ISABE-2001-1055

Adjoint Methods for Turbomachinery Design

M. S. Campobasso *
M. C. Duta †
M. B. Giles ‡
Oxford University Computing Laboratory
Oxford, OX1 3QD, U.K.

Abstract

This paper discusses the use of both steady and unsteady discrete adjoint methods for the design of turbomachinery blades. Steady adjoint methods give the linear sensitivity of steady-state quantities such as the mass flow and the average exit flow angle to arbitrary changes in the geometry of the blades. This linear sensitivity information can then be used as part of a nonlinear optimisation procedure. The unsteady adjoint method is based on a single frequency of unsteadiness and gives the generalised force for a particular structural mode of vibration due to arbitrary incoming wakes. This can be used to tailor the radial variation in the incoming wakes to greatly reduce the level of forced vibration they induce.

The paper presents an overview of the discrete adjoint approach (which follows the work of Elliott and Anderson for external aerodynamic applications), explaining why it gives exactly the same results as linear perturbation methods, but at a greatly reduced computational cost. The key issues in the numerical implementation of the adjoint methods are discussed for both the Euler and the Reynolds-averaged Navier-Stokes equations. The correctness of the implementation is validated by comparison to both nonlinear and linear perturbation calculations.

Introduction

Modern turbomachinery has to meet exacting standards of efficiency resulting in low weight and highly loaded engine components. For this reason, techniques for the optimisation of the design of fans, compressors and turbines are becoming increasingly popular in the turbomachinery industry. Multidisciplinary design systems allow the designer to modify blade and end wall geometries in order to optimise the steady aerodynamic performance,[15] possibly fulfilling prescribed mechanical constraints. For example the minimum cross section of the blade cannot be reduced below a minimum threshold to prevent the steady working stress from exceeding the material strength.

However, even if the redesigned blade fulfils the steady stress requirements, the reduced stiffness may lead to critical unsteady stresses due to the inherent unsteadiness of turbomachinery flows. The relative motion of adjacent rotors and stators transforms spatial variations of the flow variables like the static pressure into periodically time-varying forces acting on the blades. The consequent vibration may result in the phenomenon of *High Cycle Fatigue* (HCF), which may shorten the life of the blades below the target life of the engine. This explains the growing importance of *unsteady design* methods. By this expression, one means designing components which can better withstand unsteady aeroelastic loads, like those due to forced response.

Several functionals can be chosen for the optimisation of the steady design. One obvious choice would be the stage efficiency, which in turn is linked to the exit loss. However, the secondary kinetic energy is often preferred, being less affected than the loss by possible inaccuracies associated with the turbulence models. Other object functions include the mass flow and the exit angle. The formulation of the unsteady design problem is less trivial. Over the past two decades, a number of methods have emerged to carry out the analysis of turbomachinery aeroelasticity, varying from uncoupled linearised potential flow solvers[8,17] to fully-coupled nonlinear three-dimensional unsteady viscous methods.[12]Within this range, the uncoupled linear harmonic Euler and Navier-Stokes (NS) methods have proved to be a successful compromise between accuracy and cost and are now widely preferred in industry as a fast, accurate tool for aeroelastic predictions. Indeed, a growing body of evidence indicates that linear viscous calculations are adequate for a surprisingly large range of applications.[1,9,16] For the prediction of the level of structural vibrations, the most important output from such linear unsteady analyses is a quantity known

*email: msc@comlab.ox.ac.uk
†email: mcd@comlab.ox.ac.uk
‡email: giles@comlab.ox.ac.uk

as the "worksum".[2] In the context of Lagrangian mechanics, the worksum corresponds to the generalised force due to the linear unsteady aerodynamics acting on a particular structural mode of vibration and it is therefore the obvious choice for the object function to be minimised in the unsteady design problem.

In nonlinear gradient based optimisation, one has to determine the sensitivities of the object function to all the n design parameters at each step of the optimisation. One way of accomplishing this, is to perform $n+1$ non-linear NS calculations at each step. The *adjoint method* is a mathematical technique which allows the determination of all n components of the gradient with a single computation, at a cost comparable with that of a single solution of the non-linear NS equations. Therefore the computational benefit of the adjoint approach increases with the number of design parameters n.

The adjoint technique for optimal aeronautical design has been developed by Jameson.[10,11] The use of the adjoint method for the optimisation of the unsteady turbomachinery design is a novel technique being developed at the Oxford University Computing Laboratory.[2,3,6] In ref. [3] the *harmonic adjoint approach* is successfully applied for the minimisation of the blade forced response by varying the shape of the incoming wakes, which ultimately requires a 3D re-design of the upstream blade row.

This paper summarises the main aspects of the theory behind the implementation of the *HYDRA* suite of non-linear, linear and adjoint NS codes, demonstrates how the gradients of steady and unsteady object functions can be determined equivalently with the linear or adjoint methods and proves the effectiveness of the adjoint approach for turbomachinery design with two practical examples.

Nonlinear Flow Analysis

We begin with the discrete nonlinear analysis of the time-averaged turbulent flow within a single turbomachinery blade row in its frame of reference (*i.e.* stationary for a stator, rotating for a rotor). The flow is described by the Reynolds–averaged NS equations coupled with the Spalart–Allmaras turbulence model. Due to rotation, centrifugal and Coriolis forces, source terms appear in the momentum equations. The analysis computes the vector $\mathbf{U}$ of primitive flow variables (including the turbulence variables) corresponding to a computational grid with nodal coordinates $\mathbf{X}$, on which the nonlinear flow equations can be expressed as

$$\mathbf{N}(\mathbf{U},\mathbf{X}) = 0. \qquad (1)$$

The vector $\mathbf{N}$ represents the spatially discretised residuals, a nonlinear function of the discrete flow variables and, due to the discretisation, also a function of the grid node coordinates. Because the governing equations are approximated on an unstructured grid using an edge-based algorithm,[13,14] the residual vector $\mathbf{N}$ is a sum of contributions from all of the edges of the grid, with each edge contributing only to the residuals corresponding to the two nodes at either end.

For turbomachinery, the boundary conditions are of three types; inflow/outflow, periodic and inviscid/viscous wall. The inflow and outflow boundaries are handled through fluxes which incorporate the appropriate far-field information. Thus these boundary conditions become part of the residual vector $\mathbf{N}$. Periodicity is treated very simply through the use of matching pairs of periodic nodes, one on the lower and one on the upper periodic boundary, at which the flow is defined to be identical apart from the appropriate rotation of the velocity vectors to account for the annular nature of the turbomachinery flow domain. By combining flux residuals at the two periodic nodes in an appropriate manner to maintain periodicity, this boundary condition again just requires minor changes to the definition of the flux residual vector $\mathbf{N}$. Further details are given in references.[2,13]

It is the wall boundary condition which requires a more substantial change in the form of the discrete equations. For viscous flows, a no-slip boundary condition is applied by discarding the momentum residuals and replacing these equations by the specification of zero velocity at the boundary nodes. For inviscid flows, the formulation of the flux residuals for boundary nodes is based on zero mass flux through the boundary face, but in addition flow tangency is enforced by setting the normal component of the surface velocity to zero, disregarding the normal component of the momentum residuals.

These strong wall boundary conditions, in which one or more components of the momentum residuals are discarded and replaced by the specification of corresponding velocity components, can be expressed as

$$\begin{aligned} (I-B)\ \mathbf{N}(\mathbf{U},\mathbf{X}) &= 0; && (2)\\ B\,\mathbf{U} &= 0. && (3) \end{aligned}$$

Here B is a projection matrix which extracts the momentum/velocity components at the wall boundaries.

These equations are solved using a five-stage Runge-Kutta scheme, with a Jacobi preconditioner and multigrid to accelerate convergence.[13,14]

Linear Analysis

Both the changes of geometric design parameters such as blade stagger angle, thickness and camber and the flow field unsteadiness such as the periodically time-varying forces associated to the incoming wakes can be treated as small perturbations. In the steady case, in fact, one wants to use small perturbations to get an accurate estimate of the gradient. The unsteady perturbation is also small because the level of unsteadiness in turbomachines is low. In both problems, the small size of the perturbations justifies the linear analysis of the flow field.

In the steady design problem, the perturbed flow field can be assumed to be a superposition of the unperturbed nonlinear flow field $\overline{\mathbf{U}}$ and a small linear perturbation $\mathbf{u}$:

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u} \tag{4}$$

The periodic boundary conditions are the same as in the non-linear equations.
In the unsteady flow case, the time-periodicity of the unsteadiness makes possible an harmonic decomposition of the flow field. The unsteady forces can be linearly decomposed into a sum of independent harmonic components. Thus, when considering a single harmonic component, the unsteady flow field $\mathbf{U}(t)$ can be assumed to be a superposition of the steady nonlinear flow field $\overline{\mathbf{U}}$ and the real part of a small harmonic perturbation of known frequency ω and unknown complex amplitude $\mathbf{u}$:

$$\mathbf{U}(t) = \overline{\mathbf{U}} + \mathcal{R}\{\exp(\mathrm{i}\omega t)\ \mathbf{u}\}. \tag{5}$$

The periodic boundary conditions for the complex amplitude $\mathbf{u}$ are more complicated than in the steady case, due to the specification of an interblade phase angle (IBPA). This is a complex phase shift $\exp(\mathrm{i}\varphi)$ between the lower and upper periodic boundaries. In the forced response problem, it arises when the wakes and blades have different pitches and therefore there is a difference in the times at which neighbouring wakes strike neighbouring blades

The linearisation of both of the discrete steady equations (2) and (3) and their unsteady counterparts leads to the linear system

$$\begin{aligned} (I - B)\ (L\mathbf{u} - \mathbf{s}) &= 0; & (6)\\ B\ \mathbf{u} &= \mathbf{b}. & (7) \end{aligned}$$

The governing equations for the linear perturbations are formally identical in the steady and unsteady case and L is a combination of the linearisation matrix $\partial\mathbf{N}/\partial\mathbf{U}$ giving the sensitivity of the discrete nonlinear residual $\mathbf{N}$ to flow perturbations. In the steady case, however, the equations are defined in the real domain with zero frequency and IBPA, whereas in the unsteady case they are defined in the complex domain and L contains also a complex source term due to the harmonic unsteadiness. The linear harmonic equations can be viewed as the frequency domain counterpart of the non-linear unsteady equations.

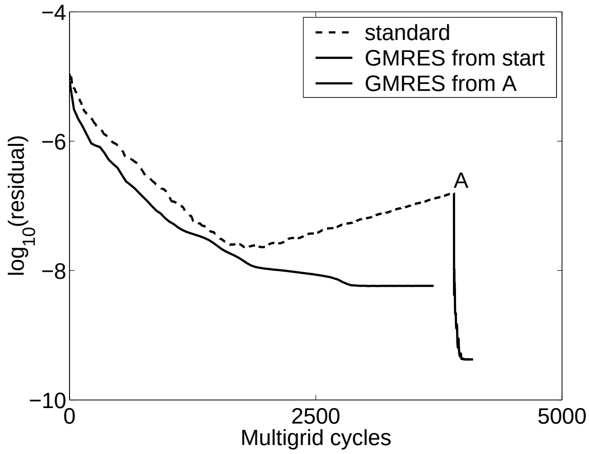

Fig. 1: Convergence histories of a turbine flutter case in which GMRES is used to stabilise an iteration, either from the original initial conditions or from a restart

For viscous walls, the wall velocity $\mathbf{b}$ is zero for both steady perturbations and forced response due to incoming wakes. For inviscid walls, however, $\mathbf{b}$ is zero only in the forced response problem; the steady geometry perturbation rotates the unit normal leading to a velocity perturbation in the normal direction.[2] The source term $\mathbf{s}$ is non-zero over the whole computational domain for steady perturbations, as a consequence of the grid deformation, whereas in the forced response analysis, it is zero throughout the flow field except at the inflow boundary where the specification of the incoming wakes enters through the boundary fluxes.

These linear equations are again solved using the five-stage Runge-Kutta scheme together with Jacobi preconditioning and multigrid. Usually this converges without difficulty, but problems have been encountered in situations in which the steady flow calculation itself failed to converge to a steady-state but instead finished in a low-level limit cycle, often related to some physical phenomenon such as vortex shedding at a blunt trailing edge. The corresponding instability in the linear calculation has been dealt with by the use of GMRES, with the usual multigrid solver being used as a very effective preconditioner, as shown in Figure 1.

The final output of the linear analysis is the variation of the object function corresponding to the prescribed perturbation, which in general is a complex inner product between a constant vector and the linear solution: $w = \mathbf{g}^H\mathbf{u}$ (where $\mathbf{g}^H$ denotes the complex conjugate transpose of $\mathbf{g}$). The elements of the vector $\mathbf{g}$ are non-zero only at nodes where the object function is defined, at the nodes on the outlet plane if the functional is the mass flow and at the nodes on the blade surface if the functional is the worksum.

Adjoint Analysis

The adjoint approach is founded on the observation that if $A\,\mathbf{u} = \mathbf{f}$ then

$$w = \mathbf{g}^H \mathbf{u} = \mathbf{g}^H A^{-1}\,\mathbf{f} = \left((A^H)^{-1}\mathbf{g}\right)^H \mathbf{f} = \mathbf{v}^H \mathbf{f}, \tag{8}$$

where $\mathbf{v}$ is the solution to the adjoint system

$$A^H \mathbf{v} = \mathbf{g}. \tag{9}$$

This adjoint approach to evaluating the object function w is beneficial when there is one $\mathbf{g}$, corresponding to a single scalar functional, but several different $\mathbf{f}$ vectors, corresponding to different geometric design parameters in the steady case and to different shapes of the incoming wakes in the case of forced response. In this situation, the usual direct approach would require a separate linear calculation for the perturbation of each design parameter or for each shape of the wake, whereas the adjoint approach needs just one adjoint calculation.

To express the linear system of equations in the required form, we add Equations (6) and (7) to give

$$\Big((I-B)L + B\Big)\;\mathbf{u} = (I-B)\,\mathbf{s} + \mathbf{b}. \tag{10}$$

The corresponding adjoint system of equations is therefore

$$\Big(L^H(I-B) + B\Big)\;\mathbf{v} = \mathbf{g}, \tag{11}$$

since the real matrix B is symmetric. To implement the adjoint method, it is convenient to split $\mathbf{v}$ into two orthogonal components [1] using the fact that B is idempotent (i.e. $B^2 = B$):

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}, \quad \mathbf{v}_{\|} = (I-B)\,\mathbf{v}, \quad \mathbf{v}_{\perp} = B\,\mathbf{v}. \tag{12}$$

Multiplying Equation (11) by $(I-B)$ yields the equation

$$(I-B)\,L^H\,\mathbf{v}_{\|} = (I-B)\,\mathbf{g} \tag{13}$$

which can be solved together with the boundary condition

$$B\,\mathbf{v}_{\|} = 0, \tag{14}$$

to determine $\mathbf{v}_{\|}$. Multiplying Equation (11) by B yields

$$\mathbf{v}_{\perp} = -BL^H\,\mathbf{v}_{\|} + B\mathbf{g}, \tag{15}$$

so $\mathbf{v}_{\perp}$ can be calculated in a post-processing step before then evaluating the linear functional as

$$w = \mathbf{v}^H\mathbf{f} = \mathbf{v}_{\|}^H\mathbf{s} + \mathbf{v}_{\perp}^H\mathbf{b}. \tag{16}$$

This equation shows that $\mathbf{v}_{\|}$ gives the dependence of the functional on the distributed source term $\mathbf{s}$, whereas $\mathbf{v}_{\perp}$ gives its dependence on the boundary velocities $\mathbf{b}$.

It is not obvious how best to solve the adjoint equations. Using the same iterative method as for the nonlinear and linear equations (except with the transpose of the preconditioning matrix) was found to work well for inviscid flows, but there were significant stability problems with viscous flows. To overcome these, Giles analysed the iterative evolution of output functional. He found that the adjoint code could be designed to give exactly the same iterative history for the functionals as with the linear code, by properly constructing an adjoint version of the usual Runge-Kutta time-marching procedure, and using adjoint restriction and prolongation operators for the multigrid.[5] This guarantees that the stability and the iterative convergence rate of the adjoint code will be identical to that of the linear code, which in turn is equal to the asymptotic convergence rate of the nonlinear code.

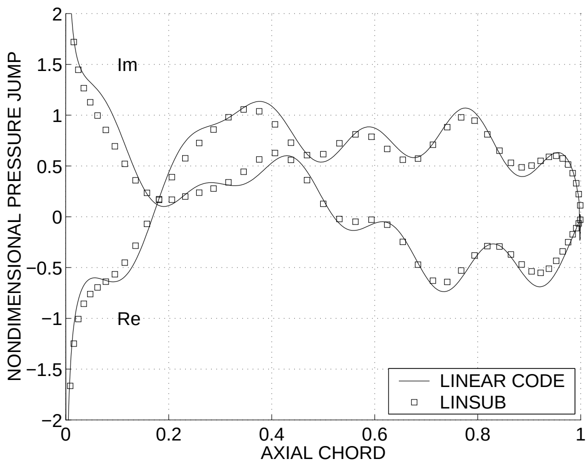

Fig. 2: Complex components of the flat plate pressure jump due to wake interaction

Validation

One difficulty in the development of an adjoint flow code is the lack of test cases for validation. For the adjoint code, the validation has been performed at two levels. At the lower level, each subroutine has been checked for consistency with its counterpart in the linear code.[6,7] At the higher level, it has been checked that the adjoint and linear codes produce the same value for both the steady and unsteady functionals, to within machine accuracy, at each step of the iterative process. This exact equivalence is one advantage of the fully discrete adjoint

[1]The reason for the choice of subscript label is that $\mathbf{v}_{\perp}$ is the part of $\mathbf{v}$ which is orthogonal to the null-space of the matrix B, whereas $\mathbf{v}_{\|}$ is the part that lies within the null-space.

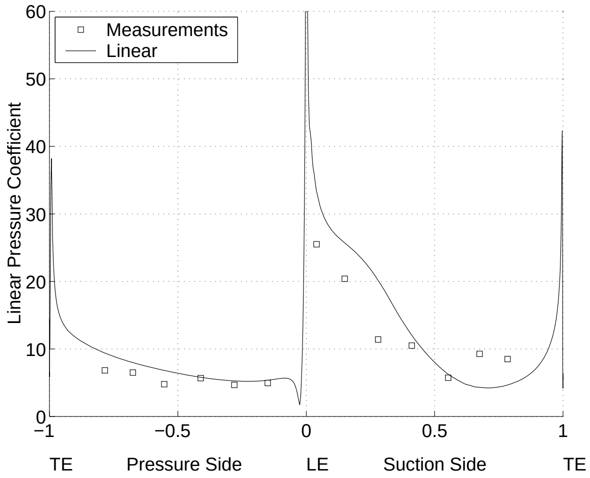

Fig. 3: First harmonic pressure variation for the 11th Standard Configuration

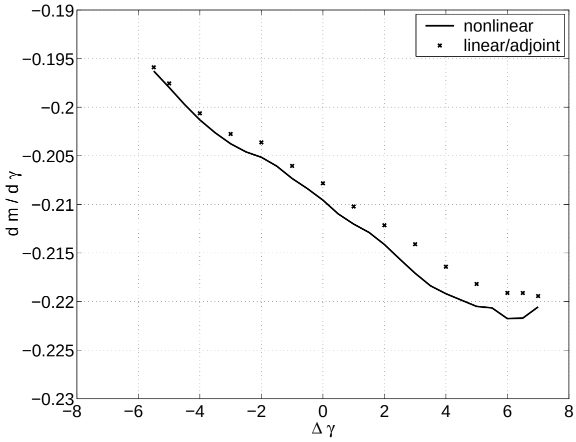

Fig. 4: Non-linear versus linear mass flow sensitivities for variation of the stagger angle γ of a 2D turbine section

approach, as opposed to the continuous adjoint approach in which one discretises the adjoint partial differential equation.

The linear code has itself been validated at a subroutine level by comparison with the subroutines in the nonlinear code.[6,7] In addition it has been checked using a range of testcases, starting with simple model problems such as inviscid flow over 2D flat plate cascades for which there is an analytic solution.[18] Figure 2 presents results for the unsteady interaction due to incoming wakes from an upstream blade row. Validation of the viscous capabilities is based on benchmark experimental testcases, such as the 11th Standard Configuration.[4] Figure 3 shows that the amplitude of the linear pressure coefficient variation agrees well with the measurements.

Example applications

In order to illustrate the efficiency of the adjoint approach, the adjoint algorithm is applied here to a steady and an unsteady example. In the first case, the linear turbine cascade shown in figure 5 with an exit Mach number of about 0.7 is used to determine the sensitivities of the outlet mass flow to the rotation of selected blade-to-blade sections around the Leading Edge (LE), that is to variations of the stagger angle γ. Positive increments $\Delta\gamma$ lead to higher angles between the blade chord and the axial direction.

Figure 4 shows the comparison between the nonlinear and the linear/adjoint sensitivities of the mass flow $\dot{m}$ for variations of the stagger angle between -5^o and 7^o. The curve of the non-linear sensitivity is made of 26 equally spaced points, each referring to a different blade and the derivative is computed with centred differences on intervals of 1^o. There is a very good agreement in the interval between -5^o and -3^o. For higher incidences, the agreement worsens slightly, but the average difference remains within about 1%. The non-linear and linear sensitivities resulting from the perturbation of the base geometry are given in the first row of table 1. For $\Delta\gamma > 5^o$, In the 3D case, we chose to perturb the section at midspan and at 20 % blade height and the last two rows of table 1 show the sensitivities for these two design parameters. Surprisingly the agreement between the nonlinear and the linear/adjoint sensitivities is better near the end wall, where one would expect more non-linearities due to 3D viscous effects, than at midspan, where the flow is cleaner. Further investigation is required on the origin of this difference. We emphasise, however, that a single calculation is required to determine the two sensitivities with the adjoint approach, whereas two non-linear or linear calculations are needed with the finite-difference approach.

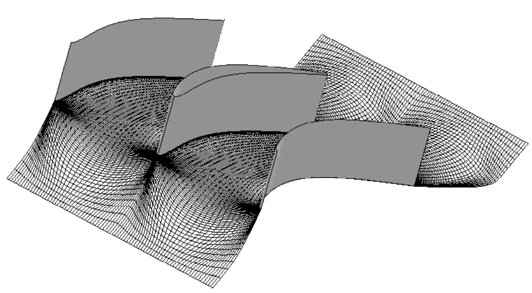

Fig. 5: 3D linear turbine cascade

	nonlin.	lin/adj
2D	0.210026	0.20783
3D-mds	2.3974E-3	2.5639E-3
3D-edw	1.8373E-3	1.8336E-3

Table 1: Non-linear versus linear mass flow sensitivities for a rotation of the turbine airfoil of 0.2^0 around the LE

The unsteady application consists of a high pressure turbine rotor subject to unsteady aerodynamic forces caused by incident wakes from an upstream row of blades. This has been previously analysed[16] and good agreement shown in the forced response predicted by linear uncoupled and nonlinear coupled methods.

The design task is to investigate the dependence of the forced vibration upon the shape of the incoming wakes. In practice, it is very difficult to significantly reduce the velocity defect in the wakes, but by changing the three-dimensional shape of the upstream blades (e.g. by moving the tip section of the blade in the circumferential direction while keeping the hub section fixed, a process known as *re-stacking*) it is possible to alter the time at which the wake shed by the tip section hits the rotor blade row, relative to that shed from the hub section. Physically, a wake hitting the blade at the same time at different radial sections will usually produce the maximum structural response, whereas allowing for time delays there may occur a phase cancellation between the forces at different radial locations leading to a reduced response.

Mathematically, the effect of re-stacking is contained in the worksum calculation. The adjoint analysis can be used to determine the worksum values corresponding to a set of different inflow wake boundary conditions in order to identify a minimum response. In this example, these boundary conditions come from the same baseline corresponding to the current design of the upstream blades and the difference between them is a complex phase shift which is defined to vary linearly with radius from zero at the hub to a maximum value at the tip. This corresponds to a linear re-stacking, leaning the entire blade in the circumferential direction.

Figure 6 shows the magnitude of the worksum corresponding to the primary torsional mode computed as a function of the maximum phase shift due to re-stacking. It indicates that within the range being considered, which is thought to be appropriate, the greater the magnitude of the phase shift, the greater the degree of phase cancellation between different parts of the blade and hence the smaller the worksum.

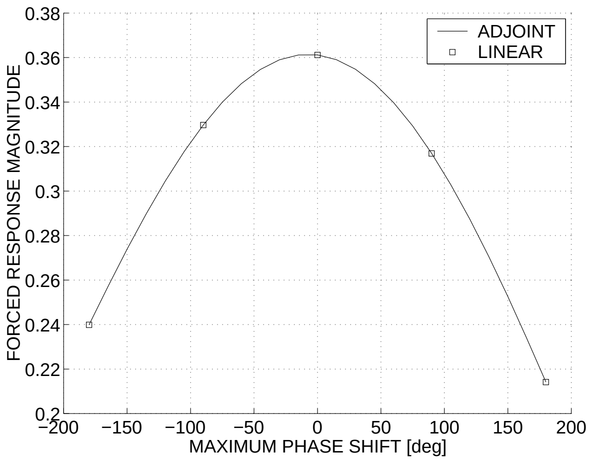

Fig. 6: Forced response magnitude versus maximum re-stacking phase shift.

The results for the full range of phase shifts were obtained from a single adjoint calculation. If the standard linear harmonic approach were used instead, each result would require a separate linear calculation since it corresponds to a different set of inflow boundary conditions. As a check, linear calculations have been performed for a variety of points and they produced identical values for the worksum output.

Conclusions

This paper has presented the application of the adjoint method for the steady and unsteady design of turbomachinery blades. The latter one is thought to be the first application of adjoint methods to the linearised analysis of periodic unsteady flows. Application of the adjoint technique to a steady design problem, consisting in determining the gradient of a selected functional to variations of geometric design variables and to an unsteady one, involving the tailoring of incoming wakes to reduce the level of forced response blade vibrations have shown the effectiveness of the approach. The capability of determining the gradient of a scalar object function depending on many design parameters with a single calculation has a big potential for application to the design practice in the turbomachinery industry.

The development of the presented adjoint methods has also involved advances in the methodology for fully-discrete adjoint methods. This includes the treatment of strong wall boundary conditions for node-based discretisations; adjoint iteration methods giving exactly the same iterative convergence as the corresponding linear method; and techniques for the validation of the adjoint solver by checking its exact equivalence to the linear solver.

The future work includes further validation work on the comparison between non-linear and linear/adjoint sensitivities for turbomachinery steady design and the application of the harmonic adjoint method to flutter prediction for the design of blades with improved flutter margins.

Acknowledgements

This research has been supported by the Engineering and Physical Sciences Research Council under grant GR/L95700, and by Rolls-Royce plc (technical monitor: Leigh Lapworth) DERA (technical monitor: John Calvert), and BAESystems plc (technical monitor: David Standingford). We also acknowledge the contributions of P. Moinier, J.-D. Müller, N.A. Pierce, L. Lapworth and M. West to the development of the HYDRA suite of nonlinear, linear and adjoint codes. Prof. Andrea Arnone of the Department of Energy Engineering, University of Florence (IT), is also kindly acknowledged for providing the grid generation software.

References

[1]W.S. Clark and K.C. Hall. A time-linearized Navier-Stokes analysis of stall flutter. *J. Turbomachinery*, 122(3):467–476, 2000.

[2]M. Duta. *(in preparation)*. PhD thesis, Oxford University, 2001.

[3]M. Duta, M.B. Giles, and M.S. Campobasso. The harmonic adjoint approach to unsteady turbomachinery design. ICFD Conference, Oxford, 2001.

[4]T.H. Fransson, M. Joecker, A. Bolcs, and P. Ott. Viscous and inviscid linear/nonlinear calculations versus quasi three-dimensional experimental cascade data for a new aeroelastic turbine standard configuration. *J. Turbomachinery*, 121:717–725, 1999.

[5]M.B. Giles. On the use of Runge-Kutta time-marching and multigrid for the solution of steady adjoint equations. Technical Report NA00/10, Oxford University Computing Laboratory, 2000. See `www.comlab.ox.ac.uk/oucl/work/mike.giles/`.

[6]M.B. Giles, M.C. Duta, and J.-D. Müller. Adjoint code developments using the exact discrete approach. AIAA Paper 2001-2596, 2001.

[7]M.B. Giles and N.A. Pierce. An introduction to the adjoint approach to design. *Flow, Turbulence and Control*, to appear, 2001.

[8]K.C. Hall. Deforming grid variational principle for unsteady small disturbance flows in cascades. *AIAA J.*, 31(5):891–900, 1993.

[9]D. Hoyniak and W.S. Clark. Aerodynamic damping predictions using a linearized Navier-Stokes analysis. ASME Paper 99-GT-207, 1999.

[10]A. Jameson. Optimum aerodynamic design using control theory. In M. Hafez and K. Oshima, editors, *Computational Fluid Dynamics Review 1995*, pages 495–528. John Wiley & Sons, 1995.

[11]A. Jameson. Re-engineering the design process through computation. *J. Aircraft*, 36:36–50, 1999.

[12]J.G. Marshall and M. Imregun. A review of aeroelasticity methods with emphasis on turbomachinery applications. *Journal of Fluids and Structures*, 10:237–267, 1996.

[13]P. Moinier. *Algorithm developments for an unstructured viscous flow solver*. PhD thesis, Oxford University, 1999.

[14]P. Moinier, J.-D. Müller, and M.B. Giles. Edge-based multigrid and preconditioning for hybrid grids. AIAA Paper 99-3339, 1999.

[15]S. Shahrokh. A comparative study of optimisation methods for aerodynamic design of turbomachinery blades. ASME Paper 2000-GT-523, 2000.

[16]M. Vahdati, J. Green, J.G. Marshall, and M. Imregun. Forced response predictions for a HP turbine rotor blade. In *Applied Vehicle Technology Panel Symposium on "Design Principles and Methods for Aircraft Gas Turbine Engines"*, Toulouse, France, May 1998. Research and Technology Organisation.

[17]J.M. Verdon and J.R. Caspar. A linearized unsteady aerodynamic analysis for transonic cascades. *J. Fluid Mech.*, 149:403–429, 1984.

[18]D. S. Whitehead. Classic two-dimensional methods. In M. Platzer and F. O. Carta, editors, *Aeroelasticity in Axial-Flow Turbomachines, AG-298*, volume 1. AGARD, 1987.