V'_\Pi)$ are disjoint. We have thus obtained that the image of $\mathfrak{d}$ contains at least $\frac{1}{2k}\frac{\epsilon^2 n^2}{\eta^2}\cdot\lambda\alpha n d$ distinct points, images of interior vertices. We proved in [BD2] that with these hypotheses the chambers are all non-collapsed and thus the area of $\mathfrak{d}$ is $\succeq n^3$.

Case 2. Now assume that at least half of the points in $\mathcal{V}$ have the property that their $\mathfrak{d}$–images are at distance $\leq \frac{\beta}{2}n$ from $X_{W_1}\times\cdots\times X_{W_k}$. Let $\mathcal{V}'$ be this new subset of vertices. Since $\mathfrak{d}(\mathcal{V}')$ avoids a δn–ball around x_0 and $\widehat{\Phi}\circ\mathfrak{d}(\mathcal{V}')$ is contained in Cube_ϵ, it follows that, for β and ϵ small enough, by post-composing $\mathfrak{d}$ with the projection onto $X_{W_1}\times\cdots\times X_{W_k}$, we obtain a filling disk $\mathfrak{d}'=\widehat{\pi}_\Delta\circ\mathfrak{d}$ such that the points in $\mathcal{V}'$ are sent at distance $\geq\frac{\beta}{2}n$ from $\Upsilon_\Delta(\mathbb{R}^k)$. Then at least $\frac{1}{k}$ of the points in $\mathcal{V}'$ have image by $\mathfrak{d}'$ at distance $\geq\frac{\beta}{2k}n$ from $\widehat{\mathfrak{p}}_i$ for some fixed $i\in\{1,2,...k\}$. Denote this subset of $\mathcal{V}'$ by $\mathcal{V}''$. For notational simplicity we assume that $i=1$.

For every vertex x_Π in $\mathcal{V}''$ there exists a point in V_Π such that given their respective images by $\mathfrak{d}'$, the projections of their respective first coordinates onto $\widehat{\mathfrak{p}}_1$ are at distance at least η. Then with the same argument as in Case 1 we deduce that $\mathfrak{d}'(V_\Pi)$ contains at least $\lambda\alpha n$ points, hence $\mathfrak{d}'$ (and therefore $\mathfrak{d}$, up to multiplication by some universal constant in $(0,1)$) has area at least $\frac{1}{6k}\frac{\epsilon^2 n^2}{\eta^2}\cdot\lambda\alpha n$.

(2) Assume that W_1 has complexity ≥ 2. The point x_0 is the image by Λ_Δ of a point $y_0=(y_0^1,\dots,y_0^k)$ in $\Upsilon_\Delta(\mathbb{R}^k)$. According to the arguments in Section 2.5, $\widehat{\mathfrak{p}}_1(-n)$ and $\widehat{\mathfrak{p}}_1(n)$ can be joined in X_{W_1} by a path $\mathfrak{c}$ with length in $[an^2,bn^2]$, where $0<a<b$, avoiding the ball centered in y_0^1, and of radius $\delta' n$. Let $\mathfrak{d}$ be a disk whose area is an affine function of n^3 filling the loop $\mathfrak{c}\cup\widehat{\mathfrak{p}}_1([-n,n])$. It can be obtained by taking, for every point x on $\mathfrak{c}$ its projection x' on $\widehat{\mathfrak{p}}_1([-n,n])$, letting x vary, using the fact that the projection on $\widehat{\mathfrak{p}}_1$ is quasi-Lipschitz and coarsely locally-constant (see Proposition 2.12) and quasi-geodesics $\mathfrak{q}_{x'x}$ given by the combing with basepoint x'.

Then $\mathfrak{c}\times\widehat{\mathfrak{p}}_2([-n,n])\times\cdots\times\widehat{\mathfrak{p}}_k([-n,n])\cup\bigcup_{i=2}^k\mathfrak{d}\times\widehat{\mathfrak{p}}_2([-n,n])\times\cdots\times\widehat{\mathfrak{p}}_{i-1}([-n,n])\times\widehat{\mathfrak{p}}_i(\pm n)\times\widehat{\mathfrak{p}}_{i+1}([-n,n])\times\cdots\times\widehat{\mathfrak{p}}_k([-n,n])$ compose a filling disk for the Υ_Δ–image of the boundary of the cube of edge $2n$ centered in 0, moreover this disk has area n^{k+1} and it is disjoint from the $\delta'' n$–ball around y_0, for an appropriate choice of δ'' and δ'. By applying Λ_Δ we obtain a disk filling the given sphere, disjoint from the δn–ball around x_0 and of area n^{k+1}. □

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