Preface

Number Theory and Geometry have been closely connected for a long time. In the first part of the XX^{th} century, Artin introduced the fundamental new idea of linking the dynamics of geodesics on the modular surface $SL(2,\mathbf{R})/SL(2,\mathbf{Z})$ to the continued fraction expansion of real numbers. In the mid-seventies the idea that the dynamics of subgroups acting on locally symmetric spaces is not only closely related to arithmetic problems, but that it is also a very effective tool for solving such problems was well established. One of the first conjectures on this dynamics, Raghunathan's conjecture on algebraic properties of orbits of unipotent subgroups, was motivated by an approach to the long-standing Oppenheim–Davenport conjecture on values of quadratic forms at integral points. Several years after it was formulated, Margulis gave a full answer to the Oppenheim–Davenport conjecture, using the dynamics of a flow on the space $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ and an answer to Raghunathan's conjecture in some particular cases. Raghunathan's conjecture in full generality, as well as its p-adic version, were proved by Ratner using ergodic theory. Ratner's theorem in its turn became a powerful tool in proving new results in Number Theory, from a quantitative answer to the Oppenheim–Davenport conjecture, due to Eskin, Margulis and Mozes, to the more recent proofs of the uniform distribution of Heegner points by Vatsal, of the Mazur conjecture on higher Heegner points by Cornut. This stream of ideas has continuously stimulated research up to present times.

The aim of this collection of papers is to present various areas of research situated at the interface between dynamical systems and Diophantine approximation and to provide surveys focused on recent progress made in these areas.

It follows from the theory of continued fractions that, for any irrational number ξ, there exist infinitely many rational numbers p/q such that $q \geq 1$ and

$$|\xi - p/q| < \Psi(q), \tag{1}$$

with $\Psi(x) = 1/x^2$ for $x > 0$. For a given positive real number τ, the real number ξ is said to be approximable by rationals at order τ if (1) with the function $\Psi(x) = 1/x^\tau$ has infinitely many solutions. Thus, every irrational number is approximable at order 2 by rationals. There exist numbers ξ for which the order 2 is essentially the best possible, for instance the quadratic irrationals. On the other hand it is easy to construct real numbers approximable at some order exceeding two. A natural question to ask is: which of these two types of real numbers is 'more frequent'? A precise answer to this was given by Khintchine in 1924, that can be viewed as the first result in metric Diophantine approximation. It asserts that if the series $\sum_{q\geq 1} q\Psi(q)$ converges, then (1) has finitely many solutions for almost all real numbers ξ, while if $\sum_{q\geq 1} q\Psi(q)$ diverges and $x \mapsto x^2\Psi(x)$ is non-increasing, then (1) has infinitely many solutions for almost all real numbers ξ. A multidimensional extension of Khintchine's result to simultaneous approximation of linear forms was established in 1938 by Groshev. Given a set of mn real numbers ξ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$, Groshev's theorem is concerned with the existence of integers $q_1, \ldots, q_m$ and $p_1, \ldots, p_n$ satisfying the inequality

$$\max_{1 \le i \le n} |\xi_{i1} q_1 + \ldots + \xi_{im} q_m - p_i| \le \Psi(\max_{1 \le j \le m} |q_j|), \tag{2}$$

and it describes the functions Ψ for which, for almost every $(\xi_{ij}) \in \mathbf{R}^{mn}$, the inequality in (2) has infinitely many integer solutions, and the functions Ψ for which only finitely many solutions can be found for a generic $(\xi_{ij}) \in \mathbf{R}^{mn}$. When $n = 1$, this becomes a question of *linear Diophantine approximation* of the vector $(\xi_{1j})_{1 \le j \le m}$ while if $m = 1$ it becomes a question of *simultaneous Diophantine approximation* of the vector $(\xi_{i1})_{1 \le i \le n}$.

Both in (1) and in (2) if the function Ψ decreases too quickly to 0 then the existence of infinitely many integer solutions can be ensured only for null sets (i.e. sets of Lebesgue measure 0) of real numbers, respectively of elements (ξ_{ij}) in $\mathbf{R}^{mn}$. Still such sets might be relatively large, in particular of power continuum. Hausdorff dimension and measure are effective tools for measuring 'how large' null sets are. It was established in 1928 by Jarník (and, independently, by Besicovitch in 1934) that for every $\tau \ge 2$, the Hausdorff dimension of the set of real numbers approximable at order τ by rationals is equal to $2/\tau$, moreover the corresponding Hausdorff measure is ∞.

In order to estimate and possibly compute the Hausdorff dimension of a null set, an upper bound is in most cases (but not always!) provided by a direct application of a covering lemma. To obtain a good lower bound is what usually requires much effort.

In his paper, Dodson begins by discussing the theorems of Khintchine and Groshev, and he shows how the torus geometry in the plane allows a relatively straightforward proof of the planar Groshev theorem. He then continues with a presentation of the idea of 'ubiquity', introduced for the first time in 1990 by himself together with Rynne and Vickers, as a systematic way of determining a lower bound for the Hausdorff dimension of exceptional sets occurring in Diophantine approximation.

Beresnevich and Velani have recently established that the Khintchine and the Jarník–Besicovitch theorems turn out to be simple consequences of the notion of 'local ubiquity'. Their framework, worked out jointly with Dickinson and presented here in a simplified and more transparent version, enables them to unify the Lebesgue and Hausdorff theories.

The classical one dimensional results of Diophantine approximation can be extended in the hyperbolic space setting. Patterson's thesis is one of the first important papers in this area. In this setting, $\mathbf{R}$ is replaced by the limit set of a geometrically finite Kleinian group with parabolics, and $\mathbf{Q}$ by the orbit of a parabolic fixed point. Note that this is consistent with the case of the modular group $SL(2, \mathbf{R})$ whose limit set can be identified with the projective line $P^1\mathbf{R}$ and whose unique orbit of parabolic fixed points identifies with the rational projective line $P^1\mathbf{Q}$. Using the notion of 'local ubiquity' in this context, Beresnevich and Velani obtain a general Hausdorff measure version of Sullivan's logarithm law on the rate of divergence of geodesics into a cuspidal end of a non-compact hyperbolic manifold with finite volume.

The contribution of Maucourant is in the same spirit. Indeed if one extends the range of study from hyperbolic manifolds (which are locally symmetric spaces of rank one) to locally symmetric spaces of arbitrary rank, and instead of the geodesic flow one considers the Weyl chamber flow, this allows to study generalized versions of the classical Diophantine approximation, in which the field $\mathbf{Q}$ is replaced by any number field (i.e., finite extension of $\mathbf{Q}$). Maucourant's paper focuses on the Lagrange spectrum for quadratic forms and the Markoff spectrum associated to a number field. The Markoff spectrum of $\mathbf{Q}$ can be interpreted in terms of the length of simple closed geodesics in a hyperbolic punctured torus. Maucourant explains how, given a number field, both spectra are related to the dynamics of the Weyl chamber flow on a locally symmetric space naturally associated to the field. He uses Busemann functions to measure the divergence of Weyl chambers into the cusp of a locally symmetric space.

Busemann functions are also at the core of the paper of Druţu. Following an idea of Dani, Druţu explains that, as vectors in $\mathbf{R}^{n-1}$ with $n \geq 2$ can be naturally identified with (locally) geodesic rays in the locally symmetric space $\mathcal{T}_n = SO(n, \mathbf{R}) \backslash SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, the way in which a vector is linearly approximated is given by the depth of the excursions of a geodesic ray in the cuspidal end of the space $\mathcal{T}_n$, depth measured by the Busemann function of a certain fixed (global) geodesic ray in $\mathcal{T}_n$. The same holds for the simultaneous Diophantine approximation, only the (global) geodesic ray defining the Busemann function is different. With these interpretations, the transference theorems (connecting the way in which a vector is linearly and, respectively, simultaneously approximated) become relations between the Busemann functions of two rays in $\mathcal{T}_n$. The latter relations are easy to establish and geometrically transparent.

As we already pointed out, Ratner's theorem is a major advance in the setting of dynamics of subgroups acting on locally symmetric spaces, seen in connection with arithmetic problems. The proof of this theorem for arbitrary Lie groups involves the development of many ideas from dynamics and group theory. The version of Ratner's results concerning dynamics on products $\mathrm{SL}(2, K)/\Gamma_1 \times \ldots \times \mathrm{SL}(2, K)/\Gamma_n$, where K is a finite extension of $\mathbf{Q}_p$ and Γ_i is a cocompact discrete subgroup of $\mathrm{SL}(2, K)$ for $i = 1, ..., n$, plays a decisive part in the work of Cornut and Vatsal on Heegner points. Motivated by this application, Shah, in his contribution, studies orbits of some unipotent subgroups acting on products $\mathrm{SL}(2, K)/\Gamma_1 \times \ldots \times \mathrm{SL}(2, K)/\Gamma_n$, where Γ_i are as above and K is a locally compact field of characteristic 0. In this setting, he gives a self contained simplified proof of Ratner's orbit closure theorem, and also deduces the corresponding equidistribution theorem from Ratner's classification of ergodic invariant mesures.

The choice of the max norm in (2) is arbitrary, and one may consider the geometric mean instead. This modified point of view leads to a new area of research called *multiplicative Diophantine approximation*, in opposition to the *standard Diophantine approximation.* Some of the classical problems become much more difficult when reformulated in multiplicative Diophantine approximation. In 1998, Kleinbock and Margulis obtained the first significant progress in this area by proving the Baker–Sprindžuk conjecture on very well multiplicatively approximable points on manifolds non-degenerate almost everywhere. This proof is a consequence of a fruitful interplay between Diophantine approximation and dynamical and ergodic properties of actions on homogeneous spaces of Lie groups.

In his contribution, Bugeaud undertakes a systematic study of multiplicative Diophantine approximation, introducing new exponents of approximation. He surveys known results, including transference theorems (see also the contribution of Druţu) and metrical theorems, establishes some new statements and makes some suggestions for further research. His text ends with a few words on an emblematic question in multiplicative approximation, namely the Littlewood conjecture, to which is devoted the paper of Queffélec. This celebrated open problem asks whether for any pair (α, β) of real numbers we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{3}$$

where $\|\cdot\|$ stands for the distance to the nearest integer. The answer is clearly positive if α (or β) has unbounded partial quotients in its continued fraction expansion; however, even to find explicit examples of pairs (α, β) of badly approximable numbers such that $1, \alpha, \beta$ are linearly independent over the rationals and (3) holds is not an easy question.

Queffélec reviews the few known results on the Littlewood conjecture, with a special emphasis on two recent metrical results. The first one, established by Pollington and Velani, asserts that, given a badly approximable number α, the set of badly approximable numbers β such that (3) holds has Hausdorff dimension one. This was sharpened by Einsiedler, Katok and Lindenstrauss, who proved that the set of pairs (α, β) for which the Littlewood conjecture does not hold has Hausdorff dimension zero. Their approach consists in establishing part of the Margulis conjecture on bounded orbits of the Weyl chamber flow on the homogeneous space $SL(3, \mathbf{R})/SL(3, \mathbf{Z})$. More generally, Queffélec provides motivations to study two-dimensional Diophantine approximation, and also discusses and compares different generalizations of the continued fractions algorithm, including the Jacobi–Perron algorithm.

The contribution of Broise focuses on this algorithm in dimension $d > 1$. Basically, the Jacobi–Perron algorithm associates to a point x in the unit d-dimensional cube a sequence of simplices $\sigma_n(x)$ with rational vertices in $\mathbf{Q}^d$ which all contain the point x and converge to it when n tends to infinity. The speed of convergence of the algorithm is almost surely measured by the characteristic exponents, usually denoted by $\lambda_1, \ldots, \lambda_{d+1}$, and satisfying $\lambda_1 \geq \ldots \geq \lambda_{d+1}$ and $\lambda_1 + \ldots + \lambda_{d+1} = 0$. The latter relations between characteristic exponents were recently improved by Broise and Guivarc'h, who established that $\lambda_1 > \ldots > \lambda_{d+1}$ and $\lambda_1 + \lambda_{d+1} > 0$, two results discussed in the present text. Using the same method, analogous results were proved recently by Avila, Viana and Forni for the generalization of the continued fractions algorithm associated to interval exchange transformations. Such transformations are closely related to billiard trajectories.

The billiard flow is the topic of the contribution of Troubetzkoy. He studies asymptotic properties of this flow such as transitivity and mixing. The core of the paper is the use of techniques of Diophantine approximation to study this dynamical system. As an example, he explains the proof of the classical result of Katok and Zemlyakov asserting that the billiard in a typical polygon is topologically transitive. He continues with a survey of old and new results for convex smooth tables. Moreover, he explains how new results in inhomogeneous Diophantine approximation allow one to prove that for certain irrational polygonal billiard tables there are many periodic trajectories.

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